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FACULTY OF MECHANICAL ENGINEERING
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SCIENTIFIC STUDENT CONFERENCE THESIS

Non-Linear Stability Studies of Unsymmetrically Supported Arches

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Statement of the supervisor

I, the undersigned, László Péter Kiss the lecturer in the Institute of Applied Mechanics at the University of Miskolc I consider the Scientific Student thesis (TDK thesis) entitled Non-Linear Stability Studies of Unsymmetrically Supported Arches to be suitable for submission.

Miskolc, 04 November 2024



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Declaration of authenticity

We, the undersigned Muhammad Hamza Daud and Ali Haider, students at the Faculty of Mechanical Engineering and Informatics of the University of Miskolc, declare and certify with our signature that we have prepared this thesis ourselves, we have carried out the examinations described therein, unless we indicate otherwise, and we have achieved the results described ourselves. We used data and information only from the sources listed in the bibliography. Any part that we have taken from another source, either verbatim or in the same sense, but reworded, is clearly marked with the source.

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1. INTRODUCTION

Curved structural members have long- and commonly been used in engineering applications by civil and mechanical engineers. The reason is their favourable behaviour under various load conditions. Such members, in general, serve as stiffeners, for instance, in roof structures or in arch bridges. So in many applications, arches are more suitable than straight beams. The centroidal axis of arches has an initial curvature. For circular members, this curvature is constant.

With new technologies ever being introduced, the use of non-homogeneous materials is getting more and more widespread – composites, laminates, sandwich structures, functionally graded distributions. The possible advantages of such structural members are the reduced weight, improved corrosion and fatigue resistance and higher strength. So there is a need to establish models predicting the behaviour of members made of these new materials.

The literature is a bit inconsistent when it is about distinguishing between shallow and deep arches. Some sources describe a limit in the span/altitude ratio or the included angle. We will agree with the later one this time. So shallowness requires the included angle be less than 0.5π [1]. However, the applicability of a model cannot accurately be limited in advance.

The number of open literature results is numerous for the buckling behaviour of beams or arches as it has been under the spotlight for quite a while – see books [2–4] for a comprehensive summary.

The very first results for buckling are attributed to Euler who published his well-known results for the critical (buckling) load of straight bars under compression in 1759 [5]. It is known that when a slender straight column is under compression, buckling can occur. It means that, at a certain load level, the member has multiple equilibrium shapes and can move from one to another due to a small perturbation. This phenomenon is to be avoided as the sudden change in the geometry results in an unwanted increase in the inner forces and can lead to failure. In Fig. 1, the black line is the initial shape and the red dashed is the buckled one.

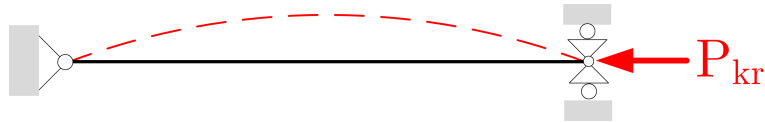


FIGURE 1. The classical Euler column

Since Euler's pioneering finding, a bunch of findings have been published for the buckling of various members. Early arch models, like [6], assumed the inextensibility of the centroidal axis. Later, it turned out [7, 8] that simplification dangerously overestimates the maximum allowable load. Afterwards, article [9] presents exact analytical solutions for fixed-fixed arches with including the extensibility of the centroidal axis. Furthermore, thesis [10] also deserve notable mention as it determines the critical loads given that the deep members are subjected to a radial dead load, whose Fourier series is known.

There are also available results about the application of finite element models. Publication [11] focuses on both deep and shallow arches, using the hypotheses of shallow arches. Multiple strain-displacement assumptions are tested and comes to the conclusion that the use of shallow elements for deep arches does not yield reliable results. Later, the author improved the theory [12] for deep members. In work [13], 3D continuum elements are discretized to find the buckling loads numerically.

In recent years, there was an intensive progress about various shallow arch buckling problems. A large number of articles have been published – see [14–21]. These all assume in-plane stability. There are various approaches and assumptions. Some of these present solutions for homogeneous or functionally graded members; the load is either concentrated or distributed along the centroidal axis; the cross-section geometry is either constant over the full arch or constant in segments, and surely, there are various support arrangements.

Based on the above, it turns out that the in-plane stability of nonhomogeneous fixed-pinned arches under an arbitrarily positioned concentrated radial load is still an open question in the professional literature. So this research aims to tackle the mentioned issue. By assumption, there is no torsion and the cross-sections rotate as rigid bodies and remain perpendicular to the centroidal axis. The one-dimensional continuous beam model is non-linear geometrically by assuming moderately large rotations and is based on the single layer Euler-Bernoulli hypothesis. The material behaviour is linearly elastic, isotropic and the material composition can vary along the thickness of the arch and is the same in every single cross-section.

The text is organized in seven sections. The Introduction overviews the literature background in short. Moreover, the aims of this work are also clarified. It is then followed by Section Basic equations. The kinematical hypothesis and the constitutive equation are shown the reader there. In the Section Governing equations, the principle of virtual work is used to find the linear differential equations of static equilibrium. Closed-form solutions are presented. A non-linear relationship is set between the external load and strain parameters. In Section Results; the evaluation of the model is delivered through parametric studies. Section Conclusions gathers the most important properties and results. Finally, the Appendix and References conclude the work.

2. BASIC EQUATIONS

2.1. Kinematical and constitutive equations.

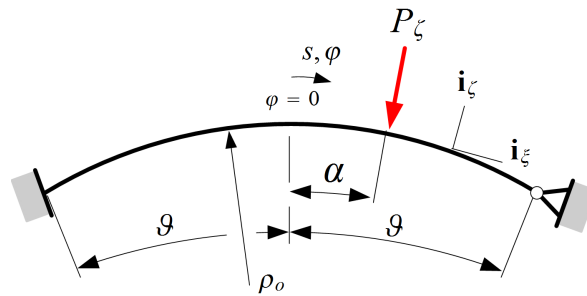


FIGURE 2. 1D beam model

For the mentioned arch (curved beam) problem, a new mechanical model is established [22]. The equations are given in the local curvilinear base. The coordinate system is attached to the undeformed centroidal axis of initial constant radius ρ_o of the arch and is determined by unit vectors $\mathbf{i}_\xi; \mathbf{i}_\eta; \mathbf{i}_\zeta$. Coordinate axes ξ, ζ lay in the plane of the centroidal axis - the former is tangential and the later one is normal in direction. The third (cross-sectional) coordinate η is perpendicular to the later two. The angle coordinate is φ and the arc coordinate is $s = \rho_o \varphi$. For practical reasons it is sometimes worthy of changing derivatives with respect to s to derivatives with respect to φ . For the sake of brevity, the following notational convention is introduced:

$$\frac{d^n(\dots)}{ds^n} = \frac{1}{\rho_o^n} \frac{d^n(\dots)}{d\varphi^n} = \frac{1}{\rho_o^n} (\dots)^{(n)} ; \quad n \in \mathbb{Z} . \quad (1)$$

In this local system, the following relations hold:

$$\frac{d\mathbf{i}_\xi}{ds} = -\frac{1}{\rho_o} \mathbf{i}_\zeta , \quad \frac{d\mathbf{i}_\zeta}{ds} = \frac{1}{\rho_o} \mathbf{i}_\xi . \quad (2)$$

At the same time, the Hamilton operator is

$$\nabla = \frac{\rho_o}{\rho_o + \zeta} \frac{\partial}{\partial s} \mathbf{i}_\xi + \frac{\partial}{\partial \eta} \mathbf{i}_\eta + \frac{\partial}{\partial \zeta} \mathbf{i}_\zeta . \quad (3)$$

Material description will be used during the investigations. Out-of-plane motion is restricted so there is no torsion. The arch cross-sections are assumed to rotate as rigid bodies and remain perpendicular to the centroidal axis during deformations (Euler-Bernoulli hypothesis). Thus, the arch displacement field becomes

$$\mathbf{u} = \mathbf{u}_o + \psi_{o\eta} \zeta \mathbf{i}_\xi = w_o \mathbf{i}_\zeta + (u_o + \psi_{o\eta} \zeta) \mathbf{i}_\xi \quad (4)$$

with

$$\mathbf{u}_o = u_o \mathbf{i}_\xi + w_o \mathbf{i}_\zeta$$

being the displacement vector and $\psi_{o\eta}$ is the rotation on the centroidal axis. The later one is

$$\psi_{o\eta} = -\frac{dw_o}{ds} \quad (5)$$

in terms of the displacements and follows from the Ψ tensor of infinitesimal rotations

$$\Psi = \frac{1}{2} (\mathbf{u} \circ \nabla - \nabla \circ \mathbf{u}) , \quad \psi_\eta = \psi_\eta|_{\zeta=0} = \psi_{o\eta} = \mathbf{i}_\xi \cdot \Psi \cdot \mathbf{i}_\eta = -\frac{dw_o}{ds} . \quad (6)$$

