

OTDK-dolgozat

2025

BUDAPEST UNIVERSITY OF TECHNOLOGY AND ECONOMICS

FACULTY OF ECONOMIC AND SOCIAL SCIENCES



**ELMÉLETI MODELL A JÁTÉKOK ÁRUKAPCSOLÁSÁRA A VIDE-
OJÁTÉK-KONZOL PLATFORMOKON
A THEORETICAL MODEL FOR BUNDLING GAMES ON VIDEO
GAME CONSOLE PLATFORMS**

Varga Viktor

Konzulens: Dr. Somogyi Róbert

Kézirat lezárásának dátuma: 2024 12. 29.

TABLE OF CONTENTS

1.	INTRODUCTION	1
2.	THE VIDEOGAME INDUSTRY.....	2
2.1.	INTRODUCTION OF THE VIDEOGAME MARKET	2
2.2.	SUBSCRIPTIONS IN THE VIDEOGAME CONSOL MARKETS	2
2.3.	VIDEOGAME DEVELOPERS.....	3
2.4.	THE PARTICIPANTS OF THE INDUSTRY	3
3.	TRADITIONAL MODEL.....	5
3.1.	BASE UTILITY MODEL	5
3.2.	TRADITIONAL MODEL SETUP	6
3.3.	CONSUMER'S UTILITY MAXIMALIZATION	6
3.4.	INTRODUCTION OF TWO TYPE OF CONSUMERS.....	7
3.5.	POSSIBLE PARAMETER SETUPS	9
3.6.	SCENARIO ANALYSIS.....	10
3.7.	PROFIT-MAXIMIZING DECISIONS.....	11
4.	BUNDLING MODEL.....	13
4.1.	BUNDLING OPTION FOR THE PLATFORM	13
4.2.	PLATFORM'S PROFIT-MAXIMIZING DECISION.....	13
5.	COMPARISON.....	16
5.1.	PROFIT MAXIMUMS IN THE TRADITIONAL MODEL	16
5.2.	SUPPLIER SURPLUS IN THE TWO MODEL.....	16
5.3.	CONCLUSIONS.....	17
6.	SUMMARY	19
7.	BIBLIOGRAPHY	20
APPENDIX		I
APPENDIX 1: SIMPLIFICATION OF EQUATION (19):.....		II
APPENDIX 2: PROOF OF INEQUALITY (35).....		IV

LIST OF TABLES AND FIGURES

Table 1: Notations for the Base Utility Function.....	5
Table 2: Optimal quantities of the two consumer types	8
Table 3: Truth table of economic constraints	9
Table 4: Possible constraint combinations	10
Table 5: Possible scenarios	10
Table 6: Traditional firm's profit differences	17
 Figure 1: Value creation in the videogame industry	 4

1. INTRODUCTION

Over the past few decades, the video game industry has rapidly developed and created a significant presence in the entertainment market. This growth is coupled with intense competition, fostering innovations in business models (Osathanunkul, 2015, 35-36). Revenue structures have shifted from traditional single-purchase models to more flexible formats, including game-as-a-service (GaaS), free-to-play, and subscription-based models. Subscription services like Sony's PlayStation Plus and Microsoft's Xbox GamePass, which provide users access to extensive game libraries for a monthly fee, are increasingly popular (Tan, Chiu, 2024, 1-2).

This study examines the impact of subscription-based models that offer bundled game libraries through monthly subscriptions in a two-sided platform environment. Although a discounted price through a subscription bundle might attract more consumers, the availability of multiple games could also potentially distract them from any single title. Consumers are categorized into two groups based on their degree of substitutability between popular "Traditional" games and less popular "Indie" games. The study compares the effects of the traditional à la carte model with the bundling model. Both models are constructed within a similar theoretical framework, relying on standard economic methods and industry-specific assumptions.

The rest of the paper is structured as follows. Section 2 provides literature review and background information on the video game industry and their subscription models. Section 3 presents the traditional model and outlines the key economic assumptions. Section 4 introduces the bundling model, and Section 5 compares the two models. Finally, Section 6 summarizes the study's main conclusions.

2. THE VIDEOGAME INDUSTRY

2.1. INTRODUCTION OF THE VIDEOGAME MARKET

Since its appearance in the 1970s, the video game industry has undergone significant transformations, evolving from the bustling arcades of its early days to becoming an integral part of the digital market. Soon, over 3 billion people worldwide are anticipated to be engaged with video games (Statista, 2024). Over the decades, this transformation has been marked not only by advances in game formats and console capabilities but also by substantial shifts in business models and consumer behaviors, making the industry a subject of various academic disciplines (Goh, Al-Tabbaa, Khan, 2024, 1-4).

In both academic literature and the business world, video game consoles are often regarded as two-sided platforms. A two-sided market is defined as one that consists of at least one platform facilitating interactions between two distinct user groups, usually sellers and consumers. In this platform environment, positive network externalities are leveraged by connecting the groups and implementing appropriate pricing strategies for each side (Rochet, Tirole 2006, 645-646). According to this definition, video game developers can be considered the seller side, video game consumers the buyer side, and video game consoles like PlayStation and Xbox serve as the intermediary platform.

2.2. SUBSCRIPTIONS IN THE VIDEOGAME CONSOLE MARKETS

With the growth of the internet and subsequent digital transformation, a growing number of business models have become adaptable to a variety of operations. One of the primary e-business models is the subscription model (Nemeslaki, Urbán, Trestyén, 2008, 4-6). Glazer et al. (1982) also demonstrate that a subscription service combined with individual sales enables a monopolist to implement price discrimination, leading to higher profits and an increase in net social welfare. Netflix is a prime example of this model; more importantly for this study, leaders in the gaming industry have also adopted this method. Both Microsoft and Sony have released their own gaming subscription services (Chan, Zhang, Yeung, 2020, 1-2). These subscriptions have also undergone various changes and are nowadays known as PlayStation Plus and Xbox Game Pass.

According to the official websites of XBOX (2024) and PlayStation (2024), there are several similarities and differences between their subscription bundles. Both Microsoft and Sony offer multi-tiered bundles, but Microsoft provides more cross-platform options due to its significant market presence not only in the console sector but also in the PC market. In contrast, Sony offers a more sophisticated, layered approach to its bundles.

Despite these differences, both subscription models ultimately aim to give consumers access to a broad selection of games during the subscription period, along with the option to purchase individual titles. This approach, known as "mixed bundling," is a common economic strategy where a firm offers multiple products that can be bought individually or at a discounted price when purchased together. Mixed bundling encourages consumers to try products they might not buy on their own, while allowing the firm to maximize revenue from those who prefer to purchase specific items separately (Belleflamme & Peitz, 2015, 264-266).

2.3. VIDEOGAME DEVELOPERS

As discussed, the video game industry is a complex system with multiple interrelated participants. One of the main function is that the video game studios are responsible for producing game content compatible with the various consoles. These firms can be classified in several ways, but for simplicity, this study categorizes them as "Indie game developers" and "Traditional game developers." Indie developers are typically smaller, enjoy greater creative freedom, but operate with limited budgets. In contrast, larger studios are organized into divisions and focus on producing AAA, blockbuster games (Zackariasson, 2012, 1-23).

