

# Set theoretical aspects of Haar null and Haar meager sets

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# 1 Introduction

In almost all branches of mathematics, there are notions of a phenomenon happening very often or very rarely. For example, one can think of the probability method in combinatorics or the notion of arithmetic density in number theory.

When we study subsets of  $\mathbb{R}^n$ , the generally accepted notions of smallness are sets with Lebesgue measure zero or being of first category in the sense of Baire. For both notions, the small sets form  $\sigma$ -ideals. One often handles these  $\sigma$ -ideals as the duals of each other, see for example the Erdős-Sierpiński duality theorem, which states that assuming the Continuum Hypothesis there is a bijection  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $X$  is meager if and only if  $f(X)$  has Lebesgue measure zero and  $X$  has Lebesgue measure zero if and only if  $f(X)$  is meager. Other examples that show the duality of the two systems are Fubini's theorem for measure and Kuratowski-Ulam theorem for category, or the existence of  $G_\delta$  hull for measure and the existence of  $F_\sigma$  hull for category. These and other statements that show the analogy between category and measure can be found in [14].

The Haar measure is a well-known generalization of the Lebesgue measure for locally compact topological groups. For Polish groups that are not locally compact, there is no Haar measure, but J.P.R. Christensen introduced a generalization of sets with Haar measure zero in [3], namely the so-called *Haar null* sets.

**Definition 1.1.** Let  $G$  be a Polish group. We say that  $X \subseteq G$  is *Haar null* if there is a Borel set  $B \supseteq X$  and a Borel probability measure  $\mu$  such that  $\mu(gBh) = 0$  for all  $g, h \in G$ .

J.P.R. Christensen required in his original definition the hull  $B$  from the previous definition only to be universally measurable. When twenty years later B. Hunt, T. Sauer, and J. Yorke independently redefined Haar null sets (under the name "shy" sets), they already used the definition with Borel hulls. As it turned out, these notions do not coincide in many cases, so we handle them separately. We call the system of sets with universally measurable hulls *generalized Haar null* sets.

Since then many nice applications of Haar null and generalized Haar null sets have been found. For example, C. Rosendal proved in [17] that every universally measurable homomorphism from a Polish group to a separable topological group (in particular to a Polish group) is continuous. Another example is a generalization of the Rademacher theorem where the set of points where the function is non-differentiable turns out to be generalized Haar null (see [4]).

The notion of a set being meager remains meaningful in non-locally compact Polish groups, however, it is not a very good dual of Haar null sets. For example, there is no Fubini-type theorem for the Haar null sets which could be the dual of Kuratowski-Ulam theorem. Thus U. B. Darji introduced in [5] the concept of Haar meager sets.

The natural next step was to define generalized Haar meager sets. In order to do that, we need a category analog of the notion of universally measurable sets. M. Pálffy

introduced the (in fact choose one, as there were many) notion of universally Baire sets.

The goal of the TDK paper is to examine the system of (generalized) Haar null and (generalized) Haar meager sets. Haar null and Haar meager sets already have relatively large literature, so in Section 2 first we will state some of the most basic properties of them. Then we turn to more recent, set-theoretical results. It is a natural question whether two  $\sigma$ -ideals are isomorphic to each other. In Section 3 we introduce the definition of cardinal invariants, which are cardinals assigned to isomorphism classes of  $\sigma$ -ideals. Then in Section 4 we calculate these cardinal invariants for the systems of Haar null, generalized Haar null, Haar meager, and generalized Haar meager sets. It turns out that consistently the values of the cardinal invariants differ for these systems, therefore they are not isomorphic to each other. On the other hand, in Section 5 we prove that if we assume the Continuum Hypothesis, then there exists isomorphism between the Haar null and Haar meager sets of uncountable Polish groups, that is, consistently the  $\sigma$ -ideals of Haar null and Haar meager sets are isomorphic. We would like to emphasize that the theorems from Section 4 related to generalized Haar meager sets and Section 5 are new results of M. Elekes, M. Pálffy and the author of the TDK paper, and will be published in an upcoming paper. As the results from the other sections are known, we only include the statements and references for them.

## 2 Haar null and Haar meager sets

In this part of the TDK paper, we will introduce the definition of Haar null, Haar meager, generalized Haar null, and generalized Haar meager sets. Most of the results of this section can be found in [8], while others in [15].

### 2.1 Haar null and generalized Haar null sets

**Definition 2.1** (Haar null set). Let  $G$  be a Polish group. We say that  $X \subseteq G$  is *Haar null* if there is a Borel set  $B \supseteq X$  and a Borel probability measure  $\mu$  such that  $\mu(gBh) = 0$  for all  $g, h \in G$ . We denote by  $\mathcal{HN}(G)$  the system of Haar null subsets of  $G$ .

**Remark 2.2.** We call the measure  $\mu$  and  $B$  from the previous definition the *witness measure* of  $X$  and the *Borel hull* of  $X$ . Some authors (including Christensen) require universally measurable hull instead of Borel hull. Many proofs work for the system of sets with this alternative definition, but the two notions are not equivalent, so we will handle them separately.

**Definition 2.3** (Generalized Haar null set). Let  $G$  be a Polish group. We say that  $X \subseteq G$  is *generalized Haar null* if there is a universally measurable set  $A \supseteq X$  and a Borel probability measure  $\mu$  such that  $\mu(gAh) = 0$  for all  $g, h \in G$ . We denote by  $\mathcal{GHN}(G)$  the system of generalized Haar null subsets of  $G$ .

It is easy to see that  $\mathcal{HN}(G) \subseteq \mathcal{GHN}(G)$  holds for every Polish group  $G$ . We would

like these definitions to extend the definition of Haar measure zero, so we have to examine groups in which Haar measure exists, that is, locally compact groups.

**Theorem 2.4.** *Let  $G$  be a locally compact Polish group. Then  $\mathcal{GHN}(G) = \mathcal{HN}(G) = \mathcal{N}(G)$ , that is, the systems of generalized Haar null sets, Haar null sets, and sets with Haar measure zero coincide.*

It is natural to aim for these definitions to be notions of smallness, that is to require that  $\mathcal{HN}(G)$  and  $\mathcal{GHN}(G)$  form  $\sigma$ -ideals. With the following theorem, this turns out to be true.

**Theorem 2.5.** *Let  $G$  be an arbitrary Polish group. Then the system of Haar null sets  $\mathcal{HN}(G)$  and the system of generalized Haar null sets  $\mathcal{GHN}(G)$  form  $\sigma$ -ideals.*

## 2.2 Haar meager and generalized Haar meager sets

Now we turn to the definition of the system of Haar meager and generalized Haar meager sets and mention the same two basic facts about them: they extend the definition of meager sets and they form  $\sigma$ -ideals.

**Definition 2.6** (Haar meager set). Let  $G$  be a Polish group. We say that  $X \subseteq G$  is *Haar meager* if there is a Borel set  $B \supseteq X$ , a compact metric space  $C$  and a continuous function  $f : C \rightarrow G$  such that  $f^{-1}(gBh)$  is meager in  $C$  for all  $g, h \in G$ . We denote by  $\mathcal{HM}(G)$  the system of Haar meager subsets of  $G$ .