Moreover, the curvature is

$$\kappa_o = -\frac{d^2 w_o}{ds^2} . \quad (7)$$

As regards the strains, non-linear terms will be kept in the Green-Lagrange strain tensor $\mathbf{E}$:

$$\mathbf{E} = \frac{1}{2} (\mathbf{u} \circ \nabla + \nabla \circ \mathbf{u}) + \frac{1}{2} (\nabla \circ \mathbf{u}) \cdot (\mathbf{u} \circ \nabla) = \mathbf{E}^L + \mathbf{E}^N . \quad (8)$$

Non-linearities will be accounted through moderately large rotations for shallow arches as

$$\mathbf{E}^N = \frac{1}{2} (\nabla \circ \mathbf{u}) \cdot (\mathbf{u} \circ \nabla) = \frac{1}{2} (\mathbf{E}^L + \Psi^T) \cdot (\mathbf{E}^L + \Psi) =$$

$$= \frac{1}{2} \left(\mathbf{E}^L \cdot \mathbf{E}^L + \boldsymbol{\Psi}^T \cdot \mathbf{E}^L + \mathbf{E}^L \cdot \boldsymbol{\Psi} + \boldsymbol{\Psi}^T \cdot \boldsymbol{\Psi} \right) \approx \frac{1}{2} \boldsymbol{\Psi}^T \cdot \boldsymbol{\Psi} = \frac{1}{2} \boldsymbol{\Psi} \cdot \boldsymbol{\Psi}^T \quad (9)$$

Here, superscript T notes the transpose of a quantity.

Thus, the linear axial strain from the above is

$$\begin{aligned} \varepsilon_\xi^L &= \mathbf{i}_\xi \cdot \mathbf{E}^L \cdot \mathbf{i}_\xi = \mathbf{i}_\xi \cdot \frac{1}{2} (\mathbf{u} \circ \nabla + \nabla \circ \mathbf{u}) \cdot \mathbf{i}_\xi = \\ &= \frac{\rho_o}{\rho_o + \zeta} \left(\frac{du_o}{ds} + \frac{w_o}{\rho_o} + \frac{d\psi_{o\eta}}{ds} \zeta \right) = \frac{\rho_o}{\rho_o + \zeta} (\varepsilon_{o\xi} + \zeta \kappa_o) \end{aligned} \quad (10)$$

while the non-linear one is

$$\varepsilon_\xi^N = \mathbf{i}_\xi \cdot \mathbf{E}^N \cdot \mathbf{i}_\xi = \frac{1}{2} \psi_{o\eta}^2 \quad (11)$$

that, altogether, yields the non-linear strain at an arbitrary point:

$$\varepsilon_\xi = \frac{\rho_o}{\rho_o + \zeta} (\varepsilon_{o\xi} + \zeta \kappa_o) + \frac{1}{2} \psi_{o\eta}^2. \quad (12)$$

Consequently, the non-linear membrane ($\zeta = 0$) strain is

$$\varepsilon_m = \varepsilon_{o\xi} + \frac{1}{2} \psi_{o\eta}^2. \quad (13)$$

The centroidal axis can be found by satisfying relation

$$Q_{e\eta} = \int_A E \zeta dA = 0. \quad (14)$$

The modulus of elasticity E is constant for homogeneous materials and is a function of the cross-sectional coordinates for functionally graded distribution along the arch thickness as long as $E(\eta; \zeta) = E(-\eta; \zeta)$ is satisfied.

It is considered that the σ_ξ stress in the second Piola-Kirchhoff stress tensor is much greater than the other stress components. For the linearly elastic and isotropic material behaviour, Hooke's

$$\sigma_\xi = E \varepsilon_\xi \quad (15)$$

law is the constitutive equation for this arch problem. With this, it is possible to evaluate the axial force and bending moment in the deformed equilibrium state through

$$\begin{aligned} N &= \int_A E \varepsilon_\xi dA = \underbrace{\int_A E \frac{\rho_o}{\rho_o + \zeta} dA}_{A_{eR} \simeq A_e} \varepsilon_{o\xi} + \underbrace{\int_A E \frac{\zeta \rho_o}{\rho_o + \zeta} dA}_{Q_{eR} = -\frac{I_{e\eta}}{\rho_o}} \kappa_o + \underbrace{\int_A E dA}_{A_e} \frac{1}{2} \psi_{o\eta}^2 = \\ &= A_{eR} \varepsilon_{o\xi} + Q_{eR} \kappa_o + A_e \frac{1}{2} \psi_{o\eta}^2 \simeq A_e \underbrace{\left(\varepsilon_{o\xi} + \frac{1}{2} \psi_{o\eta}^2 \right)}_{\varepsilon_m} - \frac{I_{e\eta}}{\rho_o} \kappa_o = A_e \varepsilon_m - \frac{I_{e\eta}}{\rho_o} \kappa_o \end{aligned} \quad (16)$$

$$M = \int_A E \varepsilon_\xi \zeta dA = \int_A E \left(\frac{1}{1 + \frac{\zeta}{\rho_o}} (\varepsilon_{o\xi} + \zeta \kappa_o) + \frac{1}{2} \psi_{o\eta}^2 \right) \zeta dA =$$

$$\begin{aligned}
&= \underbrace{\int_A E \frac{\zeta}{1 + \frac{\zeta}{\rho_o}} dA \varepsilon_{o\xi}}_{Q_{eR} \simeq -\frac{I_{e\eta}}{\rho_o}} + \underbrace{\int_A E \frac{\zeta^2}{1 + \frac{\zeta}{\rho_o}} dA \kappa_o}_{I_{eR} \simeq I_{e\eta}} + \underbrace{\int_A E \zeta dA \frac{1}{2} \psi_{o\eta}^2}_{Q_{e\eta}=0} = \\
&= -\frac{I_{e\eta}}{\rho_o} \left(\frac{du_o}{ds} + \frac{w_o}{\rho_o} \right) - I_{e\eta} \frac{d}{ds} \left(\frac{dw_o}{ds} - \frac{u_o}{\rho_o} \right) = -I_{e\eta} \left(\frac{d^2 w_o}{ds^2} + \frac{w_o}{\rho_o^2} \right). \quad (17)
\end{aligned}$$

Next, it is shown that

$$\begin{aligned}
N &= A_e \underbrace{\left(\frac{du_o}{ds} + \frac{w_o}{\rho_o} + \frac{1}{2} \psi_{o\eta}^2 \right)}_{\varepsilon_m} + \frac{I_{e\eta}}{\rho_o} \frac{d}{ds} \left(\frac{dw_o}{ds} - \frac{u_o}{\rho_o} \right) = \\
&= \frac{I_{e\eta}}{\rho_o^2} \left[\left(\frac{A_e \rho_o^2}{I_{e\eta}} - 1 \right) \left(\frac{du_o}{ds} + \frac{w_o}{\rho_o} \right) + \rho_o \left(\frac{d^2 w_o}{ds^2} + \frac{w_o}{\rho_o^2} \right) \right] + \frac{1}{2} \psi_{o\eta}^2 A_e = \\
&= \frac{I_{e\eta}}{\rho_o^3} \left[\left(\frac{A_e \rho_o^2}{I_{e\eta}} - 1 \right) (u_o^{(1)} + w_o) + w_o^{(2)} + w_o \right] + \frac{1}{2} \psi_{o\eta}^2 A_e = \\
&= \frac{I_{e\eta}}{\rho_o^3} \left[\left(\frac{A_e \rho_o^2}{I_{e\eta}} - 1 \right) (u_o^{(1)} + w_o) + w_o^{(2)} + w_o \right] + \frac{1}{2} \psi_{o\eta}^2 \underbrace{\left(\frac{A_e \rho_o^2}{I_{e\eta}} - 1 + 1 \right)}_{\approx \frac{A_e \rho_o^2}{I_{e\eta}} - 1} \frac{I_{e\eta}}{\rho_o^2} \simeq \\
&\simeq \frac{I_{e\eta}}{\rho_o^2} \left(\frac{A_e \rho_o^2}{I_{e\eta}} - 1 \right) \underbrace{\left(\frac{1}{\rho_o} (u_o^{(1)} + w_o) + \frac{1}{2} \psi_{o\eta}^2 \right)}_{\varepsilon_m} + \underbrace{\frac{I_{e\eta}}{\rho_o^3} (w_o^{(2)} + w_o)}_{-\frac{M}{\rho_o}} \simeq A_e \varepsilon_m - \frac{M}{\rho_o} \quad (18)
\end{aligned}$$

thus

$$A_e \varepsilon_m = N + \frac{M}{\rho_o}. \quad (19)$$

So it can be seen clearly that the membrane strain is the result of the coupled effect of both the axial force and bending moment caused by the external radial compressive force at the crown of the arch. It is assumed that, for shallow arches

$$\frac{A_e \rho_o^2}{I_{e\eta}} - 1 \approx \frac{A_e \rho_o^2}{I_{e\eta}} = m \quad (20)$$

with m being a function of both the geometry and material distribution.

