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IRRACIONALITÁS A FERDESÉG PREFERENCIÁKBAN

IRRATIONALITY IN SKEWNESS PREFERENCES

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1. INTRODUCTION

Skewness measures of stocks have been inversely related to their past performance using systematic (non-diversifiable) and idiosyncratic skewness, which have a high predictive power over the cross-section of expected returns. The reasons behind investors' preferences for lottery-like returns are behaviorally sound, such as cumulative prospect theory (CPT). Tversky and Kahneman (1992) would predict that investors overweigh the small probability of a significant return. Barberis and Huang (2008) further argue that CPT can explain the lower returns of securities with positively skewed past returns, such as IPOs.

When calculating the skewness premium, past papers constructed a long-short portfolio of long stocks with the lowest skewness and short stocks with the highest skewness. That, however, means that when examining the behavioral explanation of the skewness premium, it is not sufficient to only look at how investors react to high skewness measures. Thaler et al. (1997) look at myopic loss aversion in the context of equity markets. The authors find that equity investors are more sensitive to losses than gains. Furthermore, they find that investors are more sensitive to short-term bad outcomes than their generally longer investment horizon would imply. Both findings are important in the context of low or negatively skewed stocks as these stocks have much more pronounced downturns. Our paper also finds that the riskiness of low-skewness stocks is much greater over shorter investment horizons.

That is a compelling behavioral explanation for investors disliking low-skewness stocks while simultaneously preferring high-skewness stocks. Determining whether any factor is based on behavioral effects or is risk-based is essential. A perfectly rational investor could gain

exposure to a behavioral factor to increase his expected return while not increasing the risk characteristics of his portfolio. Therefore, rational investors might arbitrage a behavioral factor away, decreasing its premium substantially. A risk-based factor cannot be arbitrated away when a rational investor invests in it. Their portfolios will subsequently get riskier, resulting in some rational investors preferring the market portfolio or even excluding the risk-based factor from their portfolios.

Factors based on fundamentals, such as the ones introduced in Fama and French (2015) can be derived from a model based on fundamentals, like the discounted dividend model. The authors explain that the factors (FF5) introduced are closely related to the riskiness of the fundamentals of the underlying stocks in a diversified portfolio. Similarly, Chen and Zhang (1998) have similar findings for the (FF3) even in international markets. It is, however, less clear how the factors would make the return characteristics of a portfolio riskier. Beta alone cannot explain the factor premiums outlined in the paper. Other papers suggest that similar fundamentally based factors are riskier due to their exposure to economic state variables. Simpson and Ramchander (2008) find that the first quintiles of size and value are closely related to consumer demand, inflation, and even economic growth. That means the bad timing of the returns makes them riskier because the bad returns are likely to coincide with low returns on human capital. When it comes to the skewness factor, it is different. When we know an investor's sensitivity to distinct parts of the return distribution, we can calculate the exact amount that including the skewness factor would increase his risk. In other words, mathematics can explain the skewness premium almost entirely. We can use this fact to find an appropriate skewness premium.

Our paper contributes to the existing literature by examining the skewness premium concerning where its risk comes from. Is it based on risk, or do the preferences of irrational investors cause it? To examine this question, we estimate expected premia bounds for an investor investing in a portfolio with a fixed skewness. Instead of the average investor, we looked at a hypothetical investor whose lifetime consumption needs are highly sensitive to the return distribution's left tail. We used a value-at-risk (VaR) model and a fixed withdrawal model to estimate the sensitivity of the investor to considerable losses. A negatively skewed return distribution has much more frequent significant losses than a positively skewed one. This results in the negatively skewed distribution performing worse in the tested consumption scenarios.

We would expect past studies to find a skewness premium lower than our method suggests. In an intertemporal asset pricing model (ICAPM, Merton, 1973), we expect investors

with left-tail sensitive consumption needs to avoid negatively skewed portfolios. In contrast, in our estimations, even the aforementioned investor's risk would be unaffected by a portfolio with a different skewness, as the premium would precisely compensate for the added risk. In ICAPM, rational investors who invest in a negatively skewed portfolio would have a lower left-tail sensitivity, resulting in a lower risk premium pricing, making the skewness premium unattractive for more left-tail sensitive investors. It is worth noting that skewness greatly changes the right tail of the return distribution, too, but few investors are sensitive to the right tail when the mean of the returns is controlled, which we do in our paper.

Section 2 presents an insight into the literature on cross-sectional pricing based on skewness. Section 3 goes through the methodology for creating hypothetical skewed distributions that would best represent the returns of a highly skewed portfolio. Section 4 explains how we estimate the bounds for the skewness premia and what assumptions we need to make to arrive at them. Section 5 compares our expected premia values to the skewness premia measured by other authors. Section 6 summarizes our findings and gives a short insight into the implications. Section 7 concludes our paper.

2. LITERATURE REVIEW

Following Markowitz's (1952) paper on a mean-variance-based portfolio selection theory, there have been numerous studies and books done on optimizing a portfolio for these moments, such as Yoshimoto (1996), Markowitz et al. (2000), Björk et al. (2014), and Cong and Oosterlee (2016). Optimizing only for these two moments can be problematic as we may not capture other risk factors. Including investors' state variables is vital as some factors may be risk-based while not changing the moments of the return distribution proportionally. The intertemporal capital asset pricing model (ICAPM) was introduced by Merton (1973). New factors based on state variables such as size and value were introduced by Fama et al. (1992). Bond factors can also be added (Fama and French, 1993), and these risk factors can be further expanded by adding profitability and investment factors (Fama and French, 2015). These models aim to explain the returns of a diversified portfolio based on its exposure to risk factors. Using these models, one could build a risk-appropriate stock portfolio.

A different way to tailor a stock portfolio to an investor's preferences is to in/exclude stocks from a portfolio based on how they would influence the return's probability distribution function's moments. The addition of the third standardized moment (skewness) to the mean-variance-based portfolio was first theorized in the '70s by Rubinstein (1973) and Kraus and

Litzenberger (1976). The authors argue that the mean-variance-skewness approach to portfolio construction is better because “The evidence suggests that prior empirical findings that are interpreted as inconsistent with the traditional theory can be attributed to misspecification of the capital asset pricing model by omission of systematic (non-diversifiable) skewness.” (Kraus and Litzenberger, 1976, p. 1086).

Harvey and Siddique (2000) find that conditional skewness is crucial in the cross-section of stock returns. Furthermore, it can fit well into the already established three-factor asset pricing model (Fama et al., 1992) as it improves the model's explanatory power over expected returns. They find that investors prefer a positive skewness and are willing to pay a premium for securities with a positive skewness. The premium is measured to be 3.6% annually, which increases greatly if we take stocks at both extremes of the skewness scale.