2.4. THE PARTICIPANTS OF THE INDUSTRY

To improve understanding, it is essential to clarify the roles and relationships among the core players in the video game industry. In their study, Marchand and Hennig-Thurau (2013) developed a conceptual framework for value creation in this industry. They identified two primary pathways: a vertical pathway, referred to as the "gaming environment," and a horizontal one, which represents the distribution channels. The vertical pathway connects content providers (video game developers), platform providers, and consumers (Marchand, Hennig-Thurau, 2013, 141-143).

Based on this framework, a simplified relationship map, Figure 1 is created to enhance understanding of the topic. In this model, game development studios supply content to intermediary platforms. The platform can offer this content through subscription services or individual sales via traditional a la carte method. Consumers purchase the products from the platform, generating revenue for the platform provider, who then compensates the content providers for their offerings.

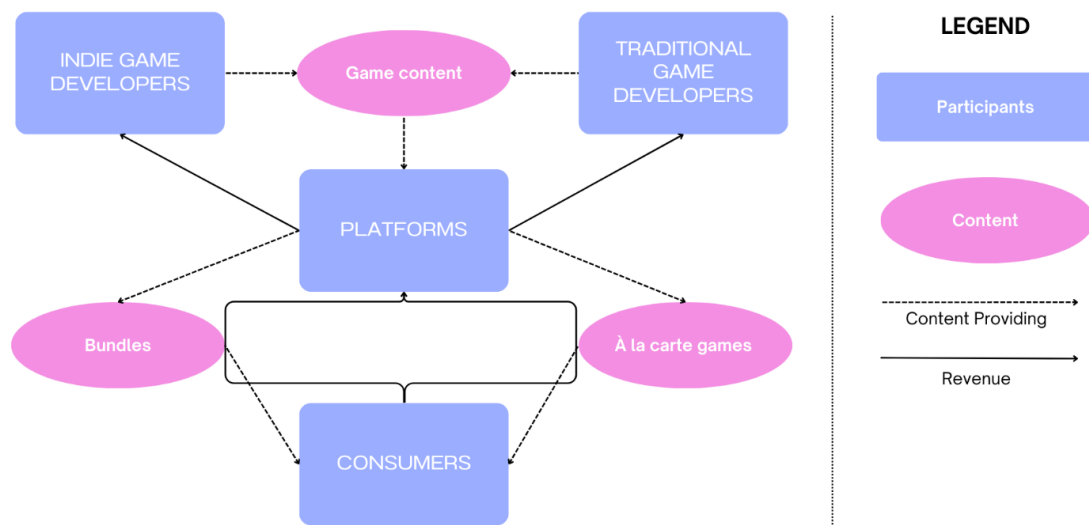


Figure 1: Value creation in the videogame industry

Source: Own illustration

3. TRADITIONAL MODEL

3.1. BASE UTILITY MODEL

In their 1984 work, Nirvikar Singh and Xavier Vives introduced the Quasilinear Quadratic Utility Model to represent the utility function of a representative consumer within an oligopoly market characterized by differentiated goods (Singh, Vives, 1984). This model serves as a foundational framework for analyzing consumer preferences and demand in markets with multiple product variations. Thanks to its appealing properties, it has been extensively used in the literature of industrial organization (Choné, Linnemer, 2020). The utility function, which will be adopted as the reference for this study, is expressed as follows:

$$U(q_1, q_2) = a_1 q_1 + a_2 q_2 - \frac{(\beta_1 q_1^2 + \beta_2 q_2^2 + 2\gamma q_1 q_2)}{2} \quad (1)$$

Upon translating this model to align with the notation of the current study, the utility function takes the following form:

$$U(q_T, q_I) = a_T q_T + a_I q_I - \frac{(q_T^2 + q_I^2 + 2\gamma q_T q_I)}{2} - P_T q_T + P_I q_I \quad (2)$$

The variables and parameters in this function are explained in Table 1:

Table 1: Notations for the Base Utility Function

$q_T > 0$	Quantity of games produced by the "Traditional" Video Game Studio
$a_T > 0$	Objective valuation of games developed the "Traditional" Video Game Studio
$q_I > 0$	Quantity of games produced by the "Indie" Video Game Studio
$a_I > 0$	Objective valuation of games developed by the "Indie" Video Game Studio
$\beta_i = 1$	Normalized to 1, not directly applied in this model
$\gamma \in [0; 1]$	Substitution rate between Indie and Traditional games, representing a subjective parameter
P_T	Price of games developed by the „Traditional” Studio
P_I	Price of games developed by the „Indie” Studio

Source: Own edit

In this model, it is assumed that $a_T > a_I$, indicating that games produced by traditional firms hold a higher objective valuation. This assumption reflects the notion that traditional firms are responsible for creating blockbuster games, often perceived as higher-quality or higher-budget products. The parameter γ represents consumer substitution, indicating the willingness of consumers to choose one game type over the other. Note that in the extreme case of $\gamma = 1$, the two types of games are perfect substitutes, whereas they are independent products when $\gamma = 0$. Meanwhile, a_T and a_I reflect the objective qualitative differences between the games produced by each type of developer.

3.2. TRADITIONAL MODEL SETUP

Following the framework outlined above, this model considers an oligopoly of Traditional and Indie game developers, each aiming to set optimal prices. The timing of the game is as follows: the developers' pricing decision in Stage 1 is followed by the consumers' quantity choices in Stage 2. Using a standard game-theoretic approach, the solution concept will be subgame-perfect Nash equilibrium, thus the model will be solved using backward induction. In particular, in this setup, the consumers' final choice represents their utility-maximizing move in each possible outcome. By substituting this utility-maximizing, optimal quantity function back into the firms' pricing, the model can derive the equilibrium price (Guo, 2015, 140-141).

In this model, the strategic decision of setting the profit-maximizing prices lies in hand of the developers. The intermediary platform generates revenue through royalty fees, where a given percentage r , of game sales is allocated to the platform, and the remainder $1-r$ percentage is allocated to the developer firms. Based on Peters' industry report (Peters, 2021), the royalty rate paid to the platform per sale is approximately $r=30\%$. For simplicity, this rate will be taken as a fixed parameter in the model rather than a variable factor subject to strategic decisions.

3.3. CONSUMER'S UTILITY MAXIMALIZATION

The optimal quantities for utility maximization can be derived from the partial derivatives of equation (2) with respect to q_T and q_I :

$$\frac{\partial U}{\partial q_T} = a_T - q_T - \gamma q_I - P_T = 0 \quad (3)$$

$$\frac{\partial U}{\partial q_I} = a_I - q_I - \gamma q_T - P_I = 0 \quad (4)$$

By rearranging equations (3) and (4) q_T and q_I can be expressed as:

$$q_T = a_T - \gamma q_I - P_T \quad (5)$$

$$q_I = a_I - \gamma q_T - P_I \quad (6)$$

From equations (5) and (6), it is evident that both the traditional and indie firms' parameters crossinfluence the optimal quantities. The optimal quantity for each firm depends positively on its game's objective valuation and negatively on its price, consistent with economic intuition. Additionally, cross-substitution effects should be considered: depending on the degree of substitution willingness, the quantity produced by one firm has a diminishing effect on the other, as consumers may choose to substitute the competing product.

By substituting equations (5) and (6), the optimal quantities can be expressed as:

$$q_T^* = \frac{a_T - P_T - \gamma(a_I - P_I)}{1 - \gamma^2} \quad (7)$$

$$q_I^* = \frac{a_I - P_I - \gamma(a_T - P_T)}{1 - \gamma^2} \quad (8)$$

These optimal quantities show that each firm's output depends primarily on its own parameters, with an adjustment based on the degree of substitution, reflecting the influence of the competing firm's parameters, as described above.