2.2. Changes for homogeneous material. When the material is homogeneous, $E = \text{const.}$ the first moment of the cross-section simplifies to

$$Q_{e\eta} = \int_A \zeta dA = 0. \quad (21)$$

The axial force and bending moment then become

$$N = AE \varepsilon_m - \frac{IE}{\rho_o} \kappa_o; \quad M = \rho_o (AE \varepsilon_m - N) \quad (22)$$

with

$$I = \int_A \zeta^2 dA, \quad A = \int_A dA \quad (23)$$

being the second moment- and area of the cross-section. Moreover, parameter m is simply

$$m = \frac{A\rho_o^2}{I} - 1 \approx \frac{A\rho_o^2}{I} \quad (24)$$

3. GOVERNING EQUATIONS

3.1. Static equilibrium conditions.

Based on the above hypotheses, the principle of virtual work is used to find the static equilibrium configuration of the pinned-fixed shallow, functionally graded arch under a radial force. The principle requires that

$$\int_V \sigma_\xi \delta \varepsilon_\xi dV = -P_\zeta \delta w_o|_{\varphi=\alpha} \quad (25)$$

be satisfied for the shallow arch of Fig. 2. Symbol δ notes the virtual, kinematically admissible quantities. Based on this principle, one can find the equilibrium conditions as well as the continuity and discontinuity conditions – see Appendix. With regard to the arbitrariness of the virtual quantities δu_o , δw_o and $\delta \psi_{o\eta}$, we have the equilibrium equations

$$\frac{dN}{ds} + \frac{1}{\rho_o} \left[\frac{dM}{ds} - \left(N + \frac{M}{\rho_o} \right) \psi_{o\eta} \right] = 0, \quad \frac{d}{ds} \left[\frac{dM}{ds} - \left(N + \frac{M}{\rho_o} \right) \psi_{o\eta} \right] - \frac{N}{\rho_o} = 0 \quad (26)$$

and the discontinuity condition at the α application point of the external force

$$\left[\frac{dM}{ds} - \left(N + \frac{M}{\rho_o} \right) \psi_{o\eta} \right] \Big|_{\varphi=+\alpha} - \left[\frac{dM}{ds} - \left(N + \frac{M}{\rho_o} \right) \psi_{o\eta} \right] \Big|_{\varphi=-\alpha} - P_\zeta = 0. \quad (27)$$

Because the arch considered is fixed-pinned, the boundary conditions at the end-supports are

$$u_o|_{\varphi(\pm\vartheta)} = 0, \quad w_o|_{\varphi(\pm\vartheta)} = 0, \quad \psi_{o\eta}|_{\varphi(-\vartheta)} = 0, \quad M|_{\varphi(+\vartheta)} = 0. \quad (28)$$

Since the basic unknown is the displacement field, the equilibrium equations should be expressed in terms of it. With As for (26)₁, first, let us plug in relation (18). Consequently, two terms vanish and the remaining is

$$\frac{d(A_e \varepsilon_m)}{ds} = 0, \quad (29)$$

meaning that the membrane strain is constant just like in [19]:

$$\varepsilon_m = \text{constant}. \quad (30)$$

Some further transformations are needed on equation (26)₂. These are detailed in Appendix A.2. Its form that is used for the stability investigation is

$$W_o^{(4)} + (2 - m\varepsilon_m) W_o^{(2)} + (1 - m\varepsilon_m) W_o = -m\varepsilon_m. \quad (31)$$

The new dimensionless normal displacement $W_o = w_o/\rho_o$ is introduced here and furthermore the strain parameter is

$$\chi^2 = 1 - m\varepsilon_m. \quad (32)$$

It is, therefore, found that Eq. (31) is actually

$$W_o^{(4)} + (\chi^2 + 1) W_o^{(2)} + \chi^2 W_o = \chi^2 - 1. \quad (33)$$

As it was mentioned, there is a number of existing models for the stability of shallow arches. Therefore, it is important to emphasise the novel properties of the current one. Comparisons are carried out in Appendix A3.

3.2. Solution to the equilibrium. The general solution of Eq. (33) is sought separately on the left (ℓ) and right (r) side of the external load because of the discontinuity in the shear force distribution. The solution of the dimensionless normal displacement satisfying equilibrium equation is

$$W_{or} = \frac{\chi^2 - 1}{\chi^2} + A_1 \cos \varphi + A_2 \sin \varphi - \frac{A_3}{\chi^2} \cos \chi \varphi - \frac{A_4}{\chi^2} \sin \chi \varphi, \quad (34a)$$

$$W_{o\ell} = \frac{\chi^2 - 1}{\chi^2} + B_1 \cos \varphi + B_2 \sin \varphi - \frac{B_3}{\chi^2} \cos \chi \varphi - \frac{B_4}{\chi^2} \sin \chi \varphi. \quad (34b)$$

Here A_i and B_i ($i = 1, \dots, 4$) are undetermined integration constants. These can be determined by using the boundary-, and discontinuity conditions, i.e., Eqs (28), (27) and continuity conditions

$$W_o|_{-\alpha} = W_o|_{+\alpha}; \quad \psi_{o\eta}|_{-\alpha} = \psi_{o\eta}|_{+\alpha}; \quad M|_{-\alpha} = M|_{+\alpha}. \quad (35)$$

These former ones are all known in terms of the displacements as shown in Eqs. (7), (17). Therefore, the previous conditions altogether determine a linear algebraic system of equations – eight equations for eight unknowns:

$$\begin{bmatrix} \cos \vartheta & \sin \vartheta & -\frac{\cos \chi \vartheta}{\chi^2} & -\frac{\sin \chi \vartheta}{\chi^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\cos \vartheta & -\sin \vartheta & \cos \chi \vartheta & \sin \chi \vartheta \\ -\sin \alpha & \cos \alpha & \chi \sin \chi \alpha & -\chi \cos \chi \alpha \\ -\cos \alpha & -\sin \alpha & \frac{\cos \chi \alpha}{\chi^2} & \frac{\sin \chi \alpha}{\chi^2} \\ \sin \alpha & -\cos \alpha & -\frac{\sin \chi \alpha}{\chi} & \frac{\cos \chi \alpha}{\chi} \\ \cos \alpha & \sin \alpha & -\cos \chi \alpha & -\sin \chi \alpha \end{bmatrix} \begin{bmatrix} 0 \\ \cos \vartheta \\ \sin \vartheta \\ 0 \\ \sin \alpha \\ \cos \alpha \\ -\sin \alpha \\ -\cos \alpha \end{bmatrix} + \begin{bmatrix} 0 \\ -\sin \vartheta \\ \cos \vartheta \\ 0 \\ -\cos \alpha \\ \sin \alpha \\ \cos \alpha \\ -\sin \alpha \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{\cos \chi \vartheta}{\chi^2} \\ -\frac{\sin \chi \vartheta}{\chi^2} \\ 0 \\ -\chi \sin \chi \alpha \\ -\frac{\cos \chi \alpha}{\chi^2} \\ \frac{\sin \chi \alpha}{\chi} \\ \cos \chi \alpha \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} = \begin{bmatrix} -\frac{\chi^2 - 1}{\chi^2} \\ -\frac{\chi^2 - 1}{\chi^2} \\ 0 \\ 0 \\ -2\frac{P}{\vartheta} \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (36)$$

It is possible to find closed-form solutions for the coefficients A_i ; B_i ($i = 1, 2, 3, 4$). However, since these are very lengthy, their publication is omitted here.

In the above coefficients, the notation

$$\mathcal{P} = \frac{P_\zeta \rho_o^2 \vartheta}{2I_{e\eta}} \quad (37)$$

was introduced and is called the dimensionless external load. For homogeneous materials, it is

$$\mathcal{P} = \frac{P_\zeta \rho_o^2 \vartheta}{2IE}. \quad (38)$$

The last issue is to reveal the relationship between the load and strain parameter. To do so, the equilibrium equation (30) is recalled. Since the membrane strain is constant, it can be replaced by its mathematical average based on the non-linear definition (13) by

$$\begin{aligned} \varepsilon_m &= \frac{1}{2\vartheta} \int_{-\vartheta}^{\vartheta} \varepsilon_m(\varphi) d\varphi = \frac{1}{2\vartheta} \int_{-\vartheta}^{\vartheta} \left(\varepsilon_{o\xi} + \frac{1}{2} \psi_{o\eta}^2 \right) d\varphi = \\ &= \frac{1}{2\vartheta} \int_{-\vartheta}^{\vartheta} \left(U_o^{(1)} + W_o + \frac{1}{2} W_o^{(1)} W_o^{(1)} \right) d\varphi = G_0 + G_1 \mathcal{P} + G_2 \mathcal{P}^2 \end{aligned} \quad (39)$$

that is simply

$$G_2 \mathcal{P}^2 + G_1 \mathcal{P} + G_0 - \varepsilon_m = 0 \quad (40)$$

with constants $G_0; G_1; G_2$ – these can be given in closed-form.