The mean-variance-skewness framework is used in a range of studies by Chunchinda et al. (1997), Sun and Yan (2003), and Joro and Na (2006). Focusing on skewness makes sense as research confirms the premia to be statistically and economically significant (Langlois, 2020). The author develops a model to predict idiosyncratic and systematic skewness and finds that a long-short portfolio based on the skewness factor has a premium of 6.37% annually, with t-ratios based on different models ranging from 2.33 to 3.06. Furthermore, he finds that ex-ante skewness can be predicted with high confidence based on ex-post data and that the skewness factor can fit well into and improve even the five-factor asset pricing model (Fama and French, 2015).

3. CREATING HYPOTHETICAL DISTRIBUTIONS

When estimating the expected skewness premia, the first step is to have return distributions with the expected returns. We need multiple different distributions to evaluate various skewness values. Due to the difficulties of targeting a specific skewness with empirical returns, we create purely hypothetical return distributions instead. The issue with creating these distributions is that they need to achieve two things simultaneously, firstly we need to change the skewness of the distribution independently, and secondly, we need the transformed distribution to have a shape that can be reasonably expected from a portfolio consisting of stocks sorted based on skewness.

Three moments do not define any particular probability density function (PDF), meaning we need to find the PDF that best suits our use case. We compare three options for PDFs whose skewness can be easily altered and whose shape comes close to a normal

distribution. Tukey (1977) and Hoaglin (2006) describe the Tukey g and h distribution, which has two parameters: g adjusts the skewness of the PDF while h adjusts the kurtosis. The parameters are, however, not independent. When adjusting g , the kurtosis also changes slightly.

Furthermore, as h is non-negative, controlling the kurtosis has limitations. Jones and Pewsey (2009) describe the sinh-arcsinh distribution, which similarly has two parameters. However, the PDF has more than two inflection points for certain parameter combinations, meaning that while the moments can be controlled well, the shape may not apply in our situation.

The PDF I settled on is the skew-normal distribution (Azzalini, 1985). In Azzalini and Dalla Valle (1996), the authors go into more detail on the distribution. It has valuable properties for our application. It describes a random variable well that has normal distribution-like properties, such as diversified stock returns. The maximum skewness that can be achieved with it is ≈ 0.9953 , much higher than the return skewness of any well-diversified stock or bond portfolio. Higher values can be found in the context of idiosyncratic skewness, but our paper looks at systematic skewness. This type of PDF has been widely studied, and therefore, its properties are much better understood. Additionally, it is advantageous to our application that when optimized for values found in stock market returns, even the distribution with the lowest skewness gave an incredibly low chance of a return of less than -100%.

In the papers, by Azzalini (2005) and Yalçinkaya et al. (2018), the authors go into more detail on the estimation of the skewness parameter. We can define the skewed normal distribution $f(x)$ with only one parameter (α) and a random variable x as the following:

$$f(x) = 2\phi(x)\Phi(\alpha x) \quad (1)$$

Where Φ and ϕ are the CDF and the PDF of a standard normal distribution, respectively.

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad (2)$$

$$\Phi(x) = \int_{-\infty}^x \phi(z) dz \quad (3)$$

We can add ξ and ω to transform and shift the coordinate system itself. Our function will be stretched by ω and then linearly shifted to the right by ξ .

$$f(x) = \frac{2}{\omega} \phi\left(\frac{x-\xi}{\omega}\right) \Phi\left(\alpha \left(\frac{x-\xi}{\omega}\right)\right) \quad (4)$$

Next, from Mudholkar and Hutson (2000), Ashour and Abdel-Hameed (2010) and the aforementioned papers, we can calculate the moments of the skew-normal distribution by introducing δ that depends only on α .

$$\delta = \frac{\alpha}{\sqrt{1+\alpha^2}} \quad (5)$$

$$\sigma^2 = \omega^2 \left(1 - \frac{2\delta^2}{\pi}\right) \quad (6)$$

$$\gamma_1 = \frac{(4-\pi)(\delta\sqrt{2/\pi})^3}{2(1-2\delta^2/\pi)^{\frac{3}{2}}} \quad (7)$$

Where σ is the standard deviation and γ_1 is the distribution's third standardized moment (skewness). From the equations, we can further infer that δ has a supremum of 1 as α goes to infinity. Plugging $\delta = 1$ into the equation, we get $\gamma_1 \approx 0.99527$, which will be the maximum skewness that we can achieve. At $\delta = 1$, the distribution is only the right tail of a normal distribution, with the left tail being completely cut off due to $\forall x < 0$ being multiplied by 0 and $\forall x > 0$ being multiplied by 1.

When creating the hypothetical skewed distributions, we must first find values for the moments we want to optimize. To avoid biases, we used the entire period US stock market time series data from Shiller (2015), available on the author's website (R. J. Shiller, 2024). We used the nominal total return figures from January 1871 to June 2024. The geometric mean of the yearly returns was 1.092814, the arithmetic mean was 1.10893, and the standard deviation was 0.19059. Interestingly, it had a non-zero skewness of 0.2344. The expected skewness premia data is not sensitive to these values, and therefore, any good proxy for stock market returns can be generalized to markets outside the United States.

We only optimized for the geometric mean of the returns as with highly skewed distributions, the difference between the arithmetic and geometric mean is large, and investors strictly earn the geometric mean of the returns. That meant that we could not use an equation to get the exact value of ξ ; instead, we used optimization techniques. We created nine different hypothetical distributions based on skewness. The values were -1; -0.75; -0.5; -0.25; 0; 0.25; 0.5; 0.75; 1. As -1 and 1 are skewness values that are not achievable with this method, when we write -1 or 1, we refer to the infimum/supremum of the achievable skewness, which is -0.99527 and 0.99527, respectively.

We go through the following steps for each distribution to get the required values for the moments. First, we can use equation (7) to solve for δ given a fixed γ_1 value. We can then

use equation (6), where we solve for ω given δ we calculated earlier, and σ is the same for all distributions. From δ , we can solve equation (5) for α . The last step is to get ζ given the now calculated ω and α values. As mentioned, we want to optimize for a given geometric mean (G) for which the equation would be non-elementary due to the nature of our distribution. We can optimize for ζ given a fixed geometric mean. γ_1 and σ are independent of ζ ; therefore, we can optimize for the mean without influencing the other moments using nonlinear optimization. Our paper explains the generalized reduced gradient (GRG) method, as Lasdon et al. (1974) explained. This method is optimal as G and ζ have no linear relationship. GRG relies on calculating the optimized value in every iteration. We could use equation (8); however, due to the non-elementary nature of the distribution, we instead use Monti Carlo simulations to estimate G. As accuracy is particularly important for every iteration to converge to an accurate result, we calculated 10,000,000 simulated yearly return figures per iteration which allowed us to solve ζ with a precision of 0.0001 resulting in the accuracy of the returns of 1 basis points. An accurate G itself is not essential, but it is key for the means of the distributions with different skewness values that have the same G. It follows that this can easily be achieved if G is very close to the benchmark value for all our distributions.