3.4. INTRODUCTION OF TWO TYPE OF CONSUMERS

We assume there is a mass one of consumers who can be categorized into two distinct groups based on their level of substitution, denoted by γ . A fraction λ of consumers is characterized by $\gamma_L = \gamma < 1$, indicating a high substitution rate. These consumers are more likely to consider playing the alternative type of game, but with the assumption that $\gamma_L \neq 1$, which excludes perfect substitution. This simplification is used to reduce the complexity of outcomes in the analysis. The remaining fraction $1 - \lambda$ is characterized by $\gamma_H = 0$, indicating indifference to the alternative game type in their purchasing decisions. This distinction categorizes them as "Low-type" and "Hig-type" consumers, respectively, based on their willingness to pay. A higher substitution rate causes a lower reservation price due to the appeal of alternative options. Both consumer types have an outside option normalized to zero.

Note that Hernandez (2011) also introduces two consumer groups differing in their level of substitution, with the crucial distinction that the baseline model Hernandez (2011, 175) uses is the Hotelling model of product differentiation.

Implementing the two consumer types leads to the following table:

Table 2: Optimal quantities of the two consumer types

	Low-type	High-type	Summarized
q_T^*	$\frac{a_T - \gamma a_I - P_T + \gamma P_I}{1 - \gamma^2}$	$a_T - P_T$	$Q_T^* = \left[\lambda * \max \left\{ \left(\frac{a_T - \gamma a_I - P_T + \gamma P_I}{1 - \gamma^2} \right); 0 \right\} + (1 - \lambda) * \max \{ (a_T - P_T); 0 \} \right]$
q_I^*	$\frac{a_I - \gamma a_T - P_I + \gamma P_T}{1 - \gamma^2}$	$a_I - P_I$	$Q_I^* = \left[\lambda * \max \left\{ \left(\frac{a_I - \gamma a_T - P_I + \gamma P_T}{1 - \gamma^2} \right); 0 \right\} + (1 - \lambda) * \max \{ (a_I - P_I); 0 \} \right]$

Source: Own edit

Let Q_T^* and Q_I^* represent the aggregate utility-maximizing decisions of each consumer type within a generalized framework. The model incorporates an economic constraint, ensuring non-negative quantities in the aggregate, which leads to the following mathematical constraints:

$$\left(\frac{a_T - \gamma a_I - P_T + \gamma P_I}{1 - \gamma^2} \right) \geq 0 \text{ or } (a_T - P_T) \geq 0 \quad (11)$$

$$\left(\frac{a_I - \gamma a_T - P_I + \gamma P_T}{1 - \gamma^2} \right) \geq 0 \text{ or } (a_I - P_I) \geq 0 \quad (12)$$

Since $1 - \gamma^2$ is always positive due to $0 \leq \gamma < 1$ the constraints for the Indie firm are $a_I - \gamma a_T + \gamma P_T \geq P_I$ or $a_I \geq P_I$ and for the Traditional firm are $a_T - \gamma a_I + \gamma P_I \geq P_T$ or $a_T \geq P_T$. In the following sections, we aim to conduct a scenario analysis on the potential combinations of these constraints, providing insights into each firm's strategic options for targeting different consumer groups.

3.5. POSSIBLE PARAMETER SETUPS

Based on the constraints calculated in the previous chapter, the number of economically feasible setups can be narrowed down using truth tables. The relationships in these economic constraints rely on logical "or" conditions, allowing for the construction of the following truth table, where „T” stands for „True” and „F” stands for „False” (Bikenaga, 2024):

Table 3: Truth table of economic constraints

OR	Traditional firm's economic constrain		Indie firm's economic constrain	
	$a_T - \gamma(a_I - P_I) \geq P_T$	$a_T \geq P_T$	$a_I - \gamma(a_T - P_T) \geq P_I$	$a_I \geq P_I$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
F	F	F	F	F

Source: Own edit

The truth table, excludes the „False” combinations where both type of consumers would chose not to consume. The valid combinations can be linearly combined into nine possible outcomes, as shown in Table 4. Each outcome's feasibility can be evaluated individually. Cells marked in the table indicate combinations that are not feasible due to mathematical contradictions. The following assumptions apply in all cases: $\gamma \in [0; 1[$ and $a_I < a_T$. . In the table, 0^+ indicates that the given term is a strictly positive value.

The outcome labeled "special" leads to a contradiction due to the utility function itself. Specifically, if for both $i = \{T; I\}$, $a_i < P_i$ then no consumers would purchase the product, as substituting these values into Q_i^* results in negative quantities leading to contradiction.

Summarized, Table 4 shows that any combination including $a_i < P_i$ for any $i \in \{T; I\}$ leads to a contradiction. This implies that "low-type" consumers are always served, and the strategic decision for the firms is whether the other group should also be served or not.

Table 4: Possible constraint combinations

$\gamma \in [0; 1[$ $a_I < a_T$	$a_I - \gamma(a_T - P_T) \geq P_I$ $a_I < P_I$	$a_I \geq P_I$ $a_I - \gamma(a_T - P_T) < P_I$	$a_I - \gamma(a_T - P_T) \geq P_I$ $a_I \geq P_I$
$a_T \geq P_T$ $a_T - \gamma(a_I - P_I) < P_T$	$a_I - (0^+) \geq P_I$ $\leftrightarrow$ $a_I < P_I$	$\frac{a_I - P_I}{\gamma} < (a_T - P_T)$ $\frac{a_T - P_T}{\gamma} < (a_I - P_I)$ $\leftrightarrow$ $\frac{a_I - P_I}{\gamma} > (a_I - P_I)$	$a_I \geq a_I - (0^+) \geq P_I$ $a_T \geq P_T > a_T - (0^+)$ $\frac{a_I - P_I}{\gamma} \geq (a_T - P_T)$ $a_I - P_I > \frac{a_T - P_T}{\gamma}$
$a_T < P_T$ $a_T - \gamma(a_I - P_I) \geq P_T$	Special	$a_T - (0^+) \geq P_T$ $\leftrightarrow$ $a_T < P_T$	$a_T - (0^+) \geq P_T$ $\leftrightarrow$ $a_T < P_T$
$a_T \geq P_T$ $a_T - \gamma(a_I - P_I) \geq P_T$	$a_I - (0^+) \geq P_I$ $\leftrightarrow$ $a_I < P_I$	$a_I \geq P_I > a_I - (0^+)$ $a_T \geq a_T - (0^+) \geq P_T$ $\frac{a_I - P_I}{\gamma} < (a_T - P_T)$ $\frac{a_T - P_T}{\gamma} \geq (a_I - P_I)$	$a_I \geq a_I - (0^+) \geq P_I$ $a_T \geq a_T - (0^+) \geq P_T$ $\frac{a_I - P_I}{\gamma} \geq (a_T - P_T)$ $\frac{a_T - P_T}{\gamma} \geq (a_I - P_I)$

Source: Own edit

3.6. SCENARIO ANALYSIS

Based on Table 4, the three possible scenarios are summarized in Table 5. These scenarios outline the conditions under which each firm targets both consumer types or only the high-type consumers. Scenario 1 represents both firms serving both consumer types, while Scenario 2 represents only the Traditional firm serving the high-type consumers, and Scenario 3 is the reverse, where only the Indie firm serves the high-type consumers.