The latest equation is in fact the non-linear relationship between the load and strain. This can be used, for instance, to investigate the stability of pinned-fixed shallow arches under a concentrated radial force.

4. RESULTS

4.1. Buckling load of arches. Evaluation of the model is presented hereinafter. Since limit points are local maxima of the equilibrium path, these points of interest can be found by differentiating (40) with respect to the strain parameter χ . Computations were carried out using mathematical software Maple. The main objective is to find the lowest critical (buckling) loads. During our efforts, the quotient of the arch length $S = 2\rho_o \vartheta$ and the E -weighted radius of gyration $r = \sqrt{I_{e\eta}/A_e}$ (or $r = \sqrt{I/A}$ for homogeneous material) and the some introduced parameters $m, \alpha/\vartheta; \vartheta$ are used.

In Fig. 3 it is shown how the lowest dimensionless buckling load varies with the semi-vertex angle of the arch when the S/r ratio is 80. Three load positions are assessed: $\alpha/\vartheta = [-0.2; 0; 0.2]$. Because the supports are unsymmetrical, the sign of the load position is relevant in terms of the load bearing abilities. The continuous curve represents that the load is in the mid-span. It shows a monotonous trend. For very small angles, there is no buckling expected. After that the critical load increases steeply until about $\vartheta \simeq 0.4$. Later, it flattens and actually the arch angle does not have so drastic effects as beforehand. When the load position parameter is set to -0.2 (load application point is moved closer to the fixed support), the critical loads exceed the former case throughout. The greatest improvement is experienced at $\vartheta \simeq 0.27$ with $+28.8\%$, while in the vicinity of $\vartheta \simeq 0.7$, the relative difference is the lowest, staying below 5% . However, when the load is moved closer to the pinned support as

$\alpha/\vartheta = 0.2$, the critical load values are much lower. The range in which it changes in relation to the crown-point load is within $-14.9\% - 32.2\%$, having the greatest discrepancy at $\vartheta = 1$.

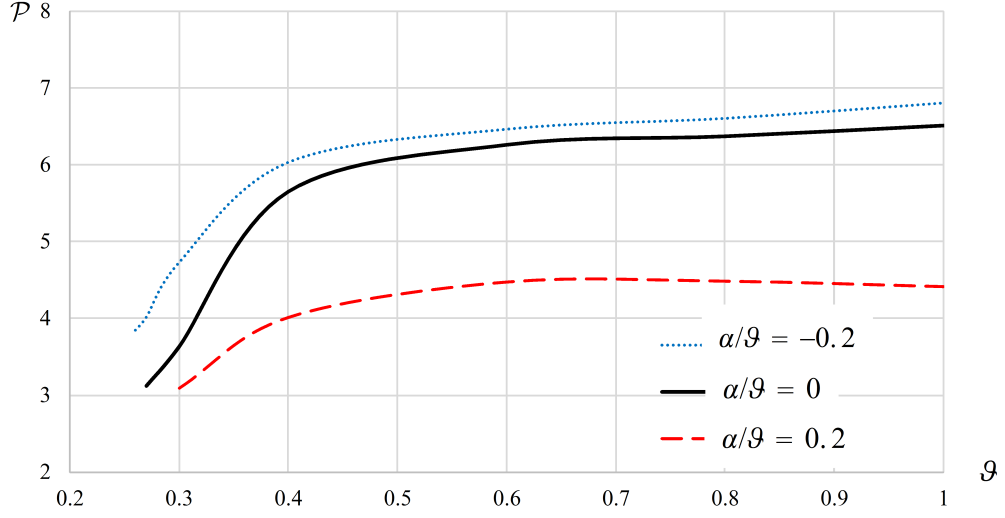


FIGURE 3. Lowest dimensionless buckling loads for $S/r = 80$

Next, the typical equilibrium paths (dimensionless load $\mathcal{P}$ - dimensionless displacement at the load application point $W_0 = W(\alpha)$) plots are made for arches with $m = 10^4$ and 10^5 . In Fig. 4, $m = 10000$ and $\vartheta = 0.25$, so these arches are very shallow. The graphs show monotonous paths for $\alpha/\vartheta = 0$ and 0.4 meaning that no buckling can be expected. However, when the load position parameter is -0.4 , there are three branches. There is a primary stable, a remote stable and an unstable between them. It means the existence of two limit points: one upper and one lower. So limit point buckling can be expected at $\mathcal{P} \simeq 5.68$. Exceeding this load would result in a sudden change in the displacement, making it increase by about 50%. The related dimensionless load - strain parameter plots are made in Fig. 5.

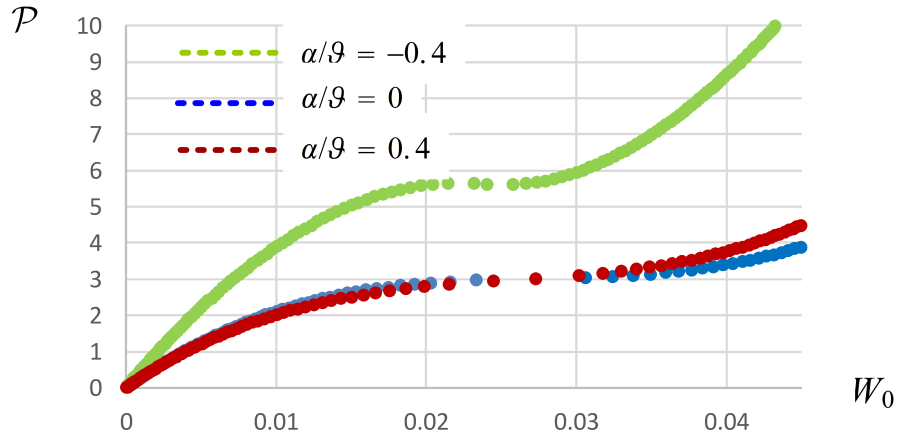


FIGURE 4. Equilibrium path when $m = 10^4$, $\vartheta = 0.25$

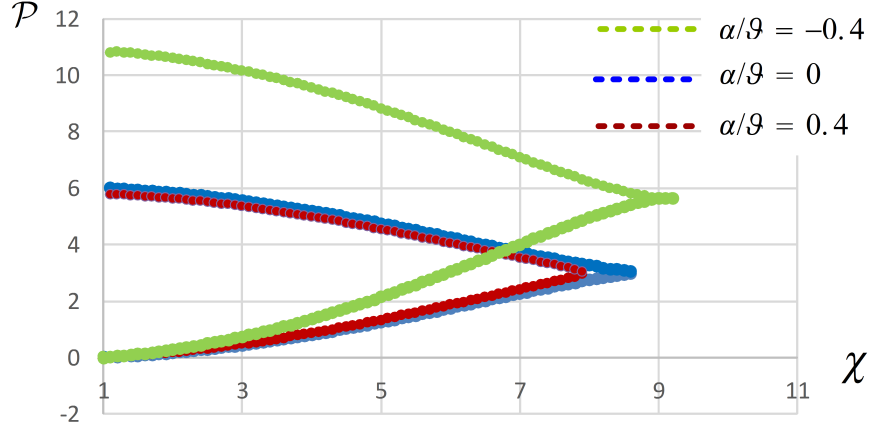


FIGURE 5. Dimensionless load - strain when $m = 10^4$, $v = 0.25$

Simply by changing m by one magnitude, there are notable differences compared to the former figure – see Figs. 6, 7. Now potentially there can be buckling for each case. If $\alpha/v = 0.4$, there are three branches, just like in Fig. 4 with $\alpha/v = -0.4$. However, the blue curve now contains two stable and two unstable branches with four limit point. While the green is somewhat even much more complicated with its four limit points between the three stable and three unstable branches. Shifting the load position from the right to the left yields a continuous increase in both the critical displacement and critical load values.

