

PRODUCT REPRESENTATION OF PERFECT CUBES

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ABSTRACT. Let us define $F_{k,d}(n)$ to be the maximal size of a set $A \subseteq [n]$ such that the equation

$$a_1 a_2 \dots a_k = x^d, \quad a_1 < a_2 < \dots < a_k$$

has no solution with $a_1, a_2, \dots, a_k \in A$ and integer x . Erdős, Sárközy and T. Sós studied $F_{k,2}$, and gave bounds for special cases $k = 2, 3, 4, 6$ and also for the general case. We study the problem in the case $d = 3$, and provide bounds for $k = 2, 3, 4, 6$ and 9. We also introduce another function $f_{k,d}$ closely related to $F_{k,d}$: While the original problem requires $a_1, \dots, a_k$ to all be distinct, we can relax this and only require that the multiset of the a_i 's can not be partitioned into d -tuples where each d -tuple consists of d copies of the same number. Accounting for these trivial solutions, this leads to an analogous definition for $f_{k,d}$.

1. INTRODUCTION

The problem of the solvability of equations of the form

$$a_1 a_2 \dots a_k = x^2, \quad a_1 < a_2 < \dots < a_k$$

in a set $A \subseteq [n]$ first appeared in a 1995 paper of Erdős, Sárközy and T. Sós [7]. They investigated the maximal size of a set A such that the equation cannot be solved in A , that is, there are no $a_1, \dots, a_k \in A$ whose product is a perfect square. This motivates the following definitions:

Let $F_{k,d}(n)$ be the maximal size of a set $A \subseteq [n]$ such that

$$a_1 a_2 \dots a_k = x^d, \quad a_1 < a_2 < \dots < a_k$$

has no solution with $a_1, a_2, \dots, a_k \in A$ and integer x . Similarly, let $f_{k,d}(n)$ be the maximal size of a set $A \subseteq [n]$ such that

$$a_1 a_2 \dots a_k = x^d$$

has no solution with $a_1, a_2, \dots, a_k \in A$ and integer x , except for *trivial* solutions that we specify below. In this case, some trivial solutions do arise: it is clear, for instance, that $a_1 = \dots = a_d$ will yield a trivial solution to the equation $a_1 \dots a_d = x^d$. Let us call a solution trivial if the multiset of the a_i 's can be partitioned into d -tuples where each d -tuple consists of d copies of the same number: see for example $(a_1 a_1 a_1)(a_2 a_2 a_2)(a_3 a_3 a_3) = x^3$ for $k = 9, d = 3$. Note that trivial solutions arise only if $d \mid k$. Clearly, $f_{k,d} \leq F_{k,d}$.

With our notation, Erdős, Sárközy and T. Sós [7] proved that $F_{2k+1,2}(n) \gg n$ for all k and $F_{2k,2}(n) = o(n)$ for $k \geq 2$. They also showed that $F_{2,2}(n)$ is equal to the number of squarefree integers not exceeding n , and gave bounds for $k = 3, 4, 6$. Based on their work, Verstraëte [13] studied a similar problem: He aimed to find the maximal size of a set $A \subseteq [n]$ such that no product of k distinct elements in A is in the value set of a given polynomial $f \in \mathbb{Z}[x]$. He showed that for a certain class of polynomials, the answer is $\Theta(n)$, for another class it is $\Theta(\pi(n))$, and conjectured that these are the only two possibilities. For further related results, see [9, 11] and Terence Tao's recent paper [12].

We investigated the original problem in the case $d = 3$, and we provided bounds on both $F_{k,3}$ and $f_{k,3}$ for the values of $k = 2, 3, 4, 6, 9$. We summarize our results below.

For $k = 2$, the following bounds hold:

Theorem 1. *There exist positive constants c_1 and c_2 such that*

$$c_1 n^{2/3} < n - F_{2,3}(n) \leq n - f_{2,3}(n) < c_2 n^{2/3}.$$

For the case $k = 3$ we prove that $f_{3,3}(n)/n$ converges to $c_{3,3} \in (0, 1)$ that we can approximate (theoretically to arbitrary precision):

Theorem 2. *There exists a constant $0.618 \leq c_{3,3} \leq 0.651$ such that*

$$f_{3,3}(n) = (c_{3,3} + o(1))n.$$

An analogous result holds for $F_{3,3}(n)$ as well:

Theorem 3. *There exists a constant $0.6894 \leq C_{3,3} \leq 0.7258$ such that*

$$F_{3,3}(n) = (C_{3,3} + o(1))n.$$

In the case $k = 4$ we show that for large n , the following bounds hold. Our proofs generalize and extend ideas from [7], $k = 3$.

Theorem 4. *Let $\varepsilon > 0$. There exists some $n_0(\varepsilon)$ such that for every $n \geq n_0(\varepsilon)$ we have*

$$\frac{n}{(\log n)^{2+\varepsilon}} < n - F_{4,3}(n) \leq n - f_{4,3}(n) < \frac{n}{(\log n)^{1-\frac{\varepsilon \log 3}{2\sqrt{3}}-\varepsilon}}.$$

For $k = 6$ we obtained the following results:

Theorem 5. *There exist positive constants c_1 and c_2 such that*

$$c_1 \frac{n^{3/4}}{(\log n)^{3/2}} < f_{6,3}(n) - \pi(n) < c_2 \frac{n^{3/4}}{(\log n)^{3/2}}.$$

Theorem 6. *There exist positive constants c_1 and c_2 such that*

$$c_1 \frac{n \log \log n}{\log n} < F_{6,3}(n) < c_2 \frac{n \log \log n}{\log n}.$$

However, since in case of the last two results all the key ideas came from our supervisors, we decided to only mention Theorem 5 and Theorem 6, and omit the proofs in this thesis. Note that Theorem 6 refutes the above-mentioned conjecture of Verstraëte [13].

Theorem 7. *There exist positive constants c_1 and c_2 such that for every sufficiently large n*

$$c_1 \frac{n^{2/3}}{(\log n)^{4/3}} < f_{9,3}(n) - \pi(n) < c_2 n^{5/6}.$$

Theorem 8. *There exist positive constants c_1 and c_2 such that for every sufficiently large n*

$$c_1 \frac{n^{5/6}}{(\log n)^{5/3}} < F_{9,3}(n) - \left(\pi(n) + \pi\left(\frac{n}{2}\right) \right) < c_2 n^{5/6}.$$

Notations. Throughout this thesis, we denote by $[n]$ the set $\{1, 2, \dots, n\}$. The standard notation $\ll, \gg$ and O is applied to positive quantities in the usual way. That is, $X \gg Y$, $Y \ll X$, and $Y = O(X)$ all mean that $X \geq cY$, for some absolute constant $c > 0$. If the constant c depends on a quantity t , we write $X \ll_t Y$, $Y = O_t(X)$. Analogously to squarefree numbers, we call an integer a cubefree if there is no integer $b > 1$ such that $b^3 \mid a$. The cubefree part of an integer a is a/b^3 , where b^3 is the largest perfect cube dividing a . In this thesis by the interval $[a, b]$ we mean only the integers in $[a, b]$. We denote by $\Omega(m)$ the total number of prime factors of m with multiplicity.

2. CASE $k = 2$

Proof of Theorem 1. First, we will show that $F_{2,3}(n) = f_{2,3}(n) + 1$.

If $A \subseteq [n]$ is a set such that there is no solution for $a_1 a_2 = x^3$ in A with $a_i \neq a_j$, then there is at most one perfect cube in A . If we omit this cube from the set, then there will be no solution for $a_1 a_2 = x^3$ (where a_i, a_j are not necessarily different): If there was a solution, then it would have to be of the form $a_1^2 = x^3$, so a_1 would be a cube. Hence $F_{2,3}(n) - 1 \leq f_{2,3}(n)$. Vice versa, if $A \subseteq [n]$ is such that there is no solution for $a_1 a_2 = x^3$ in A (with a_i, a_j not necessarily different), then $1 \notin A$, and there is no solution for $a_1 a_2 = x^3$ in $A \cup \{1\}$ with $a_1 \neq a_2$. Hence $f_{2,3}(n) + 1 \leq F_{2,3}(n)$.

We will prove the theorem for $f_{2,3}(n)$. Let us consider a set $A \subseteq [n]$ of maximal size avoiding solutions to $a_1 a_2 = x^3$, then by definition, $|A| = f_{2,3}(n)$. We will estimate at least how many elements we have to leave out from $[n]$ to get such a set.

Suppose we have a solution $a_1 a_2 = x^3$. Let us write $a_1 = uv^2 w^3$, where u, v are squarefree and $(u, v) = 1$, so uv^2 is the cubefree part of a_1 . Notice that the cubefree part of a_2 must be vu^2 , thus $a_2 \in \{vu^2 t^3 : t \in \mathbb{Z}^+\}$. We will call a and b equivalent if their cubefree parts are the same. We will call a and b opposites if their product is a perfect cube. Clearly, if a number is in A , then there is no number in A from the opposite equivalence class. Therefore, A can be partitioned into equivalence classes, and it contains exactly one (the larger) class from every pair of opposite equivalence classes.

