

Failure Processes of Bundles of Plastic Fibers

TDK Thesis

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Chapter 1

Fracture of Plastic Fiber Reinforced Composite Materials

Plastic fiber reinforced composites (PFRCs) represent a vital class of engineered materials, characterized by their ability to combine the unique properties of plastic fibers with those of a carrier matrix, which may be metallic, ceramic, or polymeric. This combination results in composites with superior mechanical and functional properties that are tailored to meet specific application demands.

Advantages and Applications: The integration of high-performance plastic fibers into a carrier matrix enhances the strength, toughness, and flexibility of the resulting composite. These materials are particularly valued for their high strength-to-weight ratio, making them indispensable in industries like aerospace and automotive engineering. Additionally, their resistance to environmental degradation and adaptability to complex geometries allow them to outperform traditional materials in various structural and functional applications.

PFRCs are also increasingly employed in renewable energy technologies, particularly in wind turbine blades and solar panel supports, where their durability and lightweight nature are critical. Their ability to perform under extreme mechanical and environmental stress has expanded their use in construction, biomedical implants, and even advanced electronics.

In my TDK research, I investigate the failure process of plastic fiber-reinforced composite materials using the approach of fiber bundle modeling.

I. Experiments on Bundles of Plastic Fibers

Recently, interesting experiments have been performed in which the failure process of a bundle of plastic fibers has been analyzed in [16]. In the study, polyethylene fibers, whose mechanical response was measured under tensile loading, were considered. These measurements performed with single fibers, provided critical insights into their elastic and plastic properties. The experimental setup is illustrated in Fig. 1.1. The measurements were repeated with a large number of fibers. It can be seen in Fig. 1.2(a) that the experiments revealed three distinct stages of deformation: elastic deformation, plastic deformation, and failure. The slope of the stress-strain (or force-displacement) curves of different fibers in the elastic and plastic regimes proved to be nearly the same for different

fibers; however, both the yield and the breaking strengths showed a great statistical variation. Notably, the probability distribution of the yield and breaking strengths of fibers could be well approximated by Weibull distributions.

In the second set of experiments, bundles of polyethylene fibers were prepared and the tensile loading measurements were repeated several times to uncover the overall response of the bundle. The experimental setup ensured that the tensile forces were applied uniformly to the individual fibers within the bundles. The stress-strain curves of bundles of plastic fibers are demonstrated in Fig. 1.2(b). Similarly to the behavior of single fibers, the bundles also exhibit an elastic regime for small deformations followed by a well-defined plastic regime where stress is a slowly increasing function of strain. For sufficiently large strain values the bundle eventually fail, i.e. all fibers get broken.

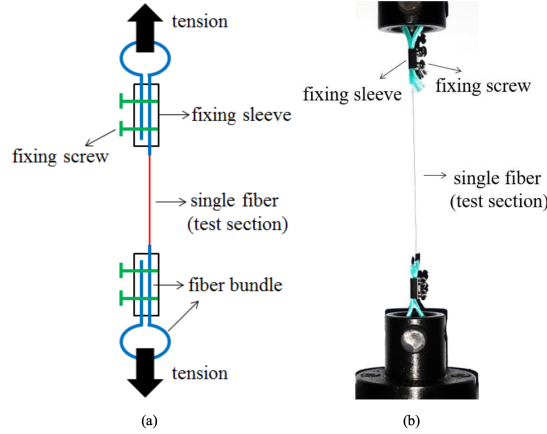


Figure 1.1: (a) Schematic diagram and (b) actual diagram of uniaxial tensile of single polythene fibers from Ref. [16].

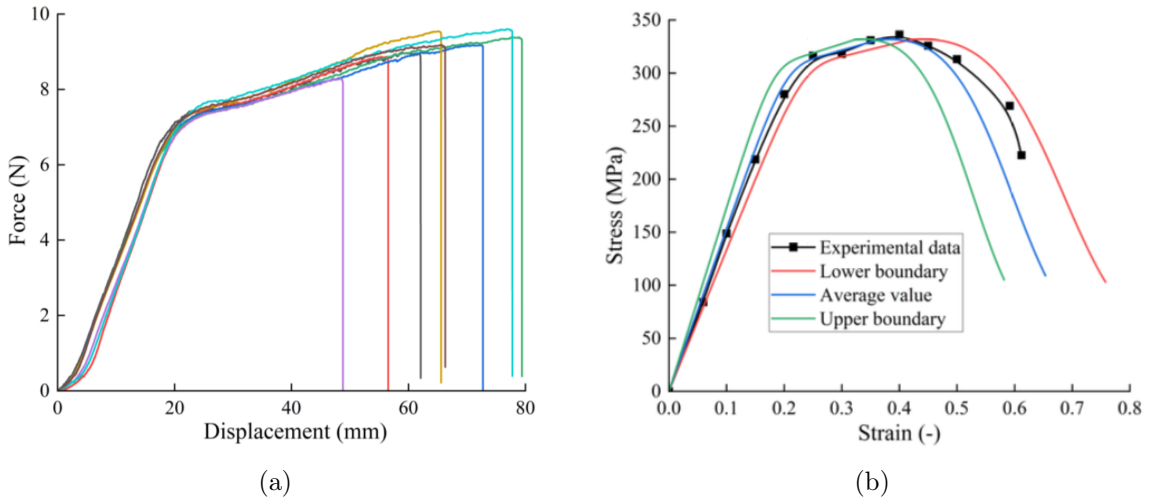


Figure 1.2: (a) Force-displacement curve of single fiber from Ref. [16]. (b) Theoretical and experimental results of tensile test of polythene fiber bundle from Ref. [16]. The experimental data are represented by dots connected by a black line. Model calculations carried out in Ref. [16] are presented by the colored lines.

II. Goals of My Research

In my TDK research I carried out a theoretical study of the failure process of bundles composed of polymer fibers with a mechanical response qualitatively similar to the ones presented in Fig. 1.2. Based on the classical fiber bundle model (FBM) of the fracture of heterogeneous materials, I worked out an FBM, where fibers exhibit elastic-plastic behavior as in the experiments of Ref. [16]. Assuming randomly distributed yielding and breaking thresholds of fibers I could cast the macroscopic response of the bundle into a closed analytical form.

