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Dynamical properties of certain piecewise linear Markov maps

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1 Introduction

Piecewise linear interval maps are one of the simplest type of discrete time dynamical systems. Thanks to their simple definition they are widely used for modelling phenomena, and by their research we can learn techniques widely applicable for more complex systems. Despite their apparent simplicity these systems can bear extremely complex and unpredictable orbits. The study of the long-term stochastic behaviour of such constructions is thus relevant, as well as examining various shadowing properties to serve as a theoretical foundation for numerical analysis. Lastly these maps can also have number theoretic applications, since a certain class of them can be thought of as a generalization of decimal expansion.

In this work we consider dynamical properties of a class of real-to-real maps which have a countable Markov partition and are linear on the elements of the partition. We start by considering the properties of a dynamical system presented in the third problem of the Miklós Schweitzer competition 2022, proposed by my advisor Zoltán Buczolich. The problem was the following, with a bit modified notation:

The transformation $T : [0, \infty) \rightarrow [0, \infty)$ is linear on every positive integer interval, and its integer values are defined as follows:

$$T(n) = \begin{cases} 0 & \text{if } 2|n \\ 4^\ell + 1 & \text{if } 2 \nmid n, 4^{\ell-1} \leq n < 4^\ell \ (\ell \in \mathbb{Z}^+). \end{cases} \quad (1)$$

Let $T^0(x) = x$ and $T^n(x) = T(T^{n-1}(x))$ for all $n > 0$. Find the $\liminf_{k \rightarrow \infty} (T^k(x))$ and the $\limsup_{k \rightarrow \infty} (T^k(x))$ of Lebesgue almost every $x \in [0, \infty)$!

We prove much stronger statements about wider classes of piecewise linear transformations. These will imply that for almost every $x \in [0, \infty)$ the orbit is dense in $\mathbb{R}_{\geq 0}$, thus

$$\liminf_{k \rightarrow \infty} (T^k(x)) = 0 \text{ and } \limsup_{k \rightarrow \infty} (T^k(x)) = \infty.$$

We define a special type of Markov partition on transformations mapping from the reals to the reals, which we call ℓ -Markov partition. This definition will be more restrictive than those in the literature in the sense that we require that the function is linear on the elements of the partition, however it will be more general in the sense that we allow countably infinite partitions of the domain. The most similar definition to ours in the literature is class $\mathcal{L}_{\mathcal{M}}$ appearing in [1]. We investigate the symbolic dynamics and the absolutely continuous invariant measures of systems possessing our newly defined type of partition. We completely describe the convergence of pushforwards of absolutely continuous measures. Some special cases of these results for finite linear partitions are present in [1], however to our knowledge neither the invariant measures on infinite partitions, neither the convergence properties of absolutely continuous measures on systems with finite or infinite ℓ -Markov partitions are present in literature. We also define a more restrictive case of ℓ -Markov partitions. For systems possessing a partition of this kind we completely describe the orbits of the rational points. To our knowledge this result is completely unique in this context. We apply these results to our original system, and we also prove that it has the shadowing property which is not implied by previous results on shadowing.

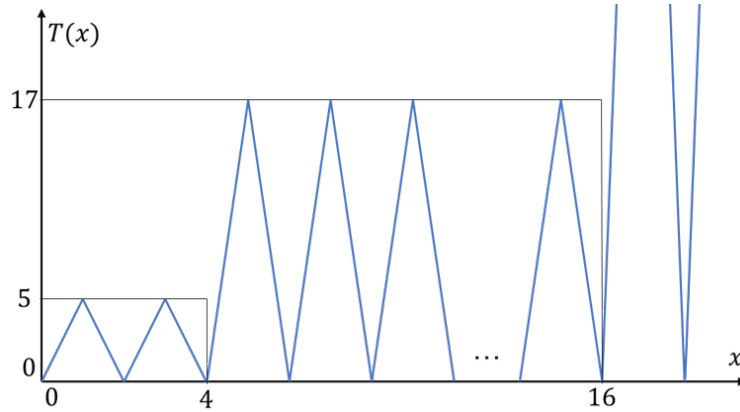


Figure 1: The transformation T .

Definition 1.1. The function $\ell : \mathbb{R}_{\geq 0} \rightarrow \mathbb{N}$ is defined as follows:

$$\ell(x) = \inf\{n \in \mathbb{Z}^+ : x \leq 4^n\}.$$

Definition 1.2. For some map $f : X \rightarrow X$ let $f^0(x) = x$ for $x \in X$ and $f^n(x) = f(f^{n-1}(x))$ for $n > 0$ integers and $x \in X$.

Definition 1.3. For a finite or infinite sequence $\underline{i} = (i_0, i_1, \dots)$ we denote by $\underline{i}|n$ the finite sequence $(i_0, \dots, i_n)$.

Definition 1.4. For any Lebesgue-measurable subset X of $\mathbb{R}^n$ we will denote the Lebesgue σ -algebra on X with $\mathcal{L}(X)$.

Definition 1.5. If a function f is linear on the interior of interval $H \subset \mathbb{R}$, then $f'(H)$ is $f'(x)$ for any $x \in \text{int}(H)$.

Definition 1.6. For some sequence of reals x_n and for some integers $a > b$ we define $\sum_{n=a}^b x_n = 0$ and $\prod_{n=a}^b x_n = 1$ as it is standard for the empty sum and product.

Throughout this paper by almost every we mean Lebesgue almost every, and by measurable we mean Lebesgue-measurable. We denote the Lebesgue measure by λ . We say that a measure μ is absolutely continuous if $\lambda \gg \mu$. If μ is an absolutely continuous measure then by the density γ of it we mean the function for which $\mu(H) = \int_H \gamma d\lambda$ for all $H \in \mathcal{L}(X)$, so its Radon-Nikodym derivative. The domain of the function γ consists of the points of the measure space and its range is $\mathbb{R}_{\geq 0} \cup \{\infty\} = \overline{\mathbb{R}}_{\geq 0}$. By null set we mean sets of Lebesgue measure zero. We define $0 \cdot \infty$ to be 0, so the measure of a null set is zero, even if the density is infinite everywhere.

2 Symbolic dynamics

In this section we describe the orbits of points in the system $(\mathbb{R}_{\geq 0}, T)$ as series of abstract symbols. The symbols will be elements of $\mathbb{Z}^+$. This kind of description of a system is standard in the study of dynamical systems, and will help us discuss important properties of the system. The properties observed here will motivate the generalizations in later sections.

Definition 2.1. We define the closed integer intervals to be $I_n = [n-1, n]$ for any $n \in \mathbb{Z}^+$.

Definition 2.2. Given a finite sequence of positive integers $(i_0, \dots, i_N)$ we denote $I_{i_0 \dots i_N} = \bigcap_{n=0}^N T^{-n}(I_{i_n})$. Given an infinite sequence of positive integers $(i_0, \dots)$ we denote $I_{i_0 \dots} = \bigcap_{n=0}^{\infty} T^{-n}(I_{i_n})$.

Lemma 2.3. *Given a finite sequence of positive integers $(i_0, \dots, i_n)$ the following are true for $I_{i_0 \dots i_n}$:*

1. *The set $(\text{int}(I_{i_0 \dots i_n k}) \cap \text{int}(I_{i_0 \dots i_n}))$ is empty if and only if $k \notin \mathbb{Z} \cap [1, 4^{\ell(i_n)} + 1]$.*
2. *The set $I_{i_0 \dots i_n}$ is an interval.*
3. *If $I_{i_0 \dots i_n}$ has a non-empty interior then $T^k(I_{i_0 \dots i_n}) = I_{i_k \dots i_n}$ for all $0 \leq k \leq n$ integers.*
4. *$I_{i_0 \dots i_n} = \bigcup_{k=1}^{4^{\ell(i_n)}+1} I_{i_0 \dots i_n k}$.*
5. *If $(\text{int}(I_{i_0 \dots i_n k}) \cap \text{int}(I_{i_0 \dots i_n})) \neq \emptyset$ then $\lambda(I_{i_0 \dots i_n k}) = \frac{1}{4^{\ell(i_n)}+1} \lambda(I_{i_0 \dots i_n})$.*
6. *For all infinite $(i_0, i_1, \dots)$ integer sequences $I_{i_0 i_1 \dots}$ is either empty, or has exactly one element.*
7. *If $(j_0 \dots j_n)$ is a finite sequence of positive integers, and it is not equal to $(i_0 \dots i_n)$ then it is true that $\text{int}(I_{i_0 \dots i_n}) \cap \text{int}(I_{j_0 \dots j_n}) = \emptyset$.*
8. *The sets $I_{i_0 \dots i_n k}$ where $k \in \mathbb{Z} \cap [1, 4^{\ell(i_n)} + 1]$ are placed in a row in increasing or decreasing order. (See Figure 2)*
9. *The above sets are in increasing order if and only if there is an even number of even numbers among $i_0 \dots i_n$. (See Figure 2)*
10. *For a point $x \in \mathbb{R}_{\geq 0}$ there exists a natural k for which $T^k(x) \in \mathbb{N}$ if and only if $x \in \partial I_{i_0 \dots i_k}$ for some finite positive integer sequence $(i_0, \dots, i_k)$.*

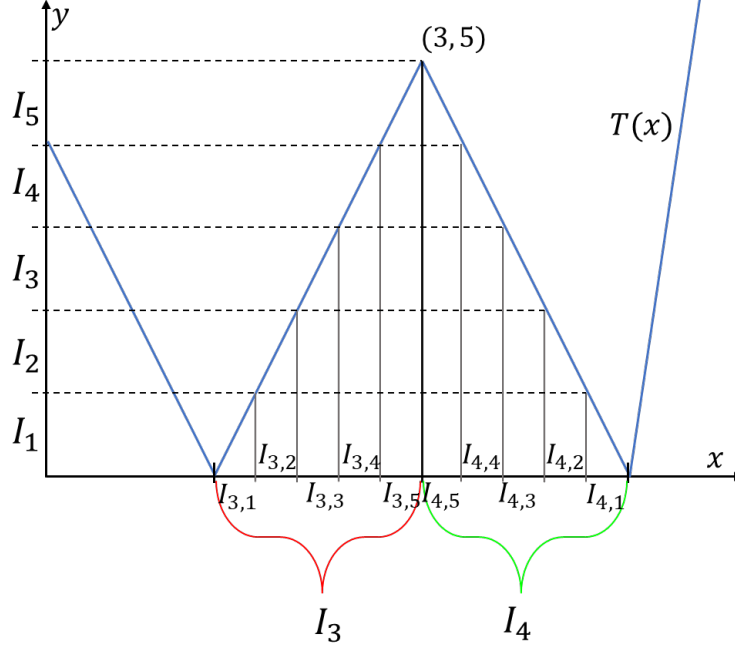


Figure 2: Some labelled intervals of $\mathbb{R}_{\geq 0}$.

Proof. These propositions all follow straight from the definition of T . It is worth noting that $I_{i_0 \dots i_n}$ is the set of points for which $T^k(x) \in I_{i_k}$ for all $0 \leq k \leq n$.

1. This follows from that the range of T on I_{i_n} is $[0, 4^{\ell(i_n)} + 1]$.
2. That $I_{i_0 \dots i_n}$ is an interval follows from the strict monotonicity and continuity of T on all integer intervals.
3. If $\text{int}(I_a) \cap \text{int}(T^{-1}(I_b))$ is non-empty then $I_b \subseteq T(I_a)$ for $a, b \in \mathbb{Z}^+$. If $\text{int}(I_{i_0 \dots i_n}) \neq \emptyset$ then for all $0 \leq k \leq n-1$ integers $\text{int}(I_{i_k}) \cap \text{int}(T^{-1}(I_{i_{k+1}})) \neq \emptyset$ so $I_{i_{k+1}} \subseteq T(I_{i_k})$. From here the statement follows.
4. This also follows from that the range of T on I_{i_n} is $[0, 4^{\ell(i_n)} + 1]$.
5. This follows from the linearity of T on all integer intervals. See Figure 2.
6. $4^{\ell(i_n)} + 1 \geq 5$ so $\lambda(I_{i_0 \dots i_k}) \leq \frac{1}{5^k}$. Either some $I_{i_0 \dots i_n}$ is empty or for all integer n $I_{i_0 \dots i_{n+1}} \subset I_{i_0 \dots i_n}$. Since all $I_{i_0 \dots i_n}$ are closed intervals $I_{i_0 \dots}$ can have only one element.

For the rest see Figure 2. □

Definition 2.4. Let the itinerary of $x \in \mathbb{R}_{\geq 0}$ be the lexicographically smallest positive integer sequence $(i_0, i_1 \dots)$ satisfying that $x \in I_{i_0 i_1 \dots}$. Let this be $\underline{i}(x)$.

Lemma 2.5. The function $\underline{i}(x) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{N}^{\mathbb{N}}$ is an injective and well defined function.

Proof. Every point x is an element of $I_{i_0 i_1 \dots}$ for at least one positive integer sequence. Since the set of positive integer sequences is well-ordered by the lexicographic ordering $\underline{i}(x)$ is well defined. Since $I_{i_0 i_1 \dots}$ either has one or zero elements for all integer sequences then $\underline{i}(x)$ must be injective. $\square$

Lemma 2.6. *If for a positive integer sequence $(i_0, i_1 \dots)$ there is some positive integer n , for which $i_n = 4^k + 1, i_{n+1} = 4^{k+1} + 1 \dots$, for $k = \ell(i_{n-1})$ or $1 = i_n = i_{n+1} = \dots$, then if $I_{i_{n-1} i_n \dots}$ is non-empty then its only element is $i_{n-1} - 1$ or i_{n-1} .*

Proof. Since $i_{n-1} - 1$ and i_{n-1} are the two endpoints of $I_{i_{n-1}}$ the image of one of them is 0, while the image of the other is $4^{\ell(i_{n-1})} + 1$. In one of these cases $1 = i_n = i_{n+1} = \dots$, while in the other case $i_n = 4^k + 1, i_{n+1} = 4^{k+1} + 1 \dots$, for $k = \ell(i_{n-1})$. The set $I_{i_{n-1} i_n \dots}$ can only have one element, and $i_{n-1} - 1$ or i_{n-1} is an element of it. This proves the statement. $\square$

Corollary 2.7. *Lemma 2.6 implies that the orbit of a point $x \in \mathbb{R}_{\geq 0}$ eventually reaches an integer if and only if there exists a sequence of positive integers $(i_0, \dots)$, such that $x \in I_{i_0 \dots}$ and there is some positive integer n , for which $i_n = 4^k + 1, i_{n+1} = 4^{k+1} + 1 \dots$, for $k = \ell(i_{n-1})$ or $1 = i_n = i_{n+1} = \dots$.*

Definition 2.8. We denote by $\Omega_T = \{\underline{i}(x) : x \in \mathbb{R}_{\geq 0}\}$ the space of allowed symbolic trajectories. This is the space of those integer sequences which occur as $\underline{i}(x)$ for some $x \in \mathbb{R}_{\geq 0}$. We define the metric $d : \Omega_T \times \Omega_T \rightarrow \mathbb{R}_{\geq 0}$ such, that $d((i_0, \dots), (j_0, \dots)) = 2^{-n}$ where n is the smallest non-negative integer for which $i_n \neq j_n$.

Theorem 2.9. *The space Ω_T is composed of exactly those sequences of positive integers $(i_0, i_1, \dots)$ which satisfy the following:*

1. $i_{n+1} \in [1, 4^{\ell(i_n)} + 1]$ for any non-negative integer n .
2. If there is some positive integer n , for which $i_n = 4^k + 1, i_{n+1} = 4^{k+1} + 1 \dots$, for $k = \ell(i_{n-1})$, then i_{n-1} is odd.
3. If there is some positive integer n , for which $1 = i_n = i_{n+1} = \dots$, then i_{n-1} is even.

Proof. The proof relies on three parts:

- a. It follows from Lemma 2.3 that for all positive integer sequences $(i_0, \dots)$ satisfying the first condition $I_{i_0 \dots}$ is non-empty, and has only one element.
- b. We show that if the positive integer sequences $(i_0, \dots)$ and $(j_0, \dots)$ satisfy all three conditions, then $I_{i_0 \dots} \cap I_{j_0 \dots} = \emptyset$.
- c. We also show that for any x there exists a positive integer sequence $(i_0 \dots)$ satisfying all three conditions for which $x \in I_{i_0 \dots}$ and $(i_0, \dots)$ is the lexicographically smallest positive integer sequence $(j_0, \dots)$ for which $x \in I_{j_0 \dots}$.

These three facts together imply that $\underline{i}(x)$ satisfies all three conditions for all x and that for all positive integer sequences $(i_0, \dots)$ satisfying all three conditions there exists an x such that $\underline{i}(x) = (i_0, \dots)$.

Let us split the remaining proof into two cases. Let us pick an $x \in \mathbb{R}_{\geq 0}$.

The first case is when for all $k \in \mathbb{N}$ the image $T^k(x) \in \text{int}(I_n)$ for some positive integer n . This is the same as saying, that there are no integers in the orbit of x from Lemma 2.3. In this case there can only be one sequence $\underline{i}$ for which $x \in I_{\underline{i}}$ is fulfilled, since for all k $T^k(x) \in \text{int}(I_n)$ for some n and the interior of integer intervals does not intersect, so n must be unique. From this if $\underline{i} = (i_0, i_1, \dots)$, then for all n the set $\text{int}(I_{i_0 \dots i_n}) \cap \text{int}(I_{i_0 \dots i_n i_{n+1}}) \neq \emptyset$ must be satisfied, which by Lemma 2.3 implies that Condition 1 is true. As a consequence of Lemma 2.6 if there are no integers in the orbit of x then Conditions 2 and 3 are automatically met.