Table 5: Possible scenarios

	Traditional firm	Indie firm
Scenario 1	$a_I - \gamma(a_T - P_T) - P_I \geq 0$	$a_T - \gamma(a_I - P_I) - P_T \geq 0$
Scenario 2	$a_T - \gamma(a_I - P_I) - P_T \geq 0$	$a_I - P_I \geq 0$
Scenario 3	$a_T - P_T \geq 0$	$a_I - \gamma(a_T - P_T) - P_I \geq 0$

Source: Own edit

This study focuses primarily on Scenario 2, where the traditional firm serves both consumer groups whereas the indie firm only serves the high-type consumers. Although analyzing all scenarios would provide a more comprehensive understanding, this is reserved for future research. Scenario 2 is emphasized here based on the intuition that bundling may positively impact indie games, which have objectively inferior products. These games may individually attract a limited audience, but bundling them with blockbuster titles could significantly enhance their demand.

3.7. PROFIT-MAXIMIZING DECISIONS

The profit-maximizing decision of indie firm, based on the condition $a_I - P_I \geq 0$, can be derived from the partial derivative of profit with respect to P_I . Where, the profit is defined as $\Pi_I = q_I * P_I - q_I * c_I$. For simplicity, $c_I = 0$ is assumed, considering it as a sunk production cost:

$$\frac{\partial \Pi_I}{\partial P_I} = (1 - \lambda)(a_I - P_I)P_I \quad (13)$$

$$(1 - \lambda)(a_I - 2P_I) = 0 \quad (14)$$

$$P_I^{2*} = \frac{a_I}{2} \quad (15)$$

$$\Pi_I^{2*} = (1 - \lambda) \left(\frac{a_I}{2} \right)^2 \quad (16)$$

This result implies that if the indie firm targets high-type consumers, the profit-maximizing decision is $P_I^{2*} = \frac{a_I}{2}$, leading to an individual pricing decision independent of the traditional firm's choice. Substituting the optimal price into the profit function shows the achievable profit maximum for the indie firm. Plugging the optimal price back into Equation (8) for the high-type consumer gives the quantity of indie games as: $q_I^*(P_I^{2*}) = \frac{a_I}{2}$.

Following similar steps, the optimal price and profit for the traditional firm can be calculated, given the indie firm's optimal pricing choice of $P_I^{2*} = \frac{a_I}{2}$:

$$\frac{\partial \Pi_T}{\partial P_T} = \lambda * \frac{a_T - P_T - \gamma \left(a_I - \frac{a_I}{2} \right)}{1 - \gamma^2} P_T + (1 - \lambda)(a_T - P_T)P_T = 0 \quad (17)$$

$$P_T^{2*} = \frac{a_T}{2} - \frac{\lambda \gamma \left(\frac{a_I}{2} \right)}{2(1 - \gamma^2 + \lambda \gamma^2)} \quad (18)$$

$$\begin{aligned} \Pi_T^{2*} = & \lambda * \frac{\frac{a_T}{2} + \frac{\lambda\gamma\left(\frac{a_I}{2}\right)}{2(1-\gamma^2 + \lambda\gamma^2)} - \gamma\left(\frac{a_I}{2}\right)}{1-\gamma^2} \left(\frac{a_T}{2} - \frac{\lambda\gamma\left(\frac{a_I}{2}\right)}{2(1-\gamma^2 + \lambda\gamma^2)} \right) \\ & + (1-\lambda) \left(\frac{a_T}{2} + \frac{\lambda\gamma\left(\frac{a_I}{2}\right)}{2(1-\gamma^2 + \lambda\gamma^2)} \right) \left(\frac{a_T}{2} - \frac{\lambda\gamma\left(\frac{a_I}{2}\right)}{2(1-\gamma^2 + \lambda\gamma^2)} \right) \end{aligned} \quad (19)$$

This result provides a generalized solution for the optimal price reaction of traditional firms in Scenario 2. From these results, we are able to conclude that by targeting both consumer types, the traditional firm gains the flexibility to reduce its prices compared to the competing firm's individual solution. The degree of optimal price reduction depends on the strictly positive parameters γ , λ and the cross-effect of the other firm's objective parameter a_I . Furthermore, this analysis reveals that a substitute product with objectively lower quality can allow for a higher optimal price, influenced by the relative weights of consumer types and the degree of substitutability between products.

4. BUNDLING MODEL

4.1. BUNDLING OPTION FOR THE PLATFORM

In the following, the outcomes of the traditional à la carte pricing model described above will be compared to a market where the platform decides to tie the different games by offering them as a bundle. Although, as the previous industry review shows, it is a common practice for platforms to use mixed bundling, this study assumes pure bundling. This assumption is made in order to simplify the model, with no à la carte option for the consumers.

A related but methodologically distinct study Lu and LaRiviere (2024, 19-21) assumes that the platform aims to maximize long-term value by enhancing consumer surplus. While Lu and LaRiviere (2024) examines the subscription effect using empirical methods, this study approaches similar dynamics through a theoretical framework. Their assumption is supported by interviews with key decision-makers at Microsoft, including Spencer (2020) and Makuch (2020). Based on this, a utility maximising combination of games could be chosen, and the price cap $P_{\text{sub cap B}}$ could be optimized by the platform accordingly.

The results of the traditional model setup lead to a mathematically complex problem. Therefore, the following simplifications are introduced: instead of selecting the purely utility-maximizing bundle, the platform sets a bundle comprising of $\{q_T^B = \frac{a_T}{2}; q_T^I = \frac{a_I}{2}\}$. Additionally, another further simplifying assumption is made. Namely, $\gamma = 0,5$. These simplifications facilitate an easier comparison between the two models, though they also lead to more limited conclusions. Without further loss of generality, I will also denote $a_T = a_I * t$, where $t > 1$.

4.2. PLATFORM'S PROFIT-MAXIMIZING DECISION

In the traditional model, videogame developers make the strategic decisions. In this model, however, the platform is allowed to optimize its own profit. As before, a 30% royalty fee is applied, while the remaining 70% is divided according to the developers' game objective qualities in proportion to the parameter a_i .

The platform's profit can be calculated as:

$$\Pi^B = \begin{cases} \lambda * P_B \\ (1 - \lambda) * P_B \\ 1 * P_B \end{cases} \quad (20)$$

where $\lambda * P^B$ represents the profit when only low-type consumers are served, $(1 - \lambda) * P^B$ when only high-type consumers are served, and $1 * P^B$ when both consumer types are served.