$$G = e^{E(\ln(f(x)))} \quad (8)$$

In equation (8), we denote the expected value as $E()$. The problem we face is that this distribution extends to negative values, which is not realistic in a real portfolio, and it also makes us unable to compute the G of function with a domain that extends to the negatives. This issue was non-existent for distribution with a skewness value of -0.25 and higher as for negative values $f(x)$ was so small that negative returns never occurred in our simulations. For the distributions with skewness values -1, -0.75, and -0.5, we unfortunately had to truncate the very end of the left tail of the PDF. Integrating $f(x)$ on $-\infty < x \leq 0$, we can calculate the CDF at point 0 before the negative results are removed. For the distributions with skewness values -1, -0.75, and -0.5, the integration gives the values 1.5E-5, 6.4E-6, and 1.6E-6, respectively. In all cases, the less than -100% theoretical returns would occur very rarely, once every 100,000 years, leading us to conclude that truncating the left tail did not change our results. It did change the standard deviation of the distribution with a skewness of -1 and the skewness of both distributions with -1 and -0.75 skewness. To mitigate this fact, we used the same optimization techniques as when optimizing ζ to recalculate α and ω in these instances.

As shown in Figure 1, the resulting distributions all have the same geometric mean, standard deviation and with distinct skewness values of -1; -0.75; -0.5; -0.25; 0; 0.25; 0.5; 0.75;

1. It is worth noting that we did not constrain higher moments due to the nature of the parameters in the skewed normal distribution. We evaluate the 4th standardized moment (kurtosis). The values are 3.87, 3.59, 3.35, 3.14, 3, 3.14, 3.35, 3.59, and 3.87 in increasing order based on the skewness of the distribution. The fact that the kurtosis increases with a higher absolute skewness value is expected as the values far from the arithmetic mean increase in probability. We could mitigate this fact using additional parameters. However, that would alter the shape of the distribution. Amaya et al. (2015) also find the same empirical relationship between kurtosis and skewness as our distribution experience.

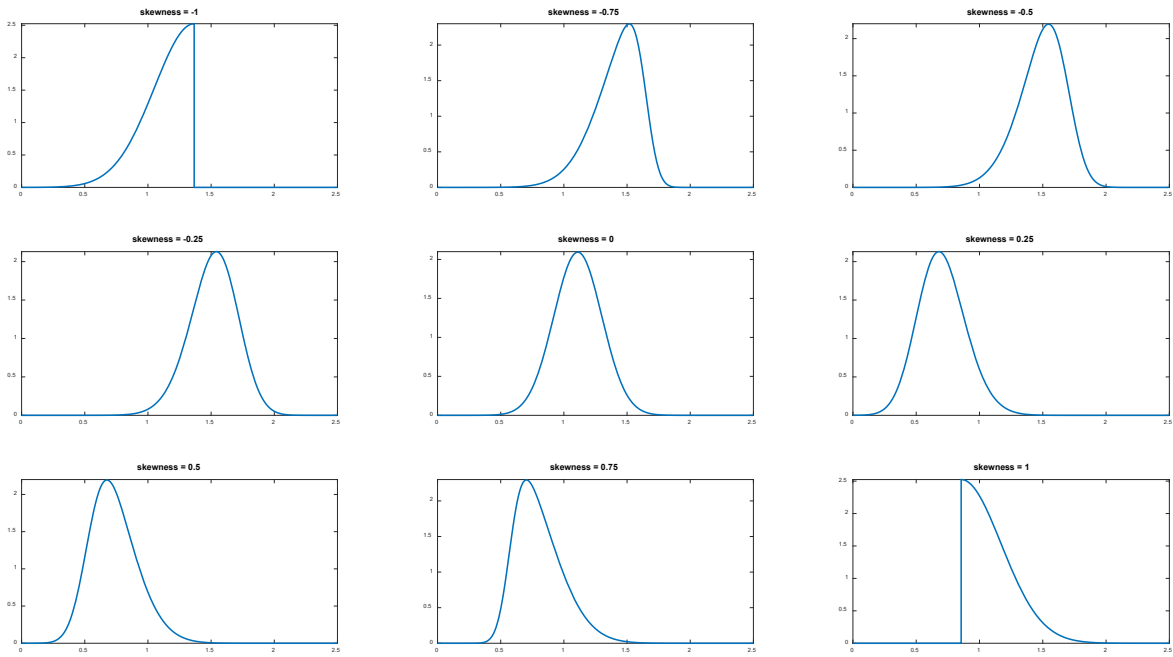


Figure 1: Skew normal distribution plots
The figure presents the PDFs of the returns of each hypothetical portfolio with different skewness measures.

4. ESTIMATING THE EXPECTED SKEWNESS PREMIA

The next step in estimating the skewness premia is to see how much excess risk more negatively skewed returns mean. In CAPM, the beta of negatively skewed stocks would be like the beta of those with positive skewness, meaning that CAPM cannot explain the widely documented return differences. To explain the excess return, we can use the ICAPM. ICAPM adds additional state variables that account for investors wanting to hedge downside risk in their future consumption needs. As the state variables are unknown (Fama and French, 1993), we can only approximate the hedging needs of investors.

A further complication is that investor's hedging needs vary greatly. A young investor might not want to hedge as much as they rationally care more about the expected return of their

portfolio, while a retiree has very high hedging needs as he cannot limit the downside risk in consumption abilities with the very steady dividends from one's human capital. An investor with a highly leveraged portfolio would similarly weigh downside risk significantly as a large drawdown would result in him permanently losing much of his ability to consume.

Negatively skewed stocks have a much higher probability of significant negative extreme outcomes than positively skewed stocks, by the definition of skewness. As an example, the probability of having a loss greater than 80% in a year is 10^{15} times larger for the distribution with a skewness of -0.75 than for the portfolio with a skewness of 0.75. That makes them undesirable for investors seeking to hedge their consumption downside risk. On the other hand, an investor might choose to invest in a positively skewed portfolio to eliminate the left tail of the distribution as much as possible. Conversely, an investor with a very long investment horizon might not care about the short-term skewness and simply choose the portfolio with the highest expected return.

Over multiple decade-long investment horizons, the distribution of ending wealth will have an economically significant, very positive skewness invested in any of the hypothetical skewed distributions. While both are very positive when the skewness of the yearly returns is higher, the ending wealth after a long period will still have a higher skewness, although to a lesser extent. For example, when 10-year samples are taken from the distribution with the skewness of -1 repeated 100,000 times using Monte Carlo randomization will have an ending wealth skewness of ≈ 1.1 , while when we take the 10-year samples from the distribution with skewness of 1, the resulting skewness of the wealth is ≈ 2.2 . That is still a higher number; however, in both cases, the left tail shrunk substantially, and the primary explanation for the resulting skewness is the relatively longer right tail when sampling from positively skewed distributions.