Hence, the minimum number of elements we have to leave out is the sum of the sizes of the smaller equivalence classes (from each opposite pair). The size of the class containing integers with cubefree part uv^2 is $|\{uv^2 t^3 : t \in \mathbb{Z}^+\} \cap [n]| = \left\lfloor \sqrt[3]{\frac{n}{uv^2}} \right\rfloor$, since $t \leq \sqrt[3]{\frac{n}{uv^2}}$ must hold. For a fixed pair u, v (satisfying $u < v$ and $\gcd(u, v) = 1$) we have the following pair of opposite equivalence classes: $\{uv^2 t^3 : t \in \mathbb{Z}^+\} \cap [n]$ and $\{vu^2 t^3 : t \in \mathbb{Z}^+\} \cap [n]$. Since $u < v$, the first class will be smaller (or of the exact same size), so the minimum number of elements we have to leave out is exactly

$$\sum_{\substack{1 \leq u < v \\ (u, v) = 1 \\ uv^2 \leq n \\ u, v \text{ squarefree}}} \left\lfloor \sqrt[3]{\frac{n}{uv^2}} \right\rfloor = n - f_{2,3}(n).$$

First we give an upper bound for $n - f_{2,3}(n)$ using this formula. Observe that

$$\sum_{\substack{1 \leq u < v \\ uv \text{ squarefree}}} \left\lfloor \sqrt[3]{\frac{n}{uv^2}} \right\rfloor = \sum_{\substack{1 \leq u < v \\ uv \text{ squarefree}}} \sum_{t=1}^{\left\lfloor \sqrt[3]{\frac{n}{uv^2}} \right\rfloor} 1 = \sum_{t=1}^{\left\lfloor \sqrt[3]{n} \right\rfloor} \sum_{\substack{1 \leq u < v \\ uv \text{ squarefree}, \\ uv^2 \leq \frac{n}{t^3}}} 1 < \sum_{t=1}^{\left\lfloor \sqrt[3]{n} \right\rfloor} \sum_{\substack{1 \leq u < v \\ uv^2 \leq \frac{n}{t^3}}} 1.$$

The inner sum can be written as

$$\sum_{\substack{1 \leq u < v, \\ uv^2 \leq \frac{n}{t^3}}} 1 = \sum_{u=1}^{\lfloor \frac{\sqrt[3]{n}}{t} \rfloor} \sum_{v=u+1}^{\lfloor \sqrt{\frac{n}{ut^3}} \rfloor} 1 \leq \sum_{u=1}^{\lfloor \frac{\sqrt[3]{n}}{t} \rfloor} \left(\sqrt{\frac{n}{ut^3}} - u \right).$$

As $\sqrt{\frac{n}{ut^3}} - u$ is monotonically decreasing in u , we get that

$$\sum_{u=1}^{\lfloor \frac{\sqrt[3]{n}}{t} \rfloor} \left(\sqrt{\frac{n}{ut^3}} - u \right) < \int_0^{\frac{\sqrt[3]{n}}{t}} \left(\sqrt{\frac{n}{ut^3}} - u \right) du = 1.5 \frac{n^{2/3}}{t^2}.$$

Hence

$$\sum_{t=1}^{\lfloor \sqrt[3]{n} \rfloor} \sum_{\substack{1 \leq u < v, \\ uv^2 \leq \frac{n}{t^3}}} 1 < \sum_{t=1}^{\lfloor \sqrt[3]{n} \rfloor} 1.5 \frac{n^{2/3}}{t^2} < 1.5 n^{2/3} \sum_{t=1}^{\infty} \frac{1}{t^2} = \frac{\pi^2}{4} n^{2/3},$$

which proves the upper bound with $c_2 = \frac{\pi^2}{4}$.

Now we provide a lower bound for the number of elements that we have to leave out.

We will construct a set A_n of disjoint pairs $\{y, z\}$ such that yz is a perfect cube. Then we have to leave out at least one element from each pair in A_n , hence $|A_n| \leq n - f_{2,3}(n)$.

Let

$$A_n = \{ \{a^2b, ab^2\} : a, b \in \mathbb{Z}^+ \text{ squarefree}, a \leq b < \sqrt[3]{n} \}.$$

Then $(a^2b)(ab^2)$ is indeed a perfect cube. The squarefree parts of ab^2 and a^2b are a and b respectively, which implies that the pairs are disjoint. It is well-known that the number of squarefree numbers less than m is $\left(\frac{6}{\pi^2} + o(1)\right)m$. Hence $|A_n| \geq \frac{1}{2} \left(\frac{6}{\pi^2} + o(1)\right)^2 n^{2/3} > 0.18n^{2/3}$, which proves the lower bound with $c_1 = 0.18$. □

3. CASE $k = 3$

Now, we prove Theorem 2 and Theorem 3 and give an approximation for the constants $c_{3,3}$ and $C_{3,3}$.

Proof of Theorem 2. Let us take a set $A \subseteq [n]$ such that the equation $a_1a_2a_3 = x^3$ has no solution with $a_1, a_2, a_3 \in A$ and integer x , except the trivial case $a_1 = a_2 = a_3$. Note that distinct elements of A have distinct cubefree parts, since $uv^3, uw^3 \in A$ with distinct v and w would yield the solution $a_1 = a_2 = uv^3, a_3 = uw^3$. Therefore, we can restrict our attention to sets of cubefree numbers, since each element of A may be replaced by its cubefree part. From now on, it is assumed that A contains only cubefree integers.

First, we give an upper bound for the size of A . Let r be a fixed positive integer and let p_i denote the i th prime. Each cubefree $a \in [n]$ can be written as

$$a = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r} a',$$

where $\alpha_1, \dots, \alpha_r \in \{0, 1, 2\}$ and a' is cubefree satisfying $\gcd(a', p_1 p_2 \dots p_r) = 1$. Here $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ is the p_r -smooth and a' is the p_{r+1} -rough part of the number a . Observe that the product of three integers is a perfect cube if and only if so are the product of their p_r -smooth parts and the product of

their p_{r+1} -rough parts. Specially, for a fixed a' there cannot be three elements in A with p_{r+1} -rough part a' such that the product of their p_r -smooth parts is a perfect cube. Note that the product of three p_r -smooth numbers is a cube if and only if the sum of their exponent vectors $(\alpha_1, \alpha_2, \dots, \alpha_r)$ add up to $(0, 0, \dots, 0)$ (calculating coordinate-wise modulo 3). (Alternatively, if we consider the exponent vectors as elements of $\mathbb{F}_3^r$, they form a nontrivial 3-term arithmetic progression.) Let $L_r(i)$ be the set of p_r -smooth cubefree integers up to i :

$$L_r(i) := \{p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r} : \alpha_1, \dots, \alpha_r \in \{0, 1, 2\}\} \cap [i],$$

and let $s_r(i)$ denote the largest possible size of a subset of $L_r(i)$ avoiding nontrivial solutions to $a_1 a_2 a_3 = x^3$. Note that $s_r(i)$ is the size of the largest 3AP-free subset of

$$\{(\alpha_1, \dots, \alpha_r) \in \{0, 1, 2\}^r : \alpha_1 \log p_1 + \dots + \alpha_r \log p_r \leq \log i\},$$

if we consider this set as a subset of $\mathbb{F}_3^r$. For brevity, we will say that $s_r(i)$ is the size of the largest 3AP-free subset of $L_r(i)$. Clearly, for every $i \geq p_1^2 \dots p_r^2$, we have $s_r(i) = s_r(p_1^2 \dots p_r^2)$ (which common value is $r_3(\mathbb{F}_3^r)$, the largest possible size of a 3AP-free subset of $\mathbb{F}_3^r$).

For a p_{r+1} -rough number a' the number of those elements of A that are of the form $a = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r} a'$ is at most $s_r(i)$, where $i = \lfloor n/a' \rfloor$. Let us partition $[n]$ into the finite union of intervals $I_i = (\frac{n}{i+1}, \frac{n}{i}]$ (where $1 \leq i \leq p_1^2 \dots p_r^2 - 1$) and $I_{p_1^2 \dots p_r^2} = [1, \frac{n}{p_1^2 \dots p_r^2}]$. Note that the number of p_{r+1} -rough elements of I_i is $|I_i| \cdot \prod_{j=1}^r \left(1 - \frac{1}{p_j}\right) + O_r(1)$, where $|I_i|$ denotes the size of I_i , since in each set of $p_1 p_2 \dots p_r$ consecutive integers the proportion of p_{r+1} -rough numbers is exactly $\prod_{j=1}^r \left(1 - \frac{1}{p_j}\right)$. Therefore,

$$|A| \leq \sum_{i=1}^{p_1^2 \dots p_r^2} |I_i| \cdot \prod_{j=1}^r \left(1 - \frac{1}{p_j}\right) s_r(i) + O_r(1).$$

Setting

$$\gamma_r := \left(\frac{s_r(p_1^2 \dots p_r^2)}{p_1^2 \dots p_r^2} + \sum_{i=1}^{p_1^2 \dots p_r^2 - 1} \frac{s_r(i)}{i(i+1)} \right) \prod_{j=1}^r \left(1 - \frac{1}{p_j}\right),$$

it is obtained that $|A| \leq \gamma_r n + O_r(1)$, consequently,

$$(1) \quad \limsup_{n \rightarrow \infty} \frac{f_{3,3}(n)}{n} \leq \inf\{\gamma_r : r \geq 1\}.$$

In the previous upper bound we used only that there is no nontrivial solution to $a_1 a_2 a_3 = x^3$, where the elements $a_1, a_2, a_3 \in A$ have the *same* p_{r+1} -rough part. However, if we consider only squarefree numbers as possible p_{r+1} -rough parts a' , then we will get a suitable set, as among the squarefree numbers $a_1 a_2 a_3 = x^3$ has only the trivial solutions, where $a_1 = a_2 = a_3$. Indeed, in the prime factorisation of a product of three squarefree numbers each exponent is 1, 2, or 3, so to get a perfect cube the three numbers must have exactly the same prime factors, thus they coincide. Let us define A in the following way. For each p_{r+1} -rough squarefree $a' \leq n$ choose $s_r(\lfloor n/a' \rfloor)$ elements $t_1, t_2, \dots, t_{s_r(\lfloor n/a' \rfloor)}$ of $L_r(\lfloor n/a' \rfloor)$ in such a way that they form a 3AP-free subset of $L_r(\lfloor n/a' \rfloor)$, and put $a' t_1, \dots, a' t_{s_r(\lfloor n/a' \rfloor)}$ into A .

