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Free-rooted packings of arborescences

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1 Introduction

In this report² we prove new results about packings of arborescences. An r -arborescence is a directed tree in which each node has an in-degree of 1 except the root node r , which has an in-degree of 0. A packing of subgraphs in a graph means a collection of subgraphs, that are edge-disjoint. The most fundamental result of the study of packings of arborescences is the following result of Edmonds ([4]):

Theorem 1.1. (*Weak Edmonds Theorem [4]*) *Let $D = (V, A)$ a digraph and $r \in V$. There exists a packing of spanning r -arborescences in G if and only if*

$$\varrho_A(X) \geq k \text{ for all } \emptyset \neq X \subseteq V - r, \quad (1)$$

where $\varrho_A(X)$ denotes the in-degree of X .

This result has been generalized in multiple ways. Durand de Gevigney, Nguyen and Szigeti characterized the existence of matroid-based packings of arborescences in [3], Cs. Király, Szigeti, Tanigawa characterized the existence of matroid-based and matroid-restricted packings of arborescences in [7] and Bérczi and Frank characterized the existence of free-rooted packings of arborescences ([2]): packings, where the roots of the arborescences are not given (for further definitions see later sections).

This report generalizes results on free-rooted packings of arborescences. In section 4 we characterize the existence of free-rooted matroid-based and matroid-restricted packings of arborescences and prove some corollaries. In section 5 we extend a result of Szigeti ([8]) about free-rooted packings of arborescences in mixed graphs. Every proof in this report is original.

2 Definitions

Given a function $f : S \rightarrow \mathbb{R}$ and a finite set $Z \subset S$, let $\tilde{f}(Z) := \sum_{s \in Z} f(s)$. Two subsets $X, Y \subseteq S$ are **intersecting**, if $X \cap Y \neq \emptyset$. A set function b on the ground set S is **subcardinal**, if $b(X) \leq |X|$ for all $X \subseteq S$, **submodular**, if

$$b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y) \text{ for all } X, Y \subseteq S, \quad (2)$$

and **supermodular**, if

$$b(X) + b(Y) \leq b(X \cap Y) + b(X \cup Y) \text{ for all } X, Y \subseteq S. \quad (3)$$

A set function is **positively intersecting submodular** (**positively intersecting supermodular**), if (2) (respectively (3)) holds for intersecting subsets of S , for which $p(X) > 0$, $p(Y) > 0$.

Let $G = (S, T, E)$ be a bipartite graph and $p : T \rightarrow \mathbb{Z}$ a positively intersecting supermodular setfunction for which $p(\emptyset) = 0$ holds. Let $|\Gamma(Y)|$ be the set of neighbours of

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Y in G . We say that G covers p , if $\forall Y \subseteq T : p(Y) \leq |\Gamma(Y)|$. Given a matroid $M = (S, r)$, we say that G M -covers p , if $\forall Y \subseteq T : p(Y) \leq r(\Gamma(Y))$.

Let $D = (V + s, A)$ be a rooted digraph, where s is called the root. The in-degree of s is 0 and the outgoing edges are called **root-edges**. We call an s -rooted arborescence an **s -arborescence**. For $X, Z \subseteq V + s$, $B \subseteq A$ let $\partial_Z(X)$ denote the set of edges that go from $Z - X$ to X and let $\varrho_Z(X) = |\partial_Z(X)|$.

Let $M_1 = (\partial_s(V), r_1)$ be a matroid on the root-edges of D . We call a packing of s -arborescences $T_1, \dots, T_k$ **M_1 -based**, if every T_i contains exactly one root-edge (e_i) and, for all vertices $v \in V$, $\{e_i : v \in V(T_i)\}$ is a basis of M_1 . Let $M_2 = (A, r_2)$ be a matroid on the edges of D . We call a packing of s -arborescences **M_2 -restricted** if the union of the edge sets of the arborescences in the packing is independent in M_2 .

3 Background results

The following theorem is a stronger version of Theorem 1.

Theorem 3.1 (Strong Edmonds Theorem [4]). *Let $D = (V + s, A)$ be a rooted digraph, and let $\{B_1, \dots, B_k\}$ be a partition of its root-edges. There exists a packing of $T_1, \dots, T_k$ spanning s -arborescences, where the root-edges of T_i are in B_i for every $i = 1, \dots, k$ if and only if $\varrho_V(X) \geq |\{i \in \{1, \dots, k\} : B_i \cap \varrho_s(X) = \emptyset\}|$ for all $\emptyset \neq X \subseteq V$.*

In [1], Bérczi and Frank characterized the existence of a packing of spanning arborescences without specified root-sets, which we call a free-rooted packing.

Theorem 3.2. (Bérczi, Frank [1]) *Let $D = (V, A)$ be a digraph and let $\mu_1, \dots, \mu_k$ positive integers. The following statements are equivalent:*

(A) *There exists in D a packing of k edge-disjoint spanning arborescences $B_1, \dots, B_k$, for which $|B_i| = \mu_i$ for all $i = 1, \dots, k$.*

(B1) *For every subpartition $\{V_1, \dots, V_q\}$ of V :*

$$\sum_{j=1}^k \max\{0, q - (n - \mu_j)\} \leq \sum_{i=1}^q \varrho(V_i) \quad (4)$$

(B2) *Let $[k] = \{1, 2, \dots, k\}$. For every subpartition $\{V_1, \dots, V_q\}$ of V and for all $X \subseteq [k]$:*

$$|[k] - X|q - \sum_{j \in [k] - X} n - \mu_j \leq \sum_{i=1}^q \varrho(V_i) \quad (5)$$

Their proof relies on the following theorem:

Theorem 3.3. (Bérczi, Frank [1]) *Let m_S be a degree-specification on S for which $\tilde{m}_S(S) = \gamma$. Let p_T be a positively intersecting supermodular function on T with $p_T(\emptyset) = 0$. Suppose that*

$$m_S(s) \leq |T| \quad \forall s \in S. \quad (6)$$

The following statements are equivalent:

(A) There exists a simple bipartite graph $G = (S, T, E)$, which covers p_T and fits the degree-specification m_S

(B1) For every subpartition $\{T_1, \dots, T_q\}$ of T and $X \subseteq S$:

$$\tilde{m}_S(X) + \sum_{i=1}^q p_T(T_i) - q|X| \leq \gamma \quad (7)$$

(B2) For every subpartition $\{T_1, \dots, T_q\}$ of T :

$$\sum_{i=1}^q p_T(T_i) \leq \sum_{s \in S} \min\{m_S(s), q\} \quad (8)$$

In [2], a generalization of Theorem 3.3 is provided:

Theorem 3.4. (Bérczi, Frank [2]) We are given a matroid $M = (S, r)$, a positively intersecting supermodular function p_T on T and a degree-specification m_S on S , for which $\tilde{m}_S(S) = \gamma$. There is a simple bigraph $G = (S, T, E)$, which M -covers p_T and fits m_S if and only if

$$m_S(s) \leq |T| \quad \forall s \in S \quad (9)$$

and for every subpartition $\{T_1, \dots, T_q\}$ of T and $X \subseteq S$:

$$\tilde{m}_S(X) + \sum_{i=1}^q p_T(T_i) - qr(X) \leq \gamma \quad (10)$$

In [3], Durand de Gevigney, Nguyen and Szigeti characterized the existence of so called matroid-based packings of arborescences:

Theorem 3.5. (Durand de Gevigney, Nguyen, Szigeti [3]) We are given a graph $D = (V + s, A)$ and a matroid $M = (\partial_s(V), r)$. There is an M -based packing of s -arborescences in D if and only if

$$\varrho_V(X) \geq r(M) - r(\partial_s(X)) \quad (11)$$

Furthermore, if we want the S to be the root set of arborescences, then the following must also hold

$$\partial_s(v) \text{ is independent in } M \text{ for every } v \in V. \quad (12)$$

The following theorem characterizes the existence of matroid-based and matroid-restricted packings of s -arborescence, which is a generalization of Theorem 3.5.

Theorem 3.6. (Cs. Király, Szigeti, Tanigawa [7]) We are given a graph $D = (V + s, A)$, a matroid $M_1 = (\partial_s(V), r_1)$ with a rank function r_1 , a matroid M_2 on A , which is the direct sum of the matroids $M_v = (\partial(v), r_v)$. There exist in D an M_1 -based M_2 -restricted packing of s -arborescences if and only if

$$r_1(F) + r_2(\partial(X) - F) \geq r_1(\partial_s(V)) \quad (13)$$

for all $\emptyset \neq X \subseteq V$ and $F \subseteq \partial_s(X)$. If on the neighbouring edges of s $M_2 = M_2|_{\partial_s(V)} \oplus M_2|_{E(V)}$ and $M_2|_{\partial_s(V)}$ is the free matroid, then the condition is the following:

$$r_1(\partial_s(X)) + r_2(\partial(X) - \partial_s(X)) \geq r_1(\partial_s(V)) \quad (14)$$

for all $\emptyset \neq X \subseteq V$.

4 Free-rooted packings of arborescences with matroid constraints

In this section we characterize the existence of free-rooted matroid-based and matroid-restricted packings of arborescences, give two characterizations of the existence of free-rooted matroid-based packings of arborescences with an in-degree prescription and provide a new characterization for the existence of a free-rooted arborescence packing with an in-degree prescription.

Using Theorem 3.6 and Theorem 3.4, we can characterize the existence of a free-rooted matroid-based and matroid restricted packing of arborescences.

Theorem 4.1. *Let $D = (V, A)$ be a digraph, let $M_1 = (S, r_1)$ be a matroid with rank function r_1 and rank k and let M_2 be a matroid on A which is the direct sum of the matroids $M_v = (\partial(v), r_v)$. Let s be a node not in V . The following statements are equivalent:*

(A) *We can add new possibly parallel arcs from s to some of the nodes of V and map the elements of S to the new edges such that there exists an M_1 -based M'_2 -restricted packing of s -arborescences, where M'_2 is defined to be the direct sum of the free matroid on the new root-edges and M_2 .*

(B) *For every subpartition $\{V_1, \dots, V_q\}$ of V and $X \subseteq S$:*

$$(k - r_1(X))q - |S - X| \leq \sum_{i=1}^q r_2(\partial(V_i)) \quad (15)$$

Proof. Necessity. Suppose that such a packing exists. Then at most $r_2(\partial(Y))$ and at least $k - \partial_s(Y)$ edges of the packing enter a set $Y \subset V$, thus

$$\sum_{i=1}^q (k - r_1(\partial_s(V_i))) \leq \sum_{i=1}^q r_2(\partial(V_i))$$

Using the properties of the rank function we can show that, for every $X \subseteq S$

$$\sum_{i=1}^q r_1(\partial_s(V_i)) \leq \sum_{i=1}^q r_1(\partial_s(V_i) \cap X) + r_1(\partial_s(V_i) - X) \leq qr_1(X) + |S - X|.$$

Hence

$$\sum_{i=1}^q r_2(\partial(V_i)) \geq \sum_{i=1}^q (k - r_1(\partial_s(V_i))) \geq qk - qr_1(X) - |S - X|.$$

Sufficiency. Let $m_S : S \rightarrow \mathbb{Z}_+$ be 1 for every element of S . Let $T := V$ and let us define the following set function on T ,

$$p_T(Y) = \begin{cases} k - r_2(\partial(Y)) & \emptyset \subsetneq Y \subseteq T, \\ 0 & Y = \emptyset. \end{cases}$$

Since r_2 is submodular, p_T is intersecting supermodular. From the conditions of the theorem,

$$(k - r_1(X))q - \sum_{i=1}^q r_2(\partial(V_i)) \leq |S - X| = \tilde{m}_S(S - X) = \tilde{m}_S(S) - \tilde{m}_S(X).$$