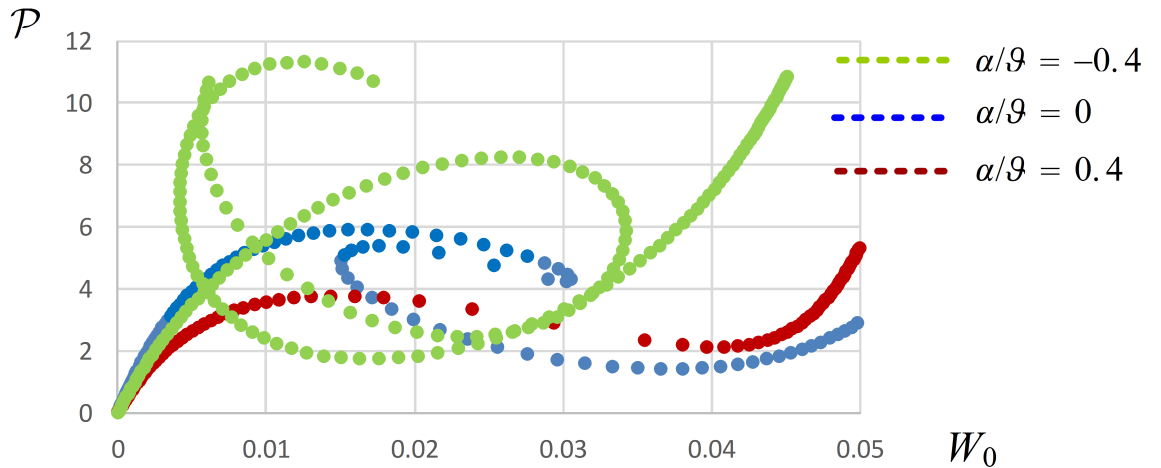


FIGURE 6. Equilibrium path when $m = 10^5$, $v = 0.25$

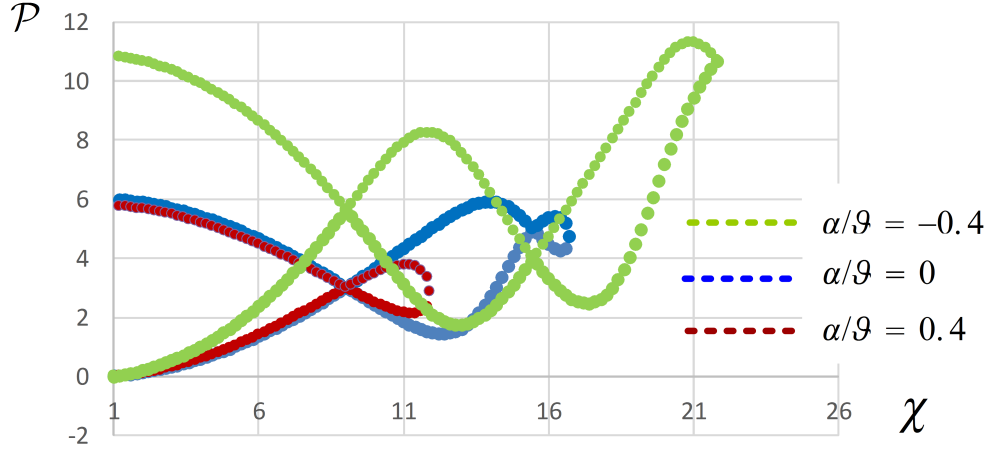


FIGURE 7. Dimensionless load - strain when $m = 10^5$, $\vartheta = 0.25$

4.2. Displacements and inner forces. In this subsection, some field distributions are presented along the arch centroidal axis, using the dimensionless coordinate $\varphi/\vartheta \in [-1; 1]$. Since the (critical) load-strain relation is known from Eq. (40), this value pair can be substituted back to the solution (34) for the radial displacement. After that, it is possible to express the inner forces according to Eqs. (16).

The selected arch for the demonstrations have the following properties. It has a homogeneous, rectangular cross section with $A = 8 \text{ mm}^2$ and $I = 2.66 \text{ mm}^4$. The modulus of elasticity is 200 GPa. Let us use $m = 10^5$ so that $\rho_o = 182.5 \text{ mm}$ and $\vartheta = 0.25$. The effect of four load positions are assessed when the related buckling load is applied – it is different for each case.

The radial displacement field is given in Fig. 8. The lowest displacements occur at the moment of buckling when $\alpha/\vartheta = 0.2$ and the greatest ones at $\alpha/\vartheta = -0.4$. This later case holds the greatest buckling load at the same time. Because of the unsymmetrical supports, even the $\alpha = 0$ case is not symmetric.

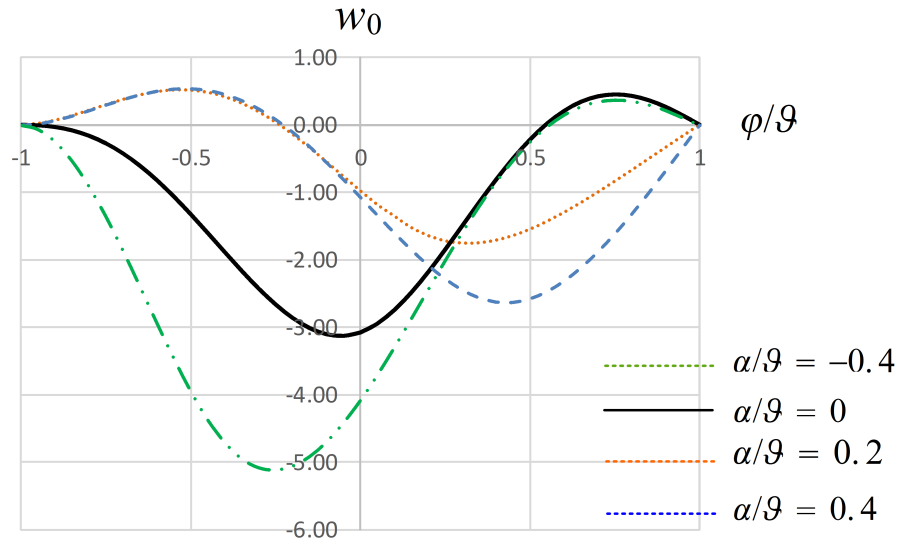


FIGURE 8. The normal displacements in [mm] along the centroidal axis

When we move on to the tangential displacements, as per Fig. 9, it can be concluded that these values are much lower than the radial displacements. It is due to the fact that the arch is shallow. Here, there is much less relative difference in the peak values, in comparison to the radial displacements.

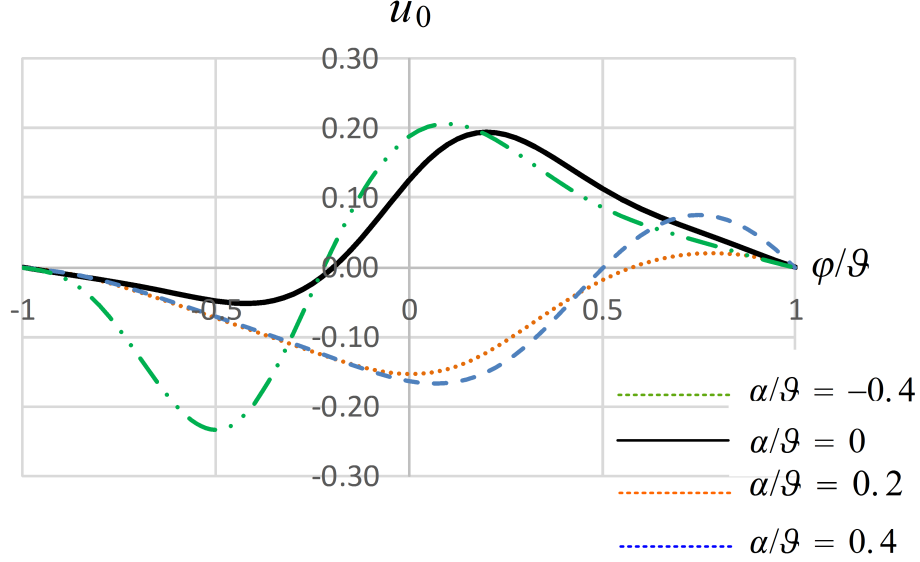


FIGURE 9. The tangential displacements in [mm] along the centroidal axis

Proceeding with the bending moment distribution, it turns out that arches when $\alpha/\vartheta = -0.4$ can bear by far the highest levels without buckling and the lowest values are found throughout if $\alpha/\vartheta = 0.2$. The difference between the peak values is more than 2.5 times. It is also clearly visible that at the right pinned support, the bending moment is always zero, while the left-sided fixed support can bear moment load.