Now, assume that $a_1 a_2 a_3$ is a perfect cube for some $a_1, a_2, a_3 \in A$. Then the product of the p_{r+1} -rough parts of a_1, a_2, a_3 is also a perfect cube, but they are all squarefree, so the p_{r+1} -rough parts of a_1, a_2, a_3 must be the same number, say a' . The product of the p_r -smooth parts is also a

perfect cube, but multiples of a' were added to A in such a way that a three-factor product is a cube only if the p_{r+1} -smooth parts are also the same, thus $a_1 = a_2 = a_3$. Hence, A satisfies the required property.

To estimate the size of A we need bounds for the number of squarefree p_{r+1} -rough numbers in the intervals I_i . The p_{r+1} -rough numbers can be partitioned into residue classes modulo $p_1 \dots p_r$. By an old result of Prachar [10] we get that

$$\begin{aligned} |\{a' \in I_i : \gcd(a', p_1 \dots p_r) = 1 \text{ and } a' \text{ is squarefree}\}| &= \\ &= |I_i| \cdot \prod_{j=1}^r \left(1 - \frac{1}{p_j}\right) \prod_{j>r} \left(1 - \frac{1}{p_j^2}\right) + O_r(\sqrt{n}). \end{aligned}$$

Therefore,

$$|A| \geq \sum_{i=1}^{p_1^2 \dots p_r^2} |I_i| \cdot \prod_{j=1}^r \left(1 - \frac{1}{p_j}\right) \prod_{j>r} \left(1 - \frac{1}{p_j^2}\right) s_r(i) - O_r(\sqrt{n}).$$

Setting

$$(2) \quad \beta_r := \gamma_r \cdot \prod_{j>r} \left(1 - \frac{1}{p_j^2}\right),$$

it is obtained that $|A| \geq \beta_r n - O_r(\sqrt{n})$, consequently,

$$\liminf_{n \rightarrow \infty} \frac{f_{3,3}(n)}{n} \geq \sup\{\beta_r : r \geq 1\}.$$

Since $\gamma_r/\beta_r \rightarrow 1$ (as $r \rightarrow \infty$), by comparing (1) and (2) we obtain that $f_{3,3}(n)/n$ converges to $\inf\{\gamma_r : r \geq 1\} = \sup\{\beta_r : r \geq 1\} =: c_{3,3}$. □

Numerically we obtained the bounds $0.618 \leq c_{3,3} \leq 0.65067$ by taking $r = 3$ and calculating by hand the values $s_3(i)$, which are given in the table below.

The interval of a'	$s_3(\lfloor n/a' \rfloor)$
$a' \in (n/2, n]$	1
$a' \in (n/3, n/2]$	2
$a' \in (n/5, n/3]$	3
$a' \in (n/6, n/5]$	4
$a' \in (n/10, n/6]$	5
$a' \in (n/15, n/10]$	6
$a' \in (n/25, n/15]$	7
$a' \in (n/30, n/25]$	8
$a' \leq n/30$	9

Proof of Theorem 3. A slight modification of the proof of Theorem 2 gives the analogous result for the function $F_{3,3}$, here we only highlight the differences. Note that here two elements of A might have the same cubefree part, since solutions like $a_1 = a_2 = uv^3, a_3 = uw^3$ are excluded here. However, *three* elements of A can not have the same cubefree part, since $uv^3, uw^3, uz^3 \in A$ with distinct v, w, z would yield the solution $a_1 = uv^3, a_2 = uw^3, a_3 = uz^3$.

Therefore, we may assume that each element of A is either cubefree, or a cubefree number multiplied by 8, which leads to the following modification of the definition of $s_r(i)$. Let $S_r(i)$ be the largest possible total weight of a 3AP-free subset of $L_r(i)$, where $s \in L_r(i)$ weights 1, if $s > \frac{i}{8}$, and weights 2, if $s \leq \frac{i}{8}$, since in this case $8s$ can also be chosen into A together with s .

Clearly, for every $i \geq 8p_1^2 \dots p_r^2$, we have $S_r(i) = S_r(p_1^2 \dots p_r^2)$. Now, partition $[n]$ into the finite union of intervals $I_i = (\frac{n}{i+1}, \frac{n}{i}]$, where $I_{8p_1^2 \dots p_r^2} = [1, \frac{n}{8p_1^2 \dots p_r^2}]$. Analogously to the proof of Theorem 2 we get that

$$|A| \leq \sum_{i=1}^{8p_1^2 \dots p_r^2} |I_i| \cdot \prod_{j=1}^r \left(1 - \frac{1}{p_j}\right) S_r(i) + O_r(1).$$

Setting

$$\Gamma_r := \left(\frac{S_r(8p_1^2 \dots p_r^2)}{8p_1^2 \dots p_r^2} + \sum_{i=1}^{8p_1^2 \dots p_r^2 - 1} \frac{S_r(i)}{i(i+1)} \right) \prod_{j=1}^r \left(1 - \frac{1}{p_j}\right),$$

it is obtained that $|A| \leq \Gamma_r n + O_r(1)$, consequently,

$$(3) \quad \limsup_{n \rightarrow \infty} \frac{F_{3,3}(n)}{n} \leq \inf\{\Gamma_r : r \geq 1\}.$$

Also, as a lower bound it is obtained that

$$|A| \geq \sum_{i=1}^{8p_1^2 \dots p_r^2} |I_i| \cdot \prod_{j=1}^r \left(1 - \frac{1}{p_j}\right) \prod_{j>r} \left(1 - \frac{1}{p_j^2}\right) S_r(i) - O_r(\sqrt{n}).$$

Setting

$$(4) \quad B_r := \Gamma_r \cdot \prod_{j>r} \left(1 - \frac{1}{p_j^2}\right),$$

it is obtained that $|A| \geq B_r n - O_r(\sqrt{n})$, consequently,

$$\liminf_{n \rightarrow \infty} \frac{F_{3,3}(n)}{n} \geq \sup\{B_r : r \geq 1\}.$$

Since $\Gamma_r/B_r \rightarrow 1$ (as $r \rightarrow \infty$), by comparing (3) and (4) we obtain that $F_{3,3}(n)/n$ converges to the limit $\inf\{\Gamma_r : r \geq 1\} = \sup\{B_r : r \geq 1\} =: C_{3,3}$. □

We approximated this constant as well, relying on the previous approximation. Note that when $a' > \frac{n}{8}$, nothing changes, but this is not necessarily true for $a' \leq \frac{n}{8}$. In the previous case, for example, we could choose 5 elements when $\frac{n}{10} \leq a' < \frac{n}{8}$, including a' . Now we can also include $8a'$ in the set, so the maximum weight will be 6. We checked every interval by computer to find the subset with the maximum weight and obtained the estimates $0.6894 \leq C_{3,3} \leq 0.7258$. (For the values of $S_3(i)$ we calculated see the table below.)

The interval of a'	$S_3(\lfloor n/a' \rfloor)$	The interval of a'	$S_3(\lfloor n/a' \rfloor)$
$a' \in (n/2, n]$	1	$a' \in (n/30, n/24]$	10
$a' \in (n/3, n/2]$	2	$a' \in (n/40, n/30]$	11
$a' \in (n/5, n/3]$	3	$a' \in (n/48, n/40]$	12
$a' \in (n/6, n/5]$	4	$a' \in (n/60, n/48]$	13
$a' \in (n/8, n/6]$	5	$a' \in (n/80, n/60]$	14
$a' \in (n/10, n/8]$	6	$a' \in (n/120, n/80]$	15
$a' \in (n/15, n/10]$	7	$a' \in (n/200, n/120]$	16
$a' \in (n/16, n/15]$	8	$a' \in (n/240, n/200]$	17
$a' \in (n/24, n/16]$	9	$a' \leq n/240$	18

4. CASE $k = 4$

We need a couple of lemmas to prove Theorem 4.