Notice that the sets with the property of Baire are often handled as the dual of measurable sets. So it is natural to define the dual of universally measurable sets, that is, universally Baire sets. Recently, M. Pálfi in his MSc thesis [15] collected various possible definitions and examined their relations to each other. It turns out that the following is the most natural:

**Definition 2.7** (Universally Baire set). Let  $X$  be a Polish space and  $A \subseteq X$ . The set  $A$  is called *universally Baire* if  $f^{-1}(A)$  has the property of Baire in  $Y$  for any Polish space  $Y$  and continuous function  $f : Y \rightarrow X$ .

Since the preimage of any  $\Sigma_1^1$  set is  $\Sigma_1^1$ , we get that every  $\Sigma_1^1$  set is universally Baire (and the same holds for  $\Pi_1^1$  sets). Thus every Borel set is universally Baire.

**Definition 2.8** (Nowhere locally constant function). Let  $X$  and  $Y$  be Polish spaces. Then we call a continuous function  $f : X \rightarrow Y$  *nowhere locally constant*, if for every  $y \in Y$  the set  $f^{-1}(y)$  has empty interior.

**Definition 2.9** (Universally meager set). Let  $X$  be a Polish space. Then the set  $M \subseteq X$  is called *universally meager* if for every Polish space  $Y$  and for every continuous nowhere locally constant function  $f : Y \rightarrow X$  the set  $f^{-1}(M)$  is meager.

**Remark 2.10.** If we only consider every nowhere locally constant continuous function  $f : \mathbb{N}^\omega \rightarrow X$  in the previous definition, we get an equivalent notion.

**Remark 2.11.** Let  $X$  be a Polish space. Then every universally meager set  $M \subseteq X$  is universally Baire.

M. Pálffy introduced generalized Haar meager sets in his MSc thesis [15]:

**Definition 2.12** (Generalized Haar meager set). Let  $G$  be a Polish group. We say that  $X \subseteq G$  is *generalized Haar meager* if there is a universally Baire set  $A \supseteq X$ , a compact metric space  $C$  and a continuous function  $f : C \rightarrow G$  such that  $f^{-1}(gAh)$  is meager in  $C$  for all  $g, h \in G$ . We denote by  $\mathcal{GHM}(G)$  the system of generalized Haar meager subsets of  $G$ .

**Remark 2.13.** Similarly, as in Remark 2.2, we call the function  $f$  and the set  $B$  (or  $A$ ) from the previous definitions the *witness function* of  $X$  and the *Borel (or universally Baire) hull* of  $X$ .

Unlike in the case of measure, the notion of meagerness remains meaningful in Polish groups which are not locally compact. In the next theorems we will examine the relation of the systems  $\mathcal{M}(G)$ ,  $\mathcal{HM}(G)$  and  $\mathcal{GHM}(G)$  in locally compact and non-locally compact Polish groups.

**Theorem 2.14.** Let  $G$  be an arbitrary Polish group. Then  $\mathcal{HM}(G) \subseteq \mathcal{GHM}(G) \subseteq \mathcal{M}(G)$ , in other words, every Haar meager set is generalized Haar meager, and every generalized Haar meager set is meager.

**Theorem 2.15.** Let  $G$  be a locally compact Polish group. Then  $\mathcal{HM}(G) = \mathcal{GHM}(G) = \mathcal{M}(G)$ , that is, the system of meager, Haar meager, and generalized Haar meager sets coincide.

First of all, we will need the following interesting (and pretty complicated) theorem of S. Solecki, the proof of which can be found in [18]. The theorem (together with Lemma 2.17) states that there are continuum many pairwise disjoint closed sets in every Polish group  $G$  with a two-sided invariant metric such that none of them are Haar null and none of them are Haar meager. (Notice that this is not the case for the  $\sigma$ -ideals  $\mathcal{N}(G)$  and  $\mathcal{M}(G)$ .)

**Theorem 2.16.** Assume that  $G$  is a non-locally compact Polish group that admits a two-sided invariant metric. Then there exists a closed set  $F \subseteq G$  and a continuous function  $\varphi : F \rightarrow 2^\omega$  such that for any  $x \in 2^\omega$  and any compact set  $C \subseteq G$  there is  $g \in G$  such that  $gC \subseteq \varphi^{-1}(x)$ .

The following lemma gives a sufficient condition for a set being not (generalized) Haar meager and not (generalized) Haar null.

**Lemma 2.17.** Let  $G$  be a Polish group and let  $X \subseteq G$  have the following property: for any compact set  $C$  there is  $g, h \in G$  such that  $C \subseteq gXh$ . Then  $X \notin \mathcal{HN}(G)$  and  $X \notin \mathcal{HM}(G)$ .

**Remark 2.18.** We call sets with the property from the previous lemma *compact catcher*. Notice that the same proof shows that compact catcher sets are neither generalized Haar null nor generalized Haar meager.

Notice that as an easy corollary of the previous theorem, we get a sufficient condition when the  $\sigma$ -ideal of meager and Haar meager sets do not coincide.

**Theorem 2.19.** *Let  $G$  be a non-locally compact Polish group that admits a two-sided invariant metric. Then  $\mathcal{GHM}(G) \subsetneq \mathcal{M}(G)$  (which implies  $\mathcal{HM}(G) \subsetneq \mathcal{M}(G)$ ).*

As in the case of null sets, we want  $\mathcal{HM}(G)$  and  $\mathcal{GHM}(G)$  to form  $\sigma$ -ideals.

**Theorem 2.20.** *Let  $G$  be an arbitrary Polish group. Then the system of Haar meager sets  $\mathcal{HM}(G)$  and the system of generalized Haar meager sets  $\mathcal{GHM}(G)$  form  $\sigma$ -ideals.*

### 3 Cardinal invariants

To avoid confusion, let us define what we call  $\sigma$ -ideal in this paper.

**Definition 3.1.** A system  $\mathcal{I} \subseteq \mathcal{P}(X)$  is a  $\sigma$ -ideal if the following conditions hold:

- $\emptyset \in \mathcal{I}, X \notin \mathcal{I}$ ,
- if  $I \in \mathcal{I}$  and  $J \subseteq I$  then  $J \in \mathcal{I}$ ,
- if  $I_1, I_2 \dots \in \mathcal{I}$  then  $\bigcup_{n \in \omega} I_n \in \mathcal{I}$

Before we turn to the notion of cardinal invariants, let us recall what we mean under two  $\sigma$ -ideals being isomorphic.

**Definition 3.2.** Let  $X$  and  $Y$  be given sets with two  $\sigma$ -ideals  $\mathcal{I} \subseteq \mathcal{P}(X)$  and  $\mathcal{J} \subseteq \mathcal{P}(Y)$ . We call a bijection  $f : X \rightarrow Y$  an *isomorphism* between the  $\sigma$ -ideals  $\mathcal{I}$  and  $\mathcal{J}$ , if  $I \in \mathcal{I}$  if and only if  $f(I) \in \mathcal{J}$ .

Cardinal invariants (or cardinal characteristics) are infinite cardinal numbers that we assign to isomorphism classes of ideals, that is, if there is an isomorphism between two ideals, then their cardinal invariants are the same. Thus a basic motivation for cardinal characteristics is that we can distinguish ideals from each other if one of their invariants do not coincide. In this TDK paper we will only focus on the below defined 4 "standard" cardinal characteristics, for a more extensive list (and a brief introduction into the topic) see [2, Chapter 6].

As we will work only with  $\sigma$ -ideals, we define additivity ( $\text{add}(\mathcal{I})$ ), uniformity ( $\text{non}(\mathcal{I})$ ), covering number ( $\text{cov}(\mathcal{I})$ ), and cofinality ( $\text{cof}(\mathcal{I})$ ) only for a  $\sigma$ -ideal  $\mathcal{I}$ .