The second case is of those points x which eventually reach an integer. Let us assume that $n - 1$ is the smallest integer for which $T^{n-1}(x)$ is an integer. Then $T^n(x) = 0$ or $T^n(x) = 4^{\ell(i_{n-1})} + 1$ and $(i_0, \dots, i_{n-2})$ is unique if $x \in I_{i_0 \dots i_{n-2}}$.

If $T^n(x) = 0$ then $T^{n-1}(x)$ is an even number. Here i_{n-1} can be chosen to be two neighbouring integers for $x \in I_{i_0 \dots i_{n-1}}$ to be true, $T^{n-1}(x)$ and $T^{n-1}(x) + 1$. Of these the even one gives a sequence lexicographically smaller. Since all other elements of the orbit of x are zero all other elements of the sequence are unique if $x \in I_{i_0 \dots}$ is met. From this Condition 3 is fulfilled, and Condition 2 automatically applies.

If $T^n(x) = 4^{\ell(i_{n-1})} + 1$ then $T^{n-1}(x)$ is an odd number. From here i_{n-1} can be chosen to be two neighbouring integers for $x \in I_{i_0 \dots i_{n-1}}$ to be true. Of these the odd one gives a sequence lexicographically smaller. In this case $T^k(x) = 4^{\ell(i_{k-1})} + 1$ for all $n < k \in \mathbb{N}$. For Condition 1 to apply, and to satisfy $x \in I_{i_0 \dots}$, the value of i_k has to be $4^{\ell(i_{k-1})} + 1$. Thus all three conditions are satisfied for the lexicographically smallest possible sequence, and there are no other sequences of positive integers $(j_0, \dots)$ satisfying $x \in I_{j_0 \dots}$ fulfilling all three conditions. $\square$

Corollary 2.10. *The map $\underline{i} : \mathbb{R}_{\geq 0} \rightarrow \Omega_T$ is bijection.*

Definition 2.11. Let $\underline{i}^{-1} : \Omega_T \rightarrow \mathbb{R}_{\geq 0}$ be the inverse of $\underline{i}$. Let $\sigma : \Omega_T \rightarrow \Omega_T$ be the shift operator.

Corollary 2.12. *For all $x \in \mathbb{R}_{\geq 0}$, $\underline{i}(T(x)) = \sigma(\underline{i}(x))$ and $T(\underline{i}^{-1}((i_0, \dots))) = \underline{i}^{-1}(\sigma((i_0, \dots)))$ for all $(i_0, \dots) \in \Omega_T$.*

Theorem 2.13. *If for some x the orbit of $\underline{i}(x)$ is periodic for the shift operator then the orbit of x is periodic as well.*

Proof. If $\underline{i}(x)$ is $i_0 \dots i_n i_0 \dots i_n \dots$, then $x \in I_{i_0 \dots i_n i_0 \dots i_n \dots}$ and $f^{n+1}(x) \in I_{i_0 \dots i_n i_0 \dots i_n \dots}$. This means that $x = f^{n+1}(x)$, so the orbit of x has period $n + 1$. $\square$

Theorem 2.14. *The function $\underline{i}^{-1}$ is continuous.*

Proof. Given an $i \in \Omega_T$ and $\varepsilon > 0$ we search for $\delta > 0$ such that $\underline{i}^{-1}(B(i, \delta)) \subseteq B(\underline{i}^{-1}(i), \varepsilon)$. Let us choose an $N \in \mathbb{Z}^+$ such that $2^{-N} < \varepsilon$. Then if $i = (i_0, i_1, \dots)$ and $j \in B(i, 2^{-N})$ then $\underline{i}^{-1}(i), \underline{i}^{-1}(j) \in I_{i_0 \dots i_N}$. By Lemma 2.3 $\lambda(I_{i_0 \dots i_N}) < 5^{-N} < 2^{-N} < \varepsilon$. Hence $\underline{i}^{-1}(j) \in B(\underline{i}^{-1}(i), \varepsilon)$. $\square$

Definition 2.15. Let $\tilde{X} = \{x \in \mathbb{R}_{\geq 0} : \nexists k \in \mathbb{N} : T^k(x) \in \mathbb{N}\}$, the set of points which have no integers in their orbit. We will call this the restricted domain.

Definition 2.16. Let the metric space $\tilde{\Omega}_T \subset \Omega_T$ be the image $\underline{i}(\tilde{X})$ with the subspace metric. We call it the space of allowed restricted trajectories.

Corollary 2.17. *The set $X \setminus \tilde{X}$ is countable. Since $\underline{i}$ is a bijection $\Omega_T \setminus \tilde{\Omega}_T$ is countable as well.*

Theorem 2.18. *The function $\underline{i}' = \underline{i}|_{\tilde{X}} : \tilde{X} \rightarrow \tilde{\Omega}_T$ is a homeomorphism.*

Proof. Since $\underline{i}$ is a bijection with a continuous inverse it is clear that $\underline{i}'$ is a bijection with a continuous inverse.

Let $\underline{i}'(x) = (i_0, \dots)$ for some $x \in \tilde{X}$. If there exists no integer k for which $T^k(x) \in \mathbb{N}$ then by Lemma 2.3 $x \in \text{int}(I_{i_0 \dots i_n})$ for all $n \in \mathbb{N}$. From here given an $\varepsilon > 0$ and an $N \in \mathbb{N}$ for which $2^{-N} < \varepsilon$ let us pick a δ for which $B(x, \delta) \subseteq I_{i_0 \dots i_N}$. This ensures that $\underline{i}'(B(x, \delta)) \subseteq B(\underline{i}'(x), \varepsilon)$. $\square$

Corollary 2.19. *The dynamical systems $(\tilde{X}, T)$ and $(\tilde{\Omega}_T, \sigma)$ are topologically conjugate.*

3 Shadowing

Shadowing of piecewise linear maps of a compact interval onto itself has been extensively studied by multiple authors. From [3] we know that even for tent maps the shadowing property does not hold for an uncountable number of examples. Our system is in many ways similar to tent maps discussed in [3], however we have a non-compact domain, which is generally not considered in the literature. We will show, that our system has the shadowing property.

For this section we will have to define the itinerary of not just a point, but a sequence of non-negative reals in general. To make it distinct from the function $\underline{i}$ defined previously we denote this by $\underline{j}$.

Definition 3.1. Let $\underline{j} : \mathbb{R}_{\geq 0}^{\mathbb{N}} \rightarrow \mathbb{Z}_{>0}^{\mathbb{N}}$ be the symbolic trajectory or itinerary of a sequence. For $x = (x_0, x_1, \dots) \in \mathbb{R}_{\geq 0}^{\mathbb{N}}$ the sequence $\underline{j}(x) = j = (j_0, j_1, \dots) \in \mathbb{Z}_{>0}^{\mathbb{N}}$ be the lexicographically smallest sequence of positive integers such that $x_k \in I_{j_k}$ for all $k \in \mathbb{N}$.

Definition 3.2. For some $H \subset \mathbb{R}$ and $\delta \geq 0$ let us define:

$$B(H, \delta) = \{x \in \mathbb{R} : \exists y \in H : |x - y| < \delta\}.$$

Lemma 3.3. *For all bounded $H \subset \mathbb{R}_{\geq 0}$ and any $\varepsilon \geq 0$ it is true that*

$$f(B(H, \varepsilon)) \subseteq B\left(f(H), \varepsilon \cdot \sup_{y \in B(H, \varepsilon)} |f'(y)|\right)$$

for any continuous $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, which is differentiable at all but finitely many points of any bounded $A \subset \mathbb{R}_{\geq 0}$.

Proof. If $k = \sup_{y \in B(H, \varepsilon)} |f'(y)|$ then f is k -Lipschitz on $B(H, \varepsilon)$. If $x \in B(H, \varepsilon)$, then there is some $y \in H$ such that $|y - x| < \varepsilon$. By the Lipschitz property $|f(x) - f(y)| \leq k|x - y| < k\varepsilon$. This means that $f(x) \in B(f(H), k\varepsilon)$. $\square$

Lemma 3.4. For all $H \subset \mathbb{R}_{\geq 0}$ and any $\varepsilon \geq 0$ it is true that $T^{-1}(B(H, \varepsilon)) \cap I_n \subseteq B(T^{-1}(H), \frac{\varepsilon}{|T'(I_n)|})$ for all $n \in \mathbb{Z}^+$. Furthermore if H is a point, then the set $T^{-1}(H)$ has in every integer interval at most one point.

Proof. In an integer interval the function T is linear with slope $|T'(I_n)|$ so if $T(x) \in B(H, \varepsilon)$ and $x \in I_n$, then $x \in B(T^{-1}(H), \frac{\varepsilon}{|T'(I_n)|})$. Since T is injective on every integer interval, if H is a point, then the set $T^{-1}(H)$ has in every integer interval at most one point. $\square$

Definition 3.5. For a dynamical system (X, f) the sequence $\{x_k\}_{k=0}^{\infty}$ of elements of X is called a δ -pseudo orbit of f , if $x_{k+1} \in B(f(x_k), \delta)$ for all $k \in \mathbb{N}$.

Definition 3.6. Given two sequences of elements of a metric space x_k and y_k it is said that the sequence y_k ε -shadows the sequence x_k , if $y_k \in B(x_k, \varepsilon)$ for all indices k .

Definition 3.7. We say that a dynamical system (X, f) has the shadowing property, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that every δ -pseudo orbit is ε -shadowed by at least one true orbit.

Definition 3.8. Let us define the space Ω'_T to be composed of exactly those sequences of positive integers $(i_0, i_1, \dots)$ which satisfy that $i_{n+1} \in [1, 4^{\ell(i_n)} + 1]$ for any non-negative integer n .

For these sequences $I_{i_0 i_1 \dots}$ is non-empty, thus $\Omega_T \subseteq \Omega'_T$.

Theorem 3.9. Let $\{x_k\}_{k=0}^{\infty}$ be a δ -pseudo orbit. Let $\underline{j}(x) = (i_0, i_1, \dots)$. If $\underline{j}(x) \in \Omega'_T$, then for $y \in I_{i_0 i_1 \dots}$ the sequence $(y, T(y), T^2(y), \dots)$, ε -shadows x_k for $\varepsilon = \frac{\delta}{4}$.

Proof. By the definition of a pseudo orbit $x_{n+1} \in B(T(x_n), \delta)$ any non-negative integer n , so $x_n \in T^{-1}(B(x_{n+1}, \delta))$. From here $x_n \in T^{-1}(B(I_{i_{n+1}}, \delta))$. Also $x_n \in I_{i_n}$. Since $|T'(x)| \geq 5$ for all $x \in \mathbb{R}_{\geq 0}$, then from Lemma 3.4 $x_n \in I_{i_n} \cap T^{-1}(B(I_{i_{n+1}}, \delta)) \subseteq I_{i_n} \cap B(T^{-1}(I_{i_{n+1}}), \frac{\delta}{5})$. This is a subset of $B(I_{i_n i_{n+1}}, \frac{\delta}{5})$.

From here we will prove by induction the following: $x_k \in B(I_{i_k i_{k+1} \dots i_{k+n}}, \sum_{i=1}^n \frac{\delta}{5^i})$ for all $n, k \in \mathbb{N}$.

We will use induction on n . For $n = 1$ we have just proven the statement for all k . We assume that for some $n < N$ the statement holds for every k , and we prove that it also holds for $n = N$. Hence $x_{k+1} \in B(I_{i_{k+1} i_{k+2} \dots i_{k+N}}, \sum_{k=1}^{N-1} \frac{\delta}{5^k})$. This is an interval. Now $x_k \in I_{i_k} \cap B(T^{-1}(x_{k+1}), \frac{\delta}{5}) \subseteq I_{i_k} \cap B(T^{-1}(B(I_{i_{k+1} i_{k+2} \dots i_{k+N}}, \sum_{i=1}^{N-1} \frac{\delta}{5^i})), \frac{\delta}{5})$. From Lemma 3.4 it follows that $T^{-1}(B(I_{i_{k+1} i_{k+2} \dots i_{k+N}}, \sum_{i=1}^{N-1} \frac{\delta}{5^i})) \subseteq B(T^{-1}(I_{i_{k+1} i_{k+2} \dots i_{k+N}}), \sum_{i=2}^N \frac{\delta}{5^i})$. From this it follows that $x_k \in B(I_{i_k i_{k+1} \dots i_{k+N}}, \sum_{i=1}^N \frac{\delta}{5^i})$.

Since it is a geometric series $\sum_{i=1}^{\infty} \frac{\delta}{5^i} = \frac{\delta}{4}$. For any k non-negative integer $\{T^k(y)\} = I_{i_k i_{k+1} \dots}$. From here $x_k \in B(T^k(y), \frac{\delta}{4})$. $\square$

Theorem 3.10. If $\frac{1}{4} > \delta > 0$, then for all δ -pseudo orbits $\{x_n\}_{n=0}^{\infty}$ we have that $\underline{j}(\{T(x_n)\}_{n=0}^{\infty}) \in \Omega'_T$.

Proof. Let $j(\{T(x_n)\}_{n=0}^\infty) = (j_0, j_1, \dots)$ and $\underline{j}(\{x_n\}_{n=0}^\infty) = (i_0, i_1, \dots)$. We want to prove that $I_{j_{n+1}} \subset T(I_{j_n})$ for all natural n . We have seen that $T(x_n) \in B(I_{i_{n+1}}, \delta)$, so if $\delta < 1$, then $i_{n+1} \in \{j_n - 1, j_n, j_n + 1\}$. If $i_{n+1} \leq j_n$, then $T(I_{i_{n+1}}) \subseteq T(I_{j_n})$, so $I_{j_{n+1}} \subset T(I_{j_n})$.

If $i_{n+1} = j_n + 1$, the only time when $T(I_{i_{n+1}}) \subseteq T(I_{j_n})$ does not hold, is when $\ell(j_n + 1) \neq \ell(j_n)$, so $j_n = 4^m$ for some positive integer m . In this case if $x_{n+1} \in [4^m, 4^m + \frac{1}{4}]$, then $T(x_{n+1}) \in T(I_{j_n})$, so $I_{j_{n+1}} \subset T(I_{j_n})$, because if $x_{n+1} = k$ integer, then $j_n = k$ since it is lexicographically smaller. From here if $\delta < \frac{1}{4}$ then $(j_0, j_1, \dots) \in \Omega'_T$. $\square$

Theorem 3.11. *Let $\{x_n\}_{n=0}^\infty$ be a δ -pseudo orbit, with $\delta < \frac{1}{4}$. Let $j(\{T(x_n)\}_{n=0}^\infty) = (j_0, j_1, \dots)$ and $\underline{j}(\{x_n\}_{n=0}^\infty) = (i_0, i_1, \dots)$. Now, if $y \in I_{j_0 j_1 \dots}$, then $\{y, T(y), T^2(y), \dots\}$ 20δ -shadows the sequence $\{T(x_n)\}_{n=0}^\infty$.*

Proof. Let us first define some parameters.

1. Let h_k be the absolute value of the slope of T on $\text{int}(I_k)$.
2. Let $\eta_k = \sup_{y \in B(x_{k+1}, \delta)} |T'(y)|$.
3. Let $a_n^{(k)} = \frac{\eta_{n+k}}{\prod_{l=k}^{n+k} h_{j_l}}$.

We know that $T(x_k) \in B(x_{k+1}, \delta)$. From this $T(T(x_k)) \in T(B(x_{k+1}, \delta))$, so by Lemma 3.3 $T(T(x_k)) \in B(T(x_{k+1}), \eta_k \delta)$. Now as a consequence of Lemma 3.4 $T(x_k) \in I_{j_k} \cap B(T^{-1}(T(x_{k+1})), \frac{\eta_k \delta}{h_{j_k}}) \subseteq B(I_{j_k j_{k+1}}, \frac{\eta_k \delta}{h_{j_k}})$. The last set is equal to $B(I_{j_k j_{k+1}}, \delta a_0^{(k)})$.

Let us assume that $T(x_k) \in B(I_{j_k j_{k+1} \dots j_{k+n}}, \delta \sum_{l=0}^n a_l^{(k)})$ for all k positive integers for all $n \leq N$ for some positive integer N . We prove that the inclusion also holds for $n = N + 1$.

From the assumption $T(x_{k+1}) \in B(I_{j_{k+1} j_{k+2} \dots j_{k+N+1}}, \delta \sum_{l=0}^N a_l^{(k+1)})$. We also know that:

$$T(x_k) \in I_{j_k} \cap B\left(T^{-1}(T(x_{k+1})), \frac{\eta_k \delta}{h_{j_k}}\right).$$

Therefore:

$$T(x_k) \in I_{j_k} \cap B\left(T^{-1}\left(B(I_{j_{k+1} j_{k+2} \dots j_{k+N+1}}, \delta \sum_{l=0}^N a_l^{(k+1)})\right), \frac{\eta_k \delta}{h_{j_k}}\right).$$

By Lemma 3.4 this is included in the following set:

$$I_{j_k} \cap B\left(T^{-1}(I_{j_{k+1} j_{k+2} \dots j_{k+N+1}}), \frac{\delta}{h_{j_k}} \sum_{l=0}^N a_l^{(k+1)} + \frac{\eta_k \delta}{h_{j_k}}\right) = B\left(I_{j_k j_{k+1} \dots j_{k+N+1}}, \delta \sum_{l=0}^{N+1} a_l^{(k)}\right).$$

This by induction proves that $T(x_k) \in B(I_{j_k j_{k+1} \dots}, \delta \sum_{l=0}^\infty a_l^{(k)}) = B(T^k(y), \delta \sum_{l=0}^\infty a_l^{(k)})$ for all k integers. If we prove that four is an upper bound of $\sum_{l=0}^\infty a_l^{(k)}$, then the proof of the theorem is finished.