To determine the feasibility of these setups, consider the following consumer-type-adjusted utility functions for a bundle:

$$U^L(q_T^B, q_T^L) = a_T q_T^B + a_I q_T^L - \frac{(q_T^{B^2} + q_T^{L^2} + 2\gamma q_T^B q_T^L)}{2} - P_B \quad (21)$$

$$U^H(q_T^B, q_T^L) = a_T q_T^B + a_I q_T^L - \frac{(q_T^{B^2} + q_T^{L^2})}{2} - P_B \quad (22)$$

Similar to the traditional model, it is not feasible to serve only low-type consumers in this setting. This is due to the term $-2\gamma q_T^B q_T^L$, which results in lower utility for low-type consumers compared to high-type consumers for the same bundles. Consequently, the platform's pricing options are limited to either a lower price $P_B^\forall$ for all consumer types or a higher price P_B^H targeting only low-type consumers. The choice of serving everyone will be made if the following condition holds:

$$(1 - \lambda) * P_B^H < 1 * P_B^\forall \quad (23)$$

The various bundle prices for the assumed bundle $\{q_T^B = \frac{a_T}{2}; q_T^L = \frac{a_I}{2}\}$ can be calculated by setting the prices to the highest level where the respective groups still make purchases:

$$P_B^\forall = U^L(q_T^B, q_T^L) = \frac{a_T^2}{2} + \frac{a_I^2}{2} - \frac{\left(\left(\frac{a_T}{2}\right)^2 + \left(\frac{a_I}{2}\right)^2 + 2\gamma \left(\frac{a_T}{2}\right) \left(\frac{a_I}{2}\right)\right)}{2} \quad (24)$$

$$P_B^H = U^H(q_T^B, q_T^L) = \frac{a_T^2}{2} + \frac{a_I^2}{2} - \frac{\left(\left(\frac{a_T}{2}\right)^2 + \left(\frac{a_I}{2}\right)^2\right)}{2} \quad (25)$$

Using the assumptions from Section 4.1, and after some simplification steps, we obtain:

$$P_B^\forall = \frac{a_I^2 t^2}{2} + \frac{a_I^2}{2} - \frac{\left(\left(\frac{a_I t}{2}\right)^2 + \left(\frac{a_I}{2}\right)^2 + \left(\frac{a_I t}{2}\right) \left(\frac{a_I}{2}\right)\right)}{2} = \frac{1}{8} a_I^2 (3t^2 + 3 - t) \quad (26)$$

$$P_B^H = \frac{a_I^2 t^2}{2} + \frac{a_I^2}{2} - \frac{\left(\left(\frac{a_I t}{2}\right)^2 + \left(\frac{a_I}{2}\right)^2\right)}{2} = \frac{1}{8} a_I^2 (3t^2 + 3) \quad (27)$$

Substituting equations (26) and (27) into equation (23) leads to the following condition:

$$\lambda > \frac{t}{3t^2 + 3} \quad (28)$$

Recall that $t > 1$. Therefore, the inequality is generally satisfied because the right-hand side of the inequality is a monotonically decreasing function with a maximum value of $\frac{1}{6}$. Thus, for $\lambda > \frac{1}{6}$ the platform will always serve both consumer types. Therefore, a sufficient ratio $\lambda > \frac{1}{6}$ of low-type consumers always results in the following platform profit, using the industry's royalty fee:

$$\Pi_B^V = \frac{1}{8} a_I^2 (3t^2 + 3 - t) * 0,3 \quad (29)$$

The developer firms' profits in the bundling model $\{\Pi_I^{B^V}; \Pi_I^{B^V}\}$ can be determined by applying the proportional rule and the remainder of the royalty fee as follows:

$$\Pi_T^{B^V} = \frac{t}{t+1} * \frac{1}{8} a_I^2 (3t^2 + 3 - t) * 0,7 \quad (30)$$

$$\Pi_I^{B^V} = \frac{1}{t+1} * \frac{1}{8} a_I^2 (3t^2 + 3 - t) * 0,7 \quad (31)$$

5. COMPARISON

5.1. PROFIT MAXIMUMS IN THE TRADITIONAL MODEL

Using the assumptions from Section 4.1, the optimal price P_T^T for the traditional game studio, as derived in equation (18), simplifies to:

$$P_T^T = \frac{1}{2} a_I \left(t - \frac{\lambda}{\lambda + 3} \right) \quad (32)$$

Consequently, as demonstrated in Appendix 1, the optimal profit for the traditional firm, based on equation (19) and incorporating a 30% royalty fee, can be expressed as:

$$\Pi_T^T = \frac{1}{12(\lambda + 3)} (a_I)^2 ((\lambda + 3)t - \lambda)^2 * 0,7 \quad (33)$$

By substituting the assumption $\gamma = 0,5$ into equation (16) and accounting for the royalty fee, the profit for the indie firm becomes:

$$\Pi_I^T = (1 - \lambda) \frac{a_I^2}{8} \quad (34)$$

Combining these results yields the comparable profit for the platform as follows:

$$\Pi_P^T = 0,3 * \left[\frac{1}{12(\lambda + 3)} (a_I)^2 ((\lambda + 3)t - \lambda)^2 + (1 - \lambda) \frac{a_I^2}{8} \right] \quad (35)$$

5.2. SUPPLIER SURPLUS IN THE TWO MODEL

To determine the potential benefits of introducing bundles for the two firms and the platform, we need to compare the profit of the firms and platforms in the traditional and bundling models. Our primary focus is on evaluating the platform's profit, as demonstrating a profit increase would confirm that bundling is a rational choice for the platform. By dividing the royalty fee of 30% on both sides of the inequality, given that the royalty fee is constant in both models, we can proceed with a simplified comparison:

$$\frac{1}{8} a_I^2 (3t^2 + 3 - t) \geq \left[\frac{1}{12(\lambda + 3)} (a_I)^2 ((\lambda + 3)t - \lambda)^2 + (1 - \lambda) \frac{a_I^2}{8} \right] \quad (36)$$

Based on the proof in Appendix 2, this inequality holds for all values $0 \leq \lambda \leq 1$ and $t > 1$, indicating that introducing the bundle is advantageous for the platform. Similarly, we can compare the indie firm's profits in the two models as follows:

$$\frac{1}{t + 1} * \frac{1}{8} a_I^2 (3t^2 + 3 - t) \geq (1 - \lambda) \frac{a_I^2}{8} \quad (37)$$

This inequality always holds because $\frac{3t^2+3-t}{t+1} > 1 \geq (1-\lambda)$ for every $0 \leq \lambda \leq 1$ and $t > 1$. This result follows from the fact that:

$$\frac{3t^2+3-t}{t+1} = \frac{3t^2+3+1}{t+1} - \frac{t+1}{t+1} = \frac{3t^2+4}{t+1} - 1 \geq 2.5 \quad (38)$$

where $\frac{3t^2+4}{t+1}$ is monotonically increasing, with a minimum value of 2.5. This result demonstrates that the indie firm always benefits from the bundling option.

Finally, we compare the traditional firm's profit similarly to the examples above:

$$\frac{t}{t+1} \frac{1}{8} (3t^2+3-t) - \left[\frac{1}{12(\lambda+3)} (a_I)^2 ((\lambda+3)t - \lambda)^2 + (1-\lambda) \frac{1}{8} \right] \geq 0 \quad (39)$$

Due to the complexity of this inequality, a numerical approach was employed for this study. Using RStudio, the following results were calculated and summarized in Table 6:

Table 6: Traditional firm's profit differences

t	lambda 0	lambda 0.05	lambda 0.1	lambda 0.15	lambda 0.2	lambda 0.25	lambda 0.3	lambda 0.35	lambda 0.4	lambda 0.45	lambda 0.5	lambda 0.6	lambda 0.7	lambda 0.8	lambda 0.9	lambda 1
1	-0.0625	-0.052151639	-0.041935484	-0.031845238	-0.021875	-0.012019231	-0.002272727	0.007369403	0.018911765	0.026358696	0.035714286	0.054166667	0.072297297	0.090131579	0.107692308	0.125
1.1	-0.065416667	-0.055109973	-0.044935484	-0.034886905	-0.024958333	-0.015144231	-0.005439394	0.00416107	0.013661765	0.023067029	0.032380952	0.05075	0.068797297	0.086548246	0.104025641	0.12125
1.2	-0.067727273	-0.057545579	-0.04749609	-0.037572511	-0.027768939	-0.018079837	-0.0085	0.000975464	0.010351159	0.019631423	0.028820346	0.046939394	0.064736691	0.08223764	0.099465035	0.116439394
1.3	-0.069184783	-0.059211422	-0.049370266	-0.039655021	-0.030059783	-0.020579013	-0.01120751	-0.00194038	0.007226982	0.016298913	0.025279503	0.042981884	0.060362515	0.077446796	0.094257525	0.110815217
1.4	-0.069583333	-0.059901639	-0.050352151	-0.040928571	-0.031625	-0.022435897	-0.013356061	-0.004380597	0.004495098	0.013275362	0.021964286	0.039083333	0.055880631	0.072381579	0.088608974	0.104583333
1.5	-0.06875	-0.059443306	-0.050268817	-0.041220238	-0.032291667	-0.023477564	-0.014772727	-0.006172264	0.002328431	0.010733696	0.019047619	0.035416667	0.051463964	0.067214912	0.082692308	0.097916667
1.6	-0.066538462	-0.057690101	-0.048973945	-0.0403837	-0.031913462	-0.023557692	-0.015311189	-0.007169059	0.000873303	0.008820234	0.016675824	0.032128205	0.047258836	0.062093117	0.076653846	0.090961538
1.7	-0.062824074	-0.05451738	-0.046342891	-0.038294312	-0.030365741	-0.022551638	-0.014846801	-0.007246338	0.000254357	0.007659622	0.014973545	0.029342593	0.04338989	0.057140838	0.070618234	0.083842593
1.8	-0.0575	-0.049818306	-0.042268817	-0.034845238	-0.027541667	-0.020352564	-0.013272727	-0.006297264	0.000578431	0.007358696	0.014047619	0.027166667	0.039963964	0.052464912	0.064692308	0.076666667
1.9	-0.050474138	-0.043500777	-0.036659622	-0.029944376	-0.023349138	-0.016868369	-0.010496865	-0.004229735	0.001937627	0.008009558	0.013990148	0.025692529	0.037073159	0.048157441	0.05896817	0.069525862
2	-0.041666667	-0.035484973	-0.029435484	-0.023511905	-0.017708333	-0.012019231	-0.006439394	-0.00096393	0.004411765	0.009692029	0.014880952	0.025	0.034797297	0.044298246	0.053525641	0.0625
2.1	-0.031008065	-0.025701371	-0.020526882	-0.015478303	-0.010549731	-0.005735629	-0.001030792	0.003569672	0.008070367	0.012475631	0.016789555	0.025158602	0.033205899	0.040956848	0.048434243	0.055658802
2.2	-0.0184375	-0.014089139	-0.009872984	-0.005782738	-0.0018125	0.002043269	0.005789773	0.009431903	0.012974265	0.016421196	0.019776786	0.026229167	0.032359797	0.038194079	0.043754808	0.0490625
2.3	-0.003901515	-0.000594821	0.002579668	0.005628247	0.008556818	0.011370921	0.014075758	0.016676221	0.019176916	0.021582181	0.023896104	0.028265152	0.032312449	0.036063397	0.039540793	0.042765152
2.4	0.012647059	0.014828753	0.016878242	0.018801821	0.020605392	0.022294495	0.023874332	0.025349795	0.026762549	0.028005754	0.029194678	0.031313725	0.033111023	0.034611971	0.035839367	0.036813725
2.5	0.03125	0.032223361	0.033064516	0.033779762	0.034375	0.034855769	0.035227273	0.035494403	0.035661765	0.035733696	0.035714286	0.035416667	0.034797297	0.033881579	0.032692308	0.03125
2.6	0.051944444	0.051626138	0.051175627	0.050599206	0.049902778	0.04909188	0.048171717	0.047147181	0.046022876	0.04480314	0.043492063	0.040611111	0.037408408	0.033909357	0.030136752	0.026111111
2.7	0.074763514	0.073070208	0.071244696	0.069293275	0.067221847	0.065035949	0.062740786	0.06034125	0.057841945	0.055247209	0.052561133	0.049301018	0.040977477	0.034728426	0.028205821	0.02143018
2.8	0.099736842	0.096585203	0.093301358	0.089891604	0.086361842	0.082717611	0.078964115	0.075106245	0.071148607	0.067095538	0.062951128	0.054403509	0.045334139	0.036368421	0.02692915	0.017236842
2.9	0.126891026	0.12219772	0.117372208	0.112420788	0.107349359	0.102163462	0.096868298	0.091468762	0.085969457	0.080374721	0.074688645	0.063057692	0.05110499	0.038855938	0.026333333	0.013557692
3	0.15625	0.149931694	0.143481183	0.136904762	0.130208333	0.123397436	0.116477273	0.109452736	0.102328431	0.095108696	0.087797619	0.072916667	0.057713964	0.042214912	0.026442308	0.010416667

Source: Own edit

From this table, we can conclude that if t and λ are sufficiently high, bundling is also advantageous for the traditional firm. However, smaller parameter values result in a relative loss compared to the traditional model.

5.3. CONCLUSIONS

As demonstrated in Section 5.2, both the platform and the indie firm are consistently able to benefit from the bundling model. For the indie firm, the primary advantage is the expanded market reach that bundling enables. In the compared traditional model scenario, the indie firm targets only a segment of consumers, whereas bundling allows it to access a broader audience. The platform sets a price that leverages the perceived quality of traditional games, which can also lead to additional profits for the indie firm.

The platform benefits from bundling because the discount offered in the bundle attracts more consumers. By setting prices that reflect both firms' characteristics, the platform can

maximize its profit, achieving a higher level of profitability than would be possible with individually set prices.

While the indie game developer and the platform always achieve higher profits under bundling compared to the traditional model, this is not necessarily true for the traditional firm. For the traditional firm, if the proportion of "low-type" consumers is low and the quality advantage of its games is marginal, bundling can result in a relative loss. In cases where the games are of relatively lower quality, the firm can attract enough consumers with lower prices. However, an increase in game quality requires a higher price, which may deter consumers, a situation that bundling can help mitigate. On the other hand, a lower proportion of high-type consumers reduces profits in the platform setup, as shown in equation (28). The platform will opt to serve both consumer types only if the proportion of low-type consumers is sufficiently high. This is because low-type consumers, given their greater willingness to substitute, are willing to pay less, making them a less profitable segment. Therefore, a larger base of low-type consumers is necessary to achieve the additional benefits of setting prices at a level that appeals to them.

6. SUMMARY

This study presents a theoretical framework for evaluating the benefits of bundling developer-supplied games on video game platforms. By comparing pricing strategies in both traditional and bundling models, the research is able to demonstrate that bundling can increase platform profits while broadening market reach for content developers.

The traditional model explores the underlying factors of individual firm pricing strategies and their interdependencies, while the bundling model captures the platform's profit-maximizing strategy through bundled price discounts. Due to the mathematical complexity of the Quasilinear Quadratic Utility Model combined with consumer differentiation, the findings are limited to a controlled environment where the consumer substitution parameter remains constant. In this context of fixed consumer differentiation, the study finds that bundling not only enhances platform profit but also supports the developer's profitability.