The main challenge of the project is to uncover how the failure process of the bundle evolves as the load gets redistributed after rupture events on the remaining intact fibers. It could be expected that load redistribution triggers cascades of failure events, i.e. failure avalanches, where starting from the breaking of a single fiber more and more fibers breaks. Such breaking avalanches have been widely studied before in the framework FBMs, however, under the constraint of equal load sharing where each fiber carries the same load. In the presence of plasticity, the randomness of the yield strength of fibers has the consequence that the load carried by the remaining intact fibers will not be the same, the stress field becomes heterogeneous. It is a great challenge to understand how this heterogeneity affects the statistics of the size of avalanches and the overall characteristic of avalanche ensemble. Such breaking avalanches can generate elastic waves in materials which could be recorded in the form of acoustic noise. Such acoustic emission measurements are the primary source of information about the microscopic process of failure. My results may help to understand the outcomes of such material tests.

Chapter 2

Modeling the Failure of Bundles of Plastic Fibers

We worked out a fiber bundle model which captures the main ingredients of the failure mechanism of fiber composites where the fibers have plastic behavior. This chapter presents the details of the model construction and the derivation of the macroscopic response of the bundle.

I. Model Construction

We consider a bundle of parallel fibers, where fibers are assumed to exhibit plastic behavior under a gradually increasing external load. The bundle is composed of N fibers which are subject to a slowly increasing tensile load parallel to the fibers' direction. The arrangement of fibers in the bundle and the loading condition is illustrated in Fig. 2.1.

Initially, each fiber behaves elastically at small deformations, following a linear stress-strain relationship characterized by the elastic modulus E . Once a fiber reaches its elastic limit, it does not fail immediately. Instead, it enters a plastic phase where it continues to support load but with a reduced stiffness. Motivated by the experiments of Ref. [16], we assume that in this plastic phase, the stress-strain relationship remains linear, but with a slope of αE , where α is the plastic coefficient whose value falls between 0 and 1. This reduced slope reflects the decreased stiffness in the plastic phase, allowing the fiber to undergo further strain under additional stress.

Each fiber ultimately breaks after reaching its maximum allowable strain in the plastic phase. Thus, the behavior of each fiber includes an initial elastic phase with a slope of E of the linear stress-strain curve $\sigma = E\varepsilon$, followed by a plastic phase with a slope of αE , and finally, irreversible failure. Beyond the yielding threshold ε_y , stress-strain response $\sigma(\varepsilon)$ of single fibers is described by the equation

$$\sigma(\varepsilon) = E\varepsilon_y + \alpha E(\varepsilon - \varepsilon_y). \quad (2.1)$$

Further increasing the strain ε , the fiber will eventually fail when ε surpasses its breaking threshold ε_b . This sequence of behaviors provides a comprehensive description of plastic fibers which can be used to form fiber reinforced composite materials [16]. The mechanical response $\sigma(\varepsilon)$ of single fibers is presented in Fig. 2.2.

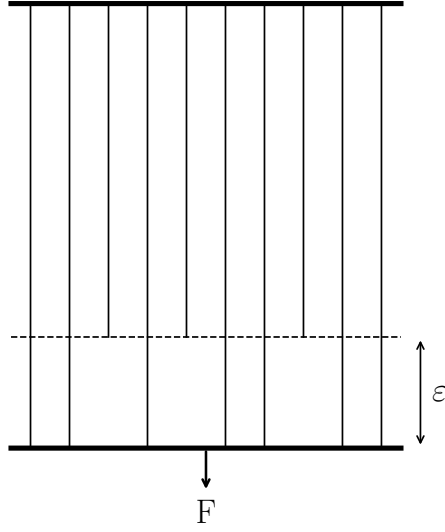


Figure 2.1: Schematic representation of our fiber bundle model. We consider a bundle of parallel fibers which is subject to tensile loading along the fibers' direction. The bundle is loaded between two stiff plates which ensures that the strain ε is the same for all the fibers.

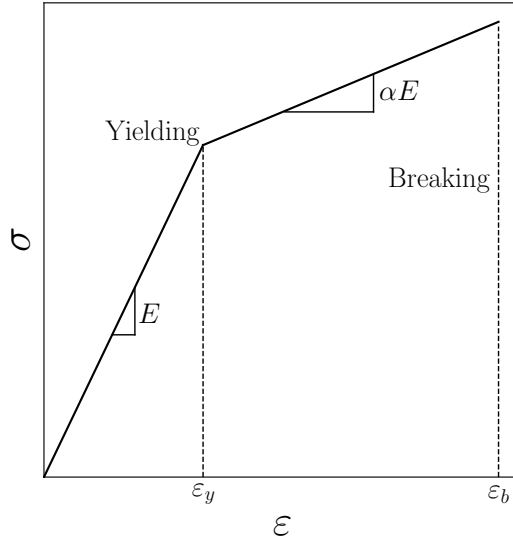


Figure 2.2: Behavior of a single fiber. Here the fiber undergoes elastic deformation (elastic modulus E) until it reaches the yielding point ε_y , after which it enters the plastic phase where the stress-strain relation remains linear with slope of αE with $0 \leq \alpha \leq 1$ be the plastic coefficient. The fiber ultimately fails when it reaches the maximum strain, ε_b , in the plastic phase.

In agreement with the experimental observations in Fig. 1.2(a), we assume that both the yielding threshold ε_y and breaking threshold ε_b of fibers have a certain degree of randomness, i.e. they are sampled from probability distributions.

To be able to carry out analytical calculations, we use uniformly distributed threshold values with the probability density functions $p_y(\varepsilon_y)$ of ε_y over the interval $\varepsilon_y \in [\langle \varepsilon_y \rangle - \Delta\varepsilon_y, \langle \varepsilon_y \rangle + \Delta\varepsilon_y]$, and the probability density function of ε_b is $p_b(\varepsilon_b)$ for the range $\varepsilon_b \in [\langle \varepsilon_b \rangle - \Delta\varepsilon_b, \langle \varepsilon_b \rangle + \Delta\varepsilon_b]$. Here the average yielding threshold of fibers is $\langle \varepsilon_y \rangle$, and the average breaking threshold of fibers is $\langle \varepsilon_b \rangle$.

$$p_y(\varepsilon_y) = \begin{cases} \frac{1}{2\Delta\varepsilon_y} & \text{if } \langle \varepsilon_y \rangle - \Delta\varepsilon_y \leq \varepsilon_y \leq \langle \varepsilon_y \rangle + \Delta\varepsilon_y \\ 0 & \text{otherwise} \end{cases} \quad (2.2)$$

$$p_b(\varepsilon_b) = \begin{cases} \frac{1}{2\Delta\varepsilon_b} & \text{if } \langle \varepsilon_b \rangle - \Delta\varepsilon_b \leq \varepsilon_b \leq \langle \varepsilon_b \rangle + \Delta\varepsilon_b \\ 0 & \text{otherwise} \end{cases} \quad (2.3)$$

