

Ideal Wall Mosaics

Research Paper for Students' Scientific Association

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1 Motivation

Walls are fundamental elements of spatial design, with a history spanning over 10,000 years [1]. They have been integral to the formation of early cities and continue to play a critical role in construction today. Given their long-standing significance, a comprehensive understanding of their structure is essential.

Masonry walls have attracted the attention of builders and architects for millennia, resulting in a long history of both construction and analysis. While numerous structural and historical investigations have been conducted, one of the earliest and most comprehensive frameworks for examining architecture comes from the Roman architect Vitruvius. His principles of stability, utility, and beauty have influenced not only the study of walls but the study of architecture as a whole [2].

Vitruvius' three pillars — stability from a structural perspective, utility from a functional perspective, and beauty from an aesthetic perspective — cover many aspects traditionally examined in architectural studies. However, one key element, the geometry of walls remains underexplored, despite its direct influence on structural stability.

The order in which Vitruvius presents these principles is noteworthy: first, a structure must be stable and resistant; second, it must be functional and useful; and finally, aesthetic considerations can be addressed. This hierarchy emphasizes the architect's primary responsibility to design durable structures. In the context of walls, *geometry* plays a significant role in determining stability. While individual walls have been analyzed in terms of their geometric characteristics, a broader correlation between geometry and the stability of walls across different types has not been fully established.

The importance of geometry in relation to stability has been recognized for thousands of years, as evidenced by early masonry structures. These structures demonstrate that ancient builders understood the critical role of bonding building elements for stability.



Figure 1: One of the oldest remaining walls in Jericho (left) built around 8500 BCE [1] with patterns similar to *brick bonding*. Inca wall in Cusco, Peru (right) in which *wall bond* is not defined yet.

In my former research papers [3] [4] [5], I analyzed wall mosaics - the pattern formed by the joints of masonry blocks - with the help of the mean field theory of convex mosaics to uncover the connection between geometry and stability. The theory is based on four fundamental variables: two versions of the respective average degrees of nodes and cells: in the first case, we count only sharp vertices, thus we distinguish between "Y" and "T" nodes and get the geometric averages $(\bar{n}^*, \bar{v}^*)$. In the second case, we do not distinguish between "Y" and "T" nodes and get the combinatorial averages $(\bar{n}, \bar{v})$.

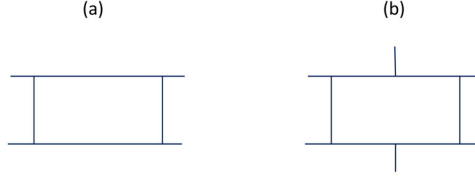


Figure 2: Example of non-ideal and ideal finite mosaic. (a) Non-ideal mosaic with $(\bar{n}_0^*, \bar{v}_0^*, \bar{n}_0, \bar{v}_0) = (2, 4, 3, 4)$, (b) Ideal mosaic with $(\bar{n}_0^*, \bar{v}_0^*, \bar{n}_0, \bar{v}_0) = (2, 4, 3, 6)$.

We will define these quantities later in more detail, however, we already note that their ratio ($\rho = \frac{\bar{n}^*}{\bar{v}^*} = \frac{\bar{n}}{\bar{v}}$, known as cell density) is an important parameter of the mosaic. One benefit of this theory is that it reveals new connections based on the position of walls on the symbolic plane $(\bar{n}^*, \bar{v}^*)$. In my previous research [3] [4] [5], I showed that cell density can be regarded as a generalization of bonding, as stable walls have high cell density ($\rho = 2.00$).

The mean field theory [6] of convex mosaics describes the properties of *infinite*, normal, balanced tilings (for definition of the latter properties see [7]). However, wall mosaics are inherently finite, meaning that the equations from the theory do not directly apply. In wall mosaics the size of the patterns and the emerging borders play a significant role, in particular, they influence the cell density. So, in order to be able to further analyze these patterns, we have to understand how finite samples behave and what borders mean.

The objective of this paper is to establish the mathematical framework for measuring and analyzing these finite mosaics. To this end, we define our central concept, the notion of *ideal mosaics*:

Definition 1. Let M_0 be a (finite or infinite) mosaic with parameters $(\bar{n}_0^*, \bar{v}_0^*, \bar{n}_0, \bar{v}_0)$. If there exists an infinite normal, balanced mosaic M with the same parameters then we say that M_0 is an ideal mosaic.

The main result of this paper is, on one hand, the construction of a function $\alpha(M_0) = \alpha(\bar{n}_0^*, \bar{v}_0^*, \bar{n}_0, \bar{v}_0) \geq 0$ which we call the *combinatorial ambiguity* of a mosaic, which is described in Theorem 1. On the other hand, we will also prove Lemma 5 claiming that $\alpha(M_0) = 0$ if and only if M_0 is an ideal mosaic. These results show that $\alpha(M)$ is a powerful tool in the geometric analysis of finite wall mosaics as it carries vital information on how well the finite sample represents infinite tilings to which the theory applies.