For the reasons outlined above, we would expect that due to the collection of unknown state variables, a portfolio with a higher (lower) systematic skewness has a lower (higher) geometric mean (G). First, instead of looking at empirical findings, we try to estimate the difference in G. As the state variables are not fully known, we can estimate an upper bound for the skewness premium by assuming that all investors have a very high sensitivity to the left tail of the outcomes. If investors are rational, this would mean that for them to invest in a negatively skewed portfolio, they would need to be compensated with a higher return for the extra risk they are taking. In this scenario where all investors have the same risk aversion, we expect all

portfolios to have the same risk-adjusted return; otherwise, everyone would prefer the portfolio with the higher risk-adjusted returns.

That is indeed an upper bound in a world where state variables and investor preferences are more diverse. We would expect only those investors to invest in negatively skewed portfolios with lower left-tail sensitivity than the average investor. A further reason why this is the upper bound is because we only adjust the risk to be as low as with a normal distribution by increasing its expected return, making it have a higher risk-adjusted return. That would result in them accepting a lower but still positive skewness premium as their consumption would be less impacted due to their lower left-tail sensitivity. The same reasoning holds for the positively skewed portfolios, where we should get a more negative skewness premium with this methodology than what is found empirically, making it a lower bound.

We must estimate the riskiness of a portfolio from the perspective of a very left-tail-sensitive investor. First, we define the normal distribution as having a skewness premium of 0. We use two models to get an estimate of the skewness premium. The first model is a Value-at-Risk (VaR) type model. In this scenario, we assume that an investor is sensitive to the size of the loss at a certain low percentile level. We test at the 1st and 5th percentiles. We further test for different investment horizons. As we created hypothetical distributions with parameters corresponding to yearly stock market figures, we can calculate the expected premium for multiples of one year. We calculated it for the following investment durations: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 years.

The estimation went as follows. As we defined the normal distribution to have a skewness premium of zero, we must first calculate the outcomes' 1st and 5th percentile in a normal distribution for all the investment durations. We took 10,000,000 randomized samples per investment horizon from our normal distribution using Monte Carlo randomization to calculate the loss at these percentiles for different numbers of yearly returns. Alternatively, one could use an algorithm to find the distribution of the product of two random variables. Glen et al. (2004) demonstrate an optimized algorithm to find the PDF of the resulting distribution while the formula was first shown in the book (Rohatgi, 1976). We did not use equation (9) as the non-elementary nature of the resulting PDF made it increasingly complex to calculate the percentile ranks. Let f_n be the PDF of n consecutive yearly returns.

$$f_n(v) = \int_{-\infty}^{\infty} f_{n-1}(x) f_1\left(\frac{v}{x}\right) \frac{1}{|x|} dx \quad (9)$$

The resulting 1st percentile returns are 0.666, 0.612, 0.587, 0.573, 0.566, 0.567, 0.572, 0.579, 0.589, 0.601 and for the 5th percentile returns 0.797, 0.768, 0.764, 0.771, 0.786, 0.806, 0.831657, 0.861, 0.894, 0.932 both in increasing order based on the length of the investment in years. These reported numbers are rounded to 3 digits. With these results calculated, we must adjust ζ by an amount that either the 1st or 5th percentile returns of a particular skewed distribution are equal to the same percentile returns of our normal distribution. In our calculations, we assume that for small numbers, the additional change in ζ equals the change in G . After our calculations, we verify that this assumption does not change the results of any of our calculations. We can then conclude that the additional amount we have had to change ζ is an estimate of the upper bound for the expected skewness premia for a given percentile, skewness, and investment horizon. As we have eight skewed portfolios, two percentile ranks and ten different investment horizons, we calculated 160 unique datapoints. These datapoints are found in Appendix 1.

We would like to fit a function to our data points; however, that would require us to know the relationship between the investment horizon and the expected skewness premia. First, from equation (9) and Monte Carlo simulations, we can observe that the difference between a low fixed percentile value on the normal distribution and a skewed distribution will be remarkably close to linear. This linear fit for this difference will have a non-zero intercept. The fit is near perfect at $R^2 > 99\%$ for both 1st and 5th percentile values and all skewness values. Now assume that when we adjust ζ , the shape of the distribution does not change even at long time horizons; we will come back to the problems with this assumption. The effect of the premium can be calculated easily, as we assumed that the premium is equal to ζ and equal to G . We can calculate the difference in the compounding effect at year t and from the way we define the premium, we can say that this compounding effect must be equal to the linear fit of the differences in percentile values. We can write this as equation (10). In the equation, let R_m The market return, as needed to calculate the effect of compounding the returns, will be constant in our case.

$$(R_m + \mu)^t - R_m^t = at + b \quad (10)$$

$$\mu(t) = \sqrt[t]{at + b + R_m^t} - R_m + e(t) \quad (11)$$

$$\mu(t) = at^b \quad (12)$$

Where μ is the dependent variable as it is the skewness premium. The regression parameters are a and b , whereas the independent variable is the time in years denoted as t . We

then add an error term $e(t)$. We then calculate the average R^2 value using the least squares method to regress to the function described in equation (11). We calculate that the average $R^2 = 99.49\%$. This value is high and shows that our assumptions were close to reality. We can notice that using a much simpler power fit with the equation described in equation (12), the power function will take remarkably similar values to the function described in equation (11), with the biggest difference being that at high t values, the power function takes lower values. They diverge greatly when $t < 1$; however, that is outside the domain of our results.

How accurate was our assumption that a change in ζ does not meaningfully change the shape of the return distribution even for high t values? A positive change in ζ will result in a more symmetrical distribution at t values close to 10, meaning their skewness will be closer to zero. This results in the right tail being shorter and the left tail being longer. All of our distributions experience an exceedingly high skewness for high t values. How does this affect μ ? As the left tail gets shorter while the mean increases, the PDF will take lower values up to our percentile value, resulting in the percentile rank representing a higher return simply due to the shape change as ζ increases. That means a lower premium is sufficient to equalize the percentile values to the normal distribution. The same will be valid for negative premium values. This effect is only meaningful at high values of t . We can correct this assumption by reducing the fitted values at high t values. (We do not change our simulated estimated skewness premia values.) The fitted function coincidentally will be remarkably similar to a power function. We can try regressing using the power function. As a result, we get $R^2 = 99.83\%$ on average, which is a near-perfect value. Therefore, in Figure (2,3), we show the points fitted using a power function to illustrate our data better. The function that perfectly describes μ is non-analytical; therefore, regressing to it would be difficult. We do not expect the power function to be extended outside of our domain, especially not for $t < 1$.