The cumulative distribution functions can be obtained as:

$$P_y(\varepsilon_y) = \begin{cases} 0 & \text{if } \varepsilon_y < \langle \varepsilon_y \rangle - \Delta\varepsilon_y \\ \frac{\varepsilon_y - (\langle \varepsilon_y \rangle - \Delta\varepsilon_y)}{2\Delta\varepsilon_y} & \text{if } \langle \varepsilon_y \rangle - \Delta\varepsilon_y \leq \varepsilon_y \leq \langle \varepsilon_y \rangle + \Delta\varepsilon_y \\ 1 & \text{if } \varepsilon_y > \langle \varepsilon_y \rangle + \Delta\varepsilon_y \end{cases} \quad (2.4)$$

$$P_b(\varepsilon_b) = \begin{cases} 0 & \text{if } \varepsilon_b < \langle \varepsilon_b \rangle - \Delta\varepsilon_b \\ \frac{\varepsilon_b - (\langle \varepsilon_b \rangle - \Delta\varepsilon_b)}{2\Delta\varepsilon_b} & \text{if } \langle \varepsilon_b \rangle - \Delta\varepsilon_b \leq \varepsilon_b \leq \langle \varepsilon_b \rangle + \Delta\varepsilon_b \\ 1 & \text{if } \varepsilon_b > \langle \varepsilon_b \rangle + \Delta\varepsilon_b \end{cases} \quad (2.5)$$

In the next section I present the derivation of the general form of the stress-strain $\sigma(\varepsilon)$ relation of the bundle. For the sake of explicit results, I will use the above uniform distribution of threshold values since it allows me to cast the results into a closed analytical form. Although, using Weibull distributed thresholds would be more realistic, they have the disadvantage that the bundle behavior could be analyzed solely numerically.

II. Constitutive Behavior of the Bundle

I present a step-by-step derivation of the macroscopic response of the bundle of the bundle of plastic fibers.

1. General form of the constitutive equation

We assume the bundle of plastic fibers is loaded between two stiff plates which ensures that the strain ε is the same for each fiber, although, the local load may have a statistical variation.

At a given strain ε during the loading process, there are 3 types of fibers present in the bundle: intact, yielding, and broken. The stress-strain relation of the bundle can be obtained by adding up the contributions of the three subsets of fibers as

$$\begin{aligned}\sigma(\varepsilon) &= E\varepsilon[1 - P_y(\varepsilon)] + \left(\int_0^\varepsilon [\alpha E(\varepsilon - \varepsilon_y) + E\varepsilon_y] \cdot p_y(\varepsilon_y) d\varepsilon_y \right) \cdot [1 - P_b(\varepsilon)] \\ &= E\varepsilon[1 - P_y(\varepsilon)] + \left[\int_0^\varepsilon [(1 - \alpha)E\varepsilon_y + \alpha E\varepsilon] \cdot p_y(\varepsilon_y) d\varepsilon_y \right] \cdot [1 - P_b(\varepsilon)]\end{aligned}\quad (2.6)$$

where α is the ratio of the elastic modulus of yielding fibers to that of intact fibers, and E is the elastic modulus of intact fibers. In the first term of the right hand side $1 - P_y(\varepsilon)$ is the fraction of those fibers which are intact, i.e. not yet yielding (and not broken, of course). The second term captures the load of those fibers which have already surpassed their yielding threshold but are not yet broken ($1 - P_b(\varepsilon)$). Here we have to integrate the contribution of yielding fibers because the load they hold at a given ε depends on the precise value of their yield threshold.

2. Explicit form for uniformly distributed threshold values

Since the behavior of the uniform distribution varies across different intervals, we need to break the process down and consider each interval separately. For the sake of simplicity, consider the two intervals $[\langle \varepsilon_y \rangle - \Delta \varepsilon_y, \langle \varepsilon_y \rangle + \Delta \varepsilon_y]$ and $[\langle \varepsilon_b \rangle - \Delta \varepsilon_b, \langle \varepsilon_b \rangle + \Delta \varepsilon_b]$ do not overlap, i.e., $\langle \varepsilon_y \rangle + \Delta \varepsilon_y < \langle \varepsilon_b \rangle - \Delta \varepsilon_b$.

Intact-only Region

For $0 \leq \varepsilon < \langle \varepsilon_y \rangle - \Delta \varepsilon_y$, all fibers are intact, i.e., no fiber has yielded yet. The stress-strain function is:

$$\sigma(\varepsilon) = E\varepsilon \quad (2.7)$$

Intact and Yielding Region

For $\langle \varepsilon_y \rangle - \Delta \varepsilon_y \leq \varepsilon \leq \langle \varepsilon_y \rangle + \Delta \varepsilon_y$, some fibers are yielding, some still are intact. Substitute (2.4), (2.2), and (2.5) into (2.6). The stress-strain function is:

$$\begin{aligned}
\sigma(\varepsilon) &= E\varepsilon \left[1 - \frac{\varepsilon - (\langle \varepsilon_y \rangle - \Delta \varepsilon_y)}{2\Delta \varepsilon_y} \right] \\
&\quad + \left[\int_0^{\langle \varepsilon_y \rangle - \Delta \varepsilon_y} 0 + \int_{\langle \varepsilon_y \rangle - \Delta \varepsilon_y}^{\varepsilon} [(1 - \alpha)E\varepsilon_y + \alpha E\varepsilon] \cdot \frac{1}{2\Delta \varepsilon_y} d\varepsilon_y \right] \cdot [1 - 0] \\
\sigma(\varepsilon) &= E\varepsilon \left(\frac{1}{2} - \frac{\varepsilon - \langle \varepsilon_y \rangle}{2\Delta \varepsilon_y} \right) \\
&\quad + \frac{E}{2\Delta \varepsilon_y} \left[\frac{1 + \alpha}{2} \varepsilon^2 - \alpha(\langle \varepsilon_y \rangle - \Delta \varepsilon_y)\varepsilon - \frac{1 - \alpha}{2} (\langle \varepsilon_y \rangle - \Delta \varepsilon_y)^2 \right] \tag{2.8}
\end{aligned}$$

Yielding-only Region

For $\langle \varepsilon_y \rangle + \Delta \varepsilon_y < \varepsilon < \langle \varepsilon_b \rangle - \Delta \varepsilon_b$, all fibers are yielding, i.e., no fiber is intact anymore, and no fiber is broken yet. Substitute (2.4), (2.2), and (2.5) into (2.6). The stress-strain function is:

$$\begin{aligned}
\sigma(\varepsilon) &= E\varepsilon(1 - 1) + \left[\int_0^{\langle \varepsilon_y \rangle - \Delta \varepsilon_y} 0 \right. \\
&\quad \left. + \int_{\langle \varepsilon_y \rangle - \Delta \varepsilon_y}^{\langle \varepsilon_y \rangle + \Delta \varepsilon_y} [(1 - \alpha)E\varepsilon_y + \alpha E\varepsilon] \cdot \frac{1}{2\Delta \varepsilon_y} d\varepsilon_y + \int_{\langle \varepsilon_y \rangle + \Delta \varepsilon_y}^0 0 \right] \cdot (1 - 0) \\
\sigma(\varepsilon) &= E[(1 - \alpha)\langle \varepsilon_y \rangle + \alpha\varepsilon] \tag{2.9}
\end{aligned}$$