From its definition $a_n^{(k)} = \frac{\eta_{n+k}}{\eta_{n+k-1} h_{j_{n+k}}} a_{n-1}^{(k)}$. Since $\ell(x_{n+k+1}) \leq \ell(x_{n+k}) + 1$, and we have $\delta < 1$, then $\frac{\eta_{n+k}}{\eta_{n+k-1}} \leq 4$. We always have $h_{j_{n+k}} \geq 5$. Now we only need an upper bound on

$a_0^{(k)} = \frac{\eta_k}{h_{j_k}}$. We know that h_{j_k} is the absolute value of the slope of T on the integer interval $T(x_k)$ is on. Since $T(x_k) \in B(x_{k+1}, \delta)$ this means that $a_0^{(k)}$ is at most four. From here $\sum_{l=0}^{\infty} a_l^{(k)} \leq \sum_{l=0}^{\infty} 4(\frac{4}{5})^l = 20$. $\square$

Theorem 3.12. *The dynamical system $(\mathbb{R}_{\geq 0}, T)$ has the shadowing property.*

Proof. We have seen in Theorem 3.11, that for every δ -pseudo orbit x_k for $\delta < \frac{1}{4}$ we have a y_x for which the sequence $T^k(y_x)$ 20δ -shadows the sequence $T(x_k)$. This means that $x_{k+1} \in B(T^k(y_x), 21\delta)$. Now letting $z_0 \in T^{-1}(x_0)$ and $z_n = x_{n-1}$ for every positive integer n , and we find y_z , then the sequence x_k is 21δ -shadowed by $T^k(y_z)$ sequence, a true orbit. $\square$

4 Frobenius-Perron operator and ℓ -Markov partitions

In this section we introduce the special kind of Markov partition we will consider, and we establish how the Frobenius-Perron operator can be represented in this case.

Definition 4.1. Given a measure space (X, Σ, μ) and a map $f : X \rightarrow X$, the pushforward of μ is $\mu \circ f^{-1}$.

Definition 4.2 (Assumptions). Let our measure space X be some countable union of closed subintervals of the reals and the σ -algebra be the corresponding Lebesgue σ -algebra. We assume that f is differentiable everywhere but at countably many points. Given a density $\rho : X \rightarrow \mathbb{R}_{\geq 0}$ let $\mu(H) = \int_H \rho d\lambda$ for all $H \in \mathcal{L}(X)$ meaning that $\mu \ll \lambda$. In that case the $(\mu \circ f^{-1})(H) = \int_{f^{-1}(H)} \rho d\lambda$.

Definition 4.3. Given a measurable subset X of the reals, which is a countable union of closed subintervals with non-empty interiors, and a measurable $f : X \rightarrow X$ the dynamical system (X, f) has an ℓ -Markov partition if there exists a countable set $\mathcal{H}$ of closed subintervals of $\mathbb{R}$ with non-empty interior such that the following are true:

1. For all $A, B \in \mathcal{H}$ if $A \neq B$ then $\text{int}(A) \cap \text{int}(B) = \emptyset$.
2. $X = \bigcup \mathcal{H}$.
3. For all $A \in \mathcal{H}$ there exists $\emptyset \neq \mathcal{A} \subseteq \mathcal{H}$ such that $\text{int}(f(A)) = \text{int}(\bigcup \mathcal{A})$.
4. The function f is linear on the interior of all $H \in \mathcal{H}$.

Note that this definition differs slightly from ones in the literature. Linearity is not generally required, and in the literature generally only finite partitions are allowed. A very similar definition to ours is the class $\mathcal{L}_{\mathcal{M}}$ in [1]. We will establish a matrix representation of the Frobenius-Perron operator. Similar ideas are present in [1] as well.

Definition 4.4. Let $D(f)$ be the domain of function f and $R(f)$ be its range. We suppose that $D(f), R(f) \subseteq \mathbb{R}$. The Frobenius-Perron operator for transformation $f : D(f) \rightarrow R(f)$ maps the space of measurable functions $\mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ which are supported on $D(f)$ into the space of measurable functions $\mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$, which are supported on $R(f)$. The operator is denoted by $\mathcal{P}_f$. If $\mu(H) = \int_H \rho d\lambda$, then the function $\mathcal{P}_f(\rho)$ is the density of the pushforward measure:

$$\mu(f^{-1}(H)) = \int_{f^{-1}(H)} \rho d\lambda = \int_H \mathcal{P}_f(\rho) d\lambda.$$

If f is a diffeomorphism from its domain to its range, then as stated abundantly in literature, for example in [5] and [1] the operator on the range can be represented in a more compact way:

$$\mathcal{P}_f(\rho)(x) = \frac{\rho(f^{-1}(x))}{|f'(f^{-1}(x))|}. \quad (2)$$

We are considering functions, which have an ℓ -Markov partition. If we restrict such a function to $\text{int}(H)$ for an $H \in \mathcal{H}$ we get that $f|_{\text{int}(H)}$ is a diffeomorphism between $\text{int}(H)$ and $f(\text{int}(H))$. Let $\mathcal{H} = \{H_i\}_{i \in \mathbb{Z}^+}$. Suppose that $\mu(A) = \int_A \rho d\lambda$ for all $A \in \mathcal{L}(X)$. Then from the σ -additivity of measures we have, that $\mu(A) = \sum_{i=1}^{\infty} \mu(A \cap H_i)$, and since μ is absolutely continuous and $\mu(H_i) = \mu(\text{int}(H_i))$, thus $\mu(A) = \sum_{i=1}^{\infty} \mu(A \cap \text{int}(H_i))$. Thus we have the following:

$$\mu(f^{-1}(A)) = \sum_{i=1}^{\infty} \int_{f^{-1}(A) \cap \text{int}(H_i)} \rho d\lambda = \sum_{i=1}^{\infty} \int_{(f|_{\text{int}(H_i)})^{-1}(A)} \rho \cdot \chi_{\text{int}(H_i)} d\lambda \quad (3)$$

$$\mu(f^{-1}(A)) = \sum_{i=1}^{\infty} \int_A \mathcal{P}_{f|_{\text{int}(H_i)}}(\rho \cdot \chi_{\text{int}(H_i)}) d\lambda. \quad (4)$$

According to Beppo Levi's theorem as stated in Corollary 21.39.b. of [7], we have that:

$$\mu(f^{-1}(A)) = \sum_{i=1}^{\infty} \int_A \mathcal{P}_{f|_{\text{int}(H_i)}}(\rho \cdot \chi_{\text{int}(H_i)}) d\lambda = \int_A \sum_{i=1}^{\infty} \mathcal{P}_{f|_{\text{int}(H_i)}}(\rho \cdot \chi_{\text{int}(H_i)}) d\lambda. \quad (5)$$

We have that $f|_{\text{int}(H_i)}$ is a diffeomorphism between its domain and its range. This along with (2) and (5) implies the following:

$$\mathcal{P}_f(\rho)(x) = \sum_{i=1}^{\infty} \mathcal{P}_{f|_{\text{int}(H_i)}}(\rho \cdot \chi_{\text{int}(H_i)})(x) = \sum_{z \in f^{-1}(x)} \frac{\rho(z)}{|f'(z)|}. \quad (6)$$

This representation for finite partitions is present in [1], however it generalizes in a straightforward way to our countably infinite case.

This representation does not make sense if $f^{-1}(x)$ has points where f is not differentiable. We assumed in Definition 4.2 that f is differentiable at everywhere but countably many points. Since for all $z \in X$, if $z \in f^{-1}(x) \cap f^{-1}(y)$, where $x, y \in X$, then $x = y$. This means that the points, where f is not differentiable can only be included in the preimages of countably many points, thus (6) defines the new density almost everywhere, thus its integral will be well-defined.

Definition 4.5. Let (X, f) be a dynamical system with an ℓ -Markov partition $\{H_k\}_{k \in \mathbb{Z}^+} = \mathcal{H}$. We say that $P = (p_{ij})$ is its transition matrix if the following are satisfied:

1. If $n = |\mathcal{H}|$ then P is an $n \times n$ matrix, where $n \in \mathbb{Z}^+$. If $\mathcal{H}$ is countably infinite, then $P = (p_{ij})_{i,j \in \mathbb{Z}^+}$.
2. For i, j where $\text{int}(H_j) \subseteq f(H_i)$ it is true that

$$p_{ij} = \frac{\lambda(H_j)}{\lambda(H_i)|f'(H_i)|} \quad (7)$$

where $f'(H_i)$ is the slope of f on the interior of H_i . Otherwise p_{ij} is zero.

For most statements we assume that a partition is countably infinite. In all cases the statements and their proofs can trivially be modified for finite partitions.

Definition 4.6. An infinite matrix is row stochastic if for any row the sum of the first n elements of it tends to one as n tends to infinity.

Corollary 4.7. *The transition matrix is always row-stochastic.*

Theorem 4.8. *Suppose that we have an absolutely continuous measure μ with a possibly infinite density ρ and it is constant c_i on the interiors of all $H_i \in \mathcal{H}$. Furthermore suppose, that for all $H_i \in \mathcal{H}$ we have that $\lambda(H_i) < \infty$. Let the row vector $[c_1, c_2, \dots] = C$, and Λ be the row vector $[\lambda(H_1), \lambda(H_2), \dots]$. In this case for all $k \in \mathbb{N}$:*

$$\mathcal{P}_f^k(\rho)(x) = \frac{((C \cdot \Lambda)P^k)_j}{\lambda(H_j)} \text{ for all } x \in \text{int}(H_j). \quad (8)$$

Where $v \cdot w$ for two vectors with identical dimensions is the coordinatewise product, such that $(v_1, \dots, v_n, \dots) \cdot (w_1, \dots, w_n, \dots) = (v_1 w_1, \dots, v_n w_n, \dots)$.

Proof. First let us assume that $\rho = 1$ on $\text{int}(H_i)$ for some i and zero otherwise. Let us start with $k = 1$. Since we have linearity on all H_n , we have that $f^{-1}(x) \cap H_n$ has at most one element, for all $H_n \in \mathcal{H}$. As a consequence of this the left hand side of (8) for all $x \in \text{int}(H_j)$ equals:

$$\mathcal{P}_f(\rho)(x) = \sum_{z \in f^{-1}(x)} \frac{c_i \chi_{\text{int}(H_i)}(z)}{|f'(H_i)|} = \begin{cases} 0 & \text{if } p_{ij} = 0 \\ \frac{c_i}{|f'(H_i)|} & \text{otherwise.} \end{cases}$$

The right hand side from (7) is as follows:

$$\frac{c_i \lambda(H_i) p_{ij}}{\lambda(H_j)} = \begin{cases} 0 & \text{if } p_{ij} = 0 \\ \frac{c_i}{|f'(H_i)|} & \text{otherwise.} \end{cases}$$

Now we can prove the statement for $k = 1$ by noting that both sides of (8) are linear in terms of C . Let us consider the densities $\{\rho_n\}_{n \in \mathbb{Z}^+}$. Their definition is as follows:

$$\rho_n(x) = \begin{cases} 1 & \text{if } x \in \text{int}(H_n) \\ 0 & \text{otherwise.} \end{cases}$$

Let the vector of constants corresponding to ρ_n be C_n , so that all coordinates of C_n are zero except for the n th which is one. Any density ρ and vector C we are considering can be written in the following way:

$$\rho = \sum_{n \in \mathbb{Z}^+} \alpha_n \rho_n, \quad C = \sum_{n \in \mathbb{Z}^+} \alpha_n C_n \text{ where } \alpha_n \in \overline{\mathbb{R}}_{\geq 0}.$$

Note that since all summands are non-negative these sums always exist in $\overline{\mathbb{R}}$. From linearity of the formulae we have that for all $x \in \text{int}(H_j)$:

$$\mathcal{P}_f(\rho)(x) = \sum_{n \in \mathbb{Z}^+} \alpha_n \mathcal{P}_f(\rho_n)(x) = \sum_{n \in \mathbb{Z}^+} \alpha_n \frac{((C_n \cdot \Lambda)P)_j}{\lambda(H_j)} = \frac{((C \cdot \Lambda)P)_j}{\lambda(H_j)}. \quad (9)$$

This proves the statement for $k = 1$. It is also clear that for a density ρ satisfying the assumptions of the theorem the density $\mathcal{P}_f(\rho)$ also satisfies them. Suppose that the statement is true for all $k < n$, we prove that it is true for $k = n$. Let $C^{(k)} = [c_0^{(k)}, \dots]$ be the vector of constants of the k th pushforward measure of μ . What remains to be proven is that for all $x \in \text{int}(H_j)$:

$$\mathcal{P}_f^n(\rho)(x) = \frac{((C \cdot \Lambda)P^n)_j}{\lambda(H_j)} \quad (10)$$

By the induction assumption:

$$C^{(n-1)} = \left[\frac{((C \cdot \Lambda)P^{n-1})_i}{\lambda(H_i)} \right]_{i \in \mathbb{Z}^+}.$$

As a consequence of the induction assumption and (9) we have the following for all $x \in \text{int}(H_j)$:

$$\mathcal{P}_f^n(\rho)(x) = \mathcal{P}_f(\mathcal{P}_f^{n-1}(\rho))(x) = \frac{((C^{(n-1)} \cdot \Lambda)P)_j}{\lambda(H_j)} = \frac{\left(\left(\left[\frac{((C \cdot \Lambda)P^{n-1})_i}{\lambda(H_i)} \right]_{i \in \mathbb{Z}^+} \cdot \Lambda \right) P \right)_j}{\lambda(H_j)}.$$

Since $\left[\frac{((C \cdot \Lambda)P^{n-1})_i}{\lambda(H_i)} \right]_{i \in \mathbb{Z}^+} \cdot \Lambda = (C \cdot \Lambda)P^{n-1}$ equation (10) is satisfied. This by induction proves the statement for all k . $\square$

This basically relates a dynamical system with an ℓ -Markov partition to a Markov chain, hence the name of this property. This is extremely useful in finding absolutely continuous invariant measures of a system.

Corollary 4.9. *A row-eigenvector v of P associated with the eigenvalue 1 defines (almost everywhere) an invariant absolutely continuous measure μ of f with density ρ in the following way:*

$$\rho(x) = \frac{v_i}{\lambda(H_i)} \text{ for all } x \in H_i.$$

The transition matrix defines a Markov chain with countable states. The elements of the matrix are called the transition probabilities. About the convergence of these values [2] writes the following:

Definition 4.10. The following are parameters of the Markov chain defined by P :

1. $p_{ij}^{(k)} = (P^k)_{i,j}$ the probability of stepping on j after k steps if we are in i .
2. $f_{ij}^{(n)} = \sum_{m \in M} p_{im_1} p_{m_1 m_2} \dots p_{m_{n-1} j}$ where $m = (m_1, \dots, m_{n-1})$ and M is the set of $n-1$ long positive integer sequences not containing j . This is the probability of starting from i and taking a path arriving at j for the first time after exactly n steps.
3. $f_{ij}^* = \sum_{n=1}^{\infty} f_{ij}^{(n)}$ the probability of eventually getting to j if we start from i .
4. $m_i = \sum_{n=1}^{\infty} n f_{ii}^{(n)}$ the average return time to i if we start from it. For i transient or null-recurrent state this is defined to be ∞ .
5. $d_i = \gcd\{n : p_{ii}^{(n)} \neq 0\}$ the period of state i , where $\gcd$ stands for greatest common divisor.

6. $f_{ij}^*(r) = \sum_{n=1}^{\infty} f_{ij}^{(nd_i+r)}$. This is the probability of eventually getting to j if we start from i given that we only consider every $(nd_i + r)$ th step.

Theorem 4.11 (According to Theorem 4 on page 33 of [2]). *For all $r \in \{1, \dots, d_j\}$*

$$\lim_{n \rightarrow \infty} p_{ij}^{(nd_j+r)} = f_{ij}^*(r) \frac{d_j}{m_j}.$$

If $m_j = \infty$, then the limit is zero.

Corollary 4.12. *If the matrix P defines an aperiodic irreducible Markov chain, then the pointwise limit of P^k exists as k tends to infinity. If additionally $f_{ii}^* = 1$ for all i then all the columns of $\lim_{k \rightarrow \infty} P^k$ are constant, so there is a row vector w such that for all stochastic row vectors v of sufficient dimension $\lim_{k \rightarrow \infty} vP^k = w$ coordinatewise.*

Proof. If P is aperiodic, then $d_j = 1$ for all j , hence by Theorem 4.11 $\lim_{n \rightarrow \infty} p_{ij}^{(n)}$ exists for all i, j . The value of d_j and m_j only depend on the value of j so they are constant throughout a column. With irreducibility we have that $p_{ij}^{(n)} \neq 0$ for all i, j for some n . Since $f_{ii}^* = 1$ we have that we return infinitely many times from i to i with probability one. From this $f_{ij}^* = 1$ for all i, j . This makes $\lim_{n \rightarrow \infty} p_{ij}^{(n)}$ independent of i . By this for all v stochastic vectors $\lim_{k \rightarrow \infty} (vP^k)_j = \lim_{k \rightarrow \infty} p_{1j}^{(k)}$. $\square$

Corollary 4.13. *Suppose that P satisfies all conditions in Corollary 4.12. Let $\pi : \mathcal{L}(X) \rightarrow \mathbb{R}_{\geq 0}$ be the absolutely continuous measure with density τ where:*

$$\tau(x) = \frac{\lim_{k \rightarrow \infty} p_{ii}^{(k)}}{\lambda(H_i)} \text{ for all } x \in \text{int}(H_i).$$

If $\mu : \mathcal{L}(X) \rightarrow \mathbb{R}_{\geq 0}$ is an absolutely continuous measure with density ρ where $\mu(X) < \infty$ and ρ is constant on $\text{int}(H_i)$ for all i , then $\lim_{k \rightarrow \infty} \mu(f^{-k}(A)) = \mu(X)\pi(A)$ for all $A \in \mathcal{L}(X)$.