Lemma 9. *If $1 < y < 2$ and $\varepsilon > 0$, then for $x > x_1(\varepsilon)$ we have*

$$|\{m : m \leq x, \Omega(m) \geq y \log \log x\}| < \frac{x}{(\log x)^{1+y \log y - y - \varepsilon}}.$$

Proof. See Corollary 2 in [6]. □

Lemma 10. *If $0 < y \leq 1$ and $\varepsilon > 0$, then for $x > x_2(\varepsilon)$ we have*

$$\frac{x}{(\log x)^{1+y \log y - y + \varepsilon}} < |\{m : m \leq x, \Omega(m) \leq y \log \log x\}| < \frac{x}{(\log x)^{1+y \log y - y - \varepsilon}}$$

Proof. This follows from a result of Hardy and Ramanujan [8]. □

Lemma 11. *Suppose that $\frac{n}{\log n} \leq a_1, a_2, a_3, a_4 \leq n$, $d^2 | a_i$ implies $d \leq \log n$, and $a_1 a_2 a_3 a_4$ is a perfect cube. Then there are integers $\frac{\sqrt[3]{n}}{(\log n)^{14}} \leq u_i, v_i, w_i \leq \sqrt[3]{n} (\log n)^{14}$ such that $a_i = u_i v_i w_i$.*

Lemma 12. *Let $B \subseteq [n]$ be the set of those $b \in \mathbb{Z}^+$ such that b can be written in the form $b = uvw$ with $u, v, w \in \mathbb{Z}^+$ and $\frac{\sqrt[3]{n}}{(\log n)^{14}} \leq u, v, w \leq \sqrt[3]{n} (\log n)^{14}$. Then for all $\varepsilon > 0$ there exists $n_0(\varepsilon)$ such that*

$$|B| \leq \frac{n}{(\log n)^{1 - \frac{\varepsilon \log 3}{2\sqrt{3}} - \varepsilon}}$$

for $n \geq n_0(\varepsilon)$.

Proof of Lemma 11. Suppose the conditions of the lemma hold for a_1, a_2, a_3, a_4 , and let $a_i = b_i c_i^2 d_i^3$, where b_i and c_i are squarefree, so $x_i = b_i c_i^2$ is the cubefree part of a_i . Then

$$x_1 x_2 x_3 x_4 = \frac{a_1 a_2 a_3 a_4}{d_1^3 d_2^3 d_3^3 d_4^3} = y^3$$

is still a perfect cube.

If p is a prime that divides the right-hand side, the multiset of the exponents of p in the x_i 's, $\{v_p(x_i) : i = 1, 2, 3, 4\}$ is exactly one of the following:

- (1) $\{2, 2, 2, 0\}$,
- (2) $\{1, 1, 1, 0\}$,
- (3) $\{2, 2, 1, 1\}$,
- (4) $\{2, 1, 0, 0\}$.

This is simply due to the fact that the x_i 's were cubefree and that the right-hand side is a cube. For given x_1, x_2, x_3, x_4 , denote by P_i the set of all primes p such that p is of type (i). We will use this notation throughout the rest of this proof.

Let $t_1 = \gcd(x_2, x_3, x_4), \dots, t_4 = \gcd(x_1, x_2, x_3)$. Then the multiset $\{v_p(t_i) : i = 1, 2, 3, 4\}$ is as follows:

- (1) $\{2, 0, 0, 0\}$ if $p \in P_1$
- (2) $\{1, 0, 0, 0\}$ if $p \in P_2$
- (3) $\{1, 1, 1, 1\}$ if $p \in P_3$
- (4) $\{0, 0, 0, 0\}$ if $p \in P_4$.

Comparing this with $\{v_p(x_i) : i \in \{1, 2, 3, 4\}\}$, it follows that

$$t_1 t_2 t_3 t_4 = \sqrt[3]{x_1 x_2 x_3 x_4} \cdot \frac{\prod_{p \in P_3} p^2}{\prod_{p \in P_4} p}.$$

We would like to estimate t_1 in the form $\frac{t_1 t_2 t_3 t_4}{t_2 t_3 t_4}$, so let us now search for a similar formula for $t_2 t_3 t_4$. If a prime p divides $t_2 t_3 t_4$, then $p \mid x_1$.

Let us assume that $p^2 \mid x_1$. Then $p \notin P_2$, and a straightforward calculation yields that $v_p(t_2 t_3 t_4) = 2$ if $p \in P_1$, $v_p(t_2 t_3 t_4) = 3$ if $p \in P_3$, and $v_p(t_2 t_3 t_4) = 0$ if $p \in P_4$. On the other hand, suppose that $p^2 \nmid x_1$. Then $p \notin P_1$, and similarly $v_p(t_2 t_3 t_4) = 1$ if $p \in P_2$, $v_p(t_2 t_3 t_4) = 3$ if $p \in P_3$, and $v_p(t_2 t_3 t_4) = 0$ if $p \in P_4$. Hence we can conclude that

$$t_2 t_3 t_4 = \frac{\prod_{p \mid x_1} p \cdot \prod_{\substack{p^2 \mid x_1 \\ p \in P_1}} p \cdot \prod_{p \in P_3} p^2}{\prod_{\substack{p \mid x_1 \\ p \in P_4}} p}.$$

In summary,

$$t_1 = \frac{t_1 t_2 t_3 t_4}{t_2 t_3 t_4} = \frac{\sqrt[3]{x_1 x_2 x_3 x_4}}{\prod_{p \mid x_1} p \cdot \prod_{\substack{p^2 \mid x_1 \\ p \in P_1}} p \cdot \prod_{\substack{p \nmid x_1 \\ p \in P_4}} p}.$$

Now let us summarize the bounds we will use to estimate t_1 . By the assumptions of the lemma, $c_i^2, d_i^2 \mid a_i$ yields that $c_i, d_i \leq \log n$. This also implies

$$x_i = \frac{a_i}{d_i^3} \geq \frac{\frac{n}{\log n}}{(\log n)^3} = \frac{n}{(\log n)^4}$$

and

$$b_i = \frac{x_i}{c_i^2} \geq \frac{\frac{n}{(\log n)^4}}{(\log n)^2} = \frac{n}{(\log n)^6}.$$

For the product of certain types of primes in our formula above, we have the following bounds:

(1) If $p^2 \mid x_1$, then $p \mid c_1$. Since this holds for all such primes p , $\prod_{\substack{p^2 \mid x_1 \\ p \in P_1}} p \mid c_1$, and therefore

$$\prod_{\substack{p^2 \mid x_1 \\ p \in P_1}} p \leq c_1 \leq \log n.$$

(2) If $p \in P_4$ and $p \nmid x_1$, then there is an index $i_p \in \{2, 3, 4\}$ such that $p^2 \mid x_{i_p}$ and thus $p \mid c_{i_p}$. Hence $\prod_{\substack{p \nmid x_1 \\ p \in P_4}} p \mid c_2 c_3 c_4$, so

$$\prod_{\substack{p \nmid x_1 \\ p \in P_4}} p \leq (\log n)^3.$$

(3) Clearly,

$$b_1 \leq \prod_{p \mid x_1} p = b_1 c_1 \leq n.$$

We can now give bounds on t_1 . First,

$$t_1 \leq \frac{n^{4/3}}{b_1 \cdot 1 \cdot 1} \leq \frac{n^{4/3}}{\frac{n}{(\log n)^6}} = \sqrt[3]{n} (\log n)^6.$$

On the other hand,

$$t_1 \geq \frac{\left(\frac{n}{(\log n)^4}\right)^{4/3}}{n \cdot \log n \cdot (\log n)^3} = \frac{\sqrt[3]{n}}{(\log n)^{28/3}}.$$

The same bounds clearly hold for t_2, t_3 and t_4 , as well.

Then the decomposition of a_1 is constructed as follows:

$$\begin{aligned} a_1 &= b_1 c_1^2 d_1^3 = x_1 d_1^3 = \prod_{p \mid x_1} p \cdot \prod_{p^2 \mid x_1} p \cdot d_1^3 = \prod_{p \mid x_1} p \cdot \prod_{\substack{p^2 \mid x_1 \\ p \in P_1}} p \cdot \prod_{\substack{p^2 \mid x_1 \\ p \in P_3}} p \cdot \prod_{\substack{p^2 \mid x_1 \\ p \in P_4}} p \cdot d_1^3 = \\ &= t_2 t_3 t_4 d_1^3 \cdot \frac{\prod_{\substack{p \mid x_1 \\ p \in P_4}} p \cdot \prod_{\substack{p^2 \mid x_1 \\ p \in P_4}} p}{\prod_{\substack{p \in P_3}} p \cdot \prod_{\substack{p^2 \nmid x_1 \\ p \in P_3}} p} = \left(\frac{t_2}{\prod_{p \in P_3} p} \prod_{\substack{p \mid x_1 \\ p \in P_4}} p \right) \left(\frac{t_3}{\prod_{\substack{p^2 \nmid x_1 \\ p \in P_3}} p} \prod_{\substack{p^2 \mid x_1 \\ p \in P_4}} p \right) (t_4 d_1^3) \end{aligned}$$

Let $u_1 = \frac{t_2}{\prod_{p \in P_3} p} \prod_{\substack{p \mid x_1 \\ p \in P_4}} p$, $v_1 = \frac{t_3}{\prod_{\substack{p^2 \nmid x_1 \\ p \in P_3}} p} \prod_{\substack{p^2 \mid x_1 \\ p \in P_4}} p$ and $w_1 = t_4 d_1^3$. Then u_1, v_1, w_1 are integers, and we show

that they satisfy the bounds in the statement of the lemma. We use the following bounds, whose proofs are analogous to those of the bounds in the estimation of t_1 :