Future research could expand these findings by examining more every and more generalized scenarios to capture the broader effects of bundling on supplier surplus. Additionally, it would be valuable to investigate welfare effects on consumers to assess the overall social welfare impact of the proposed bundling model. Further generalizing assumptions, such as zero production cost or pure bundling, could enhance the realism of the models.

7. BIBLIOGRAPHY

Literature

- Belleflamme, P., Peitz, M., (2015): Industrial organization: markets and strategies, Cambridge, 259-281.
- Ikenaga, B., (2019): Truth tables, Millersville University, <https://sites.millersville.edu/bikenaga/math-proof/truth-tables/truth-tables.html>. 2024.10.27.
- Chan, T. Y. M., Zhang, Y., Yeung, T. Y., (2020): Estimating The Effect Of Subscription based Streaming Services On The Demand For Game Consoles, Cornell University, arXiv:2012.12704.
- Choné, P., Linnemer, L., (2020): Linear demand systems for differentiated goods: Overview and user's guide, International Journal of Industrial Organization, 73, 102663.
- Goh, E., Al-Tabbaa, O., Khan, Z., (2023): Unravelling the complexity of the Video Game Industry: An integrative framework and future research directions, Telematics and Informatics Reports, 1-4.
- Guo, R., (2015): Cross-Border Behaviors as Games. In: Cross-Border Management, Springer, 135–156.
- Hernandez, M. A., (2011): Nonlinear pricing and competition intensity in a Hotelling-type model with discrete product and consumer types, Economics letters, 110(3), 174-177.
- Lu, C., H., LaRiviere, J., (2024): Welfare Impacts of Subscriptions for Digital Goods: The Case of Video Games, ResearchGate, https://www.researchgate.net/publication/378398479_Welfare_Impacts_of_Subscriptions_for_Digital_Goods_The_Case_of_Video_Games, 2024.10.27.
- Marchand, A., Hennig-Thurau, T., (2013): Value Creation in the Video Game Industry: Industry Economics, Consumer Benefits, and Research Opportunities. Journal of Interactive Marketing, 27(3), 141–157.
- Nemeslaki, A., Urbán, Z., Trestyén, A., (2008): Alapvető e-business-modellek működése és magyar-országi elterjedtségük, Vezetéstudomány-Budapest Management Review, 39 (12), 4-15.
- Osathanunkul, C., (2015): A classification of business models in video game industry, International Journal of Management Cases, 17 (1) 35-44.
- Rochet, J. C., Tirole, J., (2006). Two-sided markets: a progress report, The RAND journal of economics, 37(3), 645-667.
- Singh, N., Vives, X. (1984): Price and Quantity Competition in a Differentiated Duopoly, The RAND Journal of Economics, 15(4).
- Tan, W. K., Chiu, P. H., (2024): Theory of consumption value: A lens to examine the use and continual use intention of online game subscription services, Computers in Human Behavior, 160, 1-2.
- Zackariasson, P. (2012): The Video Game Industry: Formation, Present State, and Future, Routledge, 1-23.

Sources

- Statista, (2024): Number of video game users worldwide from 2019 to 2029, <https://www.statista.com/outlook/dmo/digital-media/video-games/worldwide>, 2024.10.27.

Web sources

- Makuch, E., (2020): Xbox game pass is not a big money-maker right now, but microsoft is thinking long term, <https://www.gamespot.com/articles/xbox-game-pass-is-not-a-big-moneymaker-right-now-b/1100-6480241/>, 2024.10.27.
- Patel, N., (2020): Microsoft's phil spencer on launching the new xbox and the future of games, <https://www.theverge.com/21611412/microsoft-phil-spencer-interview-xbox-series-x-launch-decoder-podcast>, 2024.10.27.
- Peters, J., (2021): Microsoft will let devs keep every penny their windows app makes — unless it's a game; Available at <https://www.theverge.com/2021/6/24/22549222/microsoft-store-developers-windows-11-revenues-games>, 2024.10.30.
- PlayStation; (2024): Official american website of Playstaton, <https://www.playstation.com/en-us/ps-plus/>, 2024.10.27.
- XBOX, (2024): Official american of XBOX, <https://www.xbox.com/en-US/xbox-game-pass>, 2024.10.27.

APPENDIX

APPENDIX 1: SIMPLIFICATION OF EQUATION (19):

1. Initial profit function:

$$P_T^T = \frac{1}{2}a_1 \left(t - \frac{\lambda}{\lambda+3} \right)$$

2. Profit expression Π_T^T :

$$\begin{aligned} \Pi_T^T = & \lambda \cdot \frac{\frac{a_I t}{2} - \frac{1}{2}a_I \left(\frac{\lambda}{\lambda+3} \right) - \frac{a_I}{4}}{1 - 0.25} \cdot \frac{1}{2}a_I \left(t - \frac{\lambda}{\lambda+3} \right) \\ & + (1 - \lambda) \left(\frac{a_I t}{2} + \frac{a_I}{2} \frac{\lambda}{\lambda+3} \right) \left(\frac{a_I t}{2} - \frac{a_I}{2} \frac{\lambda}{\lambda+3} \right) \end{aligned}$$

3. Expand Terms and use $(a+b)(a-b) = (a^2 - b^2)$

$$\Pi_T^T = \left[\frac{2}{3}\lambda a_I t + \frac{2}{3}\frac{\lambda^2}{\lambda+3}a_I - \frac{1}{3}\lambda a_I \right] \left[\frac{1}{2}a_I \left(t - \frac{\lambda}{\lambda+3} \right) \right] + (1 - \lambda) \left[\frac{a_I^2 t^2}{4} - \frac{a_I^2}{4} \frac{\lambda^2}{(\lambda+3)^2} \right]$$

4. Open the brackets:

$$\begin{aligned} \Pi_T^T = & \frac{2}{6}\lambda a_I^2 t^2 + \frac{2}{6}\frac{\lambda^2}{\lambda+3}a_I^2 t - \frac{1}{6}\lambda a_I^2 t - \frac{2}{6}\frac{\lambda^2}{\lambda+3}a_I^2 t - \frac{2}{6}\frac{\lambda^3}{(\lambda+3)^2}a_I^2 + \frac{1}{6}\frac{\lambda^2}{\lambda+3}a_I^2 \\ & + \frac{1}{4}a_I^2 t^2 - \frac{1}{4}\frac{\lambda^2}{(\lambda+3)^2}a_I^2 - \frac{1}{4}\lambda a_I^2 t^2 + \frac{1}{4}\frac{\lambda^3}{(\lambda+3)^2}a_I^2 \end{aligned}$$

5. Combine alike terms:

$$\Pi_T^T = \frac{1}{12}\lambda a_I^2 t^2 - \frac{1}{6}\lambda a_I^2 t - \frac{1}{12}\frac{\lambda^3}{(\lambda+3)^2}a_I^2 + \frac{1}{6}\frac{\lambda^2}{\lambda+3}a_I^2 + \frac{1}{4}a_I^2 t^2 - \frac{1}{4}\frac{\lambda^2}{(\lambda+3)^2}a_I^2$$

6. Split up the given fraction:

$$\frac{1}{6}\frac{\lambda^2}{\lambda+3}a_I^2 = \frac{1}{6}\frac{\lambda^2(\lambda+3)}{(\lambda+3)^2}a_I^2 = \frac{1}{6}\frac{\lambda^3}{(\lambda+3)^2}a_I^2 + \frac{3}{6}\frac{\lambda^2}{(\lambda+3)^2}a_I^2$$