Proof. Let the vector of constants of ρ as defined in Theorem 4.8 be $C = [c_1, c_2, \dots]$. By Theorem 4.8 we have that

$$\mathcal{P}_f^k(\rho)(x) = \frac{((C \cdot \Lambda)P^k)_i}{\lambda(H_i)} \text{ for all } x \in \text{int}(H_i).$$

Let $r = \sum_{i=1}^{\infty} c_i \lambda(H_i) = \mu(X)$. Since we have assumed that $\mu(X) < \infty$ we have that $r < \infty$. Let $q = \frac{1}{r}(C \cdot \Lambda)$, thus q is stochastic. By Corollary 4.12 we have the following:

$$\lim_{k \rightarrow \infty} ((C \cdot \Lambda)P^k)_i = r \cdot \lim_{k \rightarrow \infty} (qP^k)_i = r \cdot \lim_{k \rightarrow \infty} p_{ii}^{(k)}.$$

This implies that on all $\text{int}(H_i)$ the sequence $\mathcal{P}_f^k(\rho)$ converges uniformly to $\mu(X)\tau$. For all $\varepsilon > 0$ and for all measurable $A \subseteq X$ there exists an N_0 such that:

$$\left| \mu(A) - \mu\left(A \cap \bigcup_{i=1}^{N_0} \text{int}(H_i)\right) \right| + \left| \pi(A) - \pi\left(A \cap \bigcup_{i=1}^{N_0} \text{int}(H_i)\right) \right| < \varepsilon.$$

Let $G_\varepsilon = \bigcup_{i=1}^{N_0} \text{int}(H_i)$. On G_ε the sequence $\mathcal{P}_f^k(\rho)$ converges uniformly to $\mu(X)\tau$, and $\lambda(G_\varepsilon) < \infty$ thus $\mu(f^{-k}(A \cap G_\varepsilon)) \rightarrow \mu(X)\pi(A \cap G_\varepsilon)$. As $\varepsilon \rightarrow 0$ this tends to $\mu(X)\pi(A)$. $\square$

Corollary 4.14. *For the measure π the measure of the whole space is $0 \leq \pi(X) \leq 1$.*

5 Expansive ℓ -Markov partitions

In this section we restrict the class of systems considered so that we can meaningfully discuss the absolutely continuous invariant measures. These will be systems with an expansive ℓ -Markov partition. One of the corollaries of our results is that all of the finite absolutely continuous invariant measures of these systems are piecewise constant on the elements of the expansive ℓ -Markov partition. For finite partitions this result is present in [1] with a different proof. Our results are unique in that we focus on the convergence of the push-forwards of absolutely continuous measures, thus we prove stronger statements than just the characterization of the absolutely continuous invariant measures. Generally when discussing absolutely continuous invariant measures only σ -finite ones are considered however we consider any measure which has a $\rho : X \rightarrow \mathbb{R}_{\geq 0}$ density.

Definition 5.1. Let (X, f) be a dynamical system with an ℓ -Markov partition $\mathcal{H}$ as in Definition 4.3. Let the elements of $\mathcal{H}$ be H_i where i is a positive integer. Then for a sequence of positive integers of not necessarily finite length $(i_0, i_1, \dots)$ define:

$$H_{i_0 i_1 \dots} = \bigcap_{n=0}^{\infty} f^{-n}(H_{i_n}).$$

Note the similarity with Definition 2.2. If the sequence is finite then we only take finite intersections.

Definition 5.2. Let us define the set $D_0 = \{x \in X : \exists i \in \mathbb{Z}^+ : x \in \partial H_i\}$. Let D be the set $\bigcup_{k=0}^{\infty} f^{-k}(D_0)$. Let for all $A \subseteq X$ the set $\tilde{A}$ be $A \setminus D$.

Corollary 5.3. The set D is countable, thus $\lambda(A \Delta \tilde{A}) = 0$ for all $A \in \mathcal{L}(X)$. On all $\tilde{H}_i$ the function f is a homeomorphism between $\tilde{H}_i$ and $f(\tilde{H}_i)$.

Lemma 5.4. Given a finite sequence of positive integers $(i_0, \dots, i_n)$ the following are true for $H_{i_0 \dots i_n}$:

1. For all $H_i \in \mathcal{H}$ we have that $f(\text{int}(H_i)) = \text{int}(f(H_i))$.
2. $\text{int}(H_{i_0 \dots i_n})$ is an interval.
3. If $\text{int}(H_i) \cap f^{-1}(H_j) \neq \emptyset$ then $\text{int}(H_j) \subseteq \text{int}(f(H_i))$.
4. If $H_{i_0 \dots i_n}$ has a non-empty interior then $f^k(\text{int}(H_{i_0 \dots i_n})) = \text{int}(H_{i_k \dots i_n})$ for all integers $1 \leq k \leq n$.
5. For all $(j_0 \dots j_n) \neq (i_0 \dots i_n)$ sequences of positive integers $\text{int}(H_{i_0 \dots i_n}) \cap \text{int}(H_{j_0 \dots j_n}) = \emptyset$.
6. $\text{int}(H_i) = \text{int}(\bigcup \{H_{ik} : \text{int}(H_k) \subseteq f(H_i)\})$.
7. $\text{int}(H_{i_0 \dots i_n}) = \text{int}(\bigcup \{H_{i_0 \dots i_n k} : \text{int}(H_k) \subseteq f(H_{i_n})\})$.
8. For all $k \in \mathbb{Z}^+$ the intersection $\text{int}(H_{i_0 \dots i_n k}) \cap \text{int}(H_{i_0 \dots i_n})$ is non-empty if and only if $k \in \{m : \text{int}(H_m) \subseteq f(H_{i_n})\}$.
9. If $(\text{int}(H_{i_0 \dots i_n k}) \cap \text{int}(H_{i_0 \dots i_n})) \neq \emptyset$ then $\lambda(H_{i_0 \dots i_n k}) = \frac{\lambda(H_k)}{|f'(H_{i_n})|} \lambda(H_{i_0 \dots i_n})$.

Proof. It is worth noting that $H_{i_0 \dots i_n}$ is the set of points for which for all $0 \leq k \leq n$ integers $f^k(x) \in H_{i_k}$.

1. We have that f is a homeomorphism on $\text{int}(H_i)$ and that $f(H_i) = f(\text{int}(H_i)) \cup f(\partial H_i)$. Since $f(\partial H_i)$ consists of at most two points we have that $\text{int}(f(\text{int}(H_i)) \cup f(\partial H_i)) = \text{int}(f(\text{int}(H_i))) = f(\text{int}(H_i))$.
2. We have that $\text{int}(H_{i_0 \dots i_n}) = \text{int}(H_{i_0}) \cap \text{int}(f^{-1}(H_{i_1 \dots i_n}))$. On $\text{int}(H_{i_0})$ the function f is a homeomorphism, so we also have that $\text{int}(H_{i_0}) \cap \text{int}(f^{-1}(H_{i_1 \dots i_n})) = \text{int}(H_{i_0}) \cap f^{-1}(\text{int}(H_{i_1 \dots i_n}))$, and that the preimage of an interval is an interval, thus if $\text{int}(H_{i_1 \dots i_n})$ is an interval, then $\text{int}(H_{i_0 \dots i_n})$ is also an interval. Since $\text{int}(H_i)$ is always an interval, this proves the statement by induction for all cases.
3. We have that $\text{int}(H_i) \cap f^{-1}(H_j) \neq \emptyset$ if and only if $f(\text{int}(H_i) \cap f^{-1}(H_j)) \neq \emptyset$. Since f is a homeomorphism on $\text{int}(H_i)$ this is equivalent to $\text{int}(f(H_i)) \cap H_j \neq \emptyset$. From Definition 4.3 we have that there exists $\emptyset \neq \mathcal{A} \subseteq \mathcal{H}$, such that $\text{int}(f(H_i)) = \text{int}(\cup \mathcal{A})$. It follows, that there exists $H_k \in \mathcal{A}$ such that $\text{int}(H_k) \cap H_j \neq \emptyset$. From Definition 4.3 this implies that $H_k = H_j$ and thus $\text{int}(H_j) \subseteq \text{int}(f(H_i))$.
4. It suffices to show that if $\text{int}(H_{i_0 \dots i_n}) \neq \emptyset$ then $f(\text{int}(H_{i_0 \dots i_n})) = \text{int}(H_{i_1 \dots i_n})$. This is equivalent to $f(\text{int}(H_{i_0}) \cap \text{int}(f^{-1}(H_{i_1 \dots i_n}))) = \text{int}(H_{i_1 \dots i_n})$. Since f is a homeomorphism on $\text{int}(H_{i_0})$ we have that this is equivalent to $\text{int}(f(H_{i_0})) \cap \text{int}(H_{i_1 \dots i_n}) = \text{int}(H_{i_1 \dots i_n})$. We have that $\text{int}(H_{i_0 \dots i_n}) \neq \emptyset$, thus $\text{int}(H_{i_0}) \cap f^{-1}(H_{i_1}) \neq \emptyset$. This implies that $\text{int}(H_{i_1}) \subseteq \text{int}(f(H_{i_0}))$. This proves the statement.
5. For some k we have that $i_k \neq j_k$. In this case $\text{int}(H_{i_k}) \cap \text{int}(H_{j_k}) = \emptyset$. We have that $f^k(\text{int}(H_{i_0 \dots i_n})) \cap f^k(\text{int}(H_{j_0 \dots j_n})) = \text{int}(H_{i_k \dots i_n}) \cap \text{int}(H_{j_k \dots j_n}) = \emptyset$. This proves the statement.
6. Let $\mathcal{A} = \{H_k : \text{int}(H_k) \subseteq f(H_i)\}$. From Definition 4.3 we have that $\text{int}(f(H_i)) = \text{int}(\cup \mathcal{A})$. Let us take the preimage of both sides. Now we have that $\text{int}(H_i) \subseteq f^{-1}(f(\text{int}(H_i))) = f^{-1}(\text{int}(f(H_i))) = f^{-1}(\text{int}(\cup \mathcal{A}))$. In summary we have that $\text{int}(H_i) \subseteq f^{-1}(\text{int}(\cup \mathcal{A}))$. The left hand side of this is an open set, so by taking the interior of the right hand side, the inclusion is not violated, so $\text{int}(H_i) \subseteq \text{int}(f^{-1}(\text{int}(\cup \mathcal{A})))$. This implies that $\text{int}(H_i) \subseteq \text{int}(f^{-1}(\cup \mathcal{A}))$. Let us denote $\{H_{i_k} : \text{int}(H_k) \subseteq f(H_i)\}$ as $\{H_{i_k}\}$. We have that $\text{int}(\cup \{H_{i_k}\}) = \text{int}(H_i) \cap \text{int}(f^{-1}(\cup \mathcal{A})) = \text{int}(H_i)$.
7. We have that $\text{int}(H_{i_0 \dots i_n}) = \text{int}(H_{i_0 \dots i_{n-1}}) \cap \text{int}(f^{-n}(H_{i_n}))$. Since f^n is a homeomorphism on $\text{int}(H_{i_0 \dots i_{n-1}})$ thus we can write that $\text{int}(H_{i_0 \dots i_n}) = \text{int}(H_{i_0 \dots i_{n-1}}) \cap f^{-n}(\text{int}(H_{i_n}))$. Since $\text{int}(H_{i_n}) = \text{int}(\cup \{H_{i_{n,k}} : \text{int}(H_k) \subseteq f(H_{i_n})\})$ the statement follows easily.

The rest follows straightforward from the definitions and previous statements of this lemma. $\square$

For $A, B \subseteq X$ we use the notation $A \Delta B = (A \setminus B) \cup (B \setminus A)$ for the symmetric difference.

Definition 5.5. The dynamical system (X, f) is said to have an expansive ℓ -Markov partition, if for all $\delta > 0$ and $I \subseteq X$ interval there is a finite set of positive integer sequences of

finite length $(i_0^{(1)}, \dots, i_{n_1}^{(1)}), \dots, (i_0^{(N)}, \dots, i_{n_N}^{(N)})$ for which:

$$\lambda\left(I\Delta \bigcup_{k=1}^N H_{i_0^{(k)} \dots i_{n_k}^{(k)}}\right) < \delta.$$

We remark that if the measure of the sets of the partition is bounded and there is some $\varepsilon > 0$ that $|f'(H_i)| > 1 + \varepsilon$ for all $H_i \in \mathcal{H}$, then $\mathcal{H}$ is expansive.

Definition 5.6. Let the symbolic topology τ_σ on $\tilde{X}$ be the one generated by the empty set, and the elements of $\{\tilde{H}_{i_0 \dots i_n} : n \in \mathbb{Z}^+, (i_0, \dots, i_n) \in (\mathbb{Z}^+)^n\}$.

Theorem 5.7. *If the ℓ -Markov partition is expansive, then the symbolic topology is the standard metric topology τ on $\tilde{X}$ inherited from $\mathbb{R}$.*

Proof. All $\text{int}(H_{i_0 \dots i_n})$ are open intervals and we have that $\tilde{H}_{i_0 \dots i_n} = \text{int}(H_{i_0 \dots i_n}) \cap \tilde{X}$. This implies that $\tilde{H}_{i_0 \dots i_n} \in \tau$, thus $\tau_\sigma \subseteq \tau$. We also prove that $\tau \subseteq \tau_\sigma$.

Lemma 5.8. *For any infinite positive integer sequence $i_0, i_1, \dots$ the set $\tilde{H}_{i_0 \dots}$ is either empty or has exactly one element.*

Proof. Proceeding towards a contradiction let us suppose that $x \neq y$ and $x, y \in \tilde{H}_{i_0 \dots}$. Now $\widetilde{(x, y)} \subseteq \tilde{H}_{i_0 \dots}$. This implies that for all finite positive integer sequences $(j_0, \dots, j_n)$ we have, that either $\widetilde{(x, y)} \subseteq \tilde{H}_{j_0 \dots j_n}$ or $\widetilde{(x, y)} \cap \tilde{H}_{j_0 \dots j_n} = \emptyset$. This contradicts that $\mathcal{H}$ is an expansive Markov partition, since we cannot approximate any proper subinterval of (x, y) arbitrarily well by labelled intervals. $\square$

Let us consider a $\emptyset \neq G \in \tau$, such that for some $a, b \in \mathbb{R}$ we have that $\widetilde{(a, b)} = G$. Now for any $x \in G$ we have $c, d \in G$ such that $c < x < d$. Since $\tilde{H}_{x, f(x), f^2(x) \dots}$ can only have one element, there exists some $k \in \mathbb{N}$ such that $\tilde{H}_{x \dots f^k(x)} \cap \{c, d\} = \emptyset$ thus $\tilde{H}_{x \dots f^k(x)} \subset G$. This implies that $G \in \tau_\sigma$ thus $\tau \subseteq \tau_\sigma$. $\square$

Definition 5.9. Let the function $\underline{i} : \tilde{X} \rightarrow \mathbb{Z}^{\mathbb{N}}$ be the itinerary of a point. For a point $x \in \tilde{X}$ it gives the integer sequence $(i_0, \dots)$ for which $x \in \tilde{H}_{i_0 \dots}$.

Since we are in $\tilde{X}$ it is straightforward to see from Lemma 5.4 and Theorem 5.7 that $\underline{i}$ is a well-defined bijection between $\tilde{X}$ and $\underline{i}(\tilde{X})$.

Definition 5.10. We denote by $\Omega_f = \{\underline{i}(x) : x \in \tilde{X}\}$ the space of allowed symbolic trajectories. We define the metric $d : \Omega_f \times \Omega_f \rightarrow \tilde{X}$ such, that $d((i_0, \dots), (j_0, \dots)) = 2^{-n}$ where n is the smallest non-negative integer for which $i_n \neq j_n$.

Theorem 5.11. *The function $\underline{i}$ is a homeomorphism between $\tilde{X}$ and $\underline{i}(\tilde{X}) = \Omega_f$.*

Proof. Let $\underline{i}(x) = (i_0, \dots)$ for some $x \in \tilde{X}$. Let us start by proving the continuity of $\underline{i}$. Then $x \in \text{int}(H_{i_0 \dots i_n})$ for all $n \in \mathbb{N}$. From here given an $\varepsilon > 0$ and an $N \in \mathbb{N}$ for which $2^{-N} < \varepsilon$ let us pick a $\delta > 0$ for which $U = B(x, \delta) \subseteq \text{int}(H_{i_0 \dots i_N})$. This ensures that $\underline{i}(\tilde{U}) \subseteq B(\underline{i}(x), \varepsilon)$.