Yielding and Breaking Region

For $\langle \varepsilon_b \rangle - \Delta \varepsilon_b \leq \varepsilon \leq \langle \varepsilon_b \rangle + \Delta \varepsilon_b$, some fibers are broken, some still are yielding but no fiber is intact. Substitute (2.4), (2.2), and (2.5) into (2.6). The stress-strain function is:

$$\begin{aligned}
\sigma(\varepsilon) &= E\varepsilon(1 - 1) + \left[\int_0^{\langle \varepsilon_y \rangle - \Delta \varepsilon_y} 0 + \int_{\langle \varepsilon_y \rangle - \Delta \varepsilon_y}^{\langle \varepsilon_y \rangle + \Delta \varepsilon_y} [(1 - \alpha)E\varepsilon_y + \alpha E\varepsilon] \cdot \frac{1}{2\Delta \varepsilon_y} d\varepsilon_y + \right. \\
&\quad \left. \int_{\langle \varepsilon_y \rangle + \Delta \varepsilon_y}^0 0 \right] \cdot \left[1 - \frac{\varepsilon - (\langle \varepsilon_b \rangle - \Delta \varepsilon_b)}{2\Delta \varepsilon_b} \right] \\
\sigma(\varepsilon) &= \frac{E}{2} [(1 - \alpha)\langle \varepsilon_y \rangle + \alpha\varepsilon] \cdot \left(1 - \frac{\varepsilon - \langle \varepsilon_b \rangle}{\Delta \varepsilon_b} \right) \tag{2.10}
\end{aligned}$$

All-is-broken Region

For $\langle \varepsilon_b \rangle + \Delta \varepsilon_b < \varepsilon$, all fibers are broken. The stress-strain function is:

$$\sigma(\varepsilon) = 0 \quad (2.11)$$

3. Results

Combining the terms obtained for different strain regimes the overall response of the bundle can be cast into the final form:

$$\sigma(\varepsilon) = \begin{cases} E\varepsilon & \text{if } 0 \leq \varepsilon < \langle \varepsilon_y \rangle - \Delta \varepsilon_y \\ E\varepsilon \left(\frac{1}{2} - \frac{\varepsilon - \langle \varepsilon_y \rangle}{2\Delta \varepsilon_y} \right) + \frac{E}{2\Delta \varepsilon_y} \left[\frac{1+\alpha}{2} \varepsilon^2 - \alpha(\langle \varepsilon_y \rangle - \Delta \varepsilon_y)\varepsilon - \frac{1-\alpha}{2} (\langle \varepsilon_y \rangle - \Delta \varepsilon_y)^2 \right] & \text{if } \langle \varepsilon_y \rangle - \Delta \varepsilon_y \leq \varepsilon \leq \langle \varepsilon_y \rangle + \Delta \varepsilon_y \\ E[(1-\alpha)\langle \varepsilon_y \rangle + \alpha\varepsilon] & \text{if } \langle \varepsilon_y \rangle + \Delta \varepsilon_y < \varepsilon < \langle \varepsilon_b \rangle - \Delta \varepsilon_b \\ \frac{E}{2} [(1-\alpha)\langle \varepsilon_y \rangle + \alpha\varepsilon] \cdot \left(1 - \frac{\varepsilon - \langle \varepsilon_b \rangle}{\Delta \varepsilon_b} \right) & \text{if } \langle \varepsilon_b \rangle - \Delta \varepsilon_b \leq \varepsilon \leq \langle \varepsilon_b \rangle + \Delta \varepsilon_b \\ 0 & \text{if } \langle \varepsilon_b \rangle + \Delta \varepsilon_b < \varepsilon \end{cases}$$

The constitutive curve of the bundle is illustrated in Fig. 2.3 by plotting the above formula. The analytical results are compared to the numerically determined stress-strain curve. The algorithm of computer simulations will be presented in the next section. To obtain the simulation results, I used the parameter values as follows:

- Number of fibers: $N = 100000$
- Elastic modulus: $E = 1$
- Plastic coefficient: $\alpha = 0.2$
- Average yielding threshold of fibers: $\langle \varepsilon_y \rangle = 0.45$
- Radius of yielding thresholds interval: $\Delta \varepsilon_y = 0.1$
- Average breaking threshold of fibers: $\langle \varepsilon_b \rangle = 5.9$
- Radius of breaking thresholds interval: $\Delta \varepsilon_b = 4.0$

Good agreement of the analytical and numerical results can be observed.

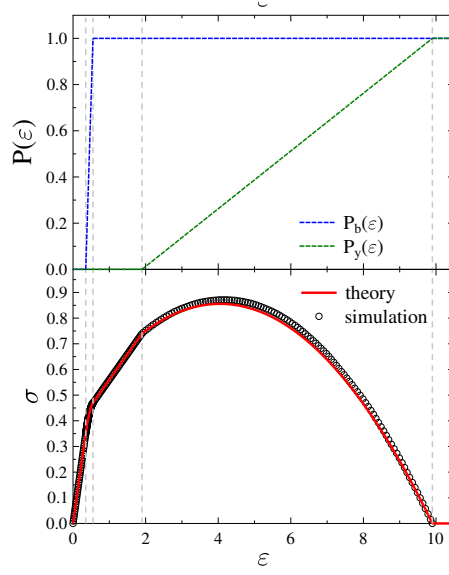


Figure 2.3: The constitutive behavior of the fiber bundle model. The stress-strain curve is composed of five regions: intact-only, intact and yielding, yielding-only, yielding and breaking, and all-is-broken. For clarity, the upper part of the figure indicates the probability distributions of the yielding and breaking thresholds.

Figure 2.4 demonstrates that at a fixed value of α , the degree of strength disorder of fibers, controlled by the value of $\Delta\epsilon_b$ has a strong effect on the overall behavior of the stress-strain curve: Decreasing $\Delta\epsilon_b$ the $\sigma(\epsilon)$ curve sharpens in the sense that less-and-less non-linearity precedes global failure (occurring at the maximum of the curve). The system gradually shifts towards a more abrupt type of failure as strength disorder is reduced.