In the following chapters I will introduce the mean field theory of convex planar mosaics in detail which I will extend with a newly formed set of fundamental formulae. Later, I will present multiple independent methods to estimate the properties of an infinite mosaic based on a finite dataset. Then I will apply the estimations and define ideal wall mosaics.

2 On the average properties of convex planar mosaics

In this section we summarize some results from [6] and expand those formulae to obtain a new set of symmetrical governing equations for convex mosaics, which we will call the fundamental equations. We use the fundamental equations to define alternative strategies for the estimation of the parameters of infinite mosaics, based on finite datasets.

2.1 Some basic concepts

A planar mosaic is a system of polygons that cover the whole plane and have pairwise no common interior points. [8] - meaning that no gaps and overlapping parts between the domains are allowed. A mosaic is convex if these domains are convex and in this case all domains are convex polygons [[8], Lemma 10.1.1].

A *convex planar mosaic* is a covering of the plane with convex polygons. Every planar mosaic consists of F cells - the polygons, and of V nodes - the points where the vertices of the polygons overlap. We assign two types of degrees both to nodes and to cells.

- The *combinatorial* degree v_i of the i th cell counts the number of nodes on the perimeter of the cell.
- The *geometric* (or *corner*) degree v_i^* of the i th cell counts the number of vertices of the polygon.
- The *combinatorial* degree n_i of the i th node counts the number of edges (half-lines) meeting at the node.
- The *geometric* (or *corner*) degree n_i^* of the i th cell counts the number of polygonal vertices overlapping at that node.

We distinguish between regular and irregular nodes. For the former, we have $n_i^* = n_i$, for the latter we have $n_i^* < n_i$. In case of a regular node, the sum of the angles $\theta_{i,j}$, $j = 1, 2, \dots, n_i$ is the following:

$$\sum_{j=1}^{n_i^*} \theta_{i,j} = 2\pi, \quad (1)$$

and in the irregular case we have :

$$\sum_{j=1}^{n_i} \theta_{i,j} = \pi. \quad (2)$$

We denote the number of regular nodes by V_R and we define the *regularity* p of the mosaic as:

$$p = \frac{V_R}{V}. \quad (3)$$

Mosaics can be described by the averages of the above quantities which we call the respective nodal and cell degrees associated with the mosaic. The combinatorial nodal and cell degrees are defined as

$$\bar{n} = \frac{\sum_{i=1}^V n_i}{V} \quad \bar{v} = \frac{\sum_{i=1}^F v_i}{F} \quad (4)$$

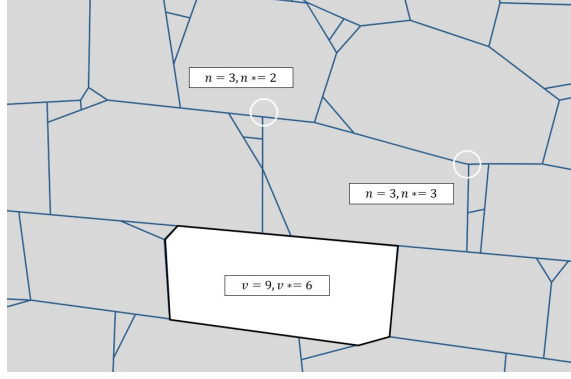


Figure 3: Graphical definition of $\bar{n}, \bar{v}$ combinatorial and $\bar{n}^*, \bar{v}^*$ geometrical degrees

and the geometric nodal and cell degrees of a mosaic are defined as

$$\bar{n}^* = \frac{\sum_{i=1}^V n_i^*}{V} \quad \bar{v}^* = \frac{\sum_{i=1}^F v_i^*}{F}. \quad (5)$$

These apply to non-convex polygons as well.

2.2 Formulae for infinite, balanced, convex tilings

The averages in (4) and (5) can be computed for any finite mosaic.

Definition 2. We call mosaics where the diameter of cells is bounded from both above and below normal tilings [7] and henceforth we only regard normal tilings.

Definition 3. If the mosaic is infinite then we regard the interior of a circle of radius R , we compute the averages inside the circle and then we compute the limits for the averages as $R \rightarrow \infty$. If the limits exist and they are independent of the center of the circle then we call these limits the nodal and cell degrees of the infinite mosaic and we call the mosaic a balanced tiling [7].

Definition 4. We call the $[\bar{n}^*; \bar{v}^*]$ plane the symbolic plane of tilings.

Definition 5. We call the ratio of the cell and nodal degrees the cell density of the mosaic and we denote this quantity by ρ . As we can see from (4) and (5), the cell density will be identical for the combinatorial and the geometric (corner) degrees:

$$\rho = \frac{\bar{v}}{\bar{n}} = \frac{\bar{v}^*}{\bar{n}^*} = \frac{V}{F}. \quad (6)$$

Next we seek a general formula linking cell and nodal degrees as introduced in [6] :