For the continuity of the inverse we have that $\underline{i}(\tilde{H}_{i_0 \dots i_n}) = B(\underline{i}(x), 2^{-n})$. From Theorem 5.7 we have that for all $\varepsilon > 0$ we have $N \in \mathbb{N}$ such that $\tilde{H}_{i_0 \dots i_N} \subset B(x, \varepsilon)$. It follows that for $0 < \delta < 2^{-N}$ we have that $\underline{i}^{-1}(B(\underline{i}(x), \delta)) \subseteq B(x, \varepsilon)$. $\square$

Corollary 5.12. *The dynamical systems $(\tilde{X}, f)$ and (Ω_f, σ) are topologically conjugate.*

Theorem 5.13. *If for some $x \in \tilde{X}$ the orbit of $\underline{i}(x)$ is periodic for the shift operator then the orbit of x is periodic as well.*

Proof. If $\underline{i}(x)$ is $i_0 \dots i_n i_0 \dots i_n \dots$, then $x \in \tilde{H}_{i_0 \dots i_n i_0 \dots i_n \dots}$ and $f^{n+1}(x) \in \tilde{H}_{i_0 \dots i_n i_0 \dots i_n \dots}$. This means that $x = f^{n+1}(x)$, so the orbit of x has period $n + 1$. $\square$

Theorem 5.14. *Suppose that we have a dynamical system (X, f) which has an expansive ℓ -Markov partition $\mathcal{H}$, defining an aperiodic irreducible Markov chain which satisfies the conditions of Corollary 4.12. Let $\pi : \mathcal{L}(X) \rightarrow \overline{\mathbb{R}}_{\geq 0}$ be the absolutely continuous stationary measure with density τ defined in Corollary 4.13. Let $\mu : \mathcal{L}(X) \rightarrow \overline{\mathbb{R}}_{\geq 0}$ be an absolutely continuous measure with $\mu(X) < \infty$. In this case for all $A \in \mathcal{L}(X)$ it is true that $\lim_{n \rightarrow \infty} \mu(f^{-n}(A)) = \mu(X)\pi(A)$.*

Proof. Let the density of μ be γ , such that $\mu(H) = \int_H \gamma d\lambda$. Let us approximate γ by a simple function ϕ in the following way:

$$\phi = \sum_{k=1}^n c_k \chi_{E_k}, \quad \left| \int_H \phi d\lambda - \int_H \gamma d\lambda \right| < \varepsilon, \quad \forall H \in \mathcal{L}(X).$$

Here E_k are measurable subsets of X and χ_{E_k} is the characteristic function of E_k . Now we approximate the set E_k by the union of finitely many closed intervals R such that $\lambda(E_k \Delta R) < \delta$.

Lemma 5.15. *For all $\delta > 0$ and E_k there is a union of finitely many closed intervals R such that $\lambda(E_k \Delta R) < \delta$.*

Proof. From the definition of the Lebesgue outer measure we can pick an $\{R_i\}_{i \in \mathbb{Z}^+}$ cover of E_k with countably many closed intervals for which $E_k \subseteq \bigcup_{i \in \mathbb{Z}^+} R_i$ and $\sum_{i \in \mathbb{Z}^+} \lambda(R_i) < \lambda(E_k) + \frac{\delta}{2}$. In this case there is a finite union $R = \bigcup_{k=1}^N R_{i_k}$ for which $R \Delta \bigcup_{i \in \mathbb{Z}^+} R_i < \frac{\delta}{2}$, which implies that $\lambda(E_k \Delta R) < \delta$. $\square$

By Lemma 5.15 γ can be approximated arbitrarily well by a function which is the linear combination of characteristic functions of finitely many closed intervals. Since $\mathcal{H}$ is expansive we can do it with intervals labelled by finite sequences as well.

Let us pick such an approximation ω of γ . Let $\Omega(H) = \int_H \omega d\lambda$. In this case the measure Ω is an approximation of μ . For all $\varepsilon > 0$ we can pick ω such that $|\Omega(H) - \mu(H)| < \varepsilon$ for all $H \in \mathcal{L}(X)$.

Lemma 5.16. *If $\mu : \mathcal{L}(X) \rightarrow \overline{\mathbb{R}}_{\geq 0}$ is such that $\mu(A) = \lambda(A \cap H_{i_0 \dots i_n})$ for a fixed sequence of positive integers $(i_0, \dots, i_n)$, then $\mu(f^{-1}(A)) = \frac{\lambda(A \cap H_{i_1 \dots i_n})}{|f'(H_{i_0})|}$.*

Proof. The density ρ of this measure μ is $\chi_{H_{i_0 \dots i_n}}$. We need to calculate $\mathcal{P}_f(\rho)$. By (6) we have the following:

$$\mathcal{P}_f(\rho)(x) = \sum_{z \in f^{-1}(x)} \frac{\chi_{H_{i_0 \dots i_n}}(z)}{|f'(z)|}.$$

If $\rho(z) \neq 0$, then $z \in H_{i_0}$. Also, if $f^{-1}(x) \cap H_{i_0 \dots i_n} \neq \emptyset$, then $x \in H_{i_1 \dots i_n}$. Since f is injective on $\text{int}(H_{i_0})$, for almost every x we have that $f^{-1}(x) \cap H_{i_0 \dots i_n}$ has at most one element. These imply the following:

$$\mathcal{P}_f(\rho)(x) = \frac{\chi_{H_{i_1 \dots i_n}}(x)}{|f'(H_{i_0})|}.$$

□

We have that ω is a linear combination of characteristic functions of finitely many labelled sets. This implies that for some $M \in \mathbb{N}$, for some $\{\eta_k\}_{k=1}^M \in \mathbb{R}_{\geq 0}^M$ and for some $\{(i_0^{(k)}, \dots, i_{n_k}^{(k)})\}_{k=1}^M$ positive integer sequences of finite length we can represent ω in the following way:

$$\omega = \sum_{k=1}^M \eta_k \chi_{H_{i_0^{(k)} \dots i_{n_k}^{(k)}}}.$$

Assuming that $n_k \geq 1$ for all k , from Lemma 5.16 we have that:

$$\mathcal{P}_f(\omega) = \sum_{k=1}^M \eta_k \frac{\chi_{H_{i_1^{(k)} \dots i_{n_k}^{(k)}}}}{|f'(H_{i_0^{(k)}})|}. \quad (11)$$

Let $N = \max\{n_k\}_{k=1}^M$. From (11) we have that the pushforward of ω is still the linear combination of characteristic functions of finitely many labelled sets, however the maximal length of these labels is $N - 1$. From the definition of the ℓ -Markov partition the assumption that $n_k \geq 1$ for all k did not violate generality since the pushforward of the characteristic function of an element of $\mathcal{H}$ is still the linear combination of characteristic functions of finitely many elements of $\mathcal{H}$. By these remarks, for $n > N$ we have that $\mathcal{P}_f^n(\omega)$ is a linear combination of characteristic functions of finitely many elements of $\mathcal{H}$.

This by 4.13 implies that for all $H \in \mathcal{L}(X)$ there is a K , such that if $n > K$ then $|\Omega(f^{-n}(H)) - \Omega(X)\pi(H)| < \varepsilon$. Hence $|\mu(f^{-n}(H)) - \Omega(X)\pi(H)| < 2\varepsilon$. Then $|\mu(X)\pi(H) - \Omega(X)\pi(H)| < \varepsilon\pi(H) \leq \varepsilon$. From these $|\mu(f^{-n}(H)) - \mu(X)\pi(H)| < 3\varepsilon$. □

Corollary 5.17. *A dynamical system which satisfies all conditions of Theorem 5.14 with the corresponding measure π is mixing.*

Proof. Define the measure $\mu(H) = \pi(H \cap B)$ for all $H \in \mathcal{L}(X)$ and some $B \in \mathcal{L}(X)$. From Theorem 5.14:

$$\lim_{k \rightarrow \infty} \mu(f^{-k}(A)) = \mu(X)\pi(A).$$

For all $A \in \mathcal{L}(X)$. This can be rewritten as:

$$\lim_{k \rightarrow \infty} \pi(f^{-k}(A) \cap B) = \pi(B)\pi(A).$$

This is the definition of mixing. □

If the Markov chain is irreducible and aperiodic, and has at least one positive recurrent state then all states are positive recurrent. This implies that $\pi(H) = 0$ implies $\lambda(H) = 0$. Otherwise, if all states are null recurrent, then $\pi(H) = 0$ for all $H \in \mathcal{L}(X)$. From now on it is assumed that there is a given dynamical system (X, f) with the properties stated in Theorem 5.14 and that all states of the Markov chain defined by its partition are positive recurrent.

Lemma 5.18. *If $\lambda(H) = 0$, then $\lambda(f^{-1}(H)) = 0$ and $\lambda(f(H)) = 0$.*

Proof. This follows straightforward from that the function is linear with a non-zero slope on every $\text{int}(H_i) \in \mathcal{H}$, where $\mathcal{H}$ is countable, and that $X \setminus \cup\{\text{int}(H_i) : H_i \in \mathcal{H}\}$ is a countable set. $\square$

Theorem 5.19. *If there exists an $E \in \mathcal{L}(X)$, such that $\lambda(f(E) \setminus E) = 0$, then either E or its complement $X \setminus E = E^c$ has Lebesgue measure zero.*

Proof. Proceeding towards a contradiction let us assume that such a set E exists with $\lambda(E) \neq 0 \neq \lambda(E^c)$. We define a measure $\mu(H) = \pi(H \cap E)$. Then we have that μ is absolutely continuous and Corollary 4.14 implies that $0 < \mu(X) = \pi(E) < \infty$. Then by Theorem 5.14 $\lim_{n \rightarrow \infty} \mu(f^{-n}(E^c)) = \pi(E^c) \cdot \mu(X) = \pi(E^c) \cdot \pi(E) > 0$.

Furthermore $\lambda(f(E) \setminus E) = \lambda(f(E) \cap E^c) = 0$. We have that $f(f^{-1}(E^c) \cap E) \subseteq E^c \cap f(E)$. This implies, that $\lambda(f(f^{-1}(E^c) \cap E)) = 0$. From Lemma 5.18 it follows that $\lambda(f^{-1}(E^c) \cap E) = 0$.

Let us suppose that $\lambda(f^{-n}(E^c) \cap E) = 0$. We show that $\lambda(f^{-n-1}(E^c) \cap E) = 0$. We have that $f^{-n-1}(E^c) = f^{-1}(f^{-n}(E^c) \cap E^c) \cup f^{-1}(f^{-n}(E^c) \cap E)$. We can deduce that $\lambda(f^{-n-1}(E^c) \cap E) \leq \lambda(f^{-1}(f^{-n}(E^c) \cap E^c) \cap E) + \lambda(f^{-1}(f^{-n}(E^c) \cap E) \cap E) \leq \lambda(f^{-1}(E^c) \cap E) = 0$.

This by induction proves that for all positive n we have $\lambda(f^{-n}(E^c) \cap E) = 0$, which implies that $\mu(f^{-n}(E^c)) = \pi(f^{-n}(E^c) \cap E) = 0$. This contradicts that $\lim_{n \rightarrow \infty} \mu(f^{-n}(E^c)) > 0$. $\square$

In the following theorem we discuss invariant measures which have a density $\rho : X \rightarrow \overline{\mathbb{R}}_{\geq 0}$. These are not necessarily σ -finite, and σ -finiteness is usually assumed when discussing absolutely continuous invariant measures, however having proven Theorem 5.19 we can describe the non σ -finite cases as well.

Theorem 5.20. *Suppose that μ is an invariant measure on $\mathcal{L}(X)$ and there exists a density $\rho : X \rightarrow \overline{\mathbb{R}}_{\geq 0}$, such that $\mu(A) = \int_A \rho d\lambda$ for all $A \in \mathcal{L}(X)$. If $\mu(X) = \infty$ and if $\pi(H) = 0$ implies $\lambda(H) = 0$, then for all $A \in \mathcal{L}(X)$, $\lambda(A) \neq 0$ it is true that $\mu(A) = \infty$.*

Proof. We start by stating a lemma.

Lemma 5.21. *Let γ be the density of μ and $O = \{x \in X : \gamma(x) = \infty\}$. If $\lambda(O) = 0$ then for all N there exists a measurable set $E \subset X$, such that $\infty > \mu(E) > N$.*

Proof. Let us first consider the sequence $\int_{[-n,n] \cap X} \gamma d\lambda$. If n goes to infinity the limit must be infinity. If there is an N such that there is no such n for which $\infty > \int_{[-n,n] \cap X} \gamma d\lambda > N$, then there exists an N_0 for which $\int_{[-N_0, N_0] \cap X} \gamma d\lambda = \infty$. Then let $A_k = \{x \in [-N_0, N_0] \cap X : \gamma(x) < k\}$. $\lim_{k \rightarrow \infty} \mu(A_k) = \infty$ but $\mu(A_k) < 2N_0k$, which implies that for all N there must be a K such that $\infty > \mu(A_K) > N$. $\square$

If $\mu(X) = \infty$ and for all N there exists a measurable set $E \subset X$, such that $\infty > \mu(E) > N$, then let $\mu'(H) = \mu(H \cap E)$. Now $\mu(H) = \mu'(H) + \mu(H \cap E^c)$ and $\mu'(X) < \infty$. Since μ is an invariant measure, for all $H \in \mathcal{L}(X)$ we have that $\lim_{n \rightarrow \infty} \mu(f^{-n}(H)) = \mu(H)$. Also $\mu(H) = \lim_{n \rightarrow \infty} \mu(f^{-n}(H)) \geq \lim_{n \rightarrow \infty} \mu'(f^{-n}(H)) > N \cdot \pi(H)$. In this case the statement

is proven. This case also implies the statement if μ is σ -finite.

If there exists an N such that we cannot find a set with a finite μ measure greater than N , then by the previous lemma $\lambda(O) > 0$ for $O = \{x \in X : \gamma(x) = \infty\}$. If $\lambda(O^c) = 0$ then the proof is finished. If O and O^c have non-zero measure, then Theorem 5.19 implies that $\lambda(f(O) \cap O^c) \neq 0$ and $\lambda(f(O^c) \cap O) \neq 0$. Hence, there exists $E \subset f(O) \setminus O$ such that $\lambda(E) \neq 0$ and $\mu(E) < \infty$. Moreover E is the image of a subset of O . Since $\lambda(E) \neq 0$ and $E \subseteq f(f^{-1}(E) \cap O)$ then $\lambda(f^{-1}(E) \cap O) \neq 0$ must be true, thus $\mu(f^{-1}(E)) = \infty$. This contradicts that μ is invariant. $\square$

6 Applications to our system

Theorem 6.1. *The closed integer intervals form an expansive ℓ -Markov partition of $(\mathbb{R}_{\geq 0}, T)$.*

Proof. The properties established in Lemma 2.3 satisfy all conditions needed. $\square$

Theorem 6.2. *Consider the closed integer intervals as the ℓ -Markov partition of our system. Then the transition matrix $P = (p_{ij})$ of $(\mathbb{R}_{\geq 0}, T)$ is the following:*

$$p_{ij} = \begin{cases} \frac{1}{4^{\ell(i)}+1} & \text{if } j \in [1, 4^{\ell(i)} + 1] \\ 0 & \text{otherwise.} \end{cases} \quad (12)$$

This Markov chain is irreducible and aperiodic. By finding a stationary distribution with positive coordinates, we will also show that all states are positive recurrent.

Definition 6.3. We define the n th step of the function. It is a closed interval. It is the union of closed integer intervals, the T image of which are identical. Explicitly:

$$L_1 = [0, 4] \quad L_n = [4^{n-1}, 4^n].$$

Let $\mathcal{L}_n$ be the set of indices of those integer intervals I_k for which $I_k \subset L_n$.

Let us construct the transition matrix S of a new Markov chain the states of which will represent the "steps" of the function T . It is interesting to study it a bit more generally. Let us introduce a parameter $E \in \mathbb{Z}_{\geq 2}$. The matrix S will describe our system if $E = 4$. For $k > 1$, let:

$$\Lambda(1) = E \text{ and } \Lambda(k) = (E - 1)E^{k-1}. \quad (13)$$

If $E = 4$ we have that $\Lambda(n) = \lambda(L_n)$. We define $S = (s_{ij})$ by:

$$s_{ij} = \begin{cases} \frac{\Lambda(j)}{E^i+1} & \text{if } j \leq i, \\ \frac{1}{E^i+1} & \text{if } j = i + 1, \\ 0 & \text{otherwise.} \end{cases} \quad (14)$$

The reader might wonder why this chain is of particular interest to us. Suppose that we have a measure μ with finite density ρ , which is constant on every integer interval. Since the image of all integer intervals in a step is identical, to calculate the density of the pushforward of μ , we only need to know $\mu(L_k)$ for every k . Let the constants of ρ be stored in a countably infinite dimensional row vector $v = (v_1, v_2, \dots)$, as in Theorem 4.8. From Theorem 4.8 we know that the constants of the pushforward of μ are in the vector vP . Let

us define the vector $w^{(v)} = (w_1^{(v)}, w_2^{(v)}, \dots)$ such that $w_k^{(v)} = \sum_{i \in \mathcal{L}_k} v_i = \mu(L_k)$. Hence we have that if $E = 4$, then $\mu(T^{-1}(L_k)) = (w^{(v)}S)_k$, thus $w^{(v)}S = w^{(vP)}$. If for an infinite dimensional row vector p we have $pP = p$, meaning that p is a stationary distribution of P , then $w^{(p)}S = w^{(p)}$, so $w^{(p)}$ is a stationary distribution of S .

Theorem 6.4. *The only possible $q = (q_1, q_2, \dots)$ stochastic, countably infinite dimensional row vector, for which $q = qS$ is given by:*

$$q_1 = \frac{1}{2}, \quad q_{k+1} = \frac{E}{E^k + 1} q_k, \quad k \in \mathbb{Z}^+.$$

This is equivalent to saying that it is the only stationary distribution of the Markov chain defined by S .