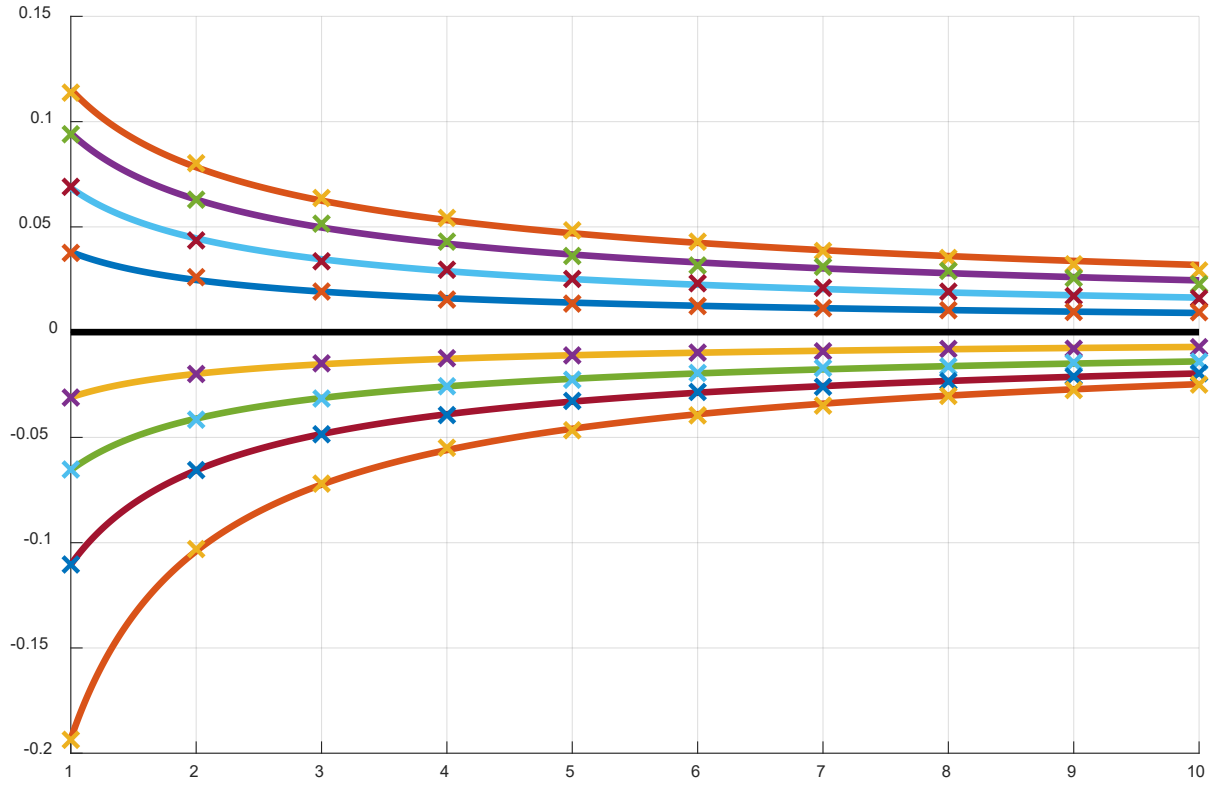


Figure 2: 1st percentile test results

The x-axis represents time in years. The y-axis represents bounds for the estimated skewness premia in terms of a return difference estimated by our 1st percentile test as detailed in section 4. The functions from top to bottom represent the skewness values in increasing order from -1 to 1.

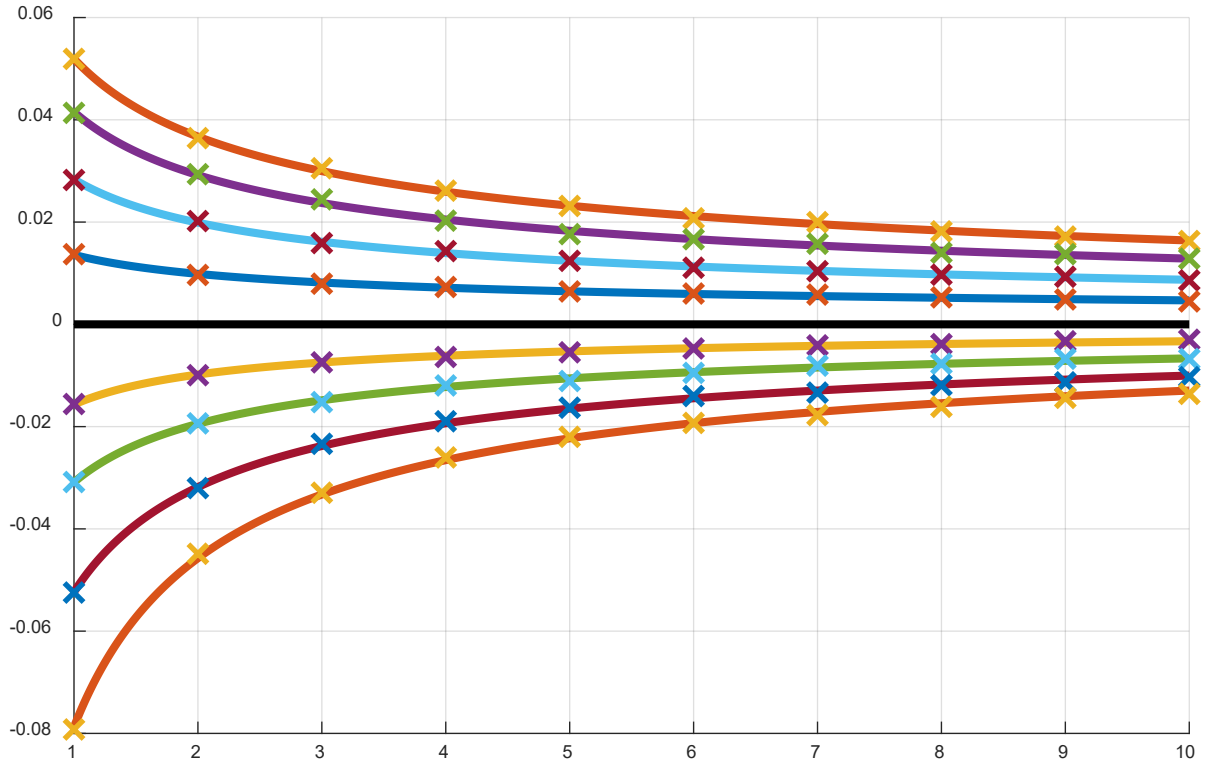


Figure 3: 5th percentile test results

The x-axis represents time in years. The y-axis represents bounds for the estimated skewness premia in terms of a return difference estimated by our 5th percentile test, as detailed in section 4. The functions from top to bottom represent the skewness values in increasing order from -1 to 1.

Using a different methodology, we also estimated an upper bound for the skewness premia. We can use the same base assumption that if all investors are very left-tail sensitive, the rationally priced risk premia will be an upper bound for what we expect in a more realistic world. We choose a long-period fixed withdrawal scenario to simulate a very left-tail sensitive consumption pattern. This type of withdrawal is often associated with retirees spending their retirement savings as they often follow a rule that allows them to take out a fixed amount per month. In this context, by fixed, we refer to a fixed real cash flow, meaning that each year, the withdrawals are increased only by the rate at which the Consumer Price Index (CPI) inflated during that year. For the withdrawal rate, we picked 4% based on Bengen (1994). The 4% rule outlined by the authors is criticized by many, including Pfau (2017), where the authors argue for a much lower withdrawal rate when considering the performance of international stocks. However, our skewness premia estimations are not sensitive to the withdrawal rate itself if we can optimize accurately across all our distributions. For this reason, we used the 4% rule as it provided a much higher chance of running out of savings, resulting in our simulations being less noisy across different investment horizons.