Lemma 1. Let M be a convex, normal, balanced mosaic. Then, for the nodal and cell degrees of M we can write

$$\bar{v}^* = \frac{2\bar{n}^*}{\bar{n}^* - p - 1}. \quad (7)$$

Proof. First, let us compute the sum of nodal and cell the degrees of the mosaic. If the mosaic is infinite then we should arrive at the same result whether we compute the sum for the nodes or for the cells. First we note that convex polygon with v^* vertices can be divided into $v^* - 2$ triangles without creating any new internal vertices, so the sum of angles in the i th cell is:

$$\sum_{j=1}^{v_i^*} \theta_{i,j} = \pi \cdot (v_i^* - 2). \quad (8)$$

Using equation (3) we can express the number of nodes as

$$V = V_R + V_I = p \cdot V + (1 - p) \cdot V. \quad (9)$$

Now we proceed by summing the angles in the mosaic both by nodes via equations (1),(2) and (9) and, on the other hand, per cells, via equation (8):

$$p \cdot V \cdot 2\pi + (1 - p) \cdot V \cdot \pi = (\bar{v}^* - 2)F\pi. \quad (10)$$

We cancel π , divide both sides by V , and using equation (6) yields:

$$\frac{\bar{v}^*}{\bar{n}^*}(p + 1) = \bar{v}^* - 2$$

By arranging this expression it results in the formula expressed in Lemma 1, so we completed the proof. $\square$

Corollary 1. *We note that from equation (7) we can also express the nodal degree:*

$$\bar{n}^* = \frac{\bar{v}^* \cdot (1 + p)}{\bar{v}^* - 2}. \quad (11)$$

As we can observe, equations (7) and (11) - as introduced in [6] - are asymmetric. Our next goal is to prove that there is a much more aesthetic set of formulae:

Lemma 2. *Let M be a convex, normal, balanced mosaic. Then, for the nodal and cell degrees of M we can write*

$$\bar{n}^* = \frac{2\bar{v}^*}{\bar{v} - 2}, \quad (12)$$

$$\bar{v}^* = \frac{2\bar{n}^*}{\bar{n} - 2}. \quad (13)$$

Proof. Based on equation (12)-(13) we can write

$$\frac{\bar{v}^*}{\bar{n}^*} = \frac{2}{\bar{n} - 2}, \quad \frac{\bar{n}^*}{\bar{v}^*} = \frac{2}{\bar{v} - 2}, \quad (14)$$

and using equation (6) this is equivalent to

$$\frac{\bar{v}}{\bar{n}} = \frac{2}{\bar{n} - 2}, \quad \frac{\bar{n}}{\bar{v}} = \frac{2}{\bar{v} - 2}. \quad (15)$$

The latter is the $p = 1$ substitution of Lemma 1, so the proof is completed. $\square$

Definition 6. We will call (12)-(13) the fundamental equations of the mosaic.

Next we prove

Lemma 3.

$$p = \bar{n}^* - \bar{n} + 1. \quad (16)$$

Proof. We first observe that each irregular node one angle is not counted, so we have

$$N - N^* = V - V_R. \quad (17)$$

Dividing by V yields

$$\bar{n} - \bar{n}^* = 1 - p, \quad (18)$$

which is equivalent to equation (16). $\square$

Remark 1. Substituting (16) into (7) and (11) we arrive at (12)-(13), so this is an alternative proof of Lemma 2.

Convex, planar, infinite mosaics only exist in a compact domain of the symbolic plane. We express this observation in

Lemma 4. Let M be a convex, normal, balanced mosaic. Then, for the nodal and cell degrees of M we can write

$$\bar{v}^* \geq 3 \quad (19)$$

$$\bar{v}^* \leq \frac{2\bar{n}^*}{\bar{n}^* - 2} \quad (20)$$

$$\bar{v}^* \geq \frac{2\bar{n}^*}{\bar{n}^* - 1} \quad (21)$$

$$\bar{v}^* \leq 2\bar{n}^*. \quad (22)$$

Proof. There is no polygon with less than 3 sides (less than 3 vertices), so $\bar{v}^* \geq 3$ and this proves (19). Equations (20) and (21) are equivalent to $0 \geq p \geq 1$, and this is evident from equation (3). Regarding (22), we consider the following: in case of regular nodes $n^* \geq 3$ and in case of irregular nodes $\bar{n}^* \geq 2$, so we have

$$N^* \geq 3V_R + 2(V - V_R) = 2V + V_R. \quad (23)$$

Dividing by V yields

$$\bar{n}^* \geq 2 + p. \quad (24)$$

From (7) we can express p as

$$p = \bar{n}^* - 1 - 2\frac{\bar{n}^*}{\bar{v}^*}. \quad (25)$$

Substituting (25) into (24) yields $\bar{v}^* \leq 2\bar{n}^*$, so we completed the proof. $\square$

Observation 1. We can observe that for convex mosaics, the maximal value of the cell density is $\rho = 2,00$.