- (1) $\prod_{p \in P_3} p \leq (\log n)^4$
- (2) $\prod_{\substack{p \mid x_1 \\ p \in P_4}} p \leq (\log n)^4$
- (3) $\prod_{\substack{p^2 \nmid x_1 \\ p \in P_3}} p \leq (\log n)^3$

$$(4) \prod_{\substack{p^2 | x_1 \\ p \in P_4}} p \leq \log n$$

Then

$$\begin{aligned} u_1 &\leq \frac{\sqrt[3]{n}(\log n)^6}{1} \cdot (\log n)^4 = \sqrt[3]{n}(\log n)^{10}, \\ u_1 &\geq \frac{\frac{\sqrt[3]{n}}{(\log n)^{28/3}}}{(\log n)^4} \cdot 1 = \frac{\sqrt[3]{n}}{(\log n)^{40/3}}, \\ v_1 &\leq \frac{\sqrt[3]{n}(\log n)^6}{1} \cdot \log n = \sqrt[3]{n}(\log n)^7, \\ v_1 &\geq \frac{\frac{\sqrt[3]{n}}{(\log n)^{28/3}}}{(\log n)^3} \cdot 1 = \frac{\sqrt[3]{n}}{(\log n)^{37/3}}, \\ w_1 &\leq \sqrt[3]{n}(\log n)^6(\log n)^3 = \sqrt[3]{n}(\log n)^9, \\ w_1 &\geq \frac{\sqrt[3]{n}}{(\log n)^{28/3}}, \end{aligned}$$

which completes the proof. $\square$

Proof of Lemma 12. Fix $\varepsilon > 0$. Let B_1 the set of all $b = uvw \in B$ such that

$$\min\{\Omega(u), \Omega(v), \Omega(w)\} > \frac{e}{3^{1.5}} \log \log n,$$

and let B_2 be the set of all $b = uvw \in B$ such that

$$\Omega(w) \leq \frac{e}{3^{1.5}} \log \log n.$$

Then $B \subseteq B_1 \cup B_2$, so $|B| \leq |B_1| + |B_2|$.

Let $n \geq n_1(\varepsilon)$ as per Lemma 9. Now if $b \in B_1$, then

$$\Omega(b) = \Omega(u) + \Omega(v) + \Omega(w) > \frac{e}{\sqrt{3}} \log \log n,$$

so it follows from Lemma 9 that

$$|B_1| \leq \frac{n}{(\log n)^{1 + \frac{e}{\sqrt{3}} \log \frac{e}{\sqrt{3}} - \frac{e}{\sqrt{3}} - \varepsilon}} = \frac{n}{(\log n)^{1 - \frac{e \log 3}{2\sqrt{3}} - \varepsilon}}.$$

For the estimation of the size of B_2 , let

$$S_n = \left\{ (u, v) : \frac{\sqrt[3]{n}}{(\log n)^{14}} \leq u, v \leq \sqrt[3]{n}(\log n)^{14} \right\}$$

and for $(u, v) \in S_n$

$$W_{u,v} = \left\{ w : q \in \mathbb{Z}^+, w \leq \frac{n}{uv}, \Omega(w) \leq \frac{e}{3^{1.5}} \log \log n \right\},$$

$$W_{u,v}^* = \left\{ w : q \in \mathbb{Z}^+, w \leq \frac{n}{uv}, \Omega(w) \leq \left(\frac{e}{3^{1.5}} + \frac{1}{\log \log n} \right) \log \log \frac{n}{uv} \right\}.$$

Then if $(u, v) \in S_n$,

$$u, v = n^{\frac{1}{3} + o(1)} \implies \frac{n}{uv} = n^{\frac{1}{3} + o(1)}$$

and therefore

$$\log \log n - \log \log \frac{n}{uv} = \log 3 + o(1),$$

so there is N such that $W_{u,v} \subseteq W_{u,v}^*$ for all $(u,v) \in S_n$ whenever $n \geq N$. Indeed,

$$(*) \quad \frac{e}{3^{1.5}} \log \log n = \frac{e}{3^{1.5}} \log \log \frac{n}{uv} + \frac{e}{3^{1.5}} \left(\log \log n - \log \log \frac{n}{uv} \right)$$

where the second term tends to $\frac{e}{3^{1.5}} \log 3$ when $n \rightarrow \infty$. Specifically for $\delta = \frac{1}{2}$,

$$(*) \leq \frac{e}{3^{1.5}} \log \log \frac{n}{uv} + \frac{e}{3^{1.5}} (\log 3 + \delta)$$

for n large enough. Then

$$\frac{\log \log \frac{n}{uv}}{\log \log n} = 1 - \frac{\log \log n - \log \log \frac{n}{uv}}{\log \log n} \geq 1 - \frac{\log 3 + \delta}{\log \log n}.$$

Here the right-hand side tends to 1 as $n \rightarrow \infty$, so for large enough n ,

$$1 - \frac{\log 3 + \delta}{\log \log n} \geq \frac{e}{3^{1.5}} (\log 3 + \delta)$$

which was what we wanted. Hence

$$|B_2| \leq \sum_{(u,v) \in S_n} |W_{u,v}| \leq \sum_{(u,v) \in S_n} |W_{u,v}^*|$$

for $n \geq N$.

From Lemma 10, it follows that

$$|W_{u,v}^*| \leq \frac{\frac{n}{uv}}{\left(\log \frac{n}{uv}\right)^{1 + \frac{e}{3^{1.5}} \log \frac{e}{3^{1.5}} - \frac{e}{3^{1.5}} - \frac{\varepsilon}{3}}} = \frac{\frac{n}{uv}}{\left(\log \frac{n}{uv}\right)^{1 - \frac{e \log 3}{2\sqrt{3}} - \frac{\varepsilon}{3}}}$$

for $n \geq n_2(\varepsilon)$. Hence

$$\begin{aligned} |B_2| &\leq \sum_{(u,v) \in S_n} |W_{u,v}^*| \leq \sum_{(u,v) \in S_n} \frac{\frac{n}{uv}}{\left(\log \frac{n}{uv}\right)^{1 - \frac{e \log 3}{2\sqrt{3}} - \frac{\varepsilon}{3}}} \\ &\ll \frac{n}{(\log n)^{1 - \frac{e \log 3}{2\sqrt{3}} - \frac{\varepsilon}{2}}} \sum_{(u,v) \in S_n} \frac{1}{uv} \\ &\leq \frac{n}{(\log n)^{1 - \frac{e \log 3}{2\sqrt{3}} - \frac{\varepsilon}{2}}} \left(\sum_{\frac{\sqrt[3]{n}}{(\log n)^{14}} \leq m \leq \sqrt[3]{n} (\log n)^{14}} \frac{1}{m} \right)^2 \\ &\leq \frac{n}{(\log n)^{1 - \frac{e \log 3}{2\sqrt{3}} - \frac{\varepsilon}{2}}} (\log \log n)^2 \ll \frac{n}{(\log n)^{1 - \frac{e \log 3}{2\sqrt{3}} - \varepsilon}} \end{aligned}$$

□

Proof of Theorem 4. First we prove the lower bound.

Let $A \subseteq [n]$ be a subset such that $a_1 a_2 a_3 a_4 \neq x^3$ if $a_i \in A$, $a_1 < a_2 < a_3 < a_4$ and let $D = \{d_1, \dots, d_t\}$ be the set of all $d \in \mathbb{Z}^+$ such that $d \leq n^{1/3}$ and $\Omega(d) \leq \frac{1}{3} \log \log n$. Then by Lemma 10,

$$\begin{aligned} t = |D| &\geq |\{d : d \leq n^{1/3}, \Omega(d) \leq \frac{1}{3} \log \log n^{1/3}\}| > \\ &> \frac{n^{1/3}}{(\frac{1}{3} \log n)^{1 + \frac{1}{3} \log \frac{1}{3} - \frac{1}{3} + \frac{\epsilon}{3}}} > \frac{n^{1/3}}{(\log n)^{1 + \frac{1}{3} \log \frac{1}{3} - \frac{1}{3} + \frac{\epsilon}{3}}} \end{aligned}$$

if $n \geq n_2(\epsilon)$.

Let H be the 3-uniform hypergraph on vertices $P_1, \dots, P_t$ such that $\{P_i, P_j, P_k\}$ is an edge in H if and only if $d_i d_j d_k \in A$. Let M be the set of all $m \in [n]$ such that $m \notin A$ and $m = d_i d_j d_k$ for some $1 \leq i < j < k \leq t$. Then $|A| \leq n - |M|$.

For a fixed $m \in M$ let $h(m)$ denote the number of triples (d_i, d_j, d_k) such that $m = d_i d_j d_k$, $1 \leq i < j < k \leq t$. If $m = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r} \in M$, then

$$\Omega(m) = \Omega(d_i) + \Omega(d_j) + \Omega(d_k) \leq \log \log n,$$

hence

$$h(m) \leq \tau_3(m) = \prod_{i=1}^r \binom{k_i + 2}{2} \leq \prod_{i=1}^r 3^{k_i} = 3^{\Omega(m)} \leq 3^{\log \log n} = (\log n)^{\log 3},$$

where $\tau_3(m)$ denotes the number of triples (a, b, c) with $a, b, c \in \mathbb{Z}^+$ such that $m = abc$.