Similarly to the VaR scenario, we must first calculate our normal distribution's failure rate. We define the failure rate as a simulation in which the withdrawal amount was higher in at least one year than the savings available. We estimate the failure rate for 20-, 30- and 40-year periods. For shorter periods, the failure rate was too low and longer periods were unreasonably longer retirement periods. We can estimate the failure rate by simulating 1,000,000 distinct retirements in a Monte Carlo randomized simulation. The resulting failure rates for the normal distribution are 5.9791%, 14.6653%, and 20.7834% for the investment horizons 20, 30, and 40 years, respectively.

We must account for the fact that the return data we used is nominal, whereas the withdrawal rate must be adjusted for inflation. To avoid biases, we use the historical inflation data from one year after the Federal Reserve's inception, meaning that we took the average starting from 1915. Using data from (Robert J. Shiller, 2024), the geometric mean of inflation was 3.18% with a standard deviation of 4.94%. We took random samples from a normal distribution with the aforementioned geometric mean and standard deviation every year to adjust the withdrawals for inflation. That assumes that long-term inflation and long-term stock market returns are independent.

In the fixed withdrawal scenario, we optimize all skewed portfolios for the same failure rates at each fixed retirement duration of t years. We achieve this by adjusting ζ until the failure

rate is within 0.1 basis point of the one observed for the same t on the normal distribution. Then, the amount we changed ζ by is the upper bound for the expected skewness premia. We calculate 200,000 Monte Carlo simulations for each iteration of the optimization. That leaves us with three times series data points for each of the eight distributions, so 24 datapoints in total. Then, similarly to the VaR-type scenario, we can fit a power function on our data. The resulting R^2 on average is 99.75%, meaning the data can be fitted similarly well using a function described in equation (12). The exact values are reported in Appendix 2. We illustrate our data with the functions fitted onto them.

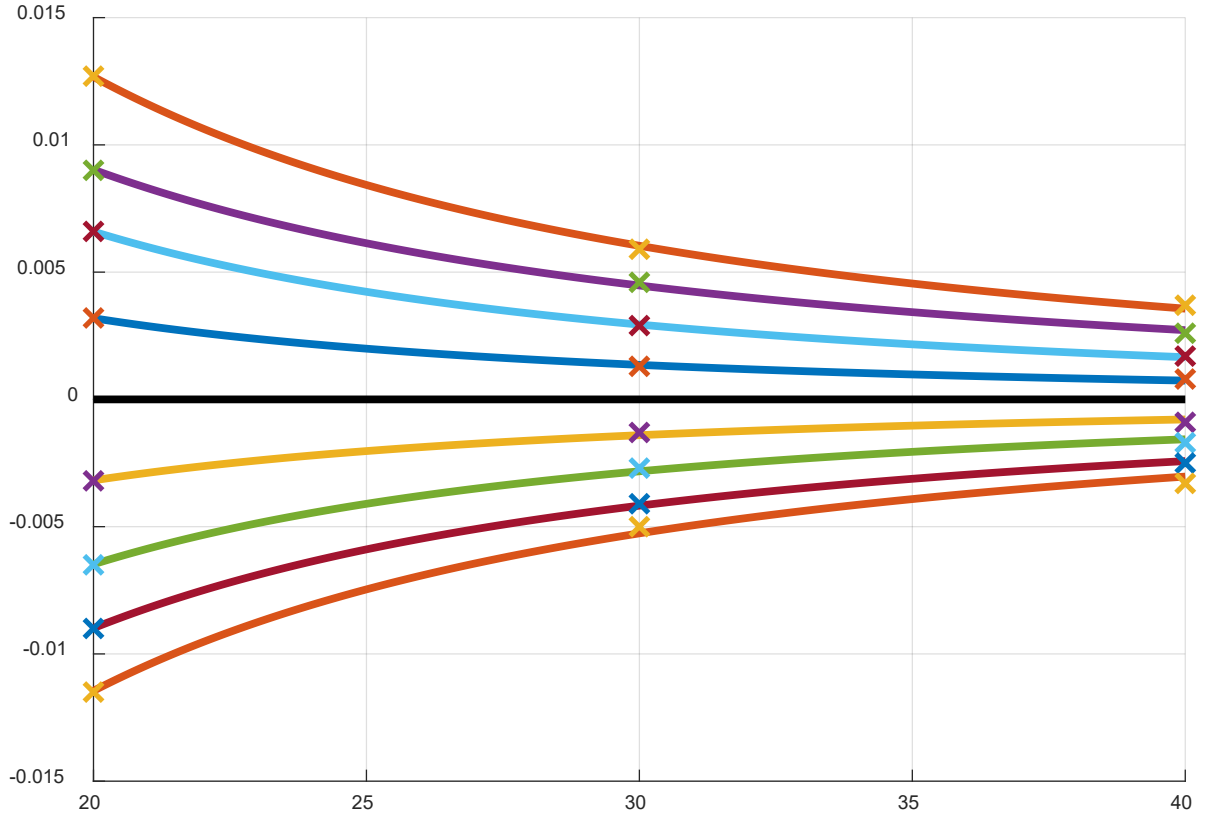


Figure 4: fixed withdrawal test results
The x-axis represents time in years. The y-axis represents bounds for the estimated skewness premia in terms of a return difference estimated by our fixed withdrawal test as detailed in section 4. The functions from top to bottom represent the skewness values in increasing order from -1 to 1.

5. DISCUSSION

Comparing our findings to those of other authors is inherently inaccurate, as the numbers are not directly comparable. To account for this, we try to make every assumption in favor of the skewness premia being able to be explained by risk, as we hypothesized that it could not be explained by rational risk-taking alone.

We gave reasons why our estimates are an upper bound for the expected skewness premia. Because we estimated an upper bound, we would expect that the empirical findings are that the historical skewness premia are less than our estimations. However, we must make some assumptions to make our results comparable to those of other studies. As we created hypothetical skewed distributions, we implied that these distributions make up the full portfolio of an investor. Whereas, if someone were to construct a factor based on skewness, they would need to account for the fact that the factor is in a real portfolio, and the investor would combine the skewness factor with something similar to the market factor. That makes it popular to optimize for co-skewness amongst researchers as it will account for the time series of the events at the far ends of the tails, which a rational investor would care greatly about. We will try to compare skewness numbers to co-skewness numbers, as we cannot create time series. We should be able to compare these figures well, as the market has a skewness close to zero. We try to account for this error in our estimations.