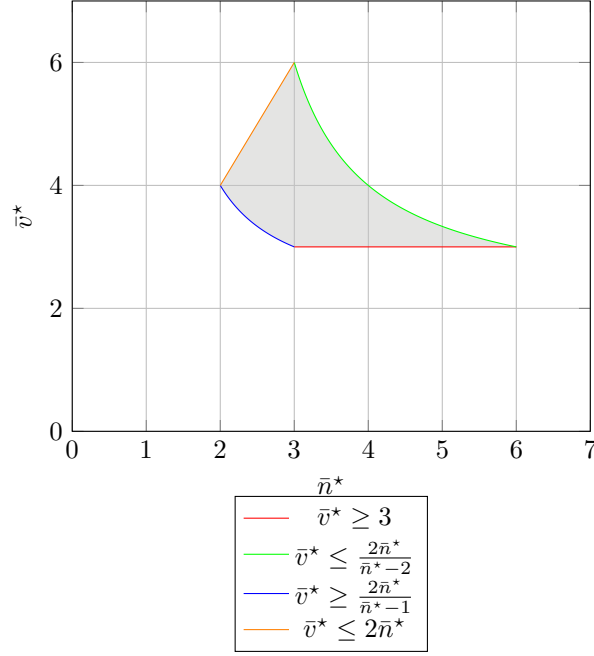


Figure 4: Domain of convex mosaics on the $[\bar{n}^*, \bar{v}^*]$ symbolic plane.

2.3 Alternative versions of the fundamental formulae

We used equation (6) to derive the fundamental formulae (12)-(13). It is evident by taking either of the two fundamental equations from (12)-(13) and combining it with equation (6) the other equation in (12)-(13) can be derived. In the following, we will demonstrate that combining the two equations in (12)-(13) also yields (6). To proceed, let us express $\bar{n}$ and $\bar{v}$ from the fundamental formulae (12)-(13) as

$$\bar{n} = 2 \frac{\bar{n}^*}{\bar{v}^*} + 2 \quad (26)$$

$$\bar{v} = 2 \frac{\bar{v}^*}{\bar{n}^*} + 2, \quad (27)$$

From equations (26)-(27) we obtain $\bar{v}^*/\bar{n}^* = \bar{v}/\bar{n}$, which is equivalent to equation (6). This also implies that

$$\bar{n} = \frac{2\bar{v}}{\bar{v} - 2} \quad (28)$$

$$\bar{v} = \frac{2\bar{n}}{\bar{n} - 2} \quad (29)$$

follow directly from the fundamental equations (12)-(13).

This implies that the fundamental equations in (12)-(13) are the only independent governing equations for mosaics. Next we use these formulae to estimate the parameters of infinite tilings.

3 Estimating the parameters of infinite mosaics based on finite datasets

In reality, infinite mosaics do not exist. All the mosaics we can study are finite, which means the equations of mean field theory, designed for infinite mosaics, do not apply. To continue analyzing these patterns, we must introduce new methods for examining mosaics. The goal of this chapter is to estimate the parameters of an infinite mosaic based on finite datasets from a finite mosaic.

Compared to the parameters of an infinite mosaic, those of a finite mosaic are affected by errors. First, the number of cells and nodes is limited, and based on equations (4) and (5) applied to finite datasets, it is evident that the degrees will deviate from those of an infinite mosaic. Second, in finite mosaics, borders emerge.

Thus, our next objective is to account for these two factors when studying mosaics. We aim to provide reliable estimates of the parameters of an infinite mosaic—so that the fundamental equations still hold—using data from the finite mosaic.

Definition 7. *Let M be a normal, balanced mosaic. The borders of M are those nodes, in which at least one of the incoming line segments belong to exactly one cell. There are V_{border} border nodes in M .*

When leaving the border nodes the degrees of M changes as the following.

$$\bar{v}_{interior} = \bar{v} \quad (30)$$

$$\bar{n}_{interior} = \frac{\sum_{i=1}^V n_i - \sum_{i=1}^V n_{i,border}}{V - V_{border}} \quad (31)$$

It is a crucial question whether we should consider borders. On one hand, a finite sample is complete with borders; on the other hand intuition tells that borders cause errors in the finite datasets, interfering with the "good" interior degrees.

As we will introduce in the next chapter through wall mosaics, there is a significant difference in the measured parameters from finite samples when borders are taken into account. It's unclear whether either dataset is reliable enough to draw further conclusions, whether all datasets need to be re-evaluated, or if they are even valid for analysis at all.

3.1 Statistical ambiguity

In order to see more clearly about the reliability of the datasets, we introduce a statistical estimation, which we will call *statistical ambiguity*.

Definition 8. *Let M be a normal tiling with the parameters $\bar{n}, \bar{v}, \bar{n}^*, \bar{v}^*$. The A_s statistical ambiguity of M is the area of the confidence-ellipse in case of combinatorial degrees:*

$$A_s = \left| \frac{(CI)_n}{2} \right| \cdot \left| \frac{(CI)_v}{2} \right| \cdot \pi \quad (32)$$

, with

$$(CI)_n = [\bar{n} + z \cdot \frac{s_n}{\sqrt{V}}; \bar{n} - z \cdot \frac{s_n}{\sqrt{V}}] \quad (33)$$

$$(CI)_v = [\bar{v} + z \cdot \frac{s_v}{\sqrt{F}}; \bar{v} - z \cdot \frac{s_v}{\sqrt{F}}] \quad (34)$$

, where

- $\bar{v}, \bar{n}$ are the measured degrees of M ,
- z is the confidence coefficient belonging to the 95% confidence level,
- s_n, s_v are the deviations of the measured degrees,
- V, F are the number of node and cell degrees in M .