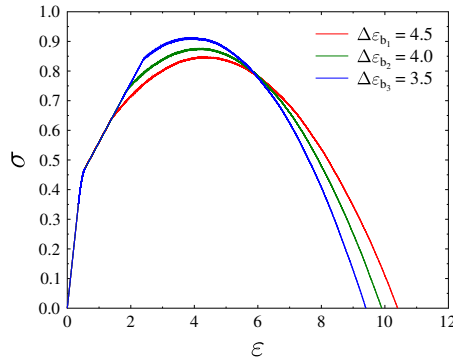


Figure 2.4: Stress-strain curves of the fiber bundle model demonstrating the dependence of the constitutive behavior on the interval of breaking thresholds. As the interval decreases in steps of 0.5, the curve sharpens, indicating a transition from gradual failure to more abrupt system-wide failure.

4. Fracture Strength of the Bundle

One of the most important macroscopic characteristics of the plastic fiber bundle is its fracture strength. Under stress controlled loading, when slowly increasing the external stress the bundle ultimately fails at the maximum of the stress-strain curve $\sigma(\varepsilon)$. Hence, the position and value of the maximum of the stress-strain curve define the critical strain ε_c and critical stress σ_c of the bundle.

It is a crucial question how these critical values depend on the plastic coefficient α , which controls the strength of plasticity of the system. For the limiting case of $\alpha = 1$ the strength values can be obtained analytically. In this limit, yielding has no role, i.e. after the yield point fibers retain their stiffness. The stress-strain curve becomes non-linear due to the disorder of the failure thresholds of fibers.

The stress can simply be expressed as a function of strain

$$\sigma(\varepsilon) = \varepsilon \cdot [1 - P_b(\varepsilon)], \quad (2.12)$$

where $P_b(\varepsilon)$ is the cumulative distribution of breaking thresholds. The critical point is determined by setting the derivative of $\sigma(\varepsilon)$ to zero

$$\frac{d\sigma(\varepsilon)}{d\varepsilon} = 1 - P(\varepsilon) - \varepsilon \cdot p(\varepsilon) = 0, \quad (2.13)$$

where $p_b(\varepsilon)$ is the probability density function of breaking thresholds. Substituting (2.5) and (2.3) into the derivative condition yields

$$1 - \frac{\varepsilon_c - (\langle \varepsilon_b \rangle - \Delta \varepsilon_b)}{2\Delta \varepsilon_b} - \frac{\varepsilon_c}{2\Delta \varepsilon_b} = 0. \quad (2.14)$$

Rearranging, we find

$$\varepsilon_c = \frac{\langle \varepsilon_b \rangle + \Delta \varepsilon_b}{2}. \quad (2.15)$$

The corresponding critical stress is obtained by substituting ε_c into $\sigma(\varepsilon)$:

$$\sigma_c = \frac{\langle \varepsilon_b \rangle + \Delta \varepsilon_b}{2} \cdot \left[1 - \frac{\varepsilon_c - (\langle \varepsilon_b \rangle - \Delta \varepsilon_b)}{2\Delta \varepsilon_b} \right] = \frac{(\langle \varepsilon_b \rangle + \Delta \varepsilon_b)^2 - \varepsilon_c (\langle \varepsilon_b \rangle + \Delta \varepsilon_b)}{4\Delta \varepsilon_b} \quad (2.16)$$

These expressions provide exact analytical results for ε_c and σ_c in the non-plastic case. They reveal that the fracture strength of the bundle is determined by the distribution of breaking thresholds.

When $\alpha < 1$, fibers exhibit a plastic regime after yielding, characterized by a reduced stiffness. Due to the randomness of yielding thresholds, after yielding starts the local load on fibers will be disordered which makes the failure process more complicated. We carried out numerical calculations to determine the critical stress and strain for arbitrary plastic coefficients α including also $\alpha = 1$. In Figure 2.5 the numerically obtained strength values are divided by the analytically obtained result of $\alpha = 1$. Calculations were repeated for three different system sizes with $N = 1000, 10.000, 100.000$ fibers.

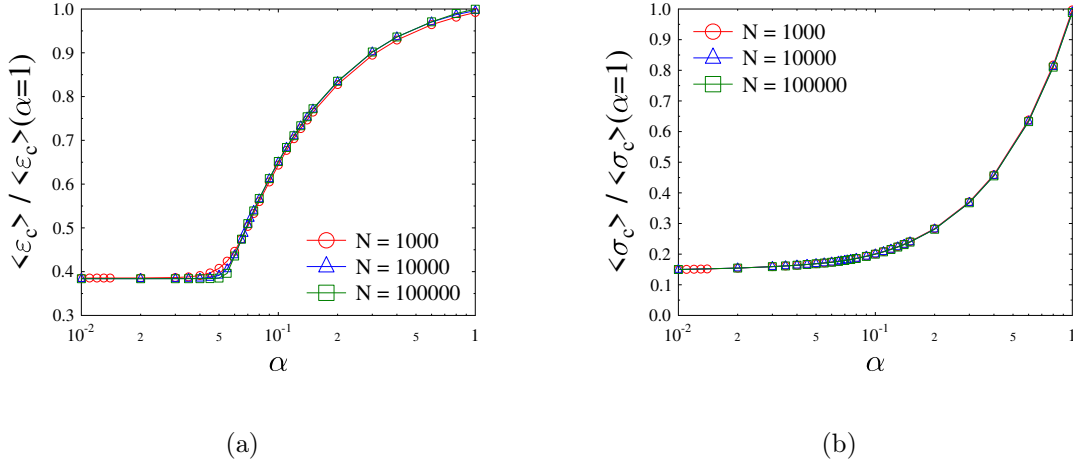


Figure 2.5: (a) Average critical strain $\langle \varepsilon_c \rangle$ and (b) critical stress $\langle \sigma_c \rangle$ as function of the plastic coefficient α for different system sizes N . The strength values are divided with the analytical result obtained for $\alpha = 1$. Note that the lowest values, the strengths take on the vertical axis, are determined by the lower bound of the interval of failure thresholds.

Good agreement can be observed between the numerical and analytical results for $\alpha = 1$. Decreasing α the strength of the bundle decreases; however, the most important outcome is that there exists a threshold value of the plastic coefficient α_c below which the strength of the bundle becomes equal to the lower bound of the failure thresholds. This means that at $\alpha < \alpha_c$, the breakage of the weakest fiber triggers the catastrophic collapse of the system. At this critical α_c , a transition occurs from ductile to abrupt failure of the system. The value of α_c can be obtained from the numerical analysis as $\alpha_c \approx 0.0536$. To understand this interesting transition, in the next section, I present a numerical algorithm and analyze the microscopic process of the failure of the bundle.