The other assumption we must make when comparing our findings to other papers is that it is impractical to construct a skewness factor based on yearly skewness figures. All the papers we compared use much higher frequency data, usually daily, weekly, or monthly, as higher frequency data allows researchers to include/exclude stocks from the portfolio more frequently, resulting in more extreme skewness values. However, we only look at yearly hypothetical portfolios as they align better with a medium-long long-term investor's horizon, making simulating a long series of yearly returns much more accurate. Thankfully, portfolios with high monthly skewness also have a strong tendency for high skewness in the yearly data. We can confirm this in some papers that report the yearly skewness figures, or in some cases, the authors provide the full time-series data, which we can use to calculate it ourselves, for example, in Langlois (2020).

The first paper we compare our results to is Harvey and Siddique (2000), the first significant paper attempting to calculate the skewness premium. The authors recently published a paper on 25-year out-of-sample results similar to those of their in-sample paper (Harvey and Siddique, 2023). They measure the skewness premium to be 3.6%. Unfortunately, their paper does not publish the skewness of the skewness factor, but we can take a reasonable estimate as they publish the range of the industry-based co-skewness at 0.51. That is a worst-case scenario for the skewness, as combining industries would diversify some of the skewness away. From our expected skewness premia figures, we can calculate that this would be a premium too high

after 2.8 years in the 1st percentile test and a premium too high over any number of years we tested in both the 5th percentile test and the fixed withdrawal test.

We chose the second paper to compare our study as it had the time series data publicly available at Langlois (2017). In Langlois (2020) the author finds a method to construct a skewness factor using only ex-ante data, whereas most other papers use only ex-post data. We can calculate the geometric mean of their skewness factor to be 4.48%. At the same time, the authors report that the CAPM alpha, which accounts for differences in market beta, is 4.02%. The authors report the co-skewness number to be -0.27. We can calculate the co-skewness value using the market return data from the Kenneth R. French data library (Kenneth R. French, 2024). We calculate the co-skewness value to be -0.39. In this case, a negative co-skewness value did not translate to an increase in the skewness of the overall portfolio when we added the skewness factor to the market factor. Over the period analyzed by the authors, the Mkt has a monthly and yearly skewness of -0.52 and -0.36, respectively, adding the skewness factor increases it to -0.2 and 0.35 for the monthly and yearly values, respectively. As the author optimized for the co-skewness measure, we also used that. Similarly to the first paper we looked at, we find that the skewness premium using the CAPM alpha as a measure of return is higher than our upper bound estimate for investment horizons longer than about two years in our 1st percentile test and that it is higher at any time horizon than the premium, we find in both our 5th percentile test and our fixed withdrawal test.

A similar paper (Conrad et al., 2013) looked at ex-ante skewness in the context of option prices of the underlying stocks. We look at the reported risk-neutral skewness sorted table in which we can analyze the 12 months to maturity options. They find that the difference between high and low skewness measures is 1.91 and that the returns are highest for the lowest skewness rank with a difference in returns of 0.45%, a 5.54% annualized rate. We could not create hypothetical skewed portfolios with skewness values this high; however, we can estimate where this annualized rate would end up compared to our skewness premia estimates. We estimate that it is lower than our estimated upper bound for all investment horizons in the 1st percentile test, around the upper bound for the 5th percentile test even at high t values, but we estimate it to be higher than our upper bound for the fixed withdrawal test for any t .

The following paper we looked at is Boyer et al. (2010). The authors give a detailed description of the quintile portfolios sorted based on predicted skewness. They find that the difference between the returns of the 1st and 5th quintiles is 8.39% annualized, and the difference between the skewness of the quintiles is 0.823. Here, we assume that the difference between

the quintile skewness numbers estimates the skewness of the long-short portfolio well. The upper bound is higher after only a 1.4-year investment horizon in the 1st percentile test. In both the 5th percentile and fixed withdrawal tests, the upper bound is higher than the premium found by the authors over any time horizon. We must add that the 4th quartile has a much higher mean return at 13.47% annualized compared to only 6.36% for the 5th quartile. In their sample, much of the return difference between low and high skewness stock can be attributed to the terrible return of the 5th percentile. If we only look at the 4th-1st percentile returns, we get a return of 1.57% annually and a skewness difference of 0.409. That is much more in line with our expectation at the 5th percentile test, which is lower than the upper bound in the 1st percentile test but is still higher than the upper bound in the fixed withdrawal test.

Our data is only slightly based on US stock market data. We took the standard deviation and the mean return from US return data; however, the expected premiums are not sensitive to these values. Therefore, we can look at international papers using our methodology. A recent paper (Dong et al., 2022) examines the skewness premium in various international markets. The authors also look at the skewness premium in developed and emerging markets. They report a CAPM alpha of 5.02%, 7.8%, and -0.55% for global, developed, and emerging markets, respectively, for the co-skewness premia based on co-skewness sorted deciles. Unfortunately, the authors do not report the co-skewness figures for the deciles; they only report the average co-skewness and the beta to the skewness factor they construct. Based on other US studies, we can give a reasonable estimate that the skewness differences should not be higher than -0.75. That would mean that the 1st percentile upper bound is larger than the premium after a 3-year investment horizon for the global portfolio and only 1.5 years for the developed markets portfolio. The 5th percentile and fixed withdrawal tests would mean larger premia than the upper bound over any horizon. In the authors' sample, they find a negative co-skewness premium for emerging markets, which would be lower than the upper bounds we find for any co-skewness difference.

In the book of Dimson et al., (2002), the authors explain that emerging markets generally have a much more negative skewness than their developed peers. The exact number is very dependent on which interval we choose our sample from, for example, including the 1940s, which brings the skewness down substantially. Using our skewness premia upper bounds, one could assign a premium to emerging markets based on the added skewness the investor would face.

6. RESULTS

We demonstrate a method to calculate an upper bound for the expected skewness. We use different assumptions to get various results as to how large the upper bound should be. These different assumptions are that investors are very sensitive to the 1st percentile of return outcomes to the 5th percentile of the outcomes, or a somewhat different approach is that we assume that for them to fulfil their consumption needs, they would need to make fixed withdrawals from their portfolio this would inherently be a very left tail sensitive scenario.

As investors' state variables in aggregate are unknown by definition and are dynamic, we can only estimate the expected skewness premia. Hence, given a fixed skewness measure, we only attempt to estimate the bounds of the skewness premia. This is also why we must make substantial assumptions to arrive at our results.

We compare our results for the bounds to skewness premia values found by other authors empirically. The literature on this topic has expanded substantially in the past decades so that we can compare our results to various findings. The problem we faced during our comparisons was that time series data is not directly comparable to hypothetical distributions. We had to assume that any time series relationships are irrelevant and that the portfolio value does indeed follow a random walk. We further assume skewness measures are directly comparable, even yearly and monthly figures. In Section 5, we state our detailed assumptions.