Thus,

$$-r_1(X)q + \sum_{i=1}^q (k - r_2(\partial(V_i))) \leq \tilde{m}_S(S) - \tilde{m}_S(X).$$

Hence,

$$\sum_{i=1}^q p_T(V_i) + \tilde{m}_S(X) - r_1(X)q \leq \tilde{m}_S(S).$$

This is the condition of Theorem 3.4 with $F_0 = \emptyset$, therefore there exists a simple bipartite graph $G = (S, V, E)$, which fits m_S and M_1 -covers p_T , that is, $r_1(\Gamma(Y)) \geq k - r_2(\partial(Y)) \forall Y \subset V$. Orient the edges of G from S to T , add the edges of D in T and contract the nodes of S into a new node s . $\Gamma(Y) = \partial_s(Y)$ holds therefore, since G M_1 -covers p_T , $r_1(\partial_s(Y)) \geq k - r_2(\partial(Y))$ holds, which is the condition of Theorem 3.6 with matroids M_1 and M'_2 . Therefore, Theorem 3.6 implies that there exists an M_1 -based M_2 -restricted packing of s -arborescences. $\square$

Using the previous theorem, we can characterize the existence of a free-rooted matroid-based packing of arborescences with an in-degree prescription:

Corollary 4.1. *Let $M = (S, r)$ be a matroid with rank function r , let $D = (V, A)$ be a digraph and let $m_{in} : V \rightarrow \mathbb{Z}^+$ be an in-degree prescription for which $0 \leq m_{in}(v) \leq \varrho(v)$, $m_{in}(v) \leq r(M)$ for all $v \in V$ and $\tilde{m}_{in}(V) = |V|r(M) - |S|$ holds. Let s be a node not in V . The following statements are equivalent:*

(A) *We can add new arcs from s to some of the nodes of V and map the elements of S to the new edges such that there exists an M -based packing of s -arborescences and if the edge set of the packing is F , then $\varrho_V^F(v) = m_{in}(v)$ holds for every $v \in V$.*

(B) *For all $X \subseteq S$ and subpartition $\{V_1, \dots, V_q\}$ of V :*

$$(r(M) - r(X))q - |S - X| \leq \sum_{i=1}^q \sum_{v \in V_i} \min\{m_{in}(v), |\partial(v) \cap \partial(V_i)|\} \quad (16)$$

Proof. Let $M_1 := M$ and $\forall v \in V$ let M_v be the uniform matroid on $\partial(v)$ with rank $m_{in}(v)$. Let M_2 be the direct sum of the matroids M_v . Then

$$r_2(\partial(V_i)) = \sum_{v \in V_i} \min\{m_{in}(v), |\partial(v) \cap \partial(V_i)|\},$$

therefore

$$\sum_{i=1}^q r_2(\partial(V_i)) = \sum_{v \in \bigcup_{i=1}^q V_i} \min\{m_{in}(v), |\partial(v) \cap \partial(V_i)|\}.$$

Hence (16) is the same as the condition of Theorem 4.1, thus there exists an M_1 -based M_2 -restricted packing of arborescences. This means that at most $m_{in}(v)$ arborescence enters every node v . Since $\tilde{m}_{in}(V) = |V|r(M) - |S|$ and the right side is the number of edges in an M -based restriction, exactly $m_{in}(v)$ edge enters every node. $\square$

Let R be a set of nodes. A graph is called an R -branching if all of its connected components are arborescences and the set of the roots of the components is R . The size of branching is the number of its edges. Using Corollary 4.1 we can prove a new characterization for the existence of a free-rooted packing of branchings with prescribed sizes and an in-degree prescription.

Collorary 4.2. *let $D = (V, A)$ be a digraph and let $m_{in} : V \rightarrow \mathbb{Z}^+$ be an in-degree pre-scription for which $0 \leq m_{in}(v) \leq \varrho_D(v)$ and $m_{in}(V) \leq k$ for all $v \in V$. Let $\mu_1, \dots, \mu_k$ be k positive integers, for which $\sum_{i=1}^k \mu_i = \tilde{m}_{in}(V)$. The following statements are equivalent:*

(A) *There exist in D a packing of spanning branchings $B_1, \dots, B_k$, for which $|B_i| = \mu_i$ and if $\bigcup_{i=1}^k B_i = F$, than $v \in V : \varrho_F(v) = m_{in}(v)$.*

(B) *For every subpartition $\{V_1, \dots, V_q\}$ of V :*

$$\sum_{i=1}^k \max\{0, q - (n - \mu_i)\} \leq \sum_{v \in \bigcup_{i=1}^q V_i} \min\{m_{in}(v), |\partial(v) \cap \partial(V_i)|\} \quad (17)$$

Proof. Let $n - \mu_j := m_j$. This is the number of roots for a spanning branching with μ_j edges.

Let M be a partition matroid with k classes, where the size of the i th class is m_i and the bound is 1 for every class. If there exists a packing satisfying the requirements of (A) then add a new node s and a root-edge from s to each of the roots of the branchings in the packing (if a node is the root of more than one branchings then we add a root-edge for each of the branchings). Then, in the resulting digraph, each of the branchings can be divided into s -arborescences containing exactly one root-edge and the packing of s -arborescences we get by dividing all of the branchings is an M -based packing. Conversely, a free-rooted M -based packing corresponds to a packing of branchings satisfying the requirements of (A).