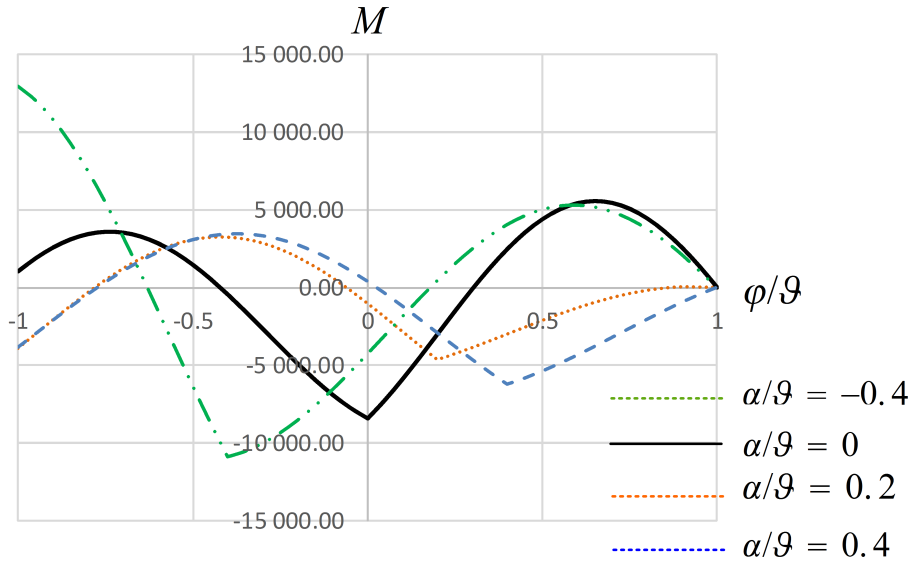


FIGURE 10. The bending moments in [Nmm] along the centroidal axis

The discontinuous shear force distribution is shown in Fig. 11 showing the actual location of the external force with its values.

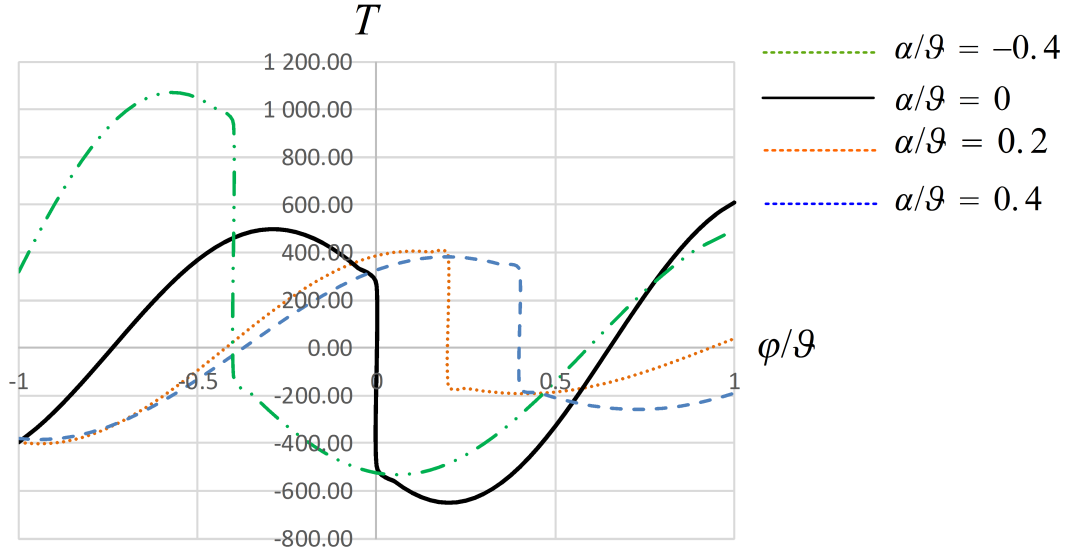


FIGURE 11. The shear force in [N] along the centroidal axis

As shown in Fig. 12, the axial force is almost evenly distributed for each case. The load position clearly affects the actual values. The maximum values on account of the load position are about 1.6 times of the minimum. Greatest values are found for crown-point load. Overall, it can be concluded that displacing the load from the crown to both directions, makes the axial force decrease, but the bending moment sometimes increases, sometimes decreases at the moment of buckling.

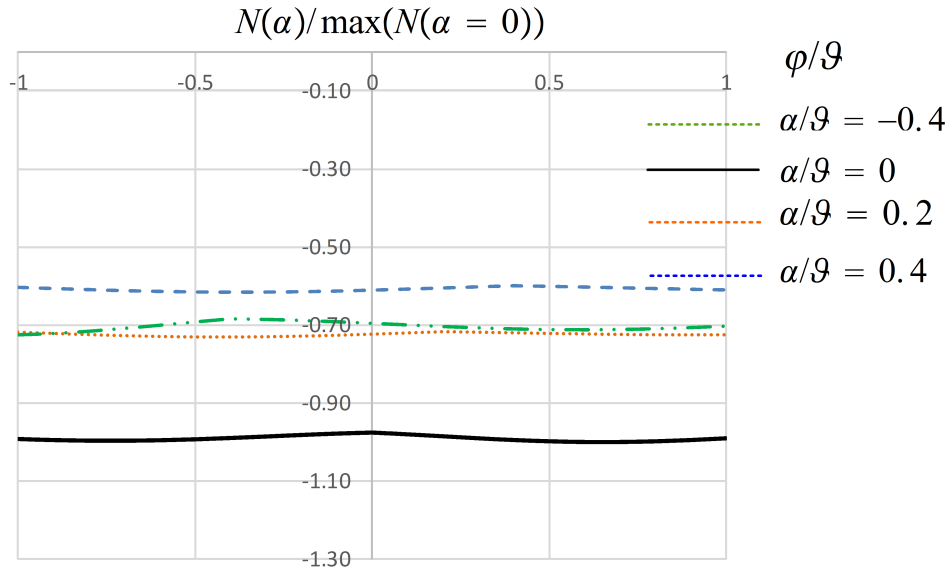


FIGURE 12. The axial force in [N] along the centroidal axis

The relative values are given in the forthcoming figures. The actual values are always divided by the maximum for mid-span load in the vertical coordinate axis.