Chapter 3

Simulation of the Failure Process of Bundles of Plastic Fibers

The macroscopic response of the bundle could be determined analytically, however, on the micro-level the complexity of the failure process of the system can only be explored numerically. We worked out a simulation technique which allowed us to study not only the macroscopic behavior of the bundle but also the failure cascades.

I. Constitutive Behavior of a FBM

We load the fiber bundle between two stiff plates, ensuring that all fibers experience the same strain ε at any given moment. However, the constitutive behavior of individual fibers Eq. (2.6) dictates that once fibers begin to yield, the load they carry becomes uneven. This means that while strain is constant across fibers, their individual responses diverge. To understand and simulate the evolution of the failure process under a gradually increasing strain, we use the following algorithm:

1. Initialization of fiber properties

- Initially, we generate the random yielding thresholds ε_y and random breaking thresholds ε_b , sampled from the given probability distributions (Eq. (2.4) and Eq. (2.5), respectively).
- Since the strain is the same for all fibers at any point in time, yielding and breaking events will occur in the order of their respective thresholds. To ensure this, we sort the yielding and breaking thresholds in ascending order.

2. Elastic-only phase

- At the beginning of the loading process, all fibers behave elastically, and the stress in the bundle increases linearly with strain according to Hooke's Law:

$$\sigma = E \cdot \varepsilon,$$

where E is the elastic modulus.

- We increment strain ε gradually, up to the smallest yielding threshold ($\langle \varepsilon_y \rangle - \Delta \varepsilon_y$). During this phase, the bundle's stress-strain response is completely linear.

3. Yielding and elastic behavior

- When the strain reaches the first yielding threshold, the first fiber begins to yield. We now account for the mixture of yielding and elastic fibers in the bundle:
 - Fibers that have already yielded contribute to the total force with both elastic and plastic components:

$$F_{\text{yielding}} = E \cdot \varepsilon_y + \alpha \cdot E \cdot (\varepsilon - \varepsilon_y),$$

where α is a coefficient describing the reduced stiffness of yielding fibers.

- Fibers that are still elastic contribute only through their elastic response:

$$F_{\text{elastic}} = E \cdot \varepsilon.$$

- To compute the stress on the bundle, we sum the forces contributed by all fibers and divide by the total number of fibers N :

$$\sigma = \frac{F_{\text{total}}}{N}.$$

- After each yielding event, strain is incremented just enough to provoke the next yielding event.

4. Yielding-only phase

- Once all fibers in the bundle have started to yield but none have yet broken, the bundle enters a yielding-only phase.
- In this phase, strain is incremented gradually from the maximum yielding strain ($\langle \varepsilon_y \rangle + \Delta \varepsilon_y$) to the smallest breaking threshold ($\langle \varepsilon_b \rangle - \Delta \varepsilon_b$).
- The total force is calculated by summing the contributions from all yielding fibers, and stress is updated accordingly.

5. Yielding and breaking behavior

- When the strain reaches the first breaking threshold, fibers begin to break. We now account for the combination of yielding and breaking fibers in the bundle:
 - Only unbroken fibers contribute to the total force. For these fibers, the force is calculated as before:

$$F = \sum (\text{Force contributions of unbroken fibers}).$$

- The strain is incremented event by event to provoke the breaking of each subsequent fiber.

6. Broken-only phase

- After all fibers in the bundle have broken, the stress drops to zero, as the bundle can no longer bear any load.

This simulation method has been used in the previous chapter to obtain the stress-strain curves of the bundle numerically. However, the main motivation for the numerical treatment of the system is to understand the microscopic process of failure under a slowly increasing external load.

II. Avalanche Dynamics in the Fiber Bundle Model

It has been shown experimentally that the failure of heterogeneous materials is not a smooth process, instead it proceeds in bursts of local breakings. Such bursts correspond to the nucleation and propagation cracks which generate acoustic noise, hence, they can be recorded in the form of noise packages [3, 4, 7, 9]. These acoustic emission measurements provide valuable information about the microscopic dynamics of failure [3, 4, 6]. For theoretical studies, it is a crucial challenge to understand the statistical and dynamical features of breaking bursts, hence, we use the simulation technique presented above to explore how failure of our elastic-plastic bundle proceeds at the level of fibers [8, 9, 11, 12].

1. Avalanche size distribution

In the fiber bundle model, the failure process is governed by the occurrence of avalanches of fiber breakings which are analogous to breaking bursts of experiments [1, 2, 3, 4, 10, 11, 13, 14]. An avalanche is defined as a sudden collective failure event during which multiple fibers in the bundle break simultaneously. Avalanches are always triggered by the breaking of a single fiber, which was provoked by an external load increment. The size Δ of an avalanche is quantified as the number of fibers that fail during such an event. Understanding the probability distribution of the size of the avalanche $p(\Delta)$ provides valuable insight into the failure dynamics of the system.

It is important to emphasize that after yielding has started, the load carried by intact (not broken) fibers is fluctuating due to the randomness of the yielding thresholds. Avalanches only occur under stress-controlled loading: Since the load is kept constant during an avalanche, failure events always increase the load on surviving fibers, which may trigger further breakings. As a consequence, through such breaking and load redistribution steps an avalanche is generated which may either stop after a few steps and the system stabilizes or it does not stop destroying the entire system. Such a destroying avalanche is called catastrophic.

The simulation algorithm presented in the previous section generates the evolution of the failure process under strain controlled loading. To reveal how the system behaves under increasing stress, we start from the numerically obtained stress-strain curve. Avalanches occur when $\sigma(\varepsilon)$ has a local "valley" where the stress falls below the stress value at which the last breaking event was provoked by an external load increase. To explore avalanches, we search for such local regimes of the $\sigma(\varepsilon)$ curve and count the number of fibers after which $\sigma(\varepsilon)$ overcomes the starting stress of the avalanche. Simulations were repeated for three different system sizes $N = 1000, 10.000, 100.000$.

Figure 3.1 illustrates the distribution of avalanche sizes $p(\Delta)$ for a fixed value of $\alpha = 1.0$ for each system size N considered averaging over $K = 1000$ bundles. It can be observed that $p(\Delta)$ has a power law functional form

$$p(\Delta) \sim \Delta^{-\tau} \quad (3.1)$$

over three orders of magnitudes of avalanche sizes in case of the largest system. The exponent τ of the power law was obtained by fitting $\tau = 5/2$.