**Definition 3.3.** Let  $X$  be an arbitrary set, and  $\mathcal{I} \subseteq \mathcal{P}(X)$  be a given  $\sigma$ -ideal on  $X$  that contains all the singletons from  $X$ . Then

- $\text{add}(\mathcal{I}) := \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{A} \notin \mathcal{I}\}$
- $\text{non}(\mathcal{I}) := \min\{|A| : A \subseteq X \text{ and } A \notin \mathcal{I}\}$
- $\text{cov}(\mathcal{I}) := \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{A} = X\}$

- $\text{cof}(\mathcal{I}) := \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \text{ and } \forall I \in \mathcal{I} \exists A \in \mathcal{A} \text{ such that } I \subseteq A\}.$

It is easy to see that these definitions are indeed invariant under isomorphism between  $\sigma$ -ideals.

**Example 3.4.** Let  $X = 2^\omega$  (or the real line  $\mathbb{R}$ ) and  $\mathcal{C}$  be the  $\sigma$ -ideal of its countable subsets. Then  $\text{add}(\mathcal{C}) = \text{non}(\mathcal{C}) = \omega_1$  and  $\text{cov}(\mathcal{C}) = \text{cof}(\mathcal{C}) = 2^\omega$ .

**Definition 3.5** (dominating number, bounding number). Let  $X = \mathbb{Z}^\mathbb{N}$ , and  $\mathcal{K}_\sigma$  be the  $\sigma$ -ideal generated by the compact sets, that is:

$$\mathcal{K}_\sigma := \{X \subseteq \mathbb{Z}^\omega : \exists C_0, C_1, \dots \text{ compact such that } \bigcup_{n \in \omega} C_n \supseteq X\}.$$

Then we define the *dominating number* as  $\mathfrak{d} = \text{cov}(\mathcal{K}_\sigma)$  and the *bounding number* as  $\mathfrak{b} = \text{non}(\mathcal{K}_\sigma)$ .

The following almost trivial lemma will be helpful in situations when a  $\sigma$ -ideal contains another.

**Lemma 3.6.** Let  $\mathcal{I} \subseteq \mathcal{P}(X)$  and  $\mathcal{J} \subseteq \mathcal{P}(X)$  be two  $\sigma$ -ideals on  $X$  such that  $\mathcal{I} \subseteq \mathcal{J}$ . Then  $\text{cov}(\mathcal{I}) \geq \text{cov}(\mathcal{J})$  and  $\text{non}(\mathcal{I}) \leq \text{non}(\mathcal{J})$ .

*Proof.* Assume that  $\text{cov}(\mathcal{I}) = \kappa$  and  $\{X_\alpha\}_{\alpha < \kappa} \subseteq \mathcal{I}$  is a covering system, that is  $\bigcup_{\alpha < \kappa} X_\alpha = X$ . Since  $\{X_\alpha\}_{\alpha < \kappa} \subseteq \mathcal{J}$ , it is a covering system of  $\mathcal{J}$  as well and thus  $\text{cov}(\mathcal{J}) \leq \kappa$ .

For the uniformity assume that  $\text{non}(\mathcal{J}) = \kappa$ , and there are elements  $x_\alpha \in X$  for every  $\alpha < \kappa$  such that  $\{x_\alpha : \alpha < \kappa\} \notin \mathcal{J}$ . But then  $\{x_\alpha : \alpha < \kappa\} \notin \mathcal{I}$  either, which shows that  $\text{non}(\mathcal{I}) \leq \kappa$ .  $\square$

## 4 Cardinal invariants of Haar null and Haar meager sets

In this section we would like to summarize some very recent results about the cardinal invariants of Haar null, generalized Haar null, Haar meager, and generalized Haar meager sets. Most of the results can be found in [9] and in [1], while the results of Section 4.5 are recent results of M. Elekes, M. Pálffy and the author of the TDK paper.

Notice that the value of the cardinal invariants (introduced in Section 3) of the  $\sigma$ -ideals  $\mathcal{HM}(G)$ ,  $\mathcal{GHM}(G)$ ,  $\mathcal{HN}(G)$ ,  $\mathcal{GHN}(G)$  may depend on the group  $G$ . It turns out that in the locally compact case the four invariants do not depend on the underlying group  $G$ . When  $G$  is not locally compact, there are results only in certain cases, e.g. when  $G$  admits a two-sided invariant metric, or when it is the countable product of locally compact groups. Notice that the results always hold for the Baer-Specker group  $\mathbb{Z}^\omega$ , which is a very widely studied special case, when one examines non-locally compact Polish groups.

## 4.1 The locally compact case

When  $G$  is locally compact, calculating the cardinal invariants turns out to be as simple as possible.

**Theorem 4.1.** *Let  $G$  be a locally compact non-discrete Polish group. Then the following hold:*

$$\begin{aligned}\text{add}(\mathcal{HN}(G)) &= \text{add}(\mathcal{GHN}(G)) = \text{add}(\mathcal{N}(\mathbb{R})) \\ \text{non}(\mathcal{HN}(G)) &= \text{non}(\mathcal{GHN}(G)) = \text{non}(\mathcal{N}(\mathbb{R})) \\ \text{cov}(\mathcal{HN}(G)) &= \text{cov}(\mathcal{GHN}(G)) = \text{cov}(\mathcal{N}(\mathbb{R})) \\ \text{cof}(\mathcal{HN}(G)) &= \text{cof}(\mathcal{GHN}(G)) = \text{cof}(\mathcal{N}(\mathbb{R}))\end{aligned}$$

The dual statement for Haar meager sets also holds.

**Theorem 4.2.** *Let  $G$  be a locally compact non-discrete Polish group. Then the following hold:*

$$\begin{aligned}\text{add}(\mathcal{HM}(G)) &= \text{add}(\mathcal{GHM}(G)) = \text{add}(\mathcal{M}(G)) = \text{add}(\mathcal{M}(\mathbb{R})) \\ \text{non}(\mathcal{HM}(G)) &= \text{non}(\mathcal{GHM}(G)) = \text{non}(\mathcal{M}(G)) = \text{non}(\mathcal{M}(\mathbb{R})) \\ \text{cov}(\mathcal{HM}(G)) &= \text{cov}(\mathcal{GHM}(G)) = \text{cov}(\mathcal{M}(G)) = \text{cov}(\mathcal{M}(\mathbb{R})) \\ \text{cof}(\mathcal{HM}(G)) &= \text{cof}(\mathcal{GHM}(G)) = \text{cof}(\mathcal{M}(G)) = \text{cof}(\mathcal{M}(\mathbb{R}))\end{aligned}$$

## 4.2 Cardinal invariants of Haar null sets

First, we will compute the covering number and the uniformity in  $\mathbb{Z}^\omega$ .

**Theorem 4.3.**

$$\begin{aligned}\text{cov}(\mathcal{HN}(\mathbb{Z}^\omega)) &= \min\{\mathfrak{b}, \text{cov}(\mathcal{N}(\mathbb{R}))\}, \\ \text{non}(\mathcal{HN}(\mathbb{Z}^\omega)) &= \max\{\mathfrak{d}, \text{non}(\mathcal{N}(\mathbb{R}))\}.\end{aligned}$$

Let us turn to the cofinality of  $\mathcal{HN}(G)$ , which was calculated in [9, Theorem 2.7].