The area of the confidence-ellipse in case of geometrical degrees can be calculated analogously.

We examine M as it was a random sample from an infinite mosaic. We do not consider the relations between the parameters of planar mosaics as introduced in the previous chapter. Instead we assume that the four variables $\bar{n}, \bar{v}, \bar{n}^*, \bar{v}^*$ are independent. The statistical analysis is carried out with the help of *confidence intervals*, which result two axes of the confidence-ellipse, of which area we will call the statistical ambiguity of M .

Confidence intervals are a fundamental tool in statistical analysis, and they depend on both the size and variability of the dataset. Yet, as we will demonstrate in the next chapter through wall mosaics, the statistical ambiguity shows no significant difference between the samples with borders and those without borders.

Since the statistical ambiguity does not explain how borders influence the parameters of the finite mosaic we need to introduce a new estimation method which considers the characteristics of planar mosaics.

3.2 Combinatorial ambiguity

As we described in the previous chapter, any planar tiling can be characterized by the 4 variables $\bar{n}, \bar{v}, \bar{n}^*, \bar{v}^*$. These 4 quantities are not independent, the fundamental equations in (12)-(13) define 2 relationships so there are 2 independent parameters determining the value of the 4 variables. Assume we have a finite mosaic and we measure the values of the 4 variables. In general, the measured values will not satisfy the fundamental equations (12)-(13).

Definition 9. A simple estimate for the 4 variables is obtained by the following strategy: based on physical considerations, we pick 2 measured variables which we regard as more reliable than the other 2. Then, using the fundamental formulae, we compute the values of the two unreliable variables from the values of the two reliable ones.

We have $\binom{3}{2} = 6$ simple estimates, which are summarized in Table 1. Since $\bar{n}$ and $\bar{v}$ are related by (28)-(29), estimate A is not an option. For the remaining 5 estimates B,C,D,E,F we summarized in Table 2 which form of the fundamental equations have to be applied to compute the two missing (unreliable) variables.

We will denote the respective estimates for the point $(\bar{n}, \bar{v})$ by B, C, D, E, F and for the point $(\bar{n}^*, \bar{v}^*)$ by B^*, C^*, D^*, E^*, F^* .

Each of the five estimates mean a geometrical construction on the symbolic plane, which results in the points B, C, D, E, F and B^*, C^*, D^*, E^*, F^* .

Definition 10. We denote the convex hull of a point set S by $C(S)$. For a 1D point set S the length of the convex hull will be denoted by $L_C(S)$ and for a 2D point set the area of the convex hull will be denoted by $A_C(S)$.

Variable	$\bar{n}$	$\bar{v}$	$\bar{n}^*$	$\bar{v}^*$
$\bar{n}$				
$\bar{v}$	A			
$\bar{n}^*$	B	C		
$\bar{v}^*$	D	E	F	

Table 1: Simple estimates

Equation Estimate	(12)	(13)	(26)	(27)	(28)	(29)
B		1.				2.
C		2.			1.	
D	1.					2.
E	1.				2.	
F			1.	2.		

Table 2: Equations for simple estimates. Serial numbers indicate the order in which the equations have to be applied.

Definition 11. Let M be a tiling with parameters $\bar{n}, \bar{v}, \bar{n}^*, \bar{v}^*$ and corresponding estimates B^*, C^*, D^*, E^*, F^* . Then we call $\alpha(M) = A_C(B^*, C^*, D^*, E^*, F^*)$ the combinatorial ambiguity of M .

To make this definition applicable to real datasets first we prove

Theorem 1.

$$\alpha(M) = \max\{A_1, A_2, A_3\},$$

where

$$A_1 = \frac{1}{4(\bar{n} - 2)}(\bar{h}_v^* - 2)^2(\bar{n}^* + \bar{v}^*)^2, \quad (35)$$

$$A_2 = \frac{\bar{n}^* \bar{v}^*}{8(\bar{n} - 2)(\bar{v} - 2)}(\bar{h} - 2)^2(\bar{n} + \bar{v})^2, \quad (36)$$

$$A_3 = \frac{1}{4(\bar{v} - 2)}(\bar{h}_n^* - 2)^2(\bar{n}^* + \bar{v}^*)^2. \quad (37)$$

and

$$\bar{h} = \frac{\bar{n}\bar{v}}{\bar{n} + \bar{v}}, \quad \bar{h}_v^* = \frac{\bar{v}^* \bar{n}}{\bar{n}^* + \bar{v}^*}, \quad \bar{h}_n^* = \frac{\bar{n}^* \bar{v}}{\bar{n}^* + \bar{v}^*}. \quad (38)$$

Proof.