7. Substitute the split up terms:

$$\Pi_T^T = \frac{1}{12}\lambda a_I^2 t^2 - \frac{1}{6}\lambda a_I^2 t + \frac{1}{12}\frac{\lambda^3}{(\lambda+3)^2}a_I^2 + \frac{1}{4}a_I^2 t^2 + \frac{3}{12}\frac{\lambda^2}{(\lambda+3)^2}a_I^2$$

8. Simplify the given fraction:

$$\begin{aligned} \frac{1}{12}\frac{\lambda^3}{(\lambda+3)^2}a_I^2 + \frac{3}{12}\frac{\lambda^2}{(\lambda+3)^2}a_I^2 &= \frac{1}{12}a_I^2 \left[\frac{\lambda^3}{(\lambda+3)^2} + \frac{3\lambda^2}{(\lambda+3)^2} \right] \\ &= \frac{1}{12}a_I^2 \frac{\lambda^3 + 3\lambda^2}{(\lambda+3)^2} = \frac{1}{12}a_I^2 \frac{\lambda^2(\lambda+3)}{(\lambda+3)^2} = \frac{1}{12}a_I^2 \frac{\lambda^2}{\lambda+3} \end{aligned}$$

9. Substitute the simplified fraction:

$$\Pi_T^T = \frac{1}{12} \lambda a_I^2 t^2 - \frac{1}{6} \lambda a_I^2 t + \frac{1}{12} a_I^2 \frac{\lambda^2}{\lambda + 3} + \frac{1}{4} a_I^2 t^2$$

10. Factorize using common terms:

$$\Pi_T^T = \frac{1}{12} a_I^2 \left[\lambda t^2 - 2\lambda t + \frac{\lambda^2}{\lambda + 3} + 3t^2 \right]$$

11. Express in Terms of $(\lambda + 3)$:

$$\Pi_T^T = \frac{1}{12} \frac{a_I^2}{\lambda + 3} [\lambda t^2 (\lambda + 3) - 2\lambda t (\lambda + 3) + \lambda^2 + 3t^2 (\lambda + 3)]$$

12. Factorize inside the bracket using $(\lambda + 3) t$:

$$\Pi_T^T = \frac{1}{12} \frac{a_I^2}{\lambda + 3} [(\lambda + 3) t [\lambda t - 2\lambda + 3t] + \lambda^2]$$

13. Simplify the inner bracket:

$$\lambda t - 2\lambda + 3t = t(\lambda + 3) - 2\lambda$$

14. Substitute the simplified term:

$$\Pi_T^T = \frac{1}{12} \frac{a_I^2}{\lambda + 3} [(\lambda + 3)^2 t^2 - 2\lambda (\lambda + 3) t + \lambda^2]$$

15. Compact Form using $(a + b)^2 = a^2 + b^2 + 2ab$:

$$\Pi_T^T = \frac{1}{12} \frac{a_I^2}{\lambda + 3} [(\lambda + 3) t - \lambda]^2$$

APPENDIX 2: PROOF OF INEQUALITY (35)

1. Initial Inequality

$$\frac{1}{8}a_I^2(3t^2 + 3 - t) \geq \frac{1}{12(\lambda + 3)}a_I^2((\lambda + 3)t - \lambda)^2 + (1 - \lambda)\frac{a_I^2}{8}$$

2. Subtract $(1 - \lambda)\frac{a_I^2}{8}$ from both sides:

$$\frac{1}{8}a_I^2(3t^2 + 3 - t - (1 - \lambda)) \geq \frac{1}{12(\lambda + 3)}a_I^2((\lambda + 3)t - \lambda)^2$$

3. Multiply both sides by $\frac{24}{a_I^2}$:

$$3(3t^2 + 3 - t - (1 - \lambda)) \geq \frac{2}{\lambda + 3}((\lambda + 3)t - \lambda)^2$$

4. Expand terms on both sides:

$$9t^2 + 6 - 3t + 3\lambda \geq \frac{2(\lambda + 3)^2 t^2}{\lambda + 3} + \frac{2\lambda^2}{\lambda + 3} - \frac{4(\lambda + 3)t\lambda}{\lambda + 3}$$

5. Multiply both sides by $(\lambda + 3)$:

$$(\lambda + 3)(9t^2 + 6 - 3t + 3\lambda) \geq 2(\lambda + 3)^2 t^2 + 2\lambda^2 - 4(\lambda + 3)t\lambda$$

6. Expand the left-hand side:

$$9(\lambda + 3)t^2 + 6(\lambda + 3) - 3(\lambda + 3)t + 3\lambda(\lambda + 3) \geq 2(\lambda + 3)^2 t^2 + 2\lambda^2 - 4(\lambda + 3)t\lambda$$

7. Further expand terms on the left-hand side:

$$9\lambda t^2 + 27t^2 + 6\lambda + 18 - 3\lambda t - 9t + 3\lambda^2 + 9\lambda \geq 2\lambda^2 t^2 + 12\lambda t^2 + 18t^2 + 2\lambda^2 - 4\lambda^2 t - 12\lambda t$$

8. Combine like terms on both sides:

$$(9\lambda + 27)t^2 + (6\lambda + 18) - (3\lambda + 9)t + (3\lambda^2 + 9\lambda) \geq 2\lambda^2 t^2 + 12\lambda t^2 + 18t^2 + 2\lambda^2 - 4\lambda^2 t - 12\lambda t$$

9. Further simplify by combining terms involving t^2 , t , and constant terms:

$$(9\lambda + 27 - 2\lambda^2 - 12\lambda - 18)t^2 + (-3\lambda - 9 + 4\lambda^2 + 12\lambda)t + (3\lambda^2 + 9\lambda - 2\lambda^2) \geq 0$$

10. Simplification of the terms:

$$(-2\lambda^2 - 3\lambda + 9)t^2 + (4\lambda^2 + 9\lambda - 9)t + (\lambda^2 + 9\lambda) \geq 0$$

11. Proving $(-2\lambda^2 - 3\lambda + 9)t^2$ is always positive:

$$-2\lambda^2 - 3\lambda \geq -5 \text{ due to the assumption } 0 \geq \lambda \geq 1$$

$$t^2 > 0 \text{ due to the assumption } t > 1$$

12. Since $(-2\lambda^2 - 3\lambda + 9)t^2$ is always positive the following holds with the assumption $t > 1$:

$$(-2\lambda^2 - 3\lambda + 9)t^2 \geq -(2\lambda^2 - 3\lambda + 9)t$$

13. Proving that a smaller function still holds the inequality proves that the original inequality holds for every λ and t

$$(-2\lambda^2 - 3\lambda + 9)t^2 + (4\lambda^2 + 9\lambda - 9)t + (\lambda^2 + 9\lambda) \geq (-2\lambda^2 - 3\lambda + 9)t + (4\lambda^2 + 9\lambda - 9)t + (\lambda^2 + 9\lambda) \geq 0$$

$$(-2\lambda^2 - 3\lambda + 9)t^2 + (4\lambda^2 + 9\lambda - 9)t + (\lambda^2 + 9\lambda) \geq (2\lambda^2 + 6\lambda)t + (\lambda^2 + 9\lambda) \geq 0$$