**Theorem 4.4.** *Let  $G$  be a non-locally compact Polish group that admits a two-sided invariant metric. Then  $\text{cof}(\mathcal{HN}(G)) = \mathfrak{c}$ .*

For computing the additivity of  $\mathcal{HN}(G)$  let us state [10, Theorem 1.4].

**Theorem 4.5.** *Let  $G$  be a non-locally compact abelian Polish group and  $1 \leq \xi < \omega_1$  an arbitrary countable ordinal. Then there exists a (Borel) Haar null set  $B \subseteq G$  that cannot be covered by a  $\Pi_\xi^0$  set.*

Now we get the additivity of Haar null sets in non-locally compact abelian Polish groups as an easy, but surprising corollary of Theorem 4.5.

**Corollary 4.6.** *Let  $G$  be a Polish group that is not locally compact and abelian. Then we have  $\text{add}(\mathcal{HN}(G)) = \omega_1$ .*

### 4.3 Cardinal invariants of Haar meager sets

First, we will compute the covering number and the uniformity in Polish groups that admit a continuous surjective homomorphism to locally compact non-discrete Polish groups.

**Theorem 4.7.** *Let  $G$  be a Polish group that admits a continuous surjective homomorphism to a non-discrete locally compact Polish group  $H$ . Then  $\text{cov}(\mathcal{HM}(G)) = \text{cov}(\mathcal{M}(\mathbb{R}))$  and  $\text{non}(\mathcal{HM}(G)) = \text{non}(\mathcal{M}(\mathbb{R}))$ .*

The cofinality of Haar meager sets was calculated in [9]. The proof for Haar meager sets is the same as the proof in the Haar null case (which we sketched in Theorem 4.4), so we only state the theorem here.

**Theorem 4.8.** *Let  $G$  be a non-locally compact Polish group that admits a two-sided invariant metric. Then  $\text{cof}(\mathcal{HM}(G)) = \mathfrak{c}$ .*

Finally, for computing the additivity of  $\mathcal{HM}(G)$  let us state the dual of Theorem 4.5, the proof of which can be found in [7].

**Theorem 4.9.** *Let  $G$  be a non-locally compact abelian Polish group and  $1 \leq \xi < \omega_1$  an arbitrary countable ordinal. Then there exists a (Borel) Haar meager set  $B \subseteq G$  that cannot be covered by a  $\Pi_\xi^0$  set.*

Using Theorem 4.9 we can derive the same corollary as in the Haar null case.

**Corollary 4.10.** *Let  $G$  be a Polish group that is not locally compact and abelian. Then  $\text{add}(\mathcal{HM}(G)) = \omega_1$ .*

### 4.4 Cardinal invariants of generalized Haar null sets

We will first examine the covering number and the uniformity of  $\mathcal{GHN}(G)$ . Let us state an equivalent characterization of Haar null sets in  $\mathbb{Z}^\omega$  from [13]. Notice that S. Solecki proved in [19, Theorem 4.1] a very similar result for countable products of locally compact amenable groups. We mention this theorem here since it might be relevant for the additivity of generalized Haar meager sets. In fact, we do not know about any natural analog of Theorem 4.11, but it would be very interesting to have one.

For any sequence of natural numbers  $a \in \mathbb{N}^\omega$ , let us denote with  $\mu_a$  the probability measure  $\prod_{n=1}^\infty \varrho_{a(n)}$  on  $\mathbb{Z}^\omega$  (where  $\varrho_{a(n)}$  denotes the uniform probability measure on  $[0, a(n)]$ ).

**Theorem 4.11.** *A subset  $X$  of  $\mathbb{Z}^\omega$  is (generalized) Haar null if and only if there is a (universally measurable) Borel set  $B \supseteq X$  and a sequence  $(a(n))_{n \in \omega} \in \mathbb{N}^\omega$  such that  $\mu_a(B + x) = 0$  for every  $x \in \mathbb{Z}^\omega$ .*

The proof of the following theorem can be found in [1]. T. Banach proved this theorem for countable products of locally compact amenable Polish groups.

**Theorem 4.12.**

$$\begin{aligned}\text{cov}(\mathcal{GHN}(\mathbb{Z}^\omega)) &= \min\{\mathfrak{b}, \text{cov}(\mathcal{N}(\mathbb{R}))\}, \\ \text{non}(\mathcal{GHN}(\mathbb{Z}^\omega)) &= \max\{\mathfrak{d}, \text{non}(\mathcal{N}(\mathbb{R}))\}.\end{aligned}$$

Now we turn to the additivity of  $\mathcal{GHN}(G)$ . T. Banach proved Theorem 4.13 in [1] for countable products of locally compact abelian groups, but for the sake of simplicity, we only state it in  $\mathbb{Z}^\omega$ .

**Theorem 4.13.**  $\text{add}(\mathcal{GHN}(\mathbb{Z}^\omega)) = \text{add}(\mathcal{N}(\mathbb{R}))$ .

Finally, in the case of cofinality we have only an inequality. Even though this result may seem weaker than the other cases, it is very interesting since cardinal invariants typically never exceed the continuum. The following theorem can be found in [1].

**Theorem 4.14.** *Let  $G$  be a Polish group that is not locally compact and admits a two-sided invariant metric. Then  $\text{cof}(\mathcal{GHN}(G)) > \min\{\mathfrak{d}, \text{non}(\mathcal{N}(\mathbb{R}))\}$ .*

## 4.5 Cardinal invariants of generalized Haar meager sets

Finally, we turn to the recent results of M. Elekes, M. Pálfi, and the author of the TDK paper.

We start this section by computing the covering number and the uniformity.

**Theorem 4.15.** *Let  $G$  be a non-locally compact Polish group that admits a continuous surjective homomorphism to a non-discrete locally compact Polish group  $H$ . Then  $\text{cov}(\mathcal{GHM}(G)) = \text{cov}(\mathcal{M}(\mathbb{R}))$  and  $\text{non}(\mathcal{GHM}(G)) = \text{non}(\mathcal{M}(\mathbb{R}))$ .*

*Proof.* From Theorem 2.14 we know that  $\mathcal{HM}(G) \subseteq \mathcal{GHM}(G) \subseteq \mathcal{M}(G)$ . Thus from Lemma 3.6 we get that  $\text{cov}(\mathcal{HM}(G)) \geq \text{cov}(\mathcal{GHM}(G)) \geq \text{cov}(\mathcal{M}(G))$  and  $\text{non}(\mathcal{HM}(G)) \leq \text{non}(\mathcal{GHM}(G)) \leq \text{non}(\mathcal{M}(G))$  hold. From Theorem 4.7 and Theorem 4.2 we know that  $\text{cov}(\mathcal{HM}(G)) = \text{cov}(\mathcal{M}(G)) = \text{cov}(\mathcal{M}(\mathbb{R}))$  and  $\text{non}(\mathcal{HM}(G)) = \text{non}(\mathcal{M}(G)) = \text{non}(\mathcal{M}(\mathbb{R}))$ , from which the statement immediately follows.  $\square$

Now let us turn to the cofinality of  $\mathcal{GHM}(G)$  in non-locally compact Polish groups. Here we only have a consistent result, that is, assuming the Continuum Hypothesis we know that  $\text{cof}(\mathcal{GHM}(G)) > \mathfrak{c}$ . This result is really peculiar on one hand since all the usual cardinal invariants are at most the continuum. On the other hand, in the generalized Haar null case the same inequality holds, thus Theorem 4.16 strengthens the connection of generalized Haar meager and generalized Haar null sets.