Therefore, by Corollary 4.1, $(r(M) - r(X))q - |S - X| \leq \sum_{v \in \bigcup_{i=1}^q V_i} \min\{m_{in}(v), |\partial(v) \cap \partial(V_i)|\}$ is equivalent to (A). We can assume that the set X contains either the entire partition class or it is disjoint from it. This is because if it intersects a class, then if we add the elements from the class that are not contained in X , then the left side increase and the right side stays the same. So if $I = \{1, \dots, k\}$, then

$$(A) \Leftrightarrow (k - |X|)q - \tilde{m}(S - X) \leq \sum_{v \in \bigcup_{i=1}^q V_i} \min\{m_{in}(v), |\partial(v) \cap \partial(V_i)|\} \forall X \subseteq I$$

The left side is maximized by $X = \{i \in I : m(i) > q\}$ and for this set $(k - |X|)q - \tilde{m}(S - X) = \sum_{i=1}^k \max\{0, q - (n - \mu_i)\}$ holds. $\square$

In [1], Bérczi and Frank provide a different characterization for the same problem with the following condition:

For all $Y \subseteq V$ and subpartition $\{V_1, \dots, V_q\}$ of $V - Y$:

$$\sum_{i=1}^k \max\{0, q + |Y| - (n - \mu_i)\} \leq \tilde{m}_{in}(Y) + \sum_{i=1}^q \varrho_D(V_i) \quad (18)$$

This result follows from the following theorem, which gives a different characterization of the existence of a free-rooted matroid-based packing of arborescences with an in-degree prescription with a seemingly weaker condition. The proof is based on Theorem 3.5 and Theorem 3.4.

Theorem 4.2. *Let $M = (S, r)$ be a matroid with rank function r , let $D = (V, A)$ be a digraph and let $m_{in} : V \rightarrow \mathbb{Z}^+$ be an in-degree prescription for which $0 \leq m_{in}(v) \leq \varrho(v)$, $m_{in}(v) \leq r(M)$ for all $v \in V$ and $\tilde{m}_{in}(V) = |V|r(M) - |S|$ holds. Let s be a node not in V . The following statements are equivalent:*

(A) *We can add new arcs from s to some of the nodes of V and map the elements of S to the new edges such that there exists an M -based s -arborescence packing and if the edge set of the packing is F , then $\varrho_V^F(v) = m_{in}(v)$ holds for every $v \in V$.*

(B) *For all $Y \subseteq V$, subpartition $\{V_1, \dots, V_q\}$ of $V - Y$, and $X \subseteq S$,*

$$(|Y| + q)(r(M) - r(X)) - |S - X| \leq \tilde{m}_{in}(Y) + \sum_{i=1}^q \varrho(V_i) \quad (19)$$

Furthermore, (16) implies (19).

Proof. First we will prove that (16) implies (19). By Theorem 4.1, this implies the necessity of (19).

Let us suppose that (16) holds and we are given a set $Y \subseteq V$ and a subpartition $\mathcal{P} = \{V_1, \dots, V_q\}$ of $V - Y$. Let us define the following partition of V : $\mathcal{P}' = \mathcal{P} \cup \bigcup_{v \in Y} \{v\}$. Then $|\mathcal{P}'| = q + |Y|$, $m_{in}(Y) = \sum_{v \in Y} \min\{m_{in}(v), |\partial(v) \cap \partial(V_i)|\}$ and $\sum_{v \in \bigcup_{i=1}^q V_i} \min\{m_{in}(v), |\partial(v) \cap \partial(V_i)|\} \leq \sum_{i=1}^q \varrho(V_i)$, so (19) holds.

To prove sufficiency let $m_S : S \rightarrow \mathbb{Z}_+$ be 1 for every element of S . Let $T := V$ and let us define the following set function on T :

$$p_T(Y) = \begin{cases} r(M) - \varrho(Y) & Y \subseteq T, |Y| \geq 2, \\ r(M) - m_{in}(v) & Y = \{v\}, v \in V \\ 0 & Y = \emptyset. \end{cases}$$

Since $m_{in}(v) \leq \varrho(v)$, $k - m_{in}(v) \geq k - \varrho(v)$ so p_T is intersecting supermodular.

Let $\mathcal{T} = \{V_1, \dots, V_q, \dots, V_{q'}\}$ be a subpartition of T , where the last $q' - q$ classes are singletons. Let $\mathcal{P} = \{V_1, \dots, V_q\}$ and $Y = V_{q+1} \cup \dots \cup V_{q'}$.

From the definition of p_T ,

$$\sum_{i=1}^{q'} p_T(V_i) = \sum_{i=1}^q [r(M) - \varrho(V_i)] + \sum_{i=q+1}^{q'} [r(M) - \tilde{m}_{in}(V_i)] = (|Y| + q)r(M) - \sum_{i=1}^q \varrho(V_i) - \tilde{m}_{in}(Y).$$

If we apply the condition of Theorem 3.4 to $\mathcal{T}$, a set $X \subset S$ and $F_0 = \emptyset$ and we use the previous equation for $\sum_{i=1}^{q'} p_T(V_i)$, we get the following.

$$\tilde{m}_S(X) + (|Y| + q)r(M) - \sum_{i=1}^q \varrho(V_i) - \tilde{m}_{in}(Y) - q'r(X) \leq \tilde{m}_S(S).$$

If we reorder the terms and use that $q' = |Y| + q$ and $\tilde{m}_S(S) - \tilde{m}_S(X) = |S - X|$ we get the condition in (19). Hence there exists a simple bipartite graph $G = (S, V, E)$ which covers p_T -t and satisfies the degree prescription. From this we get that for every $v \in V$ -re $r(M) - m_{in}(v) \leq r(\Gamma_G(v)) \leq d_G(v)$ (where $\Gamma_G(v)$ is the set of neighbours of v in G), thus

$$\sum_{v \in V} [r(M) - m_{in}(v)] \leq \sum_{v \in V} d_G(v) = \sum_{s \in S} d_G(s) = |S|.$$

Since $\tilde{m}_{in}(V) = |V|r(M) - |S|$, the left hand side of the previous equation is $|S|$, hence we have equality everywhere, thus we get $d_G(v) = r(\Gamma_G(v)) = r(M) - m_{in}(v) \forall v \in V$.