If H contains a K_4 (by K_4 in a 3-uniform hypergraph we mean a subhypergraph G with vertices $V = \{P_{i_1}, P_{i_2}, P_{i_3}, P_{i_4}\}$ such that $V \setminus \{P_{i_j}\}$ is an edge in G for all $j \in \{1, 2, 3, 4\}$), then for some $d_{i_1} < d_{i_2} < d_{i_3} < d_{i_4}$ and $a_1 = d_{i_1} d_{i_2} d_{i_3}$, $a_2 = d_{i_1} d_{i_2} d_{i_4}$, $a_3 = d_{i_1} d_{i_3} d_{i_4}$, $a_4 = d_{i_2} d_{i_3} d_{i_4}$ we have $a_1 < a_2 < a_3 < a_4$, $a_1, a_2, a_3, a_4 \in A$ and $a_1 a_2 a_3 a_4 = (d_{i_1} d_{i_2} d_{i_3} d_{i_4})^3$. Since $a_1 a_2 a_3 a_4 \neq x^3$ if $a_i \in A$ and $a_1 < a_2 < a_3 < a_4$, H does not contain K_4 . Hence there is $\delta > 0$ such that there at least δt^3 triples (i, j, k) , $1 \leq i < j < k \leq t$ such that $\{P_i, P_j, P_k\}$ is not an edge in H (see [3]).

Let $h = \max_{m \in M} h(m) \leq (\log n)^{\log 3}$. If $\{P_i, P_j, P_k\} \notin H$, $1 \leq i < j < k \leq t$, then $m = d_i d_j d_k$ has at most h decompositions as a product of three positive integers, which gives the following bound on M :

$$|M| \geq \frac{\delta t^3}{h} \gg \frac{n}{(\log n)^{3 + \log \frac{1}{3} - 1 + \epsilon} \cdot (\log n)^{\log 3}} = \frac{n}{(\log n)^{2 + \epsilon}},$$

which completes the proof of the lower bound.

Now we prove the upper bound. Let A_n denote the set of the integers a such that

- (i) $\frac{n}{\log n} \leq a \leq n$,
- (ii) $d^2 \mid a$ implies $d \leq \log n$, and
- (iii) a cannot be written in the form $a = uvw$ with integers u, v, w such that $\frac{\sqrt[3]{n}}{(\log n)^{14}} \leq u, v, w \leq \sqrt[3]{n}(\log n)^{14}$.

It follows directly from Lemma 11 that the equation

$$a_1 a_2 a_3 a_4 = x^3$$

has no solution in A_n . Hence $f_{4,3}(n) \geq |A_n|$. On the other hand, let $\epsilon > 0$ and $n \geq n_0(\epsilon)$ as per Lemma 12. Note that for $n \geq 3$, the number of integers satisfying (i) and (ii) in the definition of A_n is at least $n - \frac{3n}{\log n}$ (see [7], Proof of Theorem 2). By Lemma 12, all but $\frac{n}{(\log n)^{1 - \frac{\epsilon \log 3}{2\sqrt{3}} - \frac{\epsilon}{2}}}$ of these

integers also satisfy (iii), so we have

$$|A_n| > n - \frac{3n}{\log n} - \frac{n}{(\log n)^{1 - \frac{\epsilon \log 3}{2\sqrt{3}} - \frac{\epsilon}{2}}} > n - \frac{n}{(\log n)^{1 - \frac{\epsilon \log 3}{2\sqrt{3}} - \epsilon}},$$

which completes the proof. $\square$

5. CASE $k = 9$

Proof of Theorem 7. First, we prove the lower bound.

We will construct a set $S_n \subseteq [n]$ of the form $S_n = P_n \cup A_n$, where P_n is the set of primes from the interval $(\sqrt{n}, n]$ and each element of the set A_n is the product of two different primes.

Let us partition the primes not exceeding $\sqrt{n}$ into two sets S and T such that $|S| = |T|$ (leaving out one prime if $\pi(\sqrt{n})$ is odd), and let $G = (S, T, E)$ be a bipartite graph without 4- and 6-cycles with maximum possible number of edges. Then by [1], the number of edges in G is $\Theta((|S||T|)^{2/3}) = \Theta(|S|^{4/3})$. Let us define A_n such that $p_i p_j \in A_n$ if and only if $(p_i, p_j) \in E$. We will show that the equation has no solution in $S_n = P_n \cup A_n$.

Assume, to the contrary, that $a_1 a_2 \dots a_9 = x^3$ with $a_1, \dots, a_9 \in S_n$. Note that for $i = 1, \dots, 9$, either $a_i = p$ for some prime $p \in (\sqrt{n}, n]$, or $a_i = p_i q_i$ for primes p_i, q_i such that $p_i \in S, q_i \in T$. Since the right-hand side is a perfect cube, the number of a_i 's such that a_i is a prime is divisible by 3, that is, either 9, 6, 3 or 0. We will investigate each case to reach a contradiction.

If all the a_i 's are primes, $a_1 \leq a_2 \leq \dots \leq a_9$, then it is clear that $a_1 = a_2 = a_3, a_4 = a_5 = a_6$ and $a_7 = a_8 = a_9$, which yields a trivial solution.

If there are 6 primes among $a_1, \dots, a_9$, we can assume that $a_i = p_i q_i$ as above for $i \in \{1, 2, 3\}$. Then $\gcd(a_i, a_j) = 1$ for $i \in \{1, 2, 3\}, j \in \{4, \dots, 9\}$, so $a_1 a_2 a_3$ is also a perfect cube. Since $\gcd(p_1 p_2 p_3, q_1 q_2 q_3) = 1$, both $p_1 p_2 p_3$ and $q_1 q_2 q_3$ are perfect cubes, which implies $p_1 = p_2 = p_3$ and $q_1 = q_2 = q_3$. Hence $a_1 = a_2 = a_3$. Since $a_4, \dots, a_9$ are primes, and their product is a perfect cube, either $a_4 = \dots = a_9$ or we can assume that $a_4 = a_5 = a_6$ and $a_7 = a_8 = a_9$. Either case gives a trivial solution.

If there are 3 primes among $a_1, \dots, a_9$, we can assume that $a_1, \dots, a_6$ are not primes and $a_7 = a_8 = a_9$. Let $G' \subseteq G$ be the subgraph induced by the edges corresponding to the set $\{a_1, \dots, a_6\}$, and let G'' be the multigraph induced by the multiset $\{a_1, \dots, a_6\}$. G' has at most 6 edges (note that the number of edges may be less than 6 if some of $a_1, \dots, a_6$ coincide). Since G has no cycles of length 4 or 6, there is a vertex in G' with degree 1, say, p_1 with the only edge $p_1 q_1$. Then the multiplicity of the edge $p_1 q_1$ in G'' is either 6 or 3, that is, either $a_1 = \dots = a_6$, or we may assume that $a_1 = a_2 = a_3$ and $a_4 a_5 a_6$ is a perfect cube. We have already seen that both of these cases yield trivial solutions.

Now suppose there is no prime among $a_1, \dots, a_9$, and let G' and G'' be the graphs induced by $\{a_1, \dots, a_9\}$ as above. If there is no cycle in G' , we may assume as before that the only outgoing edge from p_1 in G' is $p_1 q_1$. Then deleting three instances of $p_1 q_1$ from G'' will yield another multigraph H'' such that H'' has 6 edges and the product of the edges of H'' is still a perfect cube, which is exactly the previous case. If there is a cycle C in G' , then $|C| = 8$, and we may assume that $C = p_1, q_1, p_2, \dots, p_4, q_4$. Then there are at least 8 distinct prime factors in $a_1 \dots a_9 = x^3$, each with multiplicity divisible by 3, hence $\Omega(x^3) \geq 24$. But $a_i = p_i q_i$ implies $\Omega(x^3) = 18$, and we have reached a contradiction.

Now we give a bound on the size of S_n . Note that it follows from the prime number theorem that

$$|S| = |T| = \left\lfloor \frac{\pi(\sqrt{n})}{2} \right\rfloor = \Theta\left(\frac{\sqrt{n}}{\log n}\right)$$

and hence

$$\begin{aligned} |S_n| &= |P_n| + |A_n| = (\pi(n) - \pi(\sqrt{n})) + \Theta\left(\left(\frac{\sqrt{n}}{\log n}\right)^{4/3}\right) > \\ &> \pi(n) - \pi(\sqrt{n}) + c_0 \left(\frac{n^{2/3}}{(\log n)^{4/3}}\right). \end{aligned}$$

Since $\pi(\sqrt{n}) = o\left(\frac{n^{2/3}}{(\log n)^{4/3}}\right)$, there exists a constant c_1 such that

$$f_{9,3}(n) - \pi(n) > c_1 \left(\frac{n^{2/3}}{(\log n)^{4/3}}\right).$$

Now we prove the upper bound.