**Theorem 4.16.** *Let  $G$  be a non-locally compact group that admits a two-sided invariant metric, furthermore let us assume the Continuum Hypothesis. Then  $\text{cof}(\mathcal{GHM}(G)) > \mathfrak{c}$ .*

To prove the theorem, we will need the following lemmas, that are interesting in their own right.

**Lemma 4.17.** *Every universally meager set  $M \subseteq G$  is generalized Haar meager.*

*Proof.* Since universally meager sets are universally Baire, we know that  $M$  is universally Baire, thus it is enough to find a function  $f : 2^\omega \rightarrow G$  such that  $f^{-1}(gMh)$  is meager in  $2^\omega$  for any  $g, h \in G$ . We claim that any continuous and nowhere locally constant function  $f : 2^\omega \rightarrow G$  is witness function. To see this, take any  $g, h \in G$ , then it is easy to check that the function  $f'(x) := g^{-1}f(x)h^{-1}$  is nowhere locally constant. Thus  $f^{-1}(gMh) = f'^{-1}(M)$  is meager in  $2^\omega$ , which concludes the proof of the lemma.  $\square$

Let us introduce the following notation.

**Notation 4.18.** Let  $\alpha < \omega_1$  be an arbitrary countable ordinal. Let us define  $X_\alpha := \{f : f \text{ is a function from } \alpha \text{ to } 2^\omega\}$  and  $X := \bigcup_{\alpha < \omega_1} X_\alpha$ . There is a natural partial ordering on  $X$ : let us say that  $f \geq g$  for any  $f, g \in X$  if  $f$  extends  $g$ .

**Lemma 4.19.** *There exists a function  $\Phi : X \rightarrow \mathcal{P}(2^\omega)$ , such that*

- 1)  $\Phi(f) \in \Pi_1^1$  for every  $f \in X$ ,
- 2)  $\Phi(f) \subsetneq \Phi(g)$  if  $f > g$ ,
- 3)  $\Phi(f) \cap \Phi(g) = \emptyset$ , if  $f \not\geq g$  and  $g \not\geq f$

*Proof.* It is clear that it is enough to find a function  $\Phi : X \rightarrow \mathcal{P}((2^\omega)^\omega \times 2^{\omega \times \omega})$  with properties 1), 2), and 3). Let us denote with  $\text{WO} \subseteq 2^{\omega \times \omega}$  the set of countable well-orderings, that is,  $\text{WO} = \{\prec \in 2^{\omega \times \omega} : \prec \text{ defines a well-ordering on } \omega\}$ .

For any  $\prec \in \text{WO}$  and  $n \in \omega$  we denote with  $\prec|_n$  the set  $\{k \in \omega : k \prec n\}$  with the well-ordering inherited from  $\prec$ . Let us fix  $\alpha$  and a function  $f \in X_\alpha$ . Then let us define:

$$\Phi(f) := \{(x, \prec) \in (2^\omega)^\omega \times \text{WO} : \alpha < \text{otp}(\prec) \wedge (\text{otp}(\prec|_n) < \alpha \Rightarrow x(n) = f(\text{otp}(\prec|_n)))\}.$$

To see that 2) holds, take any  $\alpha > \beta$  and  $f \in X_\alpha, g \in X_\beta$  such that  $f > g$ . We will show that  $(x, \prec) \in \Phi(f) \Rightarrow (x, \prec) \in \Phi(g)$ . By definition,  $\beta < \alpha < \text{otp}(\prec)$  and if  $\text{otp}(\prec|_n) < \beta$ , then  $\text{otp}(\prec|_n) < \alpha$ , therefore  $x(n) = f(\text{otp}(\prec|_n)) = g(\text{otp}(\prec|_n))$ , since  $f$  extends  $g$ . Consequently,  $(x, \prec) \in \Phi(g)$ . To see that  $\Phi(f) \setminus \Phi(g)$  is nonempty, take any  $\prec \in \text{WO}$  such that  $\beta \leq \text{otp}(\prec) < \alpha$ , and pick  $x \in (2^\omega)^\omega$  for which  $x(n) = f(\text{otp}(\prec|_n))$ . Then clearly  $(x, \prec) \in \Phi(f) \setminus \Phi(g)$ .

For 3), let us suppose to the contrary that  $f \not\geq g$  and  $g \not\geq f$  for some  $\alpha, \beta < \omega_1, f \in X_\alpha, g \in X_\beta$  and  $(x, \prec) \in \Phi(f) \cap \Phi(g)$ . Since  $f \not\geq g$  and  $g \not\geq f$ , there exists  $\gamma < \alpha, \beta$  such that  $f(\gamma) \neq g(\gamma)$ . Using that  $\prec$  defines a well-ordering, and  $\alpha, \beta < \text{otp}(\prec)$ , there exists  $n$  for which  $\text{otp}(\prec|_n) = \gamma$ . But then  $g(\gamma) = g(\text{otp}(\prec|_n)) = x(n) = f(\text{otp}(\prec|_n)) = f(\gamma)$ , which is a contradiction.

Now we have to show that  $\Phi(f)$  is  $\Pi_1^1$  for fixed  $\alpha$  and  $f \in X_\alpha$ . By [12, Theorem 32.B] WO is coanalytic in  $2^{\omega \times \omega}$ . From this and the fact that  $\{\prec \in \text{WO} : \text{otp}(\prec) \leq \alpha\}$  is Borel in  $2^{\omega \times \omega}$  (see [12, Exercise 34.17]), we get that  $\{(x, \prec) \in (2^\omega)^\omega \times \text{WO} : \alpha < \text{otp}(\prec)\}$  is coanalytic. Let us fix  $n$ , then it is enough to show that

$$\begin{aligned} & \{(x, \prec) \in (2^\omega)^\omega \times \text{WO} : \text{otp}(\prec \upharpoonright_n) \geq \alpha\} \cup \{(x, \prec) \in (2^\omega)^\omega \times \text{WO} : x(n) = f(\text{otp}(\prec \upharpoonright_n))\} \\ &= \{(x, \prec) \in (2^\omega)^\omega \times \text{WO} : \text{otp}(\prec \upharpoonright_n) \geq \alpha\} \cup \\ & \cup \bigcup_{\beta < \alpha} \left( \{(x, \prec) \in (2^\omega)^\omega \times \text{WO} : \text{otp}(\prec \upharpoonright_n) = \beta\} \cap \{(x, \prec) \in (2^\omega)^\omega \times \text{WO} : x(n) = f(\beta)\} \right) \end{aligned}$$

is coanalytic. Clearly, it is enough to show the following claim.

**Claim 4.20.** Fix  $n$  and  $\alpha < \omega_1$ . Then  $\{\prec \in \text{WO} : \text{otp}(\prec \upharpoonright_n) < \alpha\}$  is Borel in WO.