Since $\forall Y \subset V$ $r(M) - \varrho(Y) \leq r(\Gamma_G(Y))$ also holds, if we contract S , orient its outgoing edges towards $T = V$ and add the edges of D on V , then the condition of Theorem 3.5 holds for the resulting digraph, thus there exists an M -based packing of s -arborescences. Since $r(\Gamma_G(v)) = r(M) - m_{in}(v)$, at least $m_{in}(v)$ arborescence enters v with a non-root-edge. We can assume that exactly $r(\Gamma_G(v))$ root-edge is in the packing (since otherwise we can exchange certain edges of the arborescences), so there exists a packing which enters v with exactly $m_{in}(v)$ edges.

□

5 Free-rooted packings of arborescences in mixed graphs

A mixed graph is a graph which has both undirected and directed edges (arcs). In this section we prove a generalization of a result on free-rooted packings of arborescences in mixed graphs by Szigeti ([8]).

Let $F = (V, E \cup A)$ be a mixed graph where E is the set of undirected edges and A is the set of arcs. For $B \subseteq E \cup A$, let $\partial^B(\mathbf{X})$ be the set of edges in B entering X . Let $\varrho^B(\mathbf{X}) = |\partial^B(X)|$. Orienting an edge we replace it with an arc. For $\vec{Z} \subseteq A$, Z denotes the underlying undirected edges of $\vec{Z}$. For $Z \subseteq E$ and $X \subseteq V$ the set of vertices covered by Z is denoted by $\mathbf{V}(Z)$ and the set of edges in Z that are induced by X is denoted by $\mathbf{Z}(\mathbf{X})$. A mixed r -arborescence is a mixed graph that can be oriented to be an r -arborescence. For a family of sets $\mathcal{P}$ on V let $\partial_{A \cup E}(\mathcal{P})$ be the set of (directed and undirected) edges, that enter a member of $\mathcal{P}$ and $\varrho_{A \cup E}(\mathcal{P}) := |\partial_{A \cup E}(\mathcal{P})|$. If $f, v : V \rightarrow \mathbb{Z}_+$, then we call a packing of arborescences **(f, g)-bounded**, if for each $v \in V$, the number of v -arborescences in the packing is between $f(v)$ and $g(v)$. For $k, l, l' \in \mathbb{Z}_+ - \{0\}$ a packing of arborescences is **(l, l')-bounded** if the number of arborescences in the packing is between l and l' , and

k -regular, if each vertex is in exactly k arborescences in the packing. We call a packing of mixed arborescences (f, g) -bounded/ (l, l') / k -regular, if we can orient the undirected edges such that we get an (f, g) -bounded/ (l, l') / k -regular packing of arborescences.

For a graph $G = (V, E)$, let $\mathbf{M}_G$ be the graphic matroid of G , and let $\mathbf{M}_G^k$ be the k -graphic matroid of G , that is the k -sum of M_G , which is a matroid on V , where a set is independent if and only if it can be partitioned into k independent sets of M_G . Let $F = (V, E \cup A)$ be a mixed graph. For a partition $\mathcal{P}$ of V , let $\mathbf{A}(\mathcal{P})$ and $\mathbf{E}(\mathcal{P})$ be the set of directed and undirected edges entering a member of $\mathcal{P}$. Let $\mathbf{G}_F = (V, E \cup E_A)$ be the underlying undirected graph of F , and $\mathbf{D}_F = (V, A_E \cup A)$ the directed extension of F , where $A_E = \bigcup_{e \in E} A_e$, and if $e = uv$, $A_e = \{\overrightarrow{uv}, \overleftarrow{vu}\}$ ($\overrightarrow{uv}$ is an arc from u to v). The extended k -hipergraphic matroid $\mathbf{M}_F^k$ of F is a matroid on $A \cup A_E$, which we get from $M_{G_F}^k$ by replacing each edge $e \in E$ with two parallel copies of itself, associating these edges to the corresponding edges in A_E , and associating the edges of E_A with the corresponding arcs of A . It is shown in [6], that the rank function of M_F^k is the following ($Z \subseteq A \cup A_E$):

$$r_{M_F^k}(Z) = \min\{|Z \cap A(\mathcal{P})| + |\{e \in E(\mathcal{P}) : Z \cap A_e \neq \emptyset\}| + k(|V| - |\mathcal{P}|) : \mathcal{P} \text{ is a partition of } V\} \quad (20)$$

Let p and b be two set functions on S . For a vector $x \in \mathbb{R}^S$ and $Z \subseteq S$, let $\tilde{x}(Z) := \sum_{s \in S} x_s$. The polyhedron $\mathbf{Q}(p, b) = \{x \in \mathbb{R}^S : p(Z) \leq \tilde{x}(Z) \leq b(Z) \ \forall Z \subseteq S\}$ is called a generalized-polymatroid or g-polymatroid if p and b have the following properties: $p(\emptyset) = b(\emptyset)$, p is supermodular, b is submodular and $b(X) - b(Y) \geq b(X - Y) - p(Y - X)$ for all $X, Y \subseteq S$. The Minkowski sum of the n g-polymatroids $Q(p_i, b_i)$ is denoted by $\sum_1^n Q(p_i, b_i)$. For $\alpha, \beta \in \mathbb{R}$, the polyhedron $\mathbf{K}(\alpha, \beta) = \{x \in \mathbb{R}^S : \alpha \leq \tilde{x}(S) \leq \beta\}$ is called a plank. We will use the following results on g-polymatroids:

Theorem 5.1 (Frank [5]). 1. Let $Q(p, b)$ be a g-polymatroid, $K(\alpha, \beta)$ a plank and $M = Q(p, b) \cap K(\alpha, \beta)$.