Observation 2. We observe that both (B^*, C^*, F^*) and (D^*, E^*, F^*) are 1D point sets, to the former we refer as the vertical point set and to the latter as the horizontal point set. Their convex hulls $\mathcal{C}(B^*, C^*, F^*)$ and $\mathcal{C}(D^*, E^*, F^*)$ are two orthogonal line segments intersecting at F^* . This implies that

$$\alpha(M) = A_C(B^*, C^*, D^*, E^*, F^*) = \frac{1}{2} L_C(B^*, C^*, F^*) L_C(D^*, E^*, F^*). \quad (39)$$

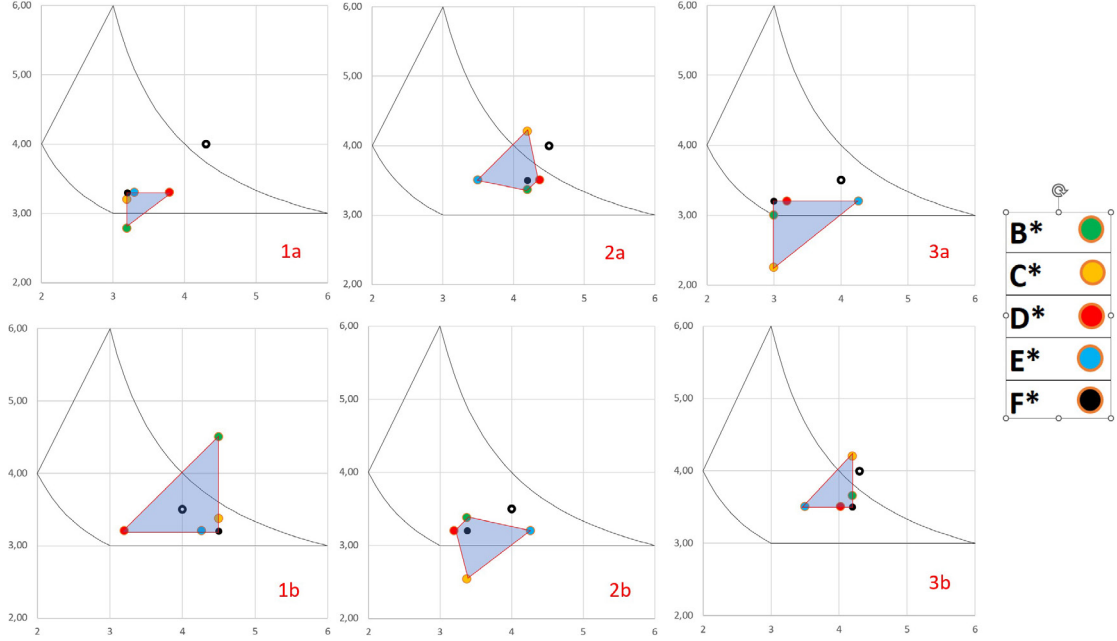


Figure 5: Patterns of the estimates B^*, C^*, D^*, E^*, F^* on the symbolic plane. Black circle denotes the point with coordinates $(\bar{n}, \bar{v})$. Observe that the formulae for the convex hull areas are pairwise identical.

Since both 1D point sets contain 3 points, equation (39) admits $3! \times 3! = 36$ alternative point patterns based on which $\alpha(M)$ can be computed. However, the patterns in the horizontal point set and the vertical point set are coupled, in fact, there are only 6 essential independent patterns (illustrated in Figure 5) which we list below.

Proposition 1. *The horizontal and vertical point sets can appear in one of the following patterns:*

$$\begin{array}{ll}
 1a. & [(B^* C^* F^*); (F^* E^* D^*)] \quad 1b. [(F^* C^* B^*); (D^* E^* F^*)] \\
 2a. & [(B^* F^* C^*); (E^* F^* D^*)] \quad 2b. [(C^* F^* B^*); (D^* F^* E^*)] \\
 3a. & [(C^* B^* F^*); (F^* D^* E^*)] \quad 3b. [(F^* B^* C^*); (E^* D^* F^*)].
 \end{array}$$

Proof. The coordinates of the individual points will be denoted as $\bar{n}_B, \bar{v}_B, \bar{n}_B^*, \bar{v}_B^*$ etc. Using Tables

1 and 2 we can write:

$$\bar{n}_B^* = \bar{n}^* \quad \bar{v}_B^* = \frac{2\bar{n}^*}{\bar{n} - 2} \quad (40)$$

$$\bar{n}_C^* = \bar{n}^* \quad \bar{v}_C^* = \frac{\bar{n}^*}{2}(\bar{v} - 2) \quad (41)$$

$$\bar{n}_D^* = \frac{\bar{v}^*}{2}(\bar{n} - 2) \quad \bar{v}_D^* = \bar{v}^* \quad (42)$$

$$\bar{n}_E^* = \frac{2\bar{v}^*}{\bar{v} - 2} \quad \bar{v}_E^* = \bar{v}^* \quad (43)$$

$$\bar{n}_F^* = \bar{n}^* \quad \bar{v}_F^* = \bar{v}^* \quad (44)$$

We first prove the Proposition 1 for pattern 1a, which can be formalized as

$$\{\bar{v}_B^* \leq \bar{v}_C^* \leq \bar{v}^*\} \iff \{\bar{n}^* \leq \bar{n}_E^* \leq \bar{n}_D^*\}. \quad (45)$$

Using equations (40)-(44), the this can be written as

$$\left\{ \frac{2\bar{n}^*}{\bar{n} - 2} \leq \frac{\bar{n}^*}{2}(\bar{v} - 2) \leq \bar{v}^* \right\} \iff \left\{ \bar{n}^* \leq \frac{2\bar{v}^*}{\bar{v} - 2} \leq \frac{\bar{v}^*}{2}(\bar{n} - 2) \right\}. \quad (46)$$

Since both inequalities are equivalent to

$$\frac{2}{\bar{n} - 2} \leq \frac{\bar{v} - 2}{2} \leq \frac{\bar{v}^*}{\bar{n}^*}, \quad (47)$$

we have completed the proof. The proof for the remaining patterns run analogously. $\square$

Now we complete the proof of Theorem 1.