*Proof.* We will proceed by transfinite induction on  $\alpha$ . For  $\alpha = 1$  this means that  $n$  is the smallest element of  $\prec$ , that is:

$$\{\prec \in \text{WO} : \text{otp}(\prec \upharpoonright_n) < 1\} = \bigcap_{k \neq n} \{\prec \in 2^{\omega \times \omega} : n \prec k\} \cap \text{WO},$$

which is clearly Borel in WO (in fact, it is closed). Assume that we have proved the claim for all  $\beta < \alpha$  and  $n \in \omega$ . If  $\alpha$  is a limit ordinal, then

$$\{\prec \in \text{WO} : \text{otp}(\prec \upharpoonright_n) < \alpha\} = \bigcup_{\beta < \alpha} \{\prec \in \text{WO} : \text{otp}(\prec \upharpoonright_n) < \beta\},$$

thus by the induction hypothesis  $\{\prec \in \text{WO} : \text{otp}(\prec \upharpoonright_n) < \alpha\} \in \mathcal{B}(\text{WO})$ . If  $\alpha = \beta + 1$  then

$$\begin{aligned} \{\prec \in \text{WO} : \text{otp}(\prec \upharpoonright_n) < \alpha\} &= \bigcap_{l \neq n} \{\prec \in \text{WO} : l \prec n \Rightarrow \text{otp}(\prec \upharpoonright_l) < \beta\} = \\ &= \bigcap_{l \neq n} \left( \{\prec \in \text{WO} : n \prec l\} \cup \{\prec \in \text{WO} : \text{otp}(\prec \upharpoonright_l) < \beta\} \right). \end{aligned}$$

The former set is closed in WO, while the latter is Borel in WO by the induction hypothesis, which completes the proof.  $\square$

$\square$

*Proof of Theorem 4.16.* Let us suppose to the contrary that  $\text{cof}(\mathcal{GHM}(G)) \leq \omega_1$  (we assumed  $\omega_1 = \mathfrak{c}$ ), so there is a cofinal system  $\{M_\alpha\}_{\alpha < \omega_1}$ . Let  $\{f_\alpha\}_{\alpha < \omega_1}$  be an enumeration of every continuous nowhere locally constant function from  $\mathbb{N}^\omega$  to  $G$  such that for any limit ordinal  $\alpha$  there exists  $\beta < \alpha$  for which  $f_\alpha = f_\beta$ . Let us recall Theorem 2.16, and let  $\varphi : F \rightarrow 2^\omega$  be the function from the theorem. Finally, let us recall the notations  $X_\alpha$  and  $\Phi$  from Notation 4.18. We will define by transfinite recursion  $g_\alpha \in X_\alpha$ ,  $H_\alpha \subseteq G$  and  $h_\alpha \in G$  for every  $\alpha < \omega_1$  such that the following hold:

- 1)  $g_\beta \leq g_\alpha$  for every  $\beta < \alpha$  (where  $\leq$  denotes extension),

- 2)  $H_\alpha = \varphi^{-1}(\Phi(g_\alpha))$ ,
- 3)  $f_\alpha^{-1}(H_\alpha)$  is meager in  $\mathbb{N}^\omega$ ,
- 4)  $h_\alpha \in H_\alpha \setminus \bigcup_{\beta \leq \alpha} M_\beta$ .

First, let us assume that we have constructed  $g_\alpha$ ,  $H_\alpha$ , and  $h_\alpha$  this way. Then let us define  $H := \{h_\alpha : \alpha < \omega_1\}$ . We claim that  $H \in \mathcal{GHM}(G)$ , but  $H \not\subseteq M_\alpha$  for every  $\alpha < \omega_1$  which is a contradiction since  $M_\alpha$  is a cofinal system. From Lemma 4.17 we know that every universally meager set is generalized Haar meager, thus for  $H \in \mathcal{GHM}(G)$  it is enough to prove the following.

**Claim 4.21.**  *$H$  is universally meager.*

*Proof.* Using Remark 2.10, it suffices to prove that for every continuous nowhere locally constant function  $f_\alpha : \mathbb{N}^\omega \rightarrow G$  the set  $f_\alpha^{-1}(H) = f_\alpha^{-1}(H \cap H_\alpha) \cup f_\alpha^{-1}(H \setminus H_\alpha)$  is meager. Notice that  $H \setminus H_\alpha$  is countable (since  $H_\alpha \subseteq H_\beta$  if  $\alpha \geq \beta$ ), thus  $f_\alpha^{-1}(H \setminus H_\alpha)$  is meager (the preimage of any singleton by  $f_\alpha$  is nowhere dense closed set in  $\mathbb{N}^\omega$ ). On the other hand by 3)  $f_\alpha^{-1}(H_\alpha \cap H) \subseteq f_\alpha^{-1}(H_\alpha)$  is meager, thus  $f_\alpha^{-1}(H)$  is meager.  $\square$

It is easy to see that  $H \not\subseteq M_\alpha$  for every  $\alpha < \omega_1$ , since  $h_\alpha \notin M_\alpha$ . Thus the proof will be finished if we show that we can define  $g_\alpha$ ,  $H_\alpha$ , and  $h_\alpha$  with the properties listed above.

So let  $\alpha < \omega_1$  be any countable ordinal, and assume that we have constructed  $g_\beta$ ,  $H_\beta$  and  $h_\beta$  for every  $\beta < \alpha$ . If  $\alpha$  is a limit ordinal, then there exists exactly one function in  $X_\alpha$  that extends  $g_\beta$  for every  $\beta < \alpha$ , let this function be  $g_\alpha$ . Let  $H_\alpha := \varphi^{-1}(\Phi(g_\alpha))$ . We assumed that  $f_\alpha = f_\beta$  for some  $\beta < \alpha$ . We know that  $g_\alpha$  extends  $g_\beta$ , thus  $H_\alpha \subseteq H_\beta$ , consequently we get that  $f_\alpha^{-1}(H_\alpha) \subseteq f_\beta^{-1}(H_\beta) \in \mathcal{M}(\mathbb{N}^\omega)$ . To define  $h_\alpha$ , notice that  $\Phi(g_\alpha)$  is nonempty, therefore by Theorem 2.16 we get that  $H_\alpha$  is a compact catcher set. On the other hand,  $\bigcup_{\beta \leq \alpha} M_\alpha$  is countable union of generalized Haar meager sets, consequently, it is generalized Haar meager itself. Using Lemma 2.17 we get that  $H_\alpha \setminus \bigcup_{\beta \leq \alpha} M_\alpha \neq \emptyset$ , so let  $h_\alpha$  be an arbitrary element of the difference.

For a successor ordinal  $\alpha = \beta + 1$  there are continuum many functions in  $X_\alpha$  that extend  $g_\beta$ . Let us call them  $g_x$  ( $x \in 2^\omega$ ) according to the value they assign to  $\alpha$ , that is,  $g_x(\alpha) = x$ . Clearly, all of the functions  $\{g_x\}_{x \in 2^\omega}$  are pairwise incompatible (that is,  $g_x \not\leq g_y$  and  $g_y \not\leq g_x$  for any  $x \neq y$ ), thus  $\Phi(g_x) \cap \Phi(g_y) = \emptyset$  for any  $x \neq y$ . Consequently  $f_\alpha^{-1}(\varphi^{-1}(\Phi(g_x))) \cap f_\alpha^{-1}(\varphi^{-1}(\Phi(g_y))) = \emptyset$ , furthermore  $f_\alpha^{-1}(\varphi^{-1}(\Phi(g_x)))$  is  $\Pi_1^1$  for every  $x \in 2^\omega$  (continuous preimage of a  $\Pi_1^1$  set is  $\Pi_1^1$ ). Recall that every  $\Pi_1^1$  set possesses the Baire property, thus we have continuum many pairwise disjoint sets in  $\mathbb{N}^\omega$  with the Baire property. From this follows that there exists  $x \in 2^\omega$  such that  $f_\alpha^{-1}(\varphi^{-1}(\Phi(g_x)))$  is meager. Accordingly, let us define  $g_\alpha := g_x$  and  $H_\alpha := \varphi^{-1}(\Phi(g_\alpha))$ . Then 1) and 3) hold due to our choice on  $g_\alpha$ , and using Theorem 2.16 we know that  $H_\alpha$  is compact catcher, thus the difference  $H_\alpha \setminus \bigcup_{\beta \leq \alpha} M_\beta$  is nonempty by Lemma 2.17. Let  $h_\alpha$  be any element in the difference. So the conditions of the transfinite recursion hold, and this completes the proof.  $\square$