- (i) $M \neq \emptyset$ if and only if $p \leq b$, $\alpha \leq \beta$, $\beta \geq p(S)$ and $\alpha \leq b(S)$.
- (ii) M is a g-polimatroid.
- (iii) If $M \neq \emptyset$, then $M = Q(p_\beta^\alpha, q_\beta^\alpha)$ with

$$p_\beta^\alpha(Z) = \max\{p(Z), \alpha - b(S - Z)\}$$

$$q_\beta^\alpha(Z) = \min\{b(Z), \beta - p(S - Z)\}$$

2. Let $Q(p_1, b_1)$ and $Q(p_2, b_2)$ be two non-empty g-polymatroids and $M = Q(p_1, b_1) \cap Q(p_2, b_2)$.

- (i) $M \neq \emptyset$ if and only if $p_1 \leq b_2$ and $p_2 \leq b_1$.
- (ii) If p_1, b_1, p_2, b_2 are integral and $M \neq \emptyset$, then M contains an integral element.

3. Let $Q(p_i, b_i)$ be n nonempty g-polimatroids. Then $\sum_1^n Q(p_i, b_i) = Q(\sum_1^n p_i, \sum_1^n b_i)$.

Theorem 5.2. (Szigeti [8]) Let $F = (V, E \cup A)$ be a mixed graph, $f, g : V \rightarrow \mathbb{Z}_+$ functions and $k, l, l' \in \mathbb{Z}_+ - \{0\}$. There exists an (f, g) -bounded k -regular (l, l') -limited packing of arborescences in F if and only if $g_k(v) \geq f(v)$ for every $v \in V$, $\min\{\tilde{g}_k(V), l'\} \geq l$ and

$$\varrho_{A \cup E}(\mathcal{P}) \geq k|\mathcal{P}| - \min\{l' - f(\overline{\cup \mathcal{P}}), \tilde{g}_k(\cup \mathcal{P})\} \text{ for every subpartition } \mathcal{P} \text{ of } V \quad (21)$$

Theorem 5.3. (Szigeti [8]) Let $F = (V, E \cup A)$ be a mixed graph, $f, g : V \rightarrow \mathbb{Z}_+$ functions and $k, l, l' \in \mathbb{Z}_+ - \{0\}$. Let $M_v := (\partial^{A \cup A_e}(v), r_v)$ be the free matroid for every $v \in V$, and let $M := \bigoplus_{v \in V} M_v$ with a rankfunction r . Let M_F^k be the extended k -graphic matroid of F on $A \cup A_e$. Let $T := (\sum_{v \in V} ((Q(0, r_v)) \cap K(k - g_k(v), k - f(v))) \cap K(k|V| - l', k|V| - l) \cap Q(0, r_{M_F^k}))$.

- (A) The characteristic vectors of the edge sets of (f, g) -bounded k -regular (l, l') -limited M -restricted packings of arborescences in orientations of F are exactly the integer points of T .
- (B) $T \neq \emptyset$ if and only if $g_k(v) \geq f(v)$ for every $v \in V$, $\min\{\tilde{g}_k(V), l'\} \geq l$ and for every $Z \subseteq A \cup A_E$,

$$\sum_{v \in V} \max\{0, k - g_k(v) - r_v(\partial^Z(v))\} \leq r_{M_F^k}(\bar{Z}) \quad (22)$$

$$k|V| - l' - \sum_{v \in V} \min\{r_v(\partial^Z(v)), k - f(v)\} \leq r_{M_F^k}(\bar{Z}) \quad (23)$$

- (C) (25) and (26) are equivalent to (24).

Theorem 5.4. Let $F = (V, E \cup A)$ be a mixed graph, $f, g : V \rightarrow \mathbb{Z}_+$ functions and $k, l, l' \in \mathbb{Z}_+ - \{0\}$. Let $M_v := (\partial^{A \cup A_e}(v), r_v)$ be a matroid for every $v \in V$, and let $M := \bigoplus_{v \in V} M_v$ with a rankfunction r . There exists an (f, g) -bounded k -regular (l, l') -limited M -restricted packing of arborescences in F if and only if $g_k(v) \geq f(v)$ for every $v \in V$, $\min\{\tilde{g}_k(V), l'\} \geq l$ and

$$R(\mathcal{P}) \geq k|\mathcal{P}| - \min\{l' - f(\overline{\cup \mathcal{P}}), \tilde{g}_k(\cup \mathcal{P})\} \text{ for every subpartition } \mathcal{P} \text{ of } V \quad (24)$$

where $R(\mathcal{P}) = \max\{r(\overrightarrow{\partial^{A \cup E}})\}$, where $\overrightarrow{\partial^{A \cup E}(\mathcal{P})}$ is an orientation of $\partial^{A \cup E}(\mathcal{P})$.

The proof of this theorem relies on the following theorem:

Theorem 5.5. Let $F = (V, E \cup A)$ be a mixed graph, $f, g : V \rightarrow \mathbb{Z}_+$ functions and $k, l, l' \in \mathbb{Z}_+ - \{0\}$. Let $M_v := (\partial^{A \cup A_e}(v), r_v)$ be a matroid for every $v \in V$, and let $M := \bigoplus_{v \in V} M_v$ with a rankfunction r . Let M_F^k be the extended k -graphic matroid of F on $A \cup A_e$. Let $T := (\sum_{v \in V} ((Q(0, r_v)) \cap K(k - g_k(v), k - f(v))) \cap K(k|V| - l', k|V| - l) \cap Q(0, r_{M_F^k}))$.

- (A) The characteristic vectors of the edge sets of (f, g) -bounded k -regular (l, l') -limited M -restricted packings of arborescences in orientations of F are exactly the integer points of T .
- (B) $T \neq \emptyset$ if and only if $g_k(v) \geq f(v)$ for every $v \in V$, $\min\{\tilde{g}_k(V), l'\} \geq l$ and for every $Z \subseteq A \cup A_E$,

$$\sum_{v \in V} \max\{0, k - g_k(v) - r_v(\partial^Z(v))\} \leq r_{M_F^k}(\bar{Z}) \quad (25)$$

$$k|V| - l' - \sum_{v \in V} \min\{r_v(\partial^Z(v)), k - f(v)\} \leq r_{M_F^k}(\bar{Z}) \quad (26)$$

(C) (24) implies (25) and (26).