Proof. Let us pick any stochastic infinite dimensional row vector $q = (q_1, q_2, \dots)$. Let $qS^n = q^{(n)} = (q_1^{(n)}, q_2^{(n)}, \dots)$. We have the following recursion:

$$q_k^{(n)} = \frac{1}{E^{k-1} + 1} q_{k-1}^{(n-1)} + \sum_{i=k}^{\infty} \frac{\Lambda(k)}{E^i + 1} q_i^{(n-1)}. \quad (15)$$

Let us rewrite this as:

$$q_{k+1}^{(n)} = \frac{\Lambda(k+1)}{\Lambda(k)} \left(q_k^{(n)} - \frac{1}{E^{k-1} + 1} q_{k-1}^{(n-1)} - \frac{\Lambda(k)}{E^k + 1} q_k^{(n-1)} \right) + \frac{1}{E^k + 1} q_k^{(n-1)}. \quad (16)$$

For a stationary distribution $q_k^{(n)} = q_k^{(m)}$, so we can omit the upper index in the recursion. If $k > 1$, then by (13) and (16) for a stationary distribution:

$$q_{k+1} = \frac{E^k + E + 1}{E^k + 1} q_k - \frac{E}{E^{k-1} + 1} q_{k-1}. \quad (17)$$

Let us sum (17) from $k = 2$ to $k = n - 1$. We get the following:

$$\sum_{k=3}^n q_k = \sum_{k=3}^{n-1} q_k + \frac{E}{E^{n-1} + 1} q_{n-1} + q_2 - \frac{E}{E + 1} q_1.$$

Let n go to infinity. Since the vector is stochastic its coordinates form an absolutely convergent series:

$$\begin{aligned} \sum_{k=3}^{\infty} q_k &= \sum_{k=3}^{\infty} q_k + q_2 - \frac{E}{E + 1} q_1. \\ \frac{E}{E + 1} q_1 &= q_2. \end{aligned}$$

From here the recursion in the proposition is satisfied for $k = 1$. Let us suppose that it is satisfied for $k = n - 1$, thus $\frac{E}{E^{n-1} + 1} q_{n-1} = q_n$. We substitute this into (17). We obtain the following:

$$q_{n+1} = \frac{E^n + E + 1}{E^n + 1} q_n - \frac{E}{E^{n-1} + 1} q_{n-1} = \frac{E^n + E + 1}{E^n + 1} q_n - q_n = \frac{E}{E^n + 1} q_n. \quad (18)$$

Hence the recursion is satisfied for $k = n$, thus the recursion follows by induction for all k . We only need to find the value of q_1 . For this we sum the elements of the distribution in

terms of q_1 .

Let $q_{k+1} = a_{k+1}q_1$. Since by (18) we have proven the recursion of q_k , it follows that:

$$a_1 = 1, \quad a_{k+1} = \frac{E}{E^k + 1} a_k = \frac{E^k}{\prod_{i=1}^k E^i + 1}.$$

Next we show by induction that $\sum_{k=1}^n a_k = 2 - \frac{a_n}{E^{n-1}}$. For $n = 1$ the statement is $1 = 2 - 1$. If the statement is true for $n \in \mathbb{Z}^+$, then:

$$\sum_{k=1}^{n+1} a_k = 2 - \frac{a_n}{E^{n-1}} + a_{n+1} = 2 + \left(1 - \frac{E^n + 1}{E^n}\right) a_{n+1}.$$

Thus it is true for $n + 1$. This implies that $\sum_{k=1}^{\infty} a_k = 2$. Since $\sum_{k=1}^{\infty} q_k = 1$ and $q_k = a_k q_1$ we have that the value of q_1 must be $1/2$ independent of E . $\square$

From here we can find the only possible stationary distribution of P as well. Let us generalize it as well.

Definition 6.5. Let $E \in \mathbb{Z}_{\geq 2}$. Let us define the function $\ell : \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 2} \rightarrow \mathbb{N}$ as follows:

$$\ell(x, E) = \inf\{n \in \mathbb{Z}^+ : x \leq E^n\}.$$

If $E = 4$, then the original definition of ℓ still applies. From now on, by $\ell(x)$ we mean $\ell(x, 4)$. Let us accordingly generalize the matrix $P = (p_{ij})$.

$$p_{ij} = \begin{cases} \frac{1}{E^{\ell(i, E)} + 1} & \text{if } j \in [1, E^{\ell(i, E)} + 1] \\ 0 & \text{otherwise.} \end{cases} \quad (19)$$

Definition 6.6. Let $L_1^E = [0, E]$ and $L_n = [E^{n-1}, E^n]$, moreover let $\mathcal{L}_n^E = \{k \in \mathbb{Z}^+ : I_k \subseteq L_n\}$.

These definitions were constructed such that $i \in \mathcal{L}_{\ell(i, E)}^E$ and that $\lambda(L_n^E) = \Lambda(n)$ for any $E \in \mathbb{Z}_{\geq 2}$.

Lemma 6.7. For fixed $i, n \in \mathbb{Z}^+$ the following equation holds:

$$\sum_{j \in \mathcal{L}_n^E} p_{ij} = s_{\ell(i, E)n}. \quad (20)$$

Proof. For this part recall the definition of S by (14). For $n > \ell(i, E) + 1$ we always have that $j > E^{\ell(i, E)} + 1$, thus $\sum_{j \in \mathcal{L}_n^E} p_{ij} = 0 = s_{\ell(i, E)n}$. For $n = \ell(i, E) + 1$ we only get a non zero transition probability p_{ij} if $j = E^{\ell(i, E)} + 1$, thus $\sum_{j \in \mathcal{L}_n^E} p_{ij} = \frac{1}{E^{\ell(i, E)} + 1} = s_{\ell(i, E)n}$. For $n \leq \ell(i, E)$ we have that for all $j \in \mathcal{L}_n^E$ the value of p_{ij} is the same, thus $\sum_{j \in \mathcal{L}_n^E} p_{ij} = \frac{\Lambda(n)}{E^{\ell(i, E)} + 1} = s_{\ell(i, E)n}$. $\square$

Lemma 6.8. Suppose that for a stochastic countably infinite dimensional row vector $p = (p_1, p_2, \dots)$ we have that $p = pP$. If $q'_n = \sum_{i \in \mathcal{L}_n^E} p_i$, then for $q' = (q'_1, q'_2, \dots)$ we have that $q' = q$.

Proof. We have that q' is a stochastic countably infinite dimensional row vector, thus $q' = q$ is equivalent to $q'S = q'$ by Theorem 6.4. We show that q' is a stationary distribution of S . Since p is a stationary distribution of P we have the following:

$$p_j = \sum_{i=1}^{\infty} p_i p_{ij}. \quad (21)$$

Thus for all $n \in \mathbb{Z}^+$:

$$\sum_{i \in \mathcal{L}_n^E} p_i = \sum_{i \in \mathcal{L}_n^E} \sum_{j=1}^{\infty} p_j p_{ji} = \sum_{i \in \mathcal{L}_n^E} \sum_{k=1}^{\infty} \sum_{j \in \mathcal{L}_k^E} p_j p_{ji}. \quad (22)$$

If $a, b \in \mathcal{L}_k^E$ for some $k \in \mathbb{Z}^+$, then $p_{ai} = p_{bi}$ for all i . From this and from the definition of q' we obtain the following from (22):

$$q'_n = \sum_{i \in \mathcal{L}_n^E} p_i = \sum_{i \in \mathcal{L}_n^E} \sum_{k=1}^{\infty} p_{E^k i} \sum_{j \in \mathcal{L}_k^E} p_j = \sum_{i \in \mathcal{L}_n^E} \sum_{k=1}^{\infty} p_{E^k i} q'_k. \quad (23)$$

Since all sums are absolutely convergent we can flip the order of them. By Lemma 6.7:

$$q'_n = \sum_{k=1}^{\infty} q'_k \sum_{i \in \mathcal{L}_n^E} p_{E^k i} = \sum_{k=1}^{\infty} q'_k s_{kn}. \quad (24)$$

By this $q' = q'S$, thus $q = q'$. $\square$

Theorem 6.9. *The only possible stochastic infinite dimensional row vector $p = (p_1, p_2, \dots)$, for which $p = pP$ is given by:*

$$p_k = \begin{cases} \frac{1}{2E} & \text{if } k \in \{1, \dots, E+1\} \\ \frac{1}{E^{n-1}(E^{n-1}+1)} q_{n-1} & \text{otherwise, if } n > 1 \text{ and } k \in \{E^{n-1}+2, \dots, E^n+1\}. \end{cases}$$

This is equivalent to saying that it is the only stationary distribution of the Markov chain defined by P .

Proof. We suppose, that p is a stationary distribution, thus (21) is true. We can deduce by (19) that if $a, b \in \{1, \dots, E+1\}$ or $a, b \in \{E^{k-1}+2, E^k+1\}$ for some $k \in \mathbb{Z}^+$, then $p_a = p_b$ must be true since $p_{ia} = p_{ib}$ for all i in this case. By Lemma 6.8 we know that:

$$q_n = \sum_{k=E^{n-1}+1}^{E^n} p_k = \sum_{k \in \mathcal{L}_n^E} p_k. \quad (25)$$

By this $p_1 = \dots = p_{E+1} = \frac{1}{2E}$.

Now let us consider a step, other than the first one. Let $p' = p_k$ where $k = E^{n-1}+1$ for some $n \in \mathbb{Z}^+$. Let $p = p_{k+1} = \dots = p_{E^n}$. From (19) we have that for $k = E^{n-1}+1$ and any $i \in \mathbb{Z}^+$ the transition probabilities $p_{i(k+1)} = p_{ik}$ if and only if $i \notin \mathcal{L}_{n-1}^E$. If $i \in \mathcal{L}_{n-1}^E$, then $p_{ik} = \frac{1}{E^{n-1}+1}$ and $p_{i(k+1)} = 0$. By this, and by applying (21) we obtain:

$$p' - p = \sum_{i=1}^{\infty} p_i p_{ik} - \sum_{i=1}^{\infty} p_i p_{i(k+1)} = \sum_{i \in \mathcal{L}_{n-1}^E} \frac{p_i}{E^{n-1}+1}. \quad (26)$$

Hence by (25):

$$p' = p + \frac{1}{E^{n-1} + 1} q_{n-1}. \quad (27)$$

By (27), by (25) and by Theorem 6.4 we obtain:

$$q_n = \frac{E}{E^{n-1} + 1} q_{n-1} = (E - 1)E^{n-1}p + \frac{1}{E^{n-1} + 1} q_{n-1}. \quad (28)$$

By simple algebra we get $p = \frac{1}{E^{n-1}(E^{n-1}+1)} q_{n-1}$ and $p' = \frac{1}{E^{n-1}} q_{n-1}$. If $n > 2$, then by Theorem 6.4 $p' = \frac{1}{E^{n-2}(E^{n-2}+1)} q_{n-2}$, if $n = 2$, then $p' = \frac{1}{2E}$. From here the proposition follows. $\square$

According to Theorem 5.14 this stationary distribution, for $E = 4$ gives an invariant measure $\pi : \mathcal{L}(\mathbb{R}_{\geq 0}) \rightarrow \mathbb{R}_{\geq 0}$ for the system $(\mathbb{R}_{\geq 0}, T)$. For this measure if μ is an absolutely continuous measure such that $\mu(\mathbb{R}_{\geq 0}) < \infty$, then $\lim_{n \rightarrow \infty} \mu(f^{-n}(H)) = \mu(\mathbb{R}_{\geq 0})\pi(H)$ for all $H \in \mathcal{L}(\mathbb{R}_{\geq 0})$. This also means that the system is mixing from Corollary 5.17. Since π is such that $\pi(H) = 0$ implies $\lambda(H) = 0$ for measurable subsets of the domain, from theorems of Section 5 the only absolutely continuous invariant measures of the system are those which are constant multiples of π , or if we consider non- σ -finite measures, infinite for all non-null sets.

Corollary 6.10. *The orbit of almost every point is dense in $\mathbb{R}_{\geq 0}$, thus for almost every $x \in \mathbb{R}_{\geq 0}$ we have that $\liminf_{k \rightarrow \infty} (f^k(x)) = 0$ and $\limsup_{k \rightarrow \infty} (f^k(x)) = \infty$.*

Proof. The absolutely continuous measure π satisfies all conditions needed for Theorem 5.17, thus the system is mixing. Thus for any $A \in \mathcal{L}(\mathbb{R}_{\geq 0})$ for all I_n there exists an $m \in \mathbb{N}$ such that $\pi(f^{-m}(I_n) \cap A) \geq 1/2(\pi(A)\pi(I_n))$. Let $A_0 = A$, $A_k = A_{k-1} \setminus f^{-m_k}(I_n)$ where:

$$m_k = \min\{m \in \mathbb{N} : \pi(f^{-m}(I_n) \cap A_{k-1}) > \frac{1}{2}(\pi(A_{k-1})\pi(I_n))\}.$$

Since $\pi(\mathbb{R}_{\geq 0}) = 1$ we have that $\pi(A_k) < \frac{\pi(I_n)}{2}\pi(A_{k-1}) < \pi(A_{k-1})$, thus $\lim_{k \rightarrow \infty} \pi(A_k) = 0$. Let $H \subset A$ be the set of those points whose orbit never intersects I_n . We have that $H \subseteq \bigcap_{k=0}^{\infty} A_k$ thus $\pi(H) = 0$ implying that $\lambda(H) = 0$. $\square$

7 Orbit of rationals

For the discussion of the orbit of rational points we will represent points of the domain by special Cantor series determined by our transformation f . This representation is a kind of generalization of the decimal expansion. We are concerned with dynamical systems (X, f) , where $X \subseteq \mathbb{R}$. About f further assumptions will be made. In this section we allow integer intervals to have negative indices. To avoid ambiguity we introduce a new notation.

Definition 7.1. Let $J_n := [n - 1, n]$ for arbitrary $n \in \mathbb{Z}$. Let $J_{i_0 i_1 \dots} := \bigcap_{k=0}^{\infty} f^{-k}(J_{i_k})$ for an arbitrary $i_0, i_1 \dots$ finite or infinite integer sequence. If the sequence is finite, then we only take finite intersection.

Definition 7.2 (Assumptions on f). For f the following assumptions hold:

1. Some subset $\mathcal{J}$ of the integer intervals forms an ℓ -Markov partition of (X, f) as in Definition 4.3.

2. For every $J_i \in \mathcal{J}$ the set $\mathcal{A} \subseteq \mathcal{J}$ from Definition 4.3, for which $\text{int}(f(J_i)) = \text{int}(\cup \mathcal{A})$ has at least two elements.

Since all integer intervals have unit Lebesgue-measure and the second assumption ensures that on every integer interval the magnitude of the slope is at least two we have that $\mathcal{J}$ is an expansive ℓ -Markov partition. In this case a bit weaker form of Lemma 2.3 is still true. We do not state facts implied by Lemma 5.4.

Lemma 7.3. *For a given finite sequence of integers $i_0, \dots, i_n$ the following are true for $J_{i_0 \dots i_n}$:*

1. *The set $\{k : \text{int}(J_k) \subseteq f(J_n)\}$ for all integer n is a set of consecutive integers.*
2. *Consider the sets $J_{i_0 \dots i_n k}$ where $k \in \{m : \text{int}(J_m) \subseteq f(J_{i_n})\}$ in the order they appear on the real line. We have that their $(n+1)$ st index, k either increases or decreases from left to right. It increases if and only if there is an even number of j indices among $i_0, \dots, i_n$ such that f is decreasing on J_j .*
3. *For $a, a+1 \in \{m : \text{int}(J_m) \subseteq f(J_{i_n})\}$ we have that $\overline{\text{int}(J_{i_0 \dots i_n a})} \cup \overline{\text{int}(J_{i_0 \dots i_n (a+1)})}$ is an interval.*
4. *For a point $x \in X$ there exists a natural k for which $f^k(x) \in \mathbb{Z}$ if and only if for that k and for some $n \in \mathbb{Z}$ we have that $f^k(x) \in \partial J_n$.*
5. *If $(\text{int}(J_{i_0 \dots i_n k}) \cap \text{int}(J_{i_0 \dots i_n})) \neq \emptyset$ then $\lambda(J_{i_0 \dots i_n k}) = \frac{1}{\#\{m : \text{int}(J_m) \subseteq f(J_{i_n})\}} \lambda(J_{i_0 \dots i_n})$.*

Proof. These statements all follow simply from our assumptions about f . □

Definition 7.4. Let us consider a finite integer sequence $i_0 i_1 \dots i_n$ for which $\text{int}(J_{i_0 \dots i_n})$ is non-empty. If the number of elements j in $i_0 \dots i_n$ for which the function f is decreasing on J_j is even then we say that the direction of $J_{i_0 \dots i_n}$ is forward, otherwise it is backward.

We would like to understand how the rationals behave in our system. Since f may be discontinuous at any $n \in \mathbb{Z}$, the orbit of integers cannot be meaningfully described. Thus we only consider rationals which have no integers in their orbit. Thus from now on we consider the $(\tilde{X}, f)$ system.

Definition 7.5. Let $L_k(x)$ and $U_k(x)$ be the k th lower and upper regressors of x meaning that:

$$L_k(x) = \inf\{y \in \tilde{X} : y \in \tilde{J}_{\underline{i}(x)|k}\}, \quad U_k(x) = \sup\{y \in \tilde{X} : y \in \tilde{J}_{\underline{i}(x)|k}\}.$$

We will only be using the lower regressor.

Definition 7.6. Let $d_k(x) = L_k(x) - L_{k-1}(x)$ for $k \geq 1$.