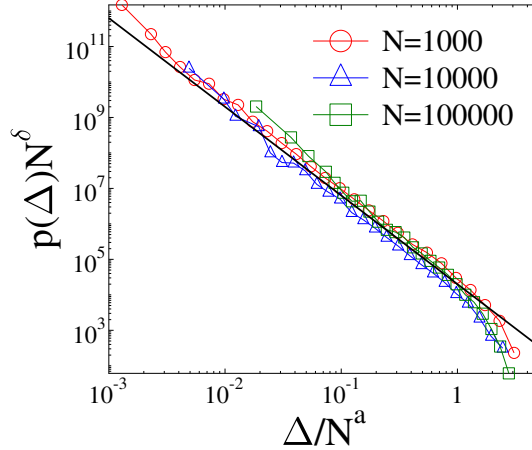


Figure 3.1: Avalanche size distribution $p(\Delta)$ of fiber bundles for three different system sizes N . The value of α was set to $\alpha = 1.0$. The straight line represents a power law of exponent $\tau = 5/2$.

Note that in the calculation of the avalanche size distribution $p(\Delta)$ the catastrophic avalanche is always excluded, since it has different statistics. Of course, in larger systems larger avalanches can be generated that's why in Fig. 3.1 we rescaled the distributions $p(\Delta)$ by appropriate powers a and δ of the number of fibers N along the horizontal and vertical axis, respectively. The good quality collapse of the curves implies that the scaling structure

$$p(\Delta, N) = N^{-\delta} \Phi(\Delta/N^a), \quad (3.2)$$

describes the N dependence of the distributions $p(\Delta, N)$. The value of the exponents that provide the best collapse is $a = 0.59 \pm 0.04$ and $\delta = 1.1 \pm 0.07$.

2. Dependence on the plastic coefficient α

At a fixed system size N , it is a crucial question how the value of the plastic coefficient α influences the microscopic process of failure, in particular, the avalanche size distribution. By systematically varying α , we can observe how the dynamics of failure and the size distribution of avalanches evolve. Most notably, I want to understand how failure avalanches change when approaching the critical α_c , where the transition to abrupt failure occurs.

Figure 3.2 shows the avalanche size distribution of the bundle for different values of α . The results reveal a clear dependence of avalanche characteristics on α : at high values of α a broad distribution $p(\Delta)$ is obtained with avalanche sizes spanning more than 3 decades in a system of $N = 10^5$ fibers. It is important to emphasize that $p(\Delta)$ can be very well described by a power law Eq. (3.1) of exponent $\tau = 5/2$. Note that $\tau = 5/2$ is the mean field avalanche size exponent when all the fibers share the same load [1, 2, 11]. It was proven analytically that this exponent is universal in the sense that it does not depend on the specific form of the strength distribution until the stress everywhere is the same in the system.

Decreasing α , first the distribution remains robust, but at about $\alpha \approx 0.1$ it starts to change in the regime of small avalanches. The functional form of $p(\Delta)$ changes in such a way that a second power law regime of a lower exponent $\tau = 3/2$ develops in the regime of small avalanches. It is important to note that as α decreases, the avalanche size Δ_c

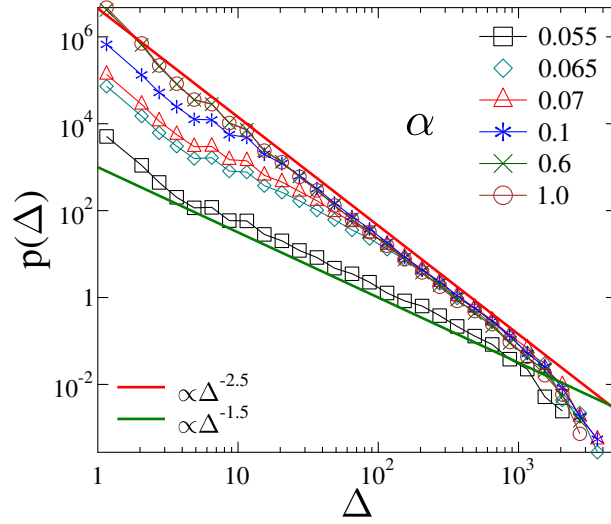


Figure 3.2: Avalanche size distribution for different values of the plastic coefficient α for a system of size $N = 10^5$. In the limit of high α a power law distribution is obtained with exponent $\tau = 5/2$; however, for small α -s, a crossover occurs to a lower exponent $\tau = 3/2$.

where the crossover occurs between the two power laws, gradually shifts to higher values. In the limit $\alpha \rightarrow \alpha_c$ the distribution becomes again a single power law with the lower exponent $\tau = 3/2$. The lower exponent implies that the distributions $p(\Delta)$ decays slower due to the increasing fraction of large avalanches. In the limit $\alpha \rightarrow \alpha_c$ the failure of the bundle becomes completely abrupt which means that the first breaking fiber triggers the catastrophic avalanche and destroys the system.

3. Finite size scaling of the largest avalanche

To understand how the transition to abrupt failure occurs when reducing the plastic coefficient, we performed a large amount of simulations at three system sizes $N = 1000, 10,000, 100,000$ varying α in a broad range. At each parameter set averages were calculated over $K = 1000$ bundles. From these simulations, I extracted the average size of the largest avalanche $\langle \Delta_{max} \rangle$ since it is a key quantity to characterize how sensitive the system is to the external load increments at given parameter sets.

Figure 3.3 presents the average size of the largest avalanche $\langle \Delta_{max} \rangle$ as a function of α for the three N values considered. In a larger system larger avalanches are triggered that's why I carried out a finite size scaling analysis: since the size of the largest avalanche is a measure of the susceptibility of the system to the external stimuli, I used the scaling relation of susceptibility well known for continuous phase transitions [17], I rescaled the two axis with proper powers of the number of fibers shifting also α with the value of the critical point α_c . The good quality data collapse in Fig. 3.3 shows that the α and N dependence of the average largest avalanche can be cast into the form [17]

$$\langle \Delta_{max} \rangle (\alpha, N) = N^{\gamma/\nu} \Psi((\alpha - \alpha_c) N^{1/\nu}). \quad (3.3)$$

Here $\Psi(x)$ denotes the scaling function. The validity of the scaling form Eq. (3.3) implies that the transition to abrupt failure exhibits analogy to continuous phase transitions occurring at α_c . From the scaling form Eq. (3.3) I could determine the correlation length exponent ν . The maximum of the