Proof. (A)

By Theorem 5.3/(A), the integer points of T are characteristic vectors of the edge sets of (f, g) -bounded k -regular (l, l') -limited packings of arborescences in orientations of F . Since an integer point of T is also in $\sum_{v \in V} Q(0, r_v)$, the corresponding packing is also M -restricted.

By the other direction of Theorem 5.3/(A) and since the characteristic vector of an M -restricted packing must be in $\sum_{v \in V} Q(0, r_v)$, the integer points of T are exactly the characteristic vectors of the edge sets of the required packings.

(B)

By Theorem 5.1.1, $Q(0, r_v) \cap K(k - g_k(v), k - f(v))$ is non empty if and only if $k - g_k(v) \leq k - f(v)$ (which is equivalent to $g_k(v) \geq f(v)$), $k - f(v) \geq 0$ (which is true because $0 \leq k - g_k(v) \leq k - f(v)$) and $k - g_k(v) \leq r_v(\partial^{A \cup A_E}(v))$ (which we will see later).

If $Q(0, r_v) \cap K(k - g_k(v), k - f(v)) \neq \emptyset$ then it is equal to $Q(p_v, b_v)$, where by Theorem 5.1.1/(iii), for $Z \subseteq A \cup A_E$ and $Z_v = Z \cap \partial^{A \cup A_E}(v)$,

$$p_v(Z_v) = \max\{0, k - g_k(v) - r(\partial^{A \cup A_E - Z_v}(v))\}$$

$$b_v(Z_v) = \min\{r(\partial^{Z_v}(v)), k - f(v)\}$$

By Theorem 5.1.3, $\sum_{v \in V} Q(p_v, b_v) = Q(p_\Sigma, b_\Sigma)$, where $p_\Sigma = \sum_{v \in V} p_v$, $b_\Sigma = \sum_{v \in V} b_v$.

By Theorem 5.1.1, $Q(0, r_{M_F^k}) \cap K(k|V| - l', k|V| - l) \neq \emptyset$ if and only if $k|V| - l' \leq k|V| - l$ (which is equivalent to $l' \geq l$), $k|V| - l \geq 0$ (which follows from $k|V| - l \geq \tilde{g}_k(V) - l \geq 0$) and $k|V| - l' \leq r_{M_F^k}(A \cup A_E)$, which is (26) for $Z = A \cup A_E$. Thus, by Theorem 5.1.1/(iii), $Q(0, r_{M_F^k}) \cap K(k|V| - l', k|V| - l) = Q(p, b)$ where $p(Z) = \max\{0, k|V| - l' - r_{M_F^k}(A \cup A_E - Z)\}$ and $b(Z) = \min\{r_{M_F^k}(Z), k|V| - l\}$.

By Theorem 5.1.2, $Q(p, b) \cap Q(p_\Sigma, b_\Sigma) \neq \emptyset$ if and only if $p_\Sigma \leq b$ and $p \leq b_\Sigma$, that is

$$\sum_{v \in V} \max\{0, k - g_k(v) - r_v(\partial^{A \cup A_E - Z_v}(v))\} \leq \min\{r_{M_F^k}(Z), k|V| - l\}$$

and

$$\max\{0, k|V| - l' - r_{M_F^k}(A \cup A_E - Z)\} \leq \sum_{v \in V} \min\{r_v(\partial^{Z_v}(v)), k - f(v)\}$$

The first inequality is equivalent to (25) by the fact that $\max\{0, k|V| - l' - r_{M_F^k}(A \cup A_E - Z)\} \leq \sum_{v \in V} k - g_k(v) \leq k|V| - l$ (here we use that $\min\{\tilde{g}_k(V), l'\} \geq l$). Since $k - f(v) \geq k - g_k(v) \geq 0$ and $r_v \geq 0$, $0 \leq \sum_{v \in V} \min\{r_v(\partial^{Z_v}(v)), k - f(v)\}$, and $k|V| - l' - r_{M_F^k}(A \cup A_E - Z) \leq \sum_{v \in V} \min\{r_v(\partial^{Z_v}(v)), k - f(v)\}$ is equivalent to (26).

Finally, $k - g_k(v) \leq r_v(\partial^{A \cup A_E}(v))$ follows from $p_\Sigma(\emptyset) \leq b(\emptyset)$ and the proof is complete.

(C)

Note, that (25) is equivalent to

$$k|V| - g_k(V) - \sum_{v \in V} \min\{r_v(\partial^Z(v)), k - g_k(v)\} \leq r_{M_F^k}(A \cup A_E - Z). \quad (27)$$

Let $Z \subseteq A \cup A_E$. By (20), there exists a partition $\mathcal{P}$ of V such that for $K = \{e \in E(\mathcal{P}) : (A \cup A_E - Z) \cap A_e \neq \emptyset\}$:

$$r_{M_F^k}(A \cup A_E - Z) = |(A \cup A_E - Z) \cap A(\mathcal{P})| + |K| + k(|V| - |\mathcal{P}|). \quad (28)$$

Let $\mathcal{P}_h := \{X \in \mathcal{P} : r_v(\partial^Z(v)) \leq k - h(v) \ (\forall v \in X)\}$, where $h \in \{f, g_k\}$. Then $\mathcal{P}_h$ is a subpartition of V and for every $X \in \mathcal{P} - \mathcal{P}_h$ there exists a $v_X \in X$ such that $r_v(\partial^Z(v)) > k - h(v)$.