Definition 7.7. Suppose that $\tilde{J}_{i_0 \dots i_n}$ is a nonempty set. We define tags of the nonempty sets of the form $\tilde{J}_{i_0 \dots i_n k}$. The tag will be defined by the function $t : \Omega \rightarrow \mathbb{Z}$, where Ω is the space of finite integer sequences $(j_0, \dots, j_n)$, for which $\tilde{J}_{j_0 \dots j_n} \neq \emptyset$. We define the value of $t(i_0, \dots, i_n, k)$, where $(i_0, \dots, i_n, k) \in \Omega_f$. We use the following variables:

$$N = \#\{p : \text{int}(J_p) \subseteq f(J_{i_n})\},$$

$$M = \max\{p : \text{int}(J_p) \subseteq f(J_{i_n})\}, \quad m = \min\{p : \text{int}(J_p) \subseteq f(J_{i_n})\}.$$

Let function t be a bijection between elements of Ω of the form $(i_0, \dots, i_n, k)$, and the integers $\{0, \dots, N-1\}$. If $\tilde{J}_{i_0 \dots i_n}$ is forward then for $a \in \{0, \dots, N-1\}$ we have $a = t(i_0, \dots, i_n, (a+m))$ and if $\tilde{J}_{i_0 \dots i_n}$ is backward then $a = t(i_0, \dots, i_n, (M-a))$. It is clear that t only depends on the sequence given as an input.

Theorem 7.8. *Let $\underline{i}(x) = (i_0, i_1 \dots)$ for some $x \in \tilde{X}$. For some $a_n, M_n \in \mathbb{Z}$ we have that $d_n(x) = \frac{a_n}{M_n}$ where $M_n = \prod_{k=0}^{n-1} \#\{p : \text{int}(J_p) \subseteq f(J_{i_k})\} = \prod_{k=0}^{n-1} |f'(J_{i_k})|$ and $a_n = t(i_0, \dots, i_n)$ is the tag of $\tilde{J}_{i_0 \dots i_n}$ as the subset of $\tilde{J}_{i_0 \dots i_{n-1}}$.*

Proof. We have by Lemma 7.3 that $\lambda(\tilde{J}_{i_0 \dots i_{n-1}})$ is always $1/M_n$ if it is nonempty. If $\overline{\tilde{J}_{i_0 \dots i_{n-1}}} = [x, y]$ for $x, y \in \mathbb{R}$ then from the definition of the tag we have that $\overline{\tilde{J}_{i_0 \dots i_n}} = [x + \frac{a_n}{M_n}, x + \frac{a_n+1}{M_n}]$, which implies the statement. $\square$

Corollary 7.9. $\lim_{k \rightarrow \infty} L_k(x) = x$ and $L_0(x) + \sum_{k=1}^{\infty} d_k(x) = x$.

Theorem 7.10. *All $x \in \tilde{X}$ which have an eventually periodic orbit are rational.*

Proof. On the interior of an integer interval $f(x) = ax + b$ where a and b are rational. This implies that the f preimage of a rational number is a set of rationals and the image of a rational is rational. This means that it suffices to show that all periodic points are rational.

Let us suppose that the orbit of $x \in \tilde{X}$ is n periodic, thus $x = f^n(x)$. Let $\underline{i}(x) = (i_0, i_1 \dots)$. From the definition of $\tilde{X}$ we have that $f^n(x) \in \text{int}(J_{i_n})$, thus the restriction of f^n to $\tilde{J}_{\underline{i}(x)|n}$ is a composition of functions of the form $ax + b$, where a and b are rational, thus f^n itself is a function of the form $ax + b$. Since $|f'(J_k)| \geq 2$ for all k , we have that $|a| > 1$. From periodicity we have that $x = ax + b$, thus $x = \frac{b}{1-a}$, a rational number. $\square$

For the following proofs recall Definition 1.6 on empty products and sums. For any x_n sequence of reals and $a > b$ integers we define $\prod_{n=a}^b x_n = 1$ and $\sum_{n=a}^b x_n = 0$.

Theorem 7.11. *The orbit of all rationals $x \in \tilde{X}$ for which $\liminf_{k \rightarrow \infty} |f^k(x)| < \infty$ is eventually periodic.*

Proof. Let $x = \frac{p}{q}$ and q be positive. Let $\underline{i}(x) = (i_0, i_1 \dots)$. Let $a_k = t(\underline{i}(x)|k)$ be the tag of $\tilde{J}_{\underline{i}(x)|k}$ as a subset of $\tilde{J}_{\underline{i}(x)|(k-1)}$, and let $a_0 = i_0 - 1 = L_0(x)$. Let $m_n = |f'(J_{i_n})| = \#\{k : \text{int}(J_k) \subseteq f(J_{i_n})\}$. By Corollary 7.9 and Theorem 7.8:

$$x = L_0(x) + \sum_{k=1}^{\infty} d_k(x) = \sum_{n=0}^{\infty} \frac{a_n}{\prod_{k=0}^{n-1} m_k}.$$

We have that $a_0 = \lfloor \frac{p}{q} \rfloor$ and $r_0 = p - qa_0$. By Theorem 7.8 the following recursion is also true:

$$a_{k+1} = \left\lfloor \frac{m_k r_k}{q} \right\rfloor, \quad r_{k+1} = m_k r_k - qa_{k+1}. \quad (29)$$

Lemma 7.12. *If for some $c, d \in \mathbb{N}$ we have that if $r_c = r_d$ and $i_c = i_d$ and $J_{\underline{i}(x)|c}$ and $J_{\underline{i}(x)|d}$ have the same direction, then $a_{c+1} = a_{d+1}$, $r_{c+1} = r_{d+1}$, $m_{c+1} = m_{d+1}$, $i_{c+1} = i_{d+1}$ and $J_{\underline{i}(x)|(c+1)}$ and $J_{\underline{i}(x)|(d+1)}$ have the same direction.*

Proof. If $i_c = i_d$, then $m_c = m_d$, thus $a_{c+1} = a_{d+1}$ and $r_{c+1} = r_{d+1}$ follow from (29). Since $i_c = i_d$ and $a_{c+1} = a_{d+1}$, and we have that $J_{\underline{i}(x)|c}$ and $J_{\underline{i}(x)|d}$ have the same direction, then $i_{c+1} = i_{d+1}$. Hence $J_{\underline{i}(x)|(c+1)}$ and $J_{\underline{i}(x)|(d+1)}$ also have the same direction. $\square$

As a consequence of the previous lemma it follows by induction that if for c and d naturals the conditions of the lemma apply, then $i_{c+n} = i_{d+n}$ for any positive integer n , thus the orbit is eventually periodic by $|c - d|$. We show that the conditions of the lemma will apply to every rational for some c and d , if its orbit has a bounded subsequence.

If the orbit has a bounded subsequence, then there is some $\alpha \in \mathbb{Z}^+$ such that the orbit has infinitely many elements in $[-\alpha, \alpha]$. In these cases i_k can have only less than 2α values. Since r_k is the remainder of an integer modulo q it is an element of the finite set $\{0, \dots, q-1\}$. There are only two directions an interval can have. By the pigeonhole principle we have that in the first $4\alpha q + 1$ instances, when $f^k(x) \in [-\alpha, \alpha]$ at least once all conditions of the lemma will apply. $\square$

Theorem 7.13. *The orbit of every rational $x \in \tilde{X}$ is eventually periodic if:*

$$0 = \liminf_{N \rightarrow \infty} \max_{n \in \{1-N, \dots, N\}} \lambda\left(f^{-1}((-\infty, -N] \cup [N, \infty)) \cap J_n\right). \quad (30)$$

Proof. Let us consider a rational

$$x = \frac{p}{q} = \sum_{n=0}^{\infty} \frac{a_n}{\prod_{k=0}^{n-1} m_k}.$$

By Theorem 7.11 it is enough to consider the case where $|f^k(x)| \rightarrow \infty$. Let us define the following:

$$\ell_{n_0} := \left(p - q \sum_{n=0}^{n_0-1} \frac{a_n}{\prod_{k=0}^{n-1} m_k}\right) \prod_{k=0}^{n_0-2} m_k.$$

By using this notation we can define the following sequence:

$$\frac{\ell_{n_0}}{q} = \frac{a_{n_0}}{m_{n_0-1}} + \frac{1}{m_{n_0-1}} \sum_{n=n_0+1}^{\infty} \frac{a_n}{\prod_{k=n_0}^{n-1} m_k}. \quad (31)$$

Since a_n is an element of $\{0, \dots, m_{n-1} - 1\}$ if $n > 0$, it is true that $\frac{a_n}{m_{n-1}} \in [0, 1]$ if $n > 0$.

Lemma 7.14. *Suppose that we have $x \in \tilde{X}$, then for all $n_0 \in \mathbb{N}$:*

$$0 < \sum_{n=n_0+1}^{\infty} \frac{a_n}{\prod_{k=n_0}^{n-1} m_k} < 1.$$

Proof. This sequence indexed by n_0 describes the position of x inside of $\tilde{J}_{\underline{i}(x)|n_0}$. Since $x = L_{n_0}(x) + \sum_{n=n_0+1}^{\infty} \frac{a_n}{\prod_{k=0}^{n-1} m_k}$ we have that:

$$(x - L_{n_0}(x)) \prod_{k=0}^{n_0-1} m_k = \sum_{n=n_0+1}^{\infty} \frac{a_n}{\prod_{k=n_0}^{n-1} m_k}.$$

We also know as a consequence of Lemma 7.3, that $\prod_{k=0}^{n_0-1} \frac{1}{m_k} = \lambda(J_{\underline{i}(x)|n_0})$. This implies that the function $\phi(y) := (y - L_{n_0}(x)) \prod_{k=0}^{n_0-1} m_k$ is the linear map which maps the two ends of $\text{int}(J_{\underline{i}(x)|n_0})$ onto the two ends of $(0, 1)$ while preserving direction. The value of the sequence at n_0 is $\phi(x)$, and if $x \in \tilde{X}$, then $x \in \text{int}(J_{\underline{i}(x)|n_0})$, thus $\phi(x) \in (0, 1)$. $\square$

Corollary 7.15. *For any $x \in \tilde{X}$ we have that $0 < \frac{\ell_{n_0}}{q} < 1$ for all $n_0 \in \mathbb{Z}^+$.*

Proof. The sequence has the following form:

$$\frac{\ell_{n_0}}{q} = \sum_{n=n_0}^{\infty} \frac{a_n}{\prod_{k=n_0-1}^{n-1} m_k} = (x - L_{n_0-1}(x)) \prod_{k=0}^{n_0-2} m_k.$$

By a change of indices, it follows from Lemma 7.14 that $0 < \frac{\ell_{n_0}}{q} < 1$. It also follows that $\frac{\ell_{n_0}}{q}$ describes the position of x inside of $\tilde{J}_{\underline{i}(x)|n_0-1}$. $\square$

Let us assume that the set of numbers which are in the sequence $\frac{\ell_{n_0}}{q}$ has a rational limit point. Let this be $\frac{c}{d}$ for $c \in \mathbb{Z}$ and $d \in \mathbb{Z}^+$. In this case for infinitely many n_0 :

$$0 < \left| \frac{\ell_{n_0}}{q} - \frac{c}{d} \right| < \frac{1}{qd}.$$

This is a contradiction, since $\left| \frac{\ell_{n_0}}{q} - \frac{c}{d} \right| = \frac{|\ell_{n_0}d - cq|}{qd}$ so if it is non-zero, then it is at least $\frac{1}{qd}$.

We will show that if the orbit of x is not eventually an integer, meaning that $x \in \tilde{X}$, and x is rational, then if $\liminf_{k \rightarrow \infty} |f^k(x)| = \infty$, then the set of numbers in the sequence $\frac{\ell_{n_0}}{q}$ has a rational limit point, and thus we have a contradiction. This will prove the proposition.

For $x \in \tilde{X} \cap \mathbb{Q}$ proceedings towards a contradiction we assume that $|f^k(x)|$ tends to infinity. Let us decipher what does (30) tell us. It basically describes that stepping out of the bounded region $[-N, N]$ becomes more and more difficult the bigger N we choose. Since the orbit of x is unbounded, we have that for every integer $N > |x|$ there is a non-negative integer k for which $f^k(x) \in [-N, N]$ and $f^{k+1}(x) \notin [-N, N]$.

For a fixed N , for this k let us consider the nonempty sets $\tilde{J}_{i_k h}$, where $h \in \mathbb{Z}$. Let us define the following set:

$$H = \{h \in \mathbb{Z} : f(\tilde{J}_{i_k h}) \cap ((-\infty, N] \cup [N, \infty)) \neq \emptyset\}.$$

We have that the set $t(\{i_k h : h \in H\})$ is either $\{0, \dots, c\}$ for some $c \in \mathbb{N}$ or it is $\{c, \dots, m_k - 1\}$. This is a consequence of that the function f is monotone on $\tilde{J}_{i_k}$. This implies that for a fixed $N > 0$ we have that:

$$\min \left(\frac{a_{k+1}}{m_k}, 1 - \frac{a_{k+1}}{m_k} \right) < \max_{n \in \{1-N, \dots, N\}} \lambda(f^{-1}((-\infty, -N] \cup [N, \infty)) \cap J_n). \quad (32)$$

By (32) the sequence $\frac{a_{n_0}}{m_{n_0-1}}$ has a subsequence which tends to zero or one. If the right hand side is $\varepsilon > 0$ this implies that $0 < \frac{1}{m_k} < \varepsilon$, thus $0 \leq \frac{1}{m_{n_0-1}} \sum_{n=n_0+1}^{\infty} \frac{a_n}{\prod_{k=n_0}^{n-1} m_k} < \varepsilon$. By (31) it follows that $\frac{\ell_{n_0}}{q}$ has a subsequence which tends to zero or one. We also have by Corollary 7.15, that $0 < \frac{\ell_{n_0}}{q} < 1$. Hence the set of numbers in the sequence has zero or one as a limit point, which is rational, thus we have a contradiction. Hence for all $x \in \tilde{X} \cap \mathbb{Q}$ we have that $\liminf_{k \rightarrow \infty} |f^k(x)| < \infty$, thus by Theorem 7.11 the statement follows. $\square$

The work [6] on Cantor series should be mentioned here, as Theorems 3. and 4. of that paper are implicitly proven and then applied in the previous proof.

Corollary 7.16. *In the system $(\mathbb{R}_{\geq 0}, T)$ all the eventually periodic points are rational and for all $x \in \mathbb{Q}_{\geq 0}$ the orbit of x is either eventually periodic or eventually reaches an integer.*

Proof. Since Theorem 7.10 applies, we have that all the eventually periodic points are rational. If $N_k = 4^k$, then:

$$\max_{n \in \{1-N_k, \dots, N_k\}} \lambda \left(f^{-1}((-\infty, -N_k] \cup [N_k, \infty)) \cap J_n \right) = \frac{1}{4^k + 1}.$$

Thus Theorem 7.13 applies. $\square$

Example 7.17. At first glance it is not obvious which Markov chains could be defined by systems satisfying the conditions of Theorem 7.13. It might seem at first that (30) is stronger than recurrence. We define a function $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ for which Theorem 7.13 applies however it only has transient states. Let us define the sequence $\{n_k\}_{k=1}^{\infty} \in (\mathbb{Z}^+)^{\mathbb{N}}$, which tends to infinity and let $s_k = \sum_{i=1}^{k-1} n_i$. If $s_k < n \leq s_{k+1}$, then let $f|_{\text{int}(I_n)}(x) = (n_k + 1)x + s_k$. The f images of integers does not matter. For an example of such a system see the green graph on Figure 3. For the sequence s_k :

$$\max_{n \in \{1-s_k, \dots, s_k\}} \lambda \left(f^{-1}((-\infty, -s_k] \cup [s_k, \infty)) \cap J_n \right) = \frac{1}{n_{k-1}}.$$

Thus Theorem 7.13 applies, thus for this system all eventually periodic points are rational and all rationals either hit an integer or are eventually periodic. The Markov chain defined by the system has no recurrent state, since for a state $s_k < n \leq s_{k+1}$ we eventually step above s_{k+1} with probability one, however we cannot step under s_k .

Example 7.18. There are positive recurrent Markov chains, which can be described by a function satisfying the assumptions of Definition 7.2 and still do not satisfy (30). An easy example is the random walk on $\mathbb{Z}^+$ where from $n \geq 3$ we step on $n+1, n, n-1$ or $n-2$ with the probability of each being $1/4$, and from one or two we step onto an element of $\{1, 2, 3, 4\}$ with the probability of each being $1/4$. For the function defining this chain see the blue graph on Figure 3. We could not yet prove for any positive recurrent Markov chain defined by some function satisfying the assumptions of Definition 7.2, that it has aperiodic rational points which do not hit an integer. It is an interesting question whether positive recurrence imply that this cannot occur.

Example 7.19. We give an example of a dynamical system satisfying the assumptions of Definition 7.2, for which the orbit of $1/2$ is aperiodic, and does not hit an integer. We define $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$. Let $f|_{[0,1)} = 3 - 3x$. Let $f(x) = (2\lceil x \rceil + 1)(x - \lfloor x \rfloor)$ for $x > 1$. (See the red graph on Figure 3.) For this function $f(1/2) = 3/2$ and for any $k \in \mathbb{Z}^+$ we have that $f(\frac{2k+1}{2}) = \frac{2(k+1)+1}{2}$, thus $f^k(1/2) = \frac{2k+1}{2}$. This system satisfies Definition 7.2. Let us consider the k th state of the Markov chain defined by the system. If $k = 1$, then the expected value of the next state is two. If $k > 1$, we can only step on the states $\{1, \dots, 2k+1\}$, with equal probability, thus the expected value of the next state is $k+1$. Thus for the corresponding transition matrix P the expected value for all non-zero stochastic row vectors v is less than the expected value of vP , thus the Markov chain cannot have a stationary distribution, thus it cannot be positive recurrent. It is easier to find systems with properties like this, if we know a Cantor series representation of a rational beforehand. This system was constructed so that it is related to the following Cantor series appearing in [4]:

$$\frac{1}{2} = \sum_{n=1}^{\infty} \frac{n}{\prod_{k=1}^n (2k+1)}. \quad (33)$$

According to Definition 7.5 and 7.6 we can represent $x = 1/2$ as $L_0(x) + \sum_{k=1}^{\infty} d_k(x)$. We have that for our function $d_n(x) = \frac{n}{\prod_{k=1}^n (2k+1)}$ and $L_0(x) = 0$, hence we get the Cantor series in (33).