Let $A \subseteq [n]$ be a set such that $a_1 \dots a_9 = x^3$ has no solution in A . Let A_0 be the set of those $a \in A$ such that a has a prime factor greater than $\sqrt{n}$, and let $A_1 = A \setminus A_0$.

It follows from the proof of [13, Theorem 2] that even if we require the a_i 's in the definition of A to be distinct, the size of A_1 is $O(n^{5/6})$, i.e. there exists a constant c_3 such that $|A_1| < c_3 n^{5/6}$. Since our condition is weaker, this bound still holds.

Observe that in A_0 every element has the form pt , where $p > \sqrt{n}$ is a prime and $t \leq \sqrt{n}$. Hence to estimate the size of A_0 , take a bipartite graph $G = (S, T, E)$ where $S = \{p : \sqrt{n} < p \leq n, p \text{ prime}\}$, $T = [\sqrt{n}]$, and $(p_i, t_j) \in E$ if and only if $p_i t_j \in A_0$. Note that the girth of G is at least 8: If there was a 4-cycle $C = (p_1, t_1, p_2, t_2)$ in G , then $(p_1 t_1)^5 (t_1 p_2) (p_2 t_2)^2 (t_2 p_1) = (p_1^2 t_1^2 p_2 t_2)^3$ would be a nontrivial solution; similarly, if there was a 6-cycle $C = (p_1, t_1, p_2, t_2, p_3, t_3)$ in G , then $(p_1 t_1)^2 (p_2 t_2)^2 (p_3 t_3)^2 (t_1 p_2) (t_2 p_3) (t_3 p_1) = (p_1 p_2 p_3 t_1 t_2 t_3)^3$ would be a nontrivial solution.

Let $S_1 = (n^{1/2}, n^{3/4}) \cap S$, $S_2 = [n^{3/4}, n] \cap S$, $T_1 = [1, n^{1/4}]$, and $T_2 = (n^{1/4}, n^{1/2}]$. Then

$$|S_1| = \pi(n^{3/4}) - \pi(n^{1/2}) = \Theta(\pi(n^{3/4})) = \Theta\left(\frac{n^{3/4}}{\log n}\right)$$

and

$$|S_2| = \pi(n) - \pi(n^{3/4}) = \Theta(\pi(n)) = \Theta\left(\frac{n}{\log n}\right).$$

Clearly, $|T_1| = \lfloor n^{1/4} \rfloor$ and $|T_2| = \lfloor n^{1/2} \rfloor - \lfloor n^{1/4} \rfloor = \Theta(n^{1/2})$.

By [5, Lemma 2], there are at most $\pi(n) + O(n^{1/2})$ edges between S_2 and T_1 , since the graph is C_4 -free. Note that there are no edges between S_2 and T_2 , so the rest of the edges are all between S_1 and T . Since $\sqrt{|S_1|} < |T| < |S_1|$, by [4, Theorem 1] there are $O((|S_1||T|)^{2/3}) = O\left(\frac{n^{5/6}}{(\log n)^{2/3}}\right)$ edges between S_1 and T . Hence $|A_0| = |E| < \pi(n) + c_4 n^{5/6}$.

Therefore $|A| = |A_0| + |A_1| < \pi(n) + c_4 n^{5/6} + c_3 n^{5/6}$, which yields the upper bound with $c_2 = c_3 + c_4$. □

For the proof of Theorem 8 we need the following graph-theoretic lemma.

Lemma 13. *For every sufficiently large n , there exists a $K_{3,3}$ -free bipartite graph $G = (S, T, E)$ with $|S| = |T| = n$ such that $|E| > cn^{5/3}$ for some $c > 0$.*

Proof of Lemma 13. Brown [2] proved that there exists a graph on $2n$ vertices with $cn^{5/3}$ edges that does not contain $K_{3,3}$ as a subgraph, where $c > 0$. Let us denote this graph by G_0 . We use a standard probabilistic argument to show that there exists such a bipartite G as the lemma requires. Let us divide the vertices of G_0 into two sets of equal sizes with uniform distribution, and delete every edge within the two sets. We delete every edge with probability $\frac{n-1}{2n-1}$, so the expected value of the number of remaining edges is at least $\frac{1}{2}cn^{5/3}$. Hence, there exists such a bipartite graph with at least $\frac{1}{2}cn^{5/3}$ edges, as we claimed. $\square$

Proof of Theorem 8. The upper bound is a special case of [13, Theorem 2] with $k = 9, d = 3$.

For the lower bound, we give a construction similar to the one in the proof of Theorem 7.

Let $A_0 = \{p : \sqrt{n} < p < n, p \text{ prime}\} \cup \{2p : \sqrt{n} < p < n/2, p \text{ prime}\}$. Let us again partition the primes not exceeding $\sqrt{n}$ into two sets S and T such that $|S| = |T|$, leaving out one prime if $\pi(\sqrt{n})$ is odd. Then $|S| = |T| = \Theta(\pi(\sqrt{n})) = \Theta\left(\frac{\sqrt{n}}{\log n}\right)$. Lemma 13 guarantees the existence of a $K_{3,3}$ -free bipartite graph $G = (S, T, E)$ with at least $c|S|^{5/3}$ edges. Let $st \in A_1$ if and only if $(s, t) \in E$, and define $A = A_0 \cup A_1$. Clearly,

$$\begin{aligned} |A| &= |A_0| + |A_1| \geq (\pi(n) - \pi(\sqrt{n})) + \left(\pi\left(\frac{n}{2}\right) - \pi(\sqrt{n})\right) + c|S|^{5/3} = \\ &= \pi(n) + \pi\left(\frac{n}{2}\right) + \Theta\left(\frac{n^{5/6}}{(\log n)^{5/3}}\right). \end{aligned}$$

Now we show that the equation

$$a_1 \dots a_9 = x^3, \quad a_1 < \dots < a_9$$

has no solution in A .

It is clear that a potential solution $a_1, \dots, a_9 \in A$ cannot contain any elements from A_0 . Suppose that the equation has a solution $a_1, \dots, a_9$ in A_1 . Then $a_1, \dots, a_9$ are represented by 9 edges in G . In order for $a_1 \dots a_9$ to be a perfect cube, these edges must form a subgraph H such that every vertex has degree divisible by 3 in H . This implies that H must contain at least 3-3 vertices from both S and T . However, since G is $K_{3,3}$ -free by construction, $H \neq K_{3,3}$, so H contains at least 4 vertices from at least one of the vertex sets. This yields that $a_1 \dots a_9$ has at least 7 prime divisors, thus $\Omega(x^3) \geq 21$. On the other hand, however, $\Omega(x^3) = \sum_{i=1}^9 \Omega(a_i) = 18$. Hence we have reached a contradiction, which concludes the proof. $\square$

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PRODUCT REPRESENTATION OF PERFECT CUBES

OTDK-DOLGOZAT

SAJÁT EREDMÉNYEK

FLEINER ZSIGMOND GYÖRGY, JUHÁSZ MÁRK HUNOR, KÖVÉR BLANKA
TÉMAVEZETŐK: PACH PÉTER PÁL, SÁNDOR CSABA

Kutatásunkra, amelynek eredménye a 2023. decemberi ELTE kari TDK-konferencián bemutatott TDK-dolgozat, a REU 2023 program keretein belül került sor. A program során a felmerülő kérdéseken hárman együtt gondolkoztunk, így az eredmények többségénél nem lehet egyértelműen meghatározni, hogy melyikünkhöz tartozik. Az alábbiakban felsoroljuk az összes saját eredményünket (és amelyiknél eldönthető, ott feltüntetjük, hogy melyikünké).

- Léteznek c_1 és c_2 pozitív konstansok úgy, hogy

$$c_1 n^{2/3} < n - F_{2,3}(n) \leq n - f_{2,3}(n) < c_2 n^{2/3}.$$

- Létezik egy $0,618 \leq c_{3,3} \leq 0,651$ konstans, amelyre

$$f_{3,3}(n) = (c_{3,3} + o(1))n$$

teljesül.

- Létezik egy $0,6894 \leq C_{3,3} \leq 0,7258$ konstans, amelyre

$$F_{3,3}(n) = (C_{3,3} + o(1))n$$

teljesül.

- Legyen $\varepsilon > 0$ rögzített. Ekkor létezik $n_0(\varepsilon)$ küszöbindex, hogy $n \geq n_0(\varepsilon)$ esetén

$$\frac{n}{(\log n)^{2+\varepsilon}} < n - F_{4,3}(n) \leq n - f_{4,3}(n) < \frac{n}{(\log n)^{1-\frac{\varepsilon \log 3}{2\sqrt{3}}-\varepsilon}}$$

teljesül. Ennél a tételnél két lemmát használunk, amelyek az [1] cikk $k = 3$ esetének egyes ötleteit általánosítják és terjesztik ki. Az egyik lemma (Lemma 11.) Kövér Blanka eredménye.

- Léteznek c_1 és c_2 pozitív konstansok úgy, hogy minden elég nagy n esetén

$$c_1 \frac{n^{2/3}}{(\log n)^{4/3}} < f_{9,3}(n) - \pi(n) < c_2 n^{5/6}$$

teljesül. Ezen korlátokat Fleiner Zsigmond bizonyította, Verstraëte [2] ötletéből kiindulva.