## 5 Isomorphism theorems for Haar null and Haar meager sets

In this section we will prove that consistently (assuming the Continuum Hypothesis) the  $\sigma$ -ideals  $\mathcal{HM}(G)$ ,  $\mathcal{HN}(G)$ ,  $\mathcal{M}(\mathbb{R})$ , and  $\mathcal{N}(\mathbb{R})$  are isomorphic for an uncountable Polish group  $G$  with two-sided invariant metric. Moreover, there is an Erdős-Sierpiński duality between the  $\sigma$ -ideals  $\mathcal{HN}(\mathbb{Z}^\omega)$  and  $\mathcal{HM}(\mathbb{Z}^\omega)$ . On the contrary, we prove that for Polish groups with two-sided invariant metric, there is neither a Borel nor an additive isomorphism. The results of this section are recent results of M. Elekes, M. Pálffy, and the author of the TDK paper.

In Section 3 we have already defined isomorphism between two  $\sigma$ -ideals. Let us now define the following special type of isomorphism:

**Definition 5.1** (Erdős-Sierpiński duality). Let  $X$  be a given set with two  $\sigma$ -ideals  $\mathcal{I}, \mathcal{J} \subseteq \mathcal{P}(X)$ . We call a bijection  $f : X \rightarrow X$  an *Erdős-Sierpiński duality* between the  $\sigma$ -ideals  $\mathcal{I}$  and  $\mathcal{J}$ , if  $f$  is an isomorphism, moreover,  $f$  is an involution, that is,  $f(f(x)) = x$  for every  $x \in X$ .

The name refers to the well-known paper of P. Erdős [11], in which he showed that there is an Erdős-Sierpiński duality between the  $\sigma$ -ideals  $\mathcal{N}(\mathbb{R})$  and  $\mathcal{M}(\mathbb{R})$ . With this, he extended the result of W. Sierpiński: the existence of an isomorphism between  $\mathcal{N}(\mathbb{R})$  and  $\mathcal{M}(\mathbb{R})$ .

Let us recall well-known sufficient conditions for the existence of an isomorphism and an Erdős-Sierpiński duality. For proof, see [14, Theorem 19.5, Theorem 19.6].

**Theorem 5.2.** *Let  $X$  be a set with cardinality  $\omega_1$  and let  $\mathcal{I} \subseteq \mathcal{P}(X)$  be a  $\sigma$ -ideal on  $X$  with the following properties:*

- 1)  $\bigcup \mathcal{I} = X$ ,
- 2)  $\text{cof}(\mathcal{I}) \leq \omega_1$ ,
- 3) *for every  $I \in \mathcal{I}$  there exists  $I' \in \mathcal{I}$  such that  $|I'| = \omega_1$  and  $I' \subseteq X \setminus I$ .*

*Then  $X$  can be decomposed into  $\omega_1$  disjoint sets  $X_\alpha$  such that  $|X_\alpha| = \omega_1$  and  $I \in \mathcal{I}$  if and only if  $I$  is contained in a countable union of them.*

**Remark 5.3.** Let  $X$  and  $Y$  be sets with cardinality  $\omega_1$  and let  $\mathcal{I} \subseteq \mathcal{P}(X)$ ,  $\mathcal{J} \subseteq \mathcal{P}(Y)$  be  $\sigma$ -ideals on  $X$  and  $Y$  satisfying the properties 1), 2), and 3) from Theorem 5.2. Then we get a decomposition of  $X$  into disjoint sets  $X_\alpha$  and respectively, a decomposition of  $Y$  into  $Y_\alpha$ . Let  $f : X \rightarrow Y$  be a bijection defined by the union of partial bijections  $f_\alpha : X_\alpha \rightarrow Y_\alpha$ . It is easy to see that  $f$  is an isomorphism of  $\mathcal{I}$  and  $\mathcal{J}$ .

**Theorem 5.4.** *Let  $X$  be a set with cardinality  $\omega_1$  and let  $\mathcal{I}, \mathcal{J} \subseteq \mathcal{P}(X)$  be two  $\sigma$ -ideals on  $X$  with properties 1), 2) and 3) from Theorem 5.2. Suppose furthermore that there exists  $I \in \mathcal{I}$  and  $J \in \mathcal{J}$  such that  $X = I \cup J$ . Then there exists an Erdős-Sierpiński duality  $f$  between  $\mathcal{I}$  and  $\mathcal{J}$ .*

Now let us prove that assuming the Continuum Hypothesis the  $\sigma$ -ideals  $\mathcal{N}(\mathbb{R})$ ,  $\mathcal{M}(\mathbb{R})$ ,  $\mathcal{HN}(G)$ , and  $\mathcal{HM}(G)$  are pairwise isomorphic.

**Theorem 5.5.** *Let  $G$  be an uncountable Polish group that admits a two-sided invariant metric. Assuming the Continuum Hypothesis, there exists an isomorphism between any two of the following  $\sigma$ -ideals:  $\mathcal{HN}(G)$ ,  $\mathcal{HM}(G)$ ,  $\mathcal{N}(\mathbb{R})$ ,  $\mathcal{M}(\mathbb{R})$ . Moreover, if  $G = \mathbb{Z}^\omega$ , then there is an Erdős-Sierpiński duality between  $\mathcal{HN}(\mathbb{Z}^\omega)$  and  $\mathcal{HM}(\mathbb{Z}^\omega)$ .*

*Proof.* We would like to apply Theorem 5.2 and Remark 5.3 after it for the  $\sigma$ -ideals  $\mathcal{HN}(G)$ ,  $\mathcal{HM}(G)$ ,  $\mathcal{N}(\mathbb{R})$ ,  $\mathcal{M}(\mathbb{R})$ . It is easy to see that the  $\sigma$ -ideals  $\mathcal{N}(\mathbb{R})$  and  $\mathcal{M}(\mathbb{R})$  satisfy the conditions of Theorem 5.2. Thus we have to show that the conditions hold for  $\mathcal{HN}(G)$  and  $\mathcal{HM}(G)$ , too. 1) trivially holds for both  $\sigma$ -ideals. Since we assumed the Continuum Hypothesis, from Theorem 4.4 and Theorem 4.8 2) also follows. Thus for the first part of the theorem it is enough to show property 3) for both  $\mathcal{HN}(G)$  and  $\mathcal{HM}(G)$ , which we manage in the following two lemmas.