We find the empirical skewness premia are very high relative to our skewness premia bounds. In all the papers that look at developed stock market returns, the skewness premia the authors find is more significant than the upper bound we find when estimating it with the 5th percentile VaR model or the fixed withdrawal rate model for any investment horizon we tested. In many papers, the upper bound estimated with the 1st percentile VaR model is not higher for all investment horizons but higher after only a short investment horizon. The longest time for the skewness premia to be higher than the upper bound was 3 years.

Option prices and emerging markets returns have behaved differently. A study that looked at options prices (Conrad et al., 2013) found an economically and statistically significant return for the skewness premium but also found that the skewness of the resulting strategy would be so negative that the premium ended up being well below our estimated upper bound. Emerging markets behaved in the opposite way. An international study (Dong et al., 2022) found that the skewness premium completely disappears when emerging markets stocks are pooled together. Looking at markets locally results in a positive premium there as well.

These results mean that overall, the skewness premia are much higher than risk could rationally explain them. That could be due to a variety of different reasons. One reason might be that the sample period in which the papers calculate the skewness premia is an outlier in the amount of skewness-affected returns. Some might further argue that the skewness premia were calculated in the sample before the widely available diversifying tools, such as a passive investing strategy. Investors who do not have the tools to diversify idiosyncratic skewness might weigh it more than systematic skewness. Outlier returns often occur when a portfolio contains only a handful of securities, resulting in a more pronounced skewness. As idiosyncratic skewness has a higher absolute value than systematic skewness, the skewness premia would be more rationally priced. The results of a recent paper (Harvey and Siddique, 2023) contradict this hypothesis as it finds a similar premium when diversifying tools were widely available.

Trading costs might have been partly why the gross premia researchers measure is relatively high. The premia were not optimized for low trading costs, nor were they reported. However, that would imply for the premia to decrease over time as trading costs for a unit of turnover have decreased drastically in the 21st century. That was the case for the momentum factor introduced in (Carhart, 1997). This factor leads to high turnover (Korajczyk and Sadka, 2004). Similarly, the results of (Harvey & Siddique, 2023) suggest that this is not the case, as the decrease in trading costs did not coincide with any decrease in the gross premium.

The most plausible explanation is that investors experience a high degree of myopic loss aversion (Thaler et al., 1997). We document that the added risk of investing in negatively skewed stocks decreases significantly over time. That is the case in the VaR-type tests and the fixed withdrawal scenario. This theory could explain all of the irrationality in skewness preferences, as when the investment horizon is even less than a year, the VaR type test would give upper bounds for the premia that are much higher than any of the premia found empirically. We did not calculate this as our methodology does not allow us to test shorter than a yearlong investment horizon. The authors also argue similarly in the context of the equity risk premium.

Our paper has great implications. Any investor worried about skewness increasing the risk of his portfolio might look at our results to find an expected premium for taking on that risk for his investment horizon. Our findings imply that even very left-tail sensitive investors such as retirees might not want to avoid highly skewed portfolios as intuition might suggest because the skewness premium would more than compensate them for the added risk. Similarly, an investor with a high leverage ratio should not optimize his portfolio for very high skewness

values. At the time of writing, only bespoke solutions exist for investing in the skewness factor, meaning that a retail investor's best choice could be to avoid preferring highly skewed stocks in any scenario.

7. CONCLUSION

We provide a methodology to estimate bounds for the expected skewness premia. We show that the bounds are relatively small compared to the skewness premia found empirically by other authors. While a very short investment horizon might give reasons to prefer stocks with high skewness, reasonable investment horizons imply that even left-tail sensitive investors should prefer negatively skewed stocks because the premium is higher than what would be needed to compensate them for taking on the additional risk.

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APPENDIX 1

Table 1: 1st percentile tests detailed results

The table shows the estimated premia bounds in our 1st percentile test rounded to 4 decimal places (in our calculations more accurate values were used).

years skew	1	2	3	4	5	6	7	8	9	10
-1	0.1138	0.0802	0.0637	0.0542	0.0484	0.0430	0.0386	0.0352	0.0322	0.0293
-0.75	0.0940	0.0629	0.0515	0.0430	0.0362	0.0317	0.0311	0.0291	0.0257	0.0227
-0.5	0.0690	0.0436	0.0337	0.0296	0.0253	0.0232	0.0211	0.0192	0.0172	0.0161
-0.25	0.0376	0.0261	0.0193	0.0155	0.0136	0.0124	0.0114	0.0104	0.0096	0.0095
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.25	-0.0311	-0.0199	-0.0148	-0.0124	-0.0110	-0.0097	-0.0091	-0.0080	-0.0079	-0.0071
0.5	-0.0652	-0.0416	-0.0315	-0.0257	-0.0226	-0.0194	-0.0170	-0.0163	-0.0145	-0.0140
0.75	-0.1104	-0.0655	-0.0485	-0.0394	-0.0327	-0.0284	-0.0260	-0.0234	-0.0211	-0.0195
1	-0.1936	-0.1030	-0.0720	-0.0549	-0.0466	-0.0395	-0.0349	-0.0303	-0.0275	-0.0249

Table 2: 5th percentile tests detailed results

The table shows the estimated premia bounds in our 5th percentile test rounded to 4 decimal places (in our calculations more accurate values were used).

years skew	1	2	3	4	5	6	7	8	9	10
-1	0.0519	0.0364	0.0305	0.0261	0.0232	0.0207	0.0200	0.0182	0.0172	0.0163
-0.75	0.0414	0.0294	0.0244	0.0203	0.0176	0.0167	0.0158	0.0140	0.0137	0.0128
-0.5	0.0282	0.0202	0.0159	0.0143	0.0124	0.0111	0.0104	0.0097	0.0093	0.0086
-0.25	0.0137	0.0097	0.0080	0.0073	0.0064	0.0060	0.0058	0.0053	0.0049	0.0045
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.25	-0.0156	-0.0099	-0.0074	-0.0064	-0.0055	-0.0047	-0.0041	-0.0039	-0.0033	-0.0030
0.5	-0.0308	-0.0194	-0.0152	-0.0120	-0.0111	-0.0095	-0.0081	-0.0077	-0.0067	-0.0068
0.75	-0.0525	-0.0320	-0.0234	-0.0190	-0.0163	-0.0140	-0.0134	-0.0119	-0.0113	-0.0103
1	-0.0792	-0.0449	-0.0329	-0.0260	-0.0220	-0.0193	-0.0177	-0.0161	-0.0144	-0.0136

APPENDIX 2

Table 3: fixed withdrawal tests detailed results

The table shows the estimated premia bounds in our fixed withdrawal test rounded to 4 decimal places (in our calculations more accurate values were used).

years skew	20	30	40
-1	0.0127	0.0059	0.0037
-0.75	0.0090	0.0046	0.0026
-0.5	0.0066	0.0029	0.0017
-0.25	0.0032	0.0013	0.0008
0	0.0000	0.0000	0.0000
0.25	-0.0032	-0.0013	-0.0009
0.5	-0.0065	-0.0027	-0.0017
0.75	-0.0090	-0.0041	-0.0025
1	-0.0115	-0.0050	-0.0033