- Léteznek c_1 és c_2 pozitív konstansok úgy, hogy minden elég nagy n esetén

$$c_1 \frac{n^{5/6}}{(\log n)^{5/3}} < F_{9,3}(n) - \left(\pi(n) + \pi\left(\frac{n}{2}\right) \right) < c_2 n^{5/6}$$

teljesül.

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- [1] P. Erdős, A. Sárközy, V. T. Sós, *On Product Representations of Powers, I.*, European Journal of Combinatorics, **16** (6) (1995) 567–588.
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PRODUCT REPRESENTATION OF PERFECT CUBES

OTDK-DOLGOZAT

MAGYAR NYELVŰ ÖSSZEFOGLALÓ

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KIVONAT. Jelöljük $F_{k,d}(n)$ -nel a legnagyobb olyan $A \subseteq [n]$ halmaz méretét, amelyre igaz, hogy az

$$a_1 a_2 \dots a_k = x^d, \quad a_1 < a_2 < \dots < a_k$$

egyenlet nem teljesül semmilyen $a_1, a_2, \dots, a_k \in A$ számokra és x egészre. Erdős, Sárközy és T. Sós az $F_{k,2}$ függvényt vizsgálták, és mind a $k = 2, 3, 4, 6$ speciális esetekben, mind az általános esetben korlátokat adtak rá. Mi a $d = 3$ esetet vizsgáljuk, és korlátokat adunk a $k = 2, 3, 4, 6$ és 9 esetekben. Bevezetünk emellett a feltétel természetes módosításával egy további, $F_{k,d}$ -vel rokon függvényt is, amelyet $f_{k,d}$ -vel jelölünk. Az eredeti problémában elvárjuk ugyanis azt, hogy $a_1, \dots, a_k$ mind különbözők legyenek, ezen a feltételen azonban enyhíthetünk: ekkor azt kötjük ki, hogy az a_i -k multihalmaza ne legyen olyan szám- d -esekre osztható, ahol minden szám- d -es egy adott szám d darab példányából áll. Ezeket a triviális megoldásokat kizárva analóg módon definiáljuk $f_{k,d}$ -t.

Először Erdős, Sárközy és T. Sós vetette fel az alábbi kérdést: mekkora a legnagyobb olyan $A \subseteq [n]$ halmaz, amelyben semelyik k szám szorzata sem négyzetszám? 1995-ös cikkükben [1] a szerzők tehát a lehető legnagyobb olyan A halmaz méretét vizsgálták, hogy az

$$a_1 a_2 \dots a_k = x^2, \quad a_1 < a_2 < \dots < a_k$$

egyenletnek ne legyen olyan megoldása $a_i \in A$ és x egész számokkal. Ez motiválja az alábbi definíciókat.

Jelölje $F_{k,d}(n)$ a legnagyobb olyan $A \subseteq [n]$ halmaz méretét, hogy az

$$a_1 a_2 \dots a_k = x^d, \quad a_1 < a_2 < \dots < a_k$$

egyenletnek nincs olyan megoldása, ahol $a_1, \dots, a_k \in A$ és x egész. Hasonlóan, jelölje $f_{k,d}(n)$ a legnagyobb olyan $A \subseteq [n]$ halmaz méretét, hogy az

$$a_1 a_2 \dots a_k = x^d$$

egyenletnek nincs olyan megoldása, ahol $a_1, \dots, a_k \in A$ és x egész, kivéve bizonyos *triviális* megoldásokat, amelyeket a következőkben definiálunk. Ha az $a_1, a_2, \dots, a_n$ számokról nem követeljük meg fel, hogy különbözők, akkor felléphetnek triviális megoldások: az $a_1 = \dots = a_d$ választás például triviális megoldását adja az $a_1 \dots a_d = x^d$ egyenletnek. Nevezzünk most egy megoldást *triviálisnak*, ha az a_i -k multihalmaza felosztható olyan szám- d -esekre, ahol minden szám- d -es egy adott szám d darab példányából áll: ilyen triviális megoldás például $k = 9, d = 3$ esetén $(a_1 a_1 a_1)(a_2 a_2 a_2)(a_3 a_3 a_3) = x^3$. Megjegyezzük, hogy triviális megoldások csak $d \mid k$ esetén léphetnek fel. A definíciókból az is rögtön adódik, hogy $f_{k,d} \leq F_{k,d}$.

Erdős, Sárközy és T. Sós [1] belátták, hogy $F_{2k+1,2}(n) \gg n$ teljesül minden k -ra, valamint hogy $F_{2k,2}(n) = o(n)$ minden $k \geq 2$ esetén. Azt is megmutatták, hogy $F_{2,2}(n)$ éppen az n -nél nem nagyobb négyzetmentes pozitív egészek számával egyenlő, illetve korlátokat adtak $k = 3, 4, 6$ esetben. Verstraëte [5] a kérdés egy általánosításával foglalkozott: a lehető legnagyobb olyan $A \subseteq [n]$ halmaz

méretét vizsgálta, amelyben semelyik k különböző elem szorzata nincs benne egy adott $f \in \mathbb{Z}[x]$ polinom értékkészletében. Belátta, hogy polinomok egy bizonyos osztályára a válasz $\Theta(n)$, egy másik osztályára pedig $\Theta(\pi(n))$, ahol Θ a szokásos módon az aszimptotikát jelöli, $\pi(n)$ pedig a prímek számát n -ig. Ezenfelül azt sejtette, hogy bármely polinomra ezen két lehetőség valamelyike teljesül. További kapcsolódó eredmények találhatóak a [2, 3] cikkekben, illetve Terence Tao friss cikkében [4].

Mi az eredeti kérdést a $d = 3$ esetben vizsgáltuk, és korlátokat adtunk $F_{k,3}$ -ra és $f_{k,3}$ -ra $k = 2, 3, 4, 6, 9$ esetén. A következőkben összefoglaljuk eredményeinket.

A $k = 2$ esetben az alábbi korlátok teljesülnek:

1. Tétel. *Léteznek c_1 és c_2 pozitív konstansok úgy, hogy*

$$c_1 n^{2/3} < n - F_{2,3}(n) \leq n - f_{2,3}(n) < c_2 n^{2/3}.$$

A $k = 3$ esetben bizonyítjuk, hogy $n \rightarrow \infty$ esetén $f_{3,3}(n)/n$ tart egy $c_{3,3} \in (0, 1)$ számhoz, amely elméletben tetszőleges pontossággal közelíthető:

2. Tétel. *Létezik egy $0,618 \leq c_{3,3} \leq 0,651$ konstans, amelyre*

$$f_{3,3}(n) = (c_{3,3} + o(1))n$$

teljesül.

Az $F_{3,3}(n)$ függvényre az alábbi hasonló eredmény áll fenn:

3. Tétel. *Létezik egy $0,6894 \leq C_{3,3} \leq 0,7258$ konstans, amelyre*

$$F_{3,3}(n) = (C_{3,3} + o(1))n$$

teljesül.

A $k = 4$ esetben azt látjuk be, hogy kellően nagy n esetén teljesülnek az alábbi korlátok. Bizonyításaink [1] $k = 3$ esetének egyes ötleteit általánosítják és terjesztik ki.

4. Tétel. *Legyen $\varepsilon > 0$ rögzített. Ekkor létezik $n_0(\varepsilon)$ küszöbindex, hogy $n \geq n_0(\varepsilon)$ esetén*

$$\frac{n}{(\log n)^{2+\varepsilon}} < n - F_{4,3}(n) \leq n - f_{4,3}(n) < \frac{n}{(\log n)^{1-\frac{\varepsilon \log 3}{2\sqrt{3}}-\varepsilon}}$$

teljesül.

A $k = 6$ esetben az alábbi eredményeket kaptuk:

5. Tétel. *Léteznek c_1 és c_2 pozitív konstansok úgy, hogy*

$$c_1 \frac{n^{3/4}}{(\log n)^{3/2}} < f_{6,3}(n) - \pi(n) < c_2 \frac{n^{3/4}}{(\log n)^{3/2}}$$

fennáll.

6. Tétel. *Léteznek c_1 és c_2 pozitív konstansok úgy, hogy*

$$c_1 \frac{n \log \log n}{\log n} < F_{6,3}(n) < c_2 \frac{n \log \log n}{\log n}$$

fennáll.

Mivel ez utóbbi két tétel esetén a legfontosabb ötletek mind a témavezetőinktől származtak, ezért a dolgozatban ezeket csak megemlítjük, de nem bizonyítjuk. Megjegyezzük továbbá, hogy a 6. Tétel cáfolja Verstraëte [5] fent említett sejtését.

Végezetül, $k = 9$ esetén az alábbi korlátok teljesülnek:

7. Tétel. *Léteznek c_1 és c_2 pozitív konstansok úgy, hogy minden elég nagy n esetén*

$$c_1 \frac{n^{2/3}}{(\log n)^{4/3}} < f_{9,3}(n) - \pi(n) < c_2 n^{5/6}$$

teljesül.

8. Tétel. *Léteznek c_1 és c_2 pozitív konstansok úgy, hogy minden elég nagy n esetén*

$$c_1 \frac{n^{5/6}}{(\log n)^{5/3}} < F_{9,3}(n) - \left(\pi(n) + \pi\left(\frac{n}{2}\right) \right) < c_2 n^{5/6}$$

teljesül.

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