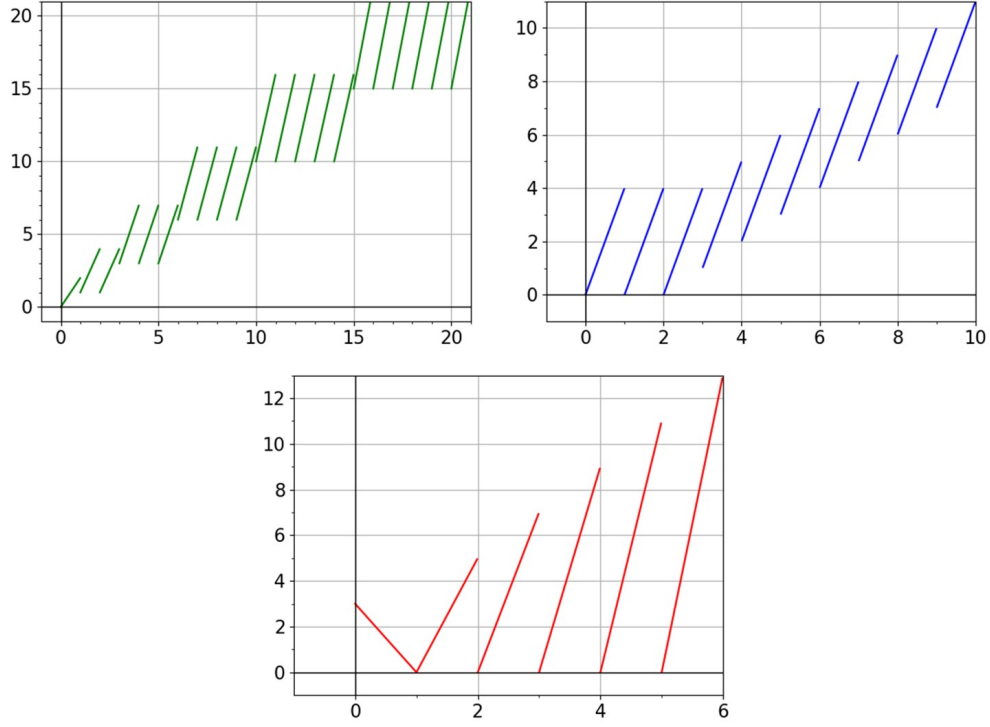


Figure 3: Illustrations of the example systems. The green one is for Example 7.17, the blue one is for Example 7.18, and the red one is for Example 7.19.

A Saját eredmények

A dolgozat legfontosabb eredménye röviden összefoglalva a szakaszonként lineáris és Markov-partícióval rendelkező egydimenziós diszkrét dinamikus rendszerek legfontosabb, és leginkább alkalmazható tulajdonságainak egységes kidolgozása a lehető legnagyobb általánosságra törekedve. Ezeket az eszközöket alkalmazzuk a közvetlen motivációul szolgáló diszkrét dinamikus rendszerre. A saját eredményeket fejezetenkénti lebontásban listázom:

Symbolic dynamics - Szimbolikus dinamika

A szimbolikus dinamika eszköztárát önállóan alkalmazzuk az $(\mathbb{R}_{\geq 0}, T)$ rendszerre.

Shadowing - Árnyékolás

A szakirodalomban szereplő árnyékolási eredményektől teljesen függetlenül belátjuk, hogy az árnyékolási tulajdonság igaz az $(\mathbb{R}_{\geq 0}, T)$ rendszerre.

Frobenius-Perron operator and ℓ -Markov partitions - Frobenius-Perron operátor és ℓ -Markov partíciók

Definiáljuk Markov partíciók egy olyan fajtáját, mely legjobban segíti az $(\mathbb{R}_{\geq 0}, T)$ -hez hasonló rendszerek vizsgálatát. A nem Markov-láncokról szóló állításokat mind a szakirodalomtól függetlenül mondjuk ki, és saját bizonyítást adunk rájuk. Ennek ellenére az, hogy szakaszonként lineáris Markov leképezések esetén a Frobenius-Perron operátor egy megszámlálható dimenziós mátrixszal reprezentálható egy széles körben ismert tény (véges partíció esetére példa [1] 9.2 fejezete), következésképpen a fejezet funkciója a számunkra releváns, ismert tulajdonságok összegyűjtése és bizonyítása.

Expansive ℓ -Markov partitions - Expanzív ℓ -Markov partíciók

A szimbolikus dinamika eszköztárát önállóan alkalmazzuk minden ℓ -Markov partícióval rendelkező rendszerre, önálló bizonyítást adunk a topologikus konjugáltságra. Bevezetjük az expanzív ℓ -Markov partíció fogalmát. Bizonyítunk egy önálló konvergenciatételt, mely teljesen leírja a vizsgált rendszerek véges előretolt mértékeinek viselkedését. Annak ellenére, hogy számos helyen kimondott állítás a hasonló rendszerek egyértelmű szakaszonként konstans invariáns sűrűségének létezése (például [1] 9.4.1 tétele), számomra nem ismert olyan tétel, mely a használt eszköztárral ilyen általánosságban előretolt mértékek konvergenciáját bizonyítja. Ezt követően megvizsgáljuk a tételünk közvetlen következményeit, valamint általánosítjuk végtelen és nem σ -véges mértékekre is, mely az abszolút folytonos invariáns mértékek vizsgálatában rendszerint figyelmen kívül van hagyva.

Applications to our system - Alkalmazások a rendszerünkre

Ebben a fejezetben kiszámoljuk az $(\mathbb{R}_{\geq 0}, T)$ egyetlen invariáns abszolút folytonos valószínűségi mértékét. A korábbi fejezetek eredményeinek következtetéseit levonva megoldjuk az eredeti Schweitzer feladatot.

Orbit of rationals - Racionális pontok pályája

Ebben a fejezetben leírjuk a racionális pontok pályáit néhány speciálisabb esetben. Összekapcsoljuk az állításokat Cantor-sorokról és számrendszerekről szóló ötletekkel. Az egyetlen hivatkozott lemmán kívül ([6] 3. és 4. tétele) minden állítás és bizonyítás saját.

B Magyar nyelvű összefoglaló

Vizsgált probléma

A dolgozatban szakaszonként lineáris valós-valós függvények bizonyos osztályainak dinamikai tulajdonságait vizsgáljuk. Elsősorban olyan rendszereket vizsgálunk, melyek rendelkeznek a Markov-partíció egy bizonyos speciális fajtájával, melyet a dolgozatban definiálunk, mint ℓ -Markov partíció. A vizsgálat közvetlen motivációjául témavezetőm, Buczolic Zoltán egy a 2022-es Schweitzer Miklós emléktversenyen kitűzött feladata, és a benne szereplő diszkrét idejű dinamikai rendszer szolgált. A feladat szövege a következőképpen szólt:

Legyen $T : [0, \infty) \rightarrow [0, \infty)$ olyan függvény, mely szomszédos egészek között lineáris, $n = 0, 1, \dots$ esetén

$$T(n) = \begin{cases} 0 & \text{ha } 2|n \\ 4^\ell + 1 & \text{ha } 2 \nmid n, 4^{\ell-1} \leq n < 4^\ell \ (\ell = 1, 2, \dots). \end{cases} \quad (34)$$

Legyen $T^1(x) = T(x)$ és $T^k(x) = T(T^{k-1}(x))$ minden $k \geq 2$ egészre. Határozzuk meg Lebesgue majdnem minden $x \in [0, \infty)$ -re a $\liminf_{k \rightarrow \infty} (T^k(x))$ és $\limsup_{k \rightarrow \infty} (T^k(x))$ értéket.

Ennek a függvénynek a leghasznosabb tulajdonsága, hogy a szakaszokat, melyeken lineáris ilyen szakaszok unióra képezi. Ez egy messzemenőig általánosítható tulajdonság, melynek segítségével az $(\mathbb{R}_{\geq 0}, T)$ rendszer, és a hozzá valamilyen értelemben hasonló rendszerek leírhatóak szimbolikus dinamikus rendszerként. A dolgozatban elsősorban az ilyen leírás következményeit vizsgáljuk. A szövegben λ jelöli a Lebesgue mértéket, valamint a mérhető és abszolút folytonos fogalmakat is mindig a Lebesgue mértékre vonatkoztatva értjük.

ℓ -Markov partíció

A dolgozatban elsősorban ℓ -Markov partícióval rendelkező rendszerekkel foglalkozunk. A fogalom a szakirodalomban is létező definíciók kisebb módosításával jött létre, hogy pontosabban leírja az általunk vizsgálni kívánt rendszereket.

Definíció B.1. Adott a valós számok egy X mérhető részhalmaza, mely előáll nemüres belsejű zárt részintervallumok uniójaként. Adott továbbá egy $f : X \rightarrow X$ függvény. Az (X, f) dinamikai rendszer rendelkezik ℓ -Markov partícióval, amennyiben létezik a valós számok zárt részintervallumainak egy olyan megszámlálható $\mathcal{H}$ halmaza, melyre igazak az alábbiak:

1. Bármely $A, B \in \mathcal{H}$ esetén, ha $A \neq B$ akkor $\text{int}(A) \cap \text{int}(B) = \emptyset$.
2. $X = \cup \mathcal{H}$.
3. Bármely $A \in \mathcal{H}$ intervallumra létezik $\emptyset \neq \mathcal{A} \subseteq \mathcal{H}$, hogy $\text{int}(f(A)) = \text{int}(\cup \mathcal{A})$.
4. Az f leképezés lineáris bármely $H \in \mathcal{H}$ intervallum belsején.

Egy másik fontos fogalom a dolgozat során az expanszív ℓ -Markov partíció, mely pontos definíciója meglehetősen technikai, viszont elégséges feltétele, ha a függvény meredeksége minden szakaszon nagyobb egy 1-nél nagyobb konstansnál.

Definíció B.2. Legyen $\mathcal{H}$ az (X, f) dinamikai rendszer ℓ -Markov partíciója. Jelöljük $\mathcal{H}$ elemeit H_i -vel ahol az i index egy pozitív egész szám. Ekkor pozitív egészek egy nem feltétlenül véges $(i_0, i_1, \dots)$ sorozatára definiáljuk az alábbi halmazt:

$$H_{i_0 i_1 \dots} = \bigcap_{n=0}^{\infty} f^{-n}(H_{i_n}).$$

Ha a sorozat véges, akkor értelemszerűen csak véges metszetet veszünk.

Definíció B.3. Az (X, f) dinamikai rendszer akkor rendelkezik expanszív ℓ -Markov partícióval, ha bármely $\delta > 0$ számhoz és $I \subseteq X$ intervallumhoz létezik a pozitív egészek véges sorozatainak $(i_0^{(1)}, \dots, i_{n_1}^{(1)}), \dots, (i_0^{(N)}, \dots, i_{n_N}^{(N)})$ véges halmaza, melyre:

$$\lambda\left(I\Delta \bigcup_{k=1}^N H_{i_0^{(k)} \dots i_{n_k}^{(k)}}\right) < \delta.$$

Bevezetjük a szimbolikus dinamika fogalomtárát, mely nagyon hasznosnak bizonyult a bizonyítások során, valamint belátjuk, hogy az expanszív ℓ -Markov partícióval rendelkező (X, f) rendszerek az értelmezési tartomány egy teljes mértékű részhalmazára vett megszorítását követően topologikusan konjugáltak egy hozzájuk tartozó szimbolikus dinamikus rendszerhez.

Árnyékolási tulajdonság

Bevezetjük a szakirodalomban használatos, az árnyékolási tulajdonsághoz tartozó alapvető fogalmakat. A szimbolikus dinamika eszköztárát alapvetően használva bizonyítjuk, hogy az $(\mathbb{R}_{\geq 0}, T)$ rendszer rendelkezik az árnyékolási tulajdonsággal.

Frobenius-Perron operátor és invariáns mértékek

Bevezetjük a Frobenius-Perron operátor fogalmát. Megmutatjuk, hogy ℓ -Markov partícióval rendelkező rendszerek esetén ez az operátor reprezentálható egy megszámlálható dimenziós mátrixszal. Megvizsgáljuk, hogy bizonyos szakaszonként konstans abszolút folytonos mértékek előretoltjai miként viselkednek a vizsgált rendszerekben. Kiderül, hogy ezt a kérdést vissza lehet vezetni, egy az adott dinamikus rendszerhez asszociált Markov-lánc stacionárius eloszlásának létezésének kérdésére és az ahhoz való konvergenciára. A legerősebb tételeket expanszív ℓ -Markov partícióval rendelkező rendszerekről sikerült bebizonyítani.

Tétel B.1. *Tegyük fel, hogy (X, f) egy olyan dinamikai rendszer, mely rendelkezik expanszív ℓ -Markov partícióval és ez a partíció egy aperiodikus irreducibilis, pozitív rekurrens Markov-láncot határoz meg. Legyen ennek a rendszernek $\pi : \mathcal{L}(X) \rightarrow \mathbb{R}_{\geq 0}$ az abszolút folytonos invariáns mértéke. Legyen $\mu : \mathcal{L}(X) \rightarrow \mathbb{R}_{\geq 0}$ egy abszolút folytonos véges mérték. Ebben az esetben, bármely $A \in \mathcal{L}(X)$ esetén teljesül, hogy $\lim_{n \rightarrow \infty} \mu(f^{-n}(A)) = \mu(X)\pi(A)$.*

Ezt a tételt bizonyos feltételek mellett tovább általánosítjuk olyan esetekre is, ahol μ nem feltétlenül véges, vagy akár nem is σ -véges.

Tétel B.2. *Tegyük fel, hogy μ egy invariáns mérték $\mathcal{L}(X)$ elemein és létezik olyan $\rho : X \rightarrow \mathbb{R}_{\geq 0}$ sűrűség, hogy bármely $A \in \mathcal{L}(X)$ halmazra $\mu(A) = \int_A \rho d\lambda$. Legyen $\mu(X) = \infty$ és, tegyük fel, hogy $\pi(H) = 0$ implikálja, hogy $\lambda(H) = 0$. Ekkor bármely $A \in \mathcal{L}(X)$, $\lambda(A) \neq 0$ halmaz esetén $\mu(A) = \infty$.*

Ezeket a tételeket közvetlenül alkalmazzuk az $(\mathbb{R}_{\geq 0}, T)$ rendszerre is. Kiszámítjuk a pontos abszolút folytonos invariáns mértékét. Ebből messzemenő következtetéseket le lehet vonni a rendszer dinamikai tulajdonságairól, például azt, hogy majdnem minden pont pályája sűrű, valamint, hogy a Schweitzer verseny feladatának eredeti kérdésére a válaszok $\liminf_{k \rightarrow \infty} (T^k(x)) = 0$ és $\limsup_{k \rightarrow \infty} (T^k(x)) = \infty$.

Racionálisok pályája

Ebben a fejezetben kifejezetten olyan rendszerekkel foglalkozunk, melyek expanszív ℓ -Markov partícióját az egész intervallumok egy $\mathcal{J}$ részhalmaza adja. Azt is feltesszük, hogy minden egész intervallum képe legalább két egész intervallumot tartalmaz. Ezeket a megkötéseket teljesíti $(\mathbb{R}_{\geq 0}, T)$. Az X értelmezési tartományt megszorítjuk egy $\tilde{X}$ részhalmazára, melyre $X \setminus \tilde{X}$ nullmértékű, és a benne levő pontok pályáit pontosan ismerjük. Használjuk a $J_n := [n-1, n]$ jelölést. A racionálisok pályáinak motivációja, hogy a szimbolikus dinamika eszköztárának segítségével, például egy szám tizedes tört alakja is értelmezhető úgy, mint az adott pont egy speciális dinamikus rendszerhez tartozó szimbolikus pályája. Érdekes kérdés, hogy mely egyéb rendszerek esetén lesznek a tizedes tört alakhoz hasonlóan pontosan a racionális számok végperiodikusak. Ezt a kérdést véges sok állapotú Markov partíció esetén teljesen, végtelen sok esetén részben megválaszoljuk. Ehhez alkalmazunk számelméleti eredményeket, mivel ez a feladat átfordítható Cantor-sorok racionalitásának kérdésére.

Tétel B.3. *Az $(\tilde{X}, f)$ rendszerben bármely végperiodikus $x \in \tilde{X}$ pont racionális.*

Tétel B.4. *Az $(\tilde{X}, f)$ rendszerben bármely $x \in \tilde{X}$ racionális pont pályája végperiodikus, amennyiben:*

$$0 = \liminf_{N \rightarrow \infty} \max_{n \in \{1-N, \dots, N\}} \lambda \left(f^{-1}((-\infty, -N] \cup [N, \infty)) \cap J_n \right). \quad (35)$$

Ezt a kritériumot teljesíti bármely véges sok állapotból álló rendszer, valamint $(\mathbb{R}_{\geq 0}, T)$ is. Ez még nem teljes karakterizáció, ezért a dolgozat végén említünk néhány rendszert, melyek érdekesek lehetnek a további vizsgálatokhoz.

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