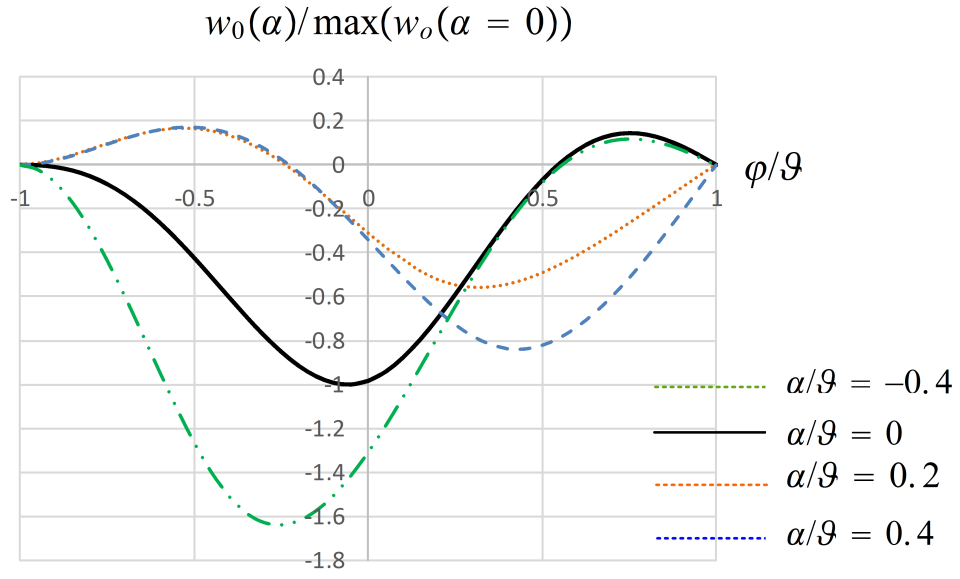


FIGURE 13. The relative radial displacement along the centroidal axis

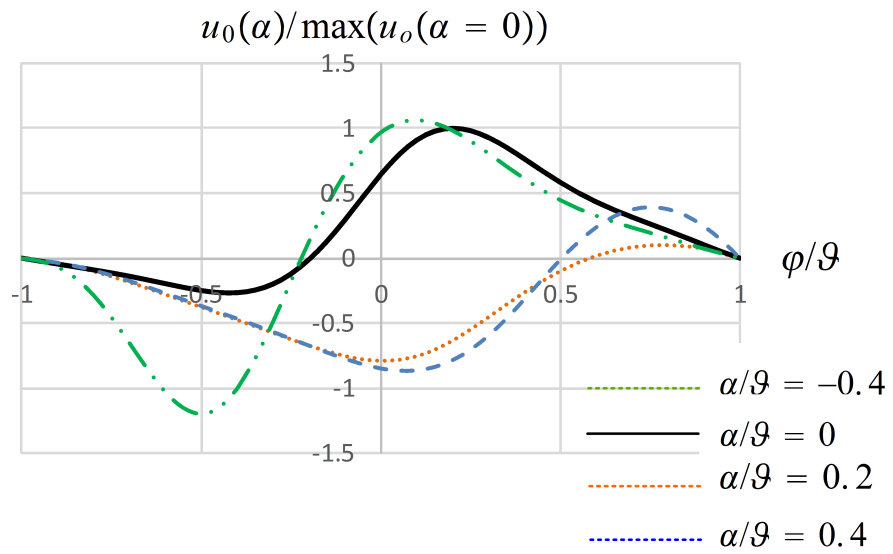


FIGURE 14. The relative tangential displacement along the centroidal axis

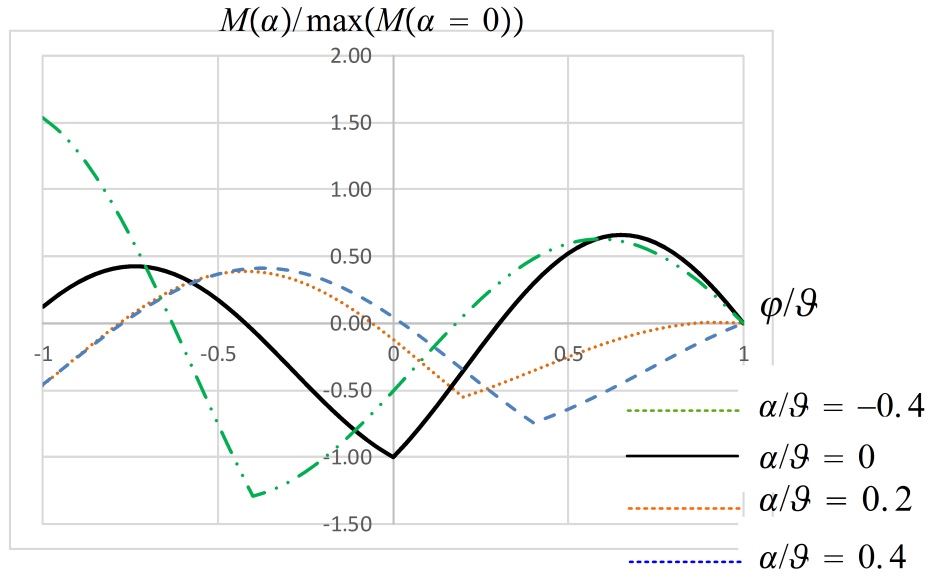


FIGURE 15. The relative bending moment along the centroidal axis

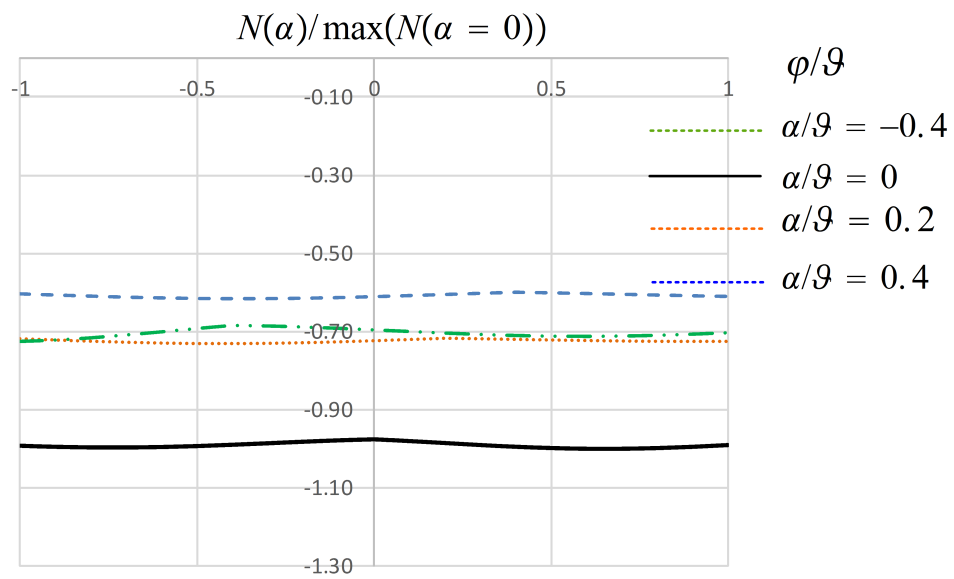


FIGURE 16. The relative axial force along the centroidal axis

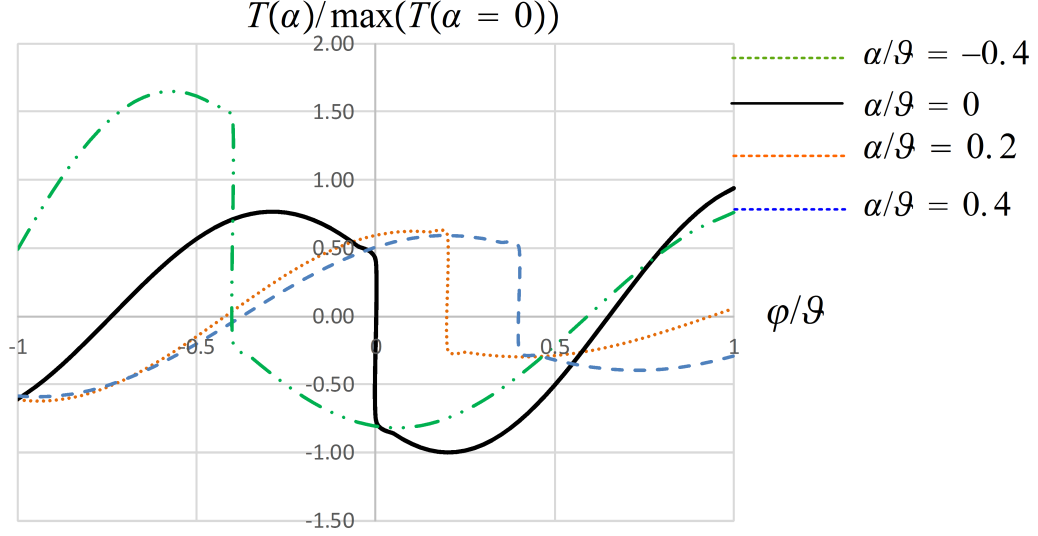


FIGURE 17. The relative shear force along the centroidal axis

5. CONCLUSIONS

The work tackles the stability fixed-pinned shallow arches. The problem is physically linear and geometrically nonlinear. The one-dimensional beam model is based on the Euler-Bernoulli hypothesis. The related coupled static equilibrium equations are derived using a variational principle. Closed-form analytical solutions are given. The buckling equations are also presented and evaluated. It was evaluated how some geometrical parameters affect the critical loads. Further, some typical equilibrium paths were also plotted to show the cross-sectional stiffness and arch angle are relevant parameters. Finally, on a simple example the typical field distributions were plotted along the arch axis to find out the effect of the load position on the displacements, bending moment and axial force. Overall, it can be concluded that the load position has substantial effect on the behaviour and buckling load.

APPENDIX A

A.1. Transformation of the principle of virtual work . Substituting [22] the corresponding kinematical quantities into the principle of virtual work (25) and taking the relation

$$dV = \left(1 + \frac{\zeta}{\rho_o}\right) ds dA \quad (41)$$

into account, which provides the infinitesimal volume element, the left side of the principle can be written as

$$\begin{aligned} & \int_V \sigma_\xi \delta \varepsilon_\xi dV = \\ & = \int_{\mathcal{L}} \int_A \left(1 + \frac{\zeta}{\rho_o}\right) \sigma_\xi \left[\frac{1}{1 + \frac{\zeta}{\rho_o}} \left(\frac{d\delta u_o}{ds} + \frac{\delta w_o}{\rho_o} + \zeta \frac{d\delta \psi_{o\eta}}{ds} \right) + \psi_{o\eta} \delta \psi_{o\eta} \right] dA ds = \end{aligned}$$

$$\begin{aligned}
&= \int_{\mathcal{L}} \left\{ \int_A \sigma_{\xi} dA \left(\frac{d\delta u_o}{ds} + \frac{\delta w_o}{\rho_o} \right) + \right. \\
&\quad \left. \int_A \zeta \sigma_{\xi} dA \frac{d\delta \psi_{o\eta}}{ds} + \int_A \left(1 + \frac{\zeta}{\rho_o} \right) \sigma_{\xi} dA \psi_{o\eta} \left(\frac{\delta u_o}{\rho_o} - \frac{d\delta w_o}{ds} \right) \right\} ds = \\
&= \int_{\mathcal{L}} \left[N \left(\frac{d\delta u_o}{ds} + \frac{\delta w_o}{\rho_o} \right) + M \frac{d\delta \psi_{o\eta}}{ds} + \left(N + \frac{M}{\rho_o} \right) \psi_{o\eta} \left(\frac{\delta u_o}{\rho_o} - \frac{d\delta w_o}{ds} \right) \right] ds ,
\end{aligned} \tag{42}$$