We can express the areas $A_C(B^*, C^*, D^*, E^*, F^*)$ associated with the 6 point patterns in Proposition 1 as

$$A_1 = A_{1a} = A_{1b} = (\bar{v}_F^* - \bar{v}_B^*)(\bar{n}_D^* - \bar{n}_F^*)/2 \quad (48)$$

$$A_2 = A_{2a} = A_{2b} = (\bar{v}_C^* - \bar{v}_B^*)(\bar{n}_D^* - \bar{n}_E^*)/2 \quad (49)$$

$$A_3 = A_{3a} = A_{3b} = (\bar{v}_C^* - \bar{v}_F^*)(\bar{n}_F^* - \bar{n}_E^*)/2 \quad (50)$$

and utilizing (40)-(43) and (38) we can rewrite (48)-(50) to obtain (35)-(37). $\square$

Next we formalize a fundamental fact: the combinatorial ambiguity $\alpha(M)$ of a finite mosaic M can only vanish if the measured 4 parameters $\bar{n}, \bar{v}, \bar{n}^*, \bar{v}^*$ are identical with those of an *infinite* tiling:

Lemma 5. *Let M be a planar tiling with parameters $\bar{n}, \bar{v}, \bar{n}^*, \bar{v}^*$. Then we have $\alpha(M) = 0$ if and only if $\bar{n}, \bar{v}, \bar{n}^*, \bar{v}^*$ satisfy the fundamental equations (26)-(27). In this case, following Definition 1, we call M an ideal mosaic.*

Proof. We first observe that, since $A_i \geq 0$, in order to have $\alpha(M) = 0$, we must have

$$A_1 = A_2 = A_3 = 0. \quad (51)$$

Equations (35)-(37) reveal that the only solution of (51) is

$$\bar{h} = \bar{h}_v^* = \bar{h}_n^* = 2. \quad (52)$$

Next we observe that, considering equation (38), (26)-(27) are equivalent to

$$\bar{h}_v^* = \bar{h}_n^* = 2. \quad (53)$$

Next we observe that considering equation (38), (28)-(29) are equivalent to

$$\bar{h} = 2. \quad (54)$$

Since (28)-(29) have been derived from (26)-(27), this completes the proof. $\square$

4 Analysis of Wall Mosaics

In this chapter we will apply the two introduced estimation methods on real-life wall samples to get closer to the main goal of this paper: reaching a better understanding about the properties of finite mosaics.

Wall mosaics are finite, normal, mostly convex planar mosaics. We have measured seventeen existing walls based on photographs. The chosen samples cover a wide range of time and space, as shown in Table 3.

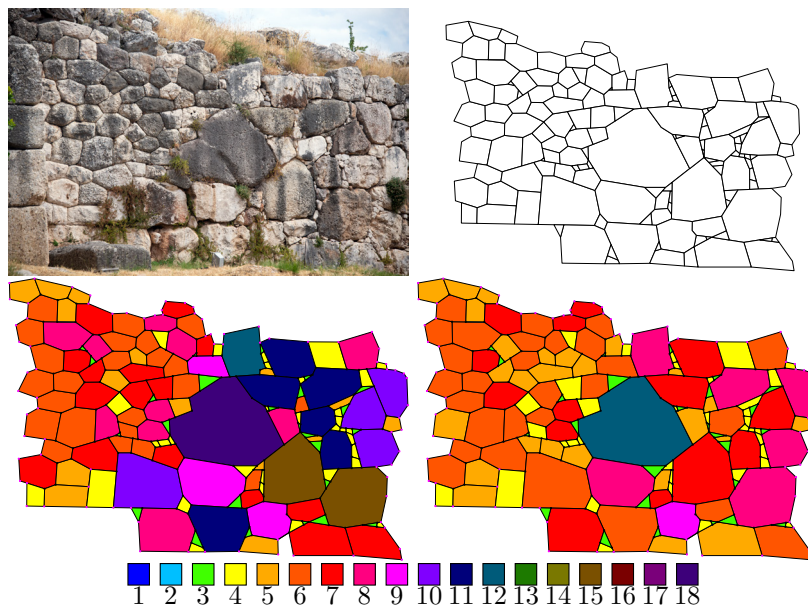


Figure 6: Photo of the City wall of Mycenae (1); The manually constructed wall mosaic (2); The automatically measured combinatorial degrees (3); The automatically measured geometrical degrees (4)

The process of analyzing wall mosaics is the following:

1. We search for a photograph where the cells and nodes are well-identifiable.
2. We construct the 2D wall mosaic manually based on the chosen photograph using a vector-graphics software.
3. These wall mosaics serve as input files for our software written in Python, which measures all the degrees and calculates the estimations for each mosaic.
4. The measured and calculated samples are then plotted.