**Lemma 5.6.** *Let  $G$  be an uncountable Polish group that admits a two-sided invariant metric. Then for every  $H \in \mathcal{HN}(G)$  there exists an uncountable Haar null set  $H' \subset G \setminus H$ .*

*Proof.* Let  $\mu$  be a witness measure for  $H$ . Without loss of generality, we may assume that  $H$  is Borel. It is well-known that in Polish groups with two-sided invariant metric every compact set is Haar null, in particular, there exists an uncountable Haar null set. So the lemma is trivial if  $H$  is countable. So we can assume that  $|H| \geq \omega_1$ , and thus by [12, Theorem 13.6]  $H$  contains a set  $C$  which is homeomorphic to  $2^\omega$ . Let  $\lambda$  be any continuous Borel probability measure on  $C$ . Let  $B = \{(x, y) \in G \times G : xy \in H\}$ . Notice that  $B$  is Borel, as it is a preimage of the Borel set  $H$  by the multiplication function, which is continuous. Thus we may apply Fubini's theorem in the product space  $G \times G$  to the product measure  $\mu \times \lambda$ .

$$\begin{aligned} (\mu \times \lambda)(B) &= \int_G \int_G \mathbb{1}_{B_x} d\lambda(y) d\mu(x) = \int_G \lambda(x^{-1}H) d\mu(x) = \\ &= \int_G \int_G \mathbb{1}_{B^y} d\mu(x) d\lambda(y) = \int_G \mu(Hy^{-1}) d\lambda(y) = 0, \end{aligned}$$

where the last equality follows from  $H$  being Haar null. Thus there exists  $x \in G$  such that  $\lambda(x^{-1}H) = 0$ , which means that  $|C \setminus x^{-1}H|$  must be uncountable. Therefore  $|xH \setminus H|$  is uncountable. Notice that  $xH \setminus H$  is Haar null ( $\mu$  is a witness measure), which completes the proof.  $\square$

**Lemma 5.7.** *Let  $G$  be an uncountable Polish group that admits a two-sided invariant metric. Then for every  $H \in \mathcal{HM}(G)$  there exists an uncountable Haar meager set  $H' \subset G \setminus H$ .*

*Proof.* Let  $C$  be a compact metric space and  $f : C \rightarrow G$  a witness function of  $H$ . Without loss of generality, we may assume that  $H$  is Borel. In Polish groups with two-sided invariant metric every compact set is Haar meager (see [7, Corollary 22]), in particular, there exists an uncountable Haar meager set. So the lemma is trivial if  $H$  is countable. So we can assume that  $|H| \geq \omega_1$  and thus by [12, Theorem 13.6]  $H$  contains a set  $K$  which is homeomorphic to  $2^\omega$ . Let  $B = \{(x, y) \in K \times C : xf(y) \in H\}$ . It is easy to see that  $B$  is Borel in the product space  $K \times C$  since it is the preimage of  $H$  by

the continuous function  $(x, y) \mapsto xf(y)$ . Thus we may apply the Kuratowski-Ulam theorem in the product space  $K \times C$  for the set  $B$ . Notice that  $B_x = f^{-1}(x^{-1}H)$  is meager in  $C$  for every  $x \in K$  by the fact that  $f$  is witness function, thus  $B$  is meager in  $K \times C$ . Consequently, there exists  $y \in C$  such that  $B^y = Hf(y)^{-1}$  is meager in  $K$ . But then  $|K \setminus Hf(y)^{-1}|$  must be uncountable, therefore  $|Hf(y) \setminus H|$  is uncountable. Notice that  $Hf(y) \setminus H$  is Haar meager (the same function  $f$  and compact space  $C$  witness this), which completes the proof.  $\square$

The second part of the theorem follows from Theorem 5.4 and [6, Example 24], which states that  $\mathbb{Z}^\omega$  is the union of a Haar null and a Haar meager set.  $\square$

We show on the other hand that there is no Borel isomorphism between Haar null and Haar meager  $\sigma$ -ideals of Polish groups that admit a two-sided invariant metric.

**Theorem 5.8.** *Let  $G$  and  $H$  be uncountable Polish groups, moreover suppose that  $H$  admits a two-sided invariant metric. Then there is no Borel isomorphism between the  $\sigma$ -ideals  $\mathcal{HM}(G)$  and  $\mathcal{HN}(H)$ , that is, there is no Borel isomorphism  $f : G \rightarrow H$  such that  $X \in \mathcal{HM}(G)$  if and only if  $f(X) \in \mathcal{HN}(H)$ .*

*Proof.* Suppose to the contrary that there is such an  $f$ . It is well-known that every Borel function between Polish spaces is continuous on a comeager set. Thus let  $U$  be a comeager set, such that  $f|_U$  is continuous. We will construct a comeager set  $V \subset U$  such that  $f(V)$  is Haar null in  $H$ . We know from Theorem 2.14 that  $\mathcal{HM}(G) \subseteq \mathcal{M}(G)$ , thus the image of the non-Haar meager set  $V$  is in  $\mathcal{HN}(H)$  which is a contradiction.

Let us fix metrics on  $G$  and  $H$  such that the metric fixed on  $H$  is two-sided invariant. Due to our assumption  $U$  is comeager, thus it contains a dense  $G_\delta$  set. Using that  $f$  is a Borel isomorphism, we get that  $f(U)$  is uncountable and Borel, thus (because of [12, Theorem 13.6]) it contains a set  $C$  that is homeomorphic with  $2^\omega$ . Let  $\mu$  be an arbitrary continuous Borel probability measure with support  $C$ , and let us fix a constant  $c > 0$ . It is easy to check that since  $\mu$  is a continuous Borel measure with compact support, for any fixed  $\varepsilon > 0$  there exists  $\delta > 0$  such that if  $\text{diam}(B) < \delta$  then  $\mu(B) < \varepsilon$ . Accordingly let us fix  $\delta_n > 0$  for  $\varepsilon = c2^{-n-1}$  for every  $n \in \omega$ . Let  $(x_n)_{n \in \omega}$  be an enumeration of a dense set in  $U$ . From the continuity of  $f$  and the invariance of the metric on  $H$  there is  $r_n$  for every  $n$  such that  $\text{diam}(hf(B(x_n, r_n))g) < \delta_n$  for all  $g, h \in H$ . Thus  $\mu(hf(\bigcup_{n \in \omega} B(x_n, r_n))g) < c$  for every  $g, h \in H$ . Notice that  $U \cap \bigcup_{n \in \omega} B(x_n, r_n) := U_c$  is a dense open set in  $U$ . Let us construct  $U_{c_n}$  for a zero-sequence  $c_n$ , and let  $V := \bigcap_n U_{c_n}$ . Then  $V$  is comeager in  $U$  (as it is dense  $G_\delta$ ). But  $f(V)$  is Haar null, as it is Borel and  $\mu(hf(V)g) = 0$  for all  $g, h \in H$ , which completes the proof.  $\square$

Finally, we finish this section with an interesting corollary of Theorem 5.8.

**Corollary 5.9.** *There is no isomorphism between the  $\sigma$ -ideals  $\mathcal{HN}(\mathbb{Z}^\omega)$  and  $\mathcal{HM}(\mathbb{Z}^\omega)$  that is a homomorphism of  $\mathbb{Z}^\omega$ .*

*Proof.* Suppose that there exists an isomorphism  $f : \mathbb{Z}^\omega \rightarrow \mathbb{Z}^\omega$  between the  $\sigma$ -ideals  $\mathcal{HM}(\mathbb{Z}^\omega)$  and  $\mathcal{HN}(\mathbb{Z}^\omega)$  which is also a homomorphism. From [16, Theorem 3.2] we

know that every homomorphism from  $\mathbb{Z}^\omega$  to  $\mathbb{Z}$  is continuous, consequently so is every homomorphism from  $\mathbb{Z}^\omega$  to  $\mathbb{Z}^\omega$ . So  $f$  is a continuous bijection, then from [12, Theorem 15.1] we know that it is a Borel isomorphism. But this contradicts Theorem 5.8, which concludes the proof.  $\square$

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