By the definition of K , we have

$$A_{E(\mathcal{P}_h)-K} \subseteq Z \cap A_{E(\mathcal{P}_h)}. \quad (29)$$

Thus, by (28), the definition of $\mathcal{P}_h$ and v_X , $r_v(\partial^Z(v)) \geq 0$, $k - h \geq 0$, $h \geq 0$ and $r(X) \leq |X|$ (that is, the subcardinality of the rank function of a matroid), we have

$$\begin{aligned} & r_{M_F^k}(A \cup A_E - Z) + \sum_{v \in V} \min\{r_v(\partial^Z(v)), k - h(V)\} \\ &= |(A \cup A_E - Z) \cap A(\mathcal{P})| + |K| + k(|V| - |\mathcal{P}|) + \sum_{v \in \cup \mathcal{P}} \min\{r_v(\partial^Z(v)), k - h(V)\} \\ & \quad + \sum_{v \in V - \cup \mathcal{P}} \min\{r_v(\partial^Z(v)), k - h(V)\} \\ & \geq |(A \cup A_E - Z) \cap A(\mathcal{P}_h)| + \sum_{v \in \cup \mathcal{P}} r_v(\partial^Z(v)) + \sum_{X \in \mathcal{P} - \mathcal{P}_h} \sum_{v \in X} \min\{r_v(\partial^Z(v)), k - h(V)\} \\ & \quad + |K| + k(|V| - |\mathcal{P}|) \\ & \geq |(A \cup A_E - Z) \cap A(\mathcal{P}_h)| + r((Z \cap A(\mathcal{P}_h)) \cup (Z \cap A_{E(\mathcal{P}_h)})) + \sum_{X \in \mathcal{P} - \mathcal{P}_h} (k - h(v_X)) \\ & \quad + |K| + k(|V| - |\mathcal{P}|) \\ & \geq r((A \cup A_E - Z) \cap A(\mathcal{P}_h)) + r((Z \cap A(\mathcal{P}_h)) \cup (Z \cap A_{E(\mathcal{P}_h)})) + \sum_{X \in \mathcal{P} - \mathcal{P}_h} (k - h(X)) \\ & \quad + |K| + k(|V| - |\mathcal{P}|) \\ & \geq r((A \cup A_E - Z) \cap A(\mathcal{P}_h)) + r((Z \cap A(\mathcal{P}_h)) \cup (Z \cap A_{E(\mathcal{P}_h)})) + k(|\mathcal{P}| - |\mathcal{P}_h|) - h(V - \cup \mathcal{P}_h) \\ & \quad + |K| + k(|V| - |\mathcal{P}|) \\ & = r((A \cup A_E - Z) \cap A(\mathcal{P}_h)) + r((Z \cap A(\mathcal{P}_h)) \cup (Z \cap A_{E(\mathcal{P}_h)})) - k|\mathcal{P}_h| - h(V - \cup \mathcal{P}_h) + |K| + k|V|. \end{aligned}$$

By (29) and the submodularity of r , we get

$$\begin{aligned} & r((A \cup A_E - Z) \cap A(\mathcal{P}_h)) + r((Z \cap A(\mathcal{P}_h)) \cup (Z \cap A_{E(\mathcal{P}_h)})) \\ & \geq r(A(\mathcal{P}_h) \cup (Z \cap A_{E(\mathcal{P}_h)})) + r(((A \cup A_E - Z) \cap A(\mathcal{P}_h)) \cap (Z \cap A_{E(\mathcal{P}_h)})) \\ & \geq r(A(\mathcal{P}_h) \cup A_{E(\mathcal{P}_h)-K}) + r(\emptyset) \\ & \geq R(\mathcal{P}_h) - |K|. \end{aligned}$$

In the last inequality we use $r(X - K) \geq r(X) - |K|$. By the previous two inequalities, we get

$$r_{M_F^k}(A \cup A_E - Z) + \sum_{v \in V} \min\{r_v(\partial^Z(v)), k - h(V)\} \geq R(\mathcal{P}_h) - k|\mathcal{P}_h| - h(V - \cup \mathcal{P}_h) + k|V|.$$

Using this inequality for $h = f$ and (24) we get (26), and if we apply it for $h = g_k$, we get (27). □

Finally, we prove Theorem 5.4.

Proof of Theorem 5.4. Necessity. The necessity of $g_k \geq f$ and $\min\{\tilde{g}_k(V), l'\} \geq l$ is trivial. Let $\mathcal{P}$ be a subpartition of V and let B be the arc set of an (f, g) -bounded k -regular (l, l') -limited M -restricted packing of arborescences in an orientation $\vec{F}$ of F . For a node v , let the number of v -arborescences in the packing be $q(v)$. Let C be a class of $\mathcal{P}$. By k -regularity, there is at least k arborescences in the packing, which have arcs induced by C . If the root of an arborescence is not in C , then it enters it. Thus, the number of edges in B that enter C is at least $k - \sum_{v \in C} q(v)$. The number of edges in B entering a class of $\mathcal{P}$ is therefore at least $k|\mathcal{P}| - \sum_{C \in \mathcal{P}} \tilde{q}(C) = k|\mathcal{P}| - \tilde{q}(\cup \mathcal{P})$. Since the packing is (f, g) -bounded and (l, l') -limited, we have $\tilde{q}(\cup \mathcal{P}) \leq \min\{l' - f(V - \cup \mathcal{P}), \tilde{g}_k(\cup \mathcal{P})\}$, therefore the right side of (24) is a lower bound on the number of edges in B , that enter a member of $\mathcal{P}$. Since B is independent in M , we get (24).

Sufficiency. Let $(F = (V, E \cup A), f, g, k, l, l')$ be an instance of Theorem 5.4, that satisfies the necessary conditions. Since (24) holds, by Theorem 5.5/(C), (25) and (26) hold. Since $g_k \geq f$ and $\min\{\tilde{g}_k(V), l'\} \geq l$ also hold, by Theorem 5.5/(B), T (as defined in Theorem 5.5) is nonempty, thus, by Theorem 5.1/2./ (ii), it contains an integral element x . By Theorem 5.5/(A), x is the characteristic vector of the edge sets of an (f, g) -bounded k -regular (l, l') -limited M -restricted packing of arborescences in an orientation $\vec{F} = (V, \vec{E} \cup A)$ of F . Replacing the arcs in $\vec{E}$ with the edges in E , we get the required packing. □

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