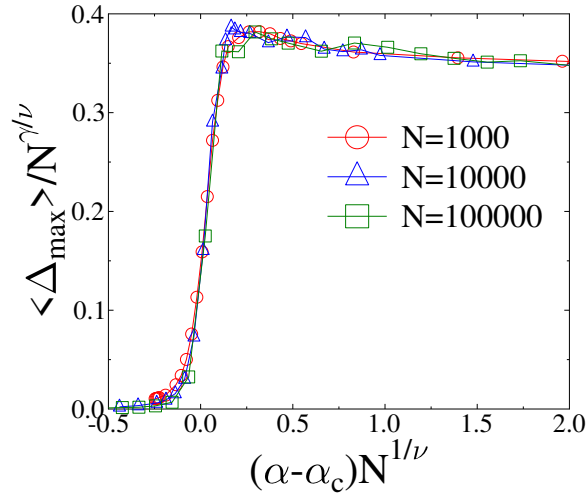


Figure 3.3: Scaling plot of the average size of the largest avalanche $\langle \Delta_{max} \rangle$ as a function of the plastic coefficient α . Simulations were carried out for three system sizes N . Rescaling the two axis with proper powers of the number of fibers N and shifting by α_c along the horizontal axis, good quality data collapse is obtained. The values of ν , γ , and α_c were adjusted to obtain best collapse.

$\langle \Delta_{max} \rangle$ curves defines the critical point of the finite size system, however, α_c in Eq. (3.3) is the critical point of the infinite system $N \rightarrow \infty$. From the scaling of the maximum

we can determine the critical exponent γ of the susceptibility. The parameter values providing best collapse are: $\nu = 3.98 \pm 0.06$, $\gamma = 2.65 \pm 0.05$, and $\alpha_c = 0.0536 \pm 0.0004$.

Chapter 4

Discussion

In my TDK research I was investigating the failure process of a bundle of plastic fibers under a slowly increasing external load. Plastic fiber reinforced composite have a broad spectrum of potential application; however, a detailed understanding of their mechanical and fracture behavior is needed to make them a reliable engineering material. Recently, experiments have been performed with single fibers of polyethylene and with bundles of them without a carrier matrix. These experiments showed that fibers have an elastic - plastic behavior where both the yielding and breaking thresholds exhibit a statistical variation, however, their elastic moduli are fairly constant. The disorder of the fiber characteristics is the source of probabilistic behavior of the bundle on the macro-scale.

To understand how the failure process of the bundle evolve under a slowly increasing load we introduced a simple fiber bundle model where fibers have a mechanical response similar to the outcomes of the experiments. The bundle was assumed to be loaded between two stiff plates which ensures that the strain is always the same for all the fibers during the loading process. First, I derived the macroscopic constitutive equation of the system. To obtain a closed analytic solution from the general stress-strain relation of the bundle, I assumed uniformly distributed threshold values. The analytical results were in a good qualitative agreement with the results of experiments. As a very important outcome I showed that decreasing the plastic coefficient, the bundle exhibits a transition to an abrupt failure, where the first fiber breaking initiates the collapse of the bundle. The transition occurs at a critical value of the plastic coefficient α_c .

On the micro-scale, failure of the bundle proceeds in bursts, which could be recorded in the form of acoustic noise. Such acoustic emissions are the main source of information about the microscopic process of failure. In our fiber bundle model, fibers break in avalanches of local breaking events, which correspond to the acoustic events in real experiments. We worked out a simulation technique, which revealed how avalanches are formed under stress controlled loading of the bundle. The simulation technique could also be used to obtain the stress-strain curve of the bundle, which was in a very good agreement with the analytical results.

We demonstrated that the statistics of avalanche sizes strongly depends on the plastic coefficient of the fibers, which determines the remaining stiffness of the fibers after the yield point. When the plastic coefficient falls close to one, power law distribution of avalanche sizes is obtained where the exponent coincides with the mean field exponent of fiber bundles $\tau = 5/2$, i.e. with the exponent obtained when all the fibers share the

same load. The result is very interesting because in the plastic bundle the load of fibers is heterogeneous, after yielding started the load must fluctuate from fiber to fiber. The results indicate that even in the heterogeneous stress field, avalanches follow power law statistics with the same exponent as in a homogeneous stress field. The origin of the common behavior is the constant strain fixed by the stiff loading plates.

Our simulations reveal that the plastic coefficient has a substantial effect on the avalanche size distribution: as the coefficient gets reduced, i.e. fibers keep less and less load beyond the yield point and the load on fibers becomes inhomogeneous, the fraction of small avalanches decreases, while the one of the large avalanches increases. The avalanche size distribution exhibits a crossover between two power law regimes of different exponents: for large avalanches the $\tau = 5/2$ exponents prevails, however, for the small ones a smaller exponent $\tau = 3/2$ occurs. As the plastic coefficient approaches the critical point of abrupt failure again a single power law is obtained with the lower exponent.

To understand how the transition to abrupt failure occurs as the plastic coefficient is reduced, I carried out a finite size scaling analysis of the average size of the largest avalanche. As the most notable outcome of my calculations I showed that the transition to abrupt failure emerges analogously to continuous phase transitions. Based on the numerical analysis I could determine two critical exponents of the transitions and the critical point in the limit of infinite system sizes.

Results were presented in the thesis for the case of uniformly distributed threshold values. Tests with other types of distributions are in progress. My expectation is that the avalanche size distribution exponents and the critical exponents are robust against the degree of disorder. However, the critical point has a non-trivial dependence on the model parameters.

Appendix: Own Results

Scientific Questions (Motivations)

- What governs the failure processes of plastic fiber bundles under increasing load?
- How do elastic-plastic behavior and statistical variations in fiber properties affect macroscopic and microscopic failure?
- How does the plastic coefficient α control the transition from gradual to abrupt failure?
- What insights do avalanche dynamics provide into microscopic failure mechanisms?

Methods Applied

- Developed a fiber bundle model (FBM) with elastic-plastic behavior and statistical thresholds.
- Derived macroscopic stress-strain curves using uniform distributions for thresholds.
- Simulated failure processes to analyze stress-strain behavior, avalanche dynamics, and plastic coefficient effects.
- Performed finite-size scaling to study transitions to abrupt failure.
- Validated results by comparing analytical, numerical, and experimental data.

Results Obtained

- Identified a critical plastic coefficient ($\alpha_c \approx 0.0536$) marking the transition to abrupt failure.
- Stress-strain curves aligned well with experiments, showing elastic, plastic, and failure regimes.
- Avalanche sizes followed power laws ($\tau = 5/2$ for large, $\tau = 3/2$ for small avalanches near α_c).
- Decreasing α increased large avalanches and reduced small ones, leading to a crossover between power-law regimes.
- Finite-size scaling revealed phase-transition-like behavior, with critical exponents $\nu = 3.98$, $\gamma = 2.65$.
- Predicted robustness of avalanche statistics to disorder, with critical points depending on parameters.

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