Integrating now by parts and performing some manipulations, we arrive at

$$\begin{aligned}
\int_V \sigma_{\xi} \delta \varepsilon_{\xi} dV &= - \int_{\mathcal{L}} \frac{dN}{ds} \delta u_o ds - N \delta u_o|_{s(-\vartheta)} + N \delta u_o|_{s(\vartheta)} + \int_{\mathcal{L}} \frac{N}{\rho_o} \delta w_o ds - \\
&- \int_{\mathcal{L}} \frac{1}{\rho_o} \frac{dM}{ds} \delta u_o ds - \int_{\mathcal{L}} \frac{d^2 M}{ds^2} \delta w_o ds - \frac{dM}{ds} \delta w_o \Big|_{s(-\vartheta)} + \frac{dM}{ds} \delta w_o \Big|_{s=-\alpha} - \frac{dM}{ds} \delta w_o \Big|_{s=+\alpha} \\
&+ \frac{dM}{ds} \delta w_o \Big|_{s(\vartheta)} + M \delta \psi_{o\eta}|_{s(\vartheta)} - M \delta \psi_{o\eta}|_{s(-\vartheta)} + \int_{\mathcal{L}} \frac{1}{\rho_o} \left(N + \frac{M}{\rho_o} \right) \psi_{o\eta} \delta u_o ds + \\
&+ \int_{\mathcal{L}} \left[\frac{d}{ds} \left(N + \frac{M}{\rho_o} \right) \psi_{o\eta} \right] \delta w_o ds + \left(N + \frac{M}{\rho_o} \right) \psi_{o\eta} \delta w_o \Big|_{s(-\vartheta)} - \\
&- \left(N + \frac{M}{\rho_o} \right) \psi_{o\eta} \delta w_o \Big|_{s=-\alpha} + \left(N + \frac{M}{\rho_o} \right) \psi_{o\eta} \delta w_o \Big|_{s=+\alpha} - \left(N + \frac{M}{\rho_o} \right) \psi_{o\eta} \delta w_o \Big|_{s(\alpha)} .
\end{aligned} \tag{43}$$

If we set (43) equal to the right side of (25), we finally get

$$\begin{aligned}
&- \int_{\mathcal{L}} \left(\frac{dN}{ds} + \frac{1}{\rho_o} \frac{dM}{ds} - \frac{1}{\rho_o} \left(N + \frac{M}{\rho_o} \right) \psi_{o\eta} \right) \delta u_o ds - \\
&- \int_{\mathcal{L}} \left(\frac{d^2 M}{ds^2} - \frac{N}{\rho_o} - \frac{d}{ds} \left(N + \frac{M}{\rho_o} \right) \psi_{o\eta} \right) \delta w_o ds - \\
&- \left[\frac{dM}{ds} - \left(N + \frac{M}{\rho_o} \right) \psi_{o\eta} \right] \delta w_o \Big|_{s(-\vartheta)} + \left[\frac{dM}{ds} - \left(N + \frac{M}{\rho_o} \right) \psi_{o\eta} \right] \delta w_o \Big|_{s(\vartheta)} - \\
&\left\{ \left[\frac{dM}{ds} - \left(N + \frac{M}{\rho_o} \right) \psi_{o\eta} \right] \Big|_{s=+\alpha} + \left[\frac{dM}{ds} - \left(N + \frac{M}{\rho_o} \right) \psi_{o\eta} \right] \Big|_{s=-0} - P_{\zeta} \right\} \delta w_o|_{s0} \\
&- N \delta u_o|_{s(-\vartheta)} + N \delta u_o|_{s(\vartheta)} + (M \psi_{o\eta})|_{s(\vartheta)} \delta \psi_{o\eta}|_{s(\vartheta)} - (M \psi_{o\eta})|_{s(-\vartheta)} \delta \psi_{o\eta}|_{s(-\vartheta)} = 0 .
\end{aligned} \tag{44}$$

A.2. Equilibrium in terms of the displacements. It follows from equation (26)₂ that

$$\frac{d^2 M}{ds^2} - \psi_{o\eta} \frac{d}{ds} \left(N + \frac{M}{\rho_o} \right) - \left(N + \frac{M}{\rho_o} \right) \frac{d\psi_{o\eta}}{ds} - \frac{N}{\rho_o} = 0 . \tag{45}$$

Substitute here now the equations for the inner forces in terms of the displacements. The first and third terms in (45) require no further manipulation at this point. The second one, however, vanishes – see (30). As for the fourth one, some transformations

need to be performed:

$$\frac{N}{\rho_o} = \frac{A_e}{\rho_o} \varepsilon_m - \frac{M}{\rho_o^2} = \frac{I_{e\eta}}{\rho_o^3} \frac{A_e \rho_o^2}{I_{e\eta}} \varepsilon_m - \frac{M}{\rho_o^2} = \frac{I_{e\eta}}{\rho_o^3} m \varepsilon_m + \frac{I_{e\eta}}{\rho_o^4} (w_o^{(2)} + w_o) .$$

Consequently, the equilibrium condition (45) can now be rewritten as

$$-\frac{I_{e\eta}}{\rho_o^4} (w_o^{(4)} + w_o^{(2)}) - \frac{I_{e\eta}}{\rho_o^4} \frac{A_e \rho_o^2}{I_{e\eta}} \rho_o \varepsilon_m \psi_{o\eta}^{(1)} - \frac{I_{e\eta}}{\rho_o^3} m \varepsilon_m - \frac{I_{e\eta}}{\rho_o^4} (w_o^{(2)} + w_o) = 0 .$$

If we multiply this formula by $(-\rho_o^4/I_{e\eta})$ we get

$$\begin{aligned} (w_o^{(4)} + w_o^{(2)}) + \frac{A_e \rho_o^2}{I_{e\eta}} \rho_o \varepsilon_m \psi_{o\eta}^{(1)} + \rho_o m \varepsilon_m + (w_o^{(2)} + w_o) &= \\ &= (w_o^{(4)} + w_o^{(2)}) + \rho_o m \varepsilon_m (\psi_{o\eta}^{(1)} + 1) + (w_o^{(2)} + w_o) = 0 . \end{aligned} \quad (46)$$

If we now substitute $\psi_{o\eta}$ and $u_o^{(1)}$ into the term $\rho_o \varepsilon_m (1 + \psi_{o\eta}^{(1)})$, we arrive at:

$$\begin{aligned} \rho_o \varepsilon_m (1 + \psi_{o\eta}^{(1)}) &= \rho_o \varepsilon_m \left[1 + \frac{1}{\rho_o} (u_o^{(1)} - w_o^{(2)}) \right] = \\ &= \rho_o \varepsilon_m \left[1 + \frac{1}{\rho_o} \left(\rho_o \varepsilon_m - w_o - \frac{1}{2} \psi_{o\eta}^2 \rho_o - w_o^{(2)} \right) \right] \approx \\ &\approx \rho_o \varepsilon_m \underbrace{(1 + \varepsilon_m)}_{\approx 1} - \varepsilon_m (w_o + w_o^{(2)}) \approx \rho_o \varepsilon_m - \varepsilon_m (w_o^{(2)} + w_o) . \end{aligned} \quad (47)$$

Plugging it back into (46) we find that the pre-buckling displacement w_o should satisfy the differential equation

$$w_o^{(4)} + 2w_o^{(2)} + w_o - m \varepsilon_m (w_o^{(2)} + w_o) = -m \rho_o \varepsilon_m . \quad (48)$$

A.3. Comparison of the current and previous literature models. When this model is compared with existing literature results, the following table gathers the differences.

Table 1: Major differences in models

Models [19, 23]	
Kinematics	$\varepsilon_\xi = \frac{du_o}{ds} + \frac{w_o}{\rho_o} - \zeta \frac{d^2 w_o}{ds^2} + \frac{1}{2} \left(\frac{dw_o}{ds} \right)^2$
Axial force	$N = AE \varepsilon_m$
Bending moment	$M = -IE \frac{d^2 w_o}{ds^2}$
Equilibrium equations	$W_o^{(4)} + (\chi^2 - 1) W_o^{(2)} = \chi^2 - 1$

Current model	
Kinematics	$\varepsilon_\xi = \frac{\rho_o}{\rho_o + \zeta} \left[\frac{du_o}{ds} + \frac{w_o}{\rho_o} - \zeta \left(\frac{d^2 w_o}{ds^2} - \frac{du_o}{\rho_o ds} \right) \right] + \frac{1}{2} \left(-\frac{dw_o}{ds} \right)^2$
Axial force	$N = A_e \varepsilon_m - \frac{M}{\rho_o}$
Bending moment	$M = -I_e \left(\frac{d^2 w_o}{ds^2} + \frac{w_o}{\rho_o^2} \right)$
Equilibrium equations	$W_o^{(4)} + (\chi^2 + 1) W_o^{(2)} + \chi^2 W_o = \chi^2 - 1$

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