All the measured degrees and calculated estimates are to be found in Appendix 1.

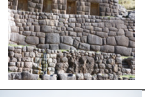
		Wall sample	Location	Built
1		Temple of Athena Nike, Acropolis	Athens, Greece	420 BCE [9]
2		Bulguksa Temple	Gyeongjeu, South Korea	751-774 [10]
3		Cusco	Cusco, Peru	1438–1471 [11]
4		Polygonal wall, Stoa of the Athenians	Delphi, Greece	478 BCE [12]
5		Hadrian's Wall	Northumberland, England	122- [13]
6		Ingapirca	Cañar Province, Ecuador	15th-16th Century [14]
7		Huaca de los monos, Kusilluchayuq	Cusco, Peru	1438–1471 [11]
8		Lubaantun	Toledo District, Belize	700-870 [15]
9		Machu Picchu	Cusco, Peru	1450–1510 [16]
10		Castelo dos Mouros	Sintra, Portugal	9th Century [17]
11		City Wall	Mycenae, Greece	13th Century BCE [18]
12		City Wall	Mycenae, Greece	13th Century BCE [18]
13		City Wall	Mycenae, Greece	13th Century BCE [18]
14		City Wall	Rhamnous, Greece	4-5th Century BCE [19]
15		Sacsayhuaman	Cusco, Peru	1438–1471 [11]
16		Tambomachay	Cusco, Peru	1438–1471 [11]
17		City Wall	Tiryns, Greece	13th Century BCE [18]

Table 3: The sampled walls.

In this chapter, we want to find the *ideal* wall mosaics, to which the fundamental formulae apply. In other words we want to find *finite* samples with parameters that could belong to *infinite* mosaics.

Fig.7 displays a significant difference between the samples with and without border. In order to analyze further these mosaics with the tools of the mean field theory we need to discover how close these real-life samples are to ideal mosaics.

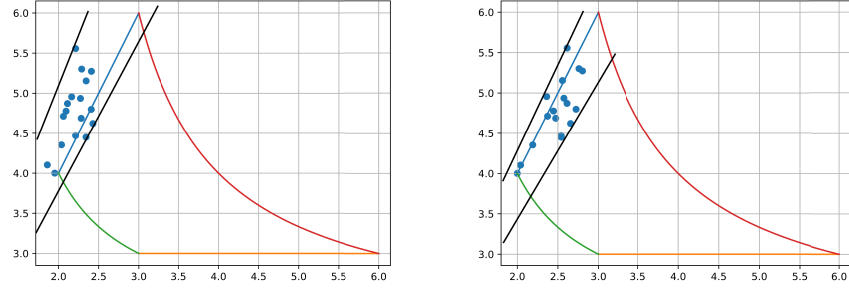


Figure 7: Degrees of the examined walls measured with (left) and without (right) borders. With borders the covered domain is $\rho = 1.90 - 2.51$ without $\rho = 1.74 - 2.13$ displayed on the $(\bar{n}; \bar{v})$ symbolic plane.

In order to do so, we will measure first the statistical and then the combinatorial ambiguities of the chosen wall samples.

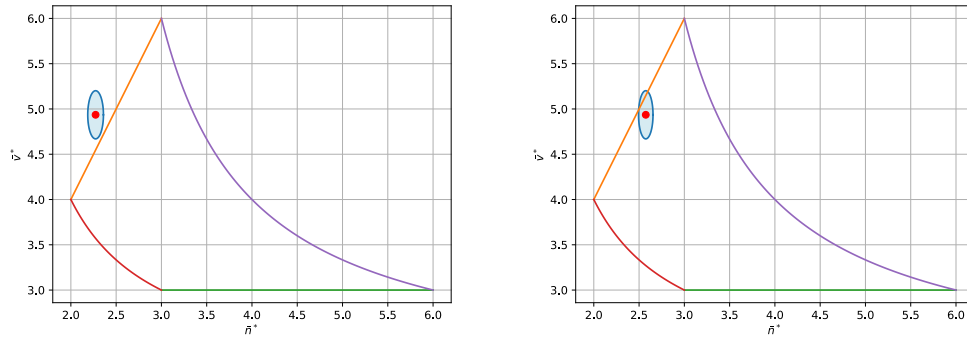


Figure 8: Statistical ambiguity of Sample 11 (City wall, Mycenae, Greece) with (left) and without (right) borders displayed on the $(\bar{n}; \bar{v})$ symbolic plane. $A_s = A_{s,interior} = 0,07$

As seen on Fig.8 and on the rest of the samples in Appendix 1, there is practically no difference between the statistical ambiguities when the sample is examined with and without borders.

Observation 3. *The statistical ambiguity is minimally sensitive to the leaving of the borders. For this reason it cannot be used for estimating parameters of wall mosaics.*

This phenomenon can be explained by the fact that the average degree of border nodes is consistently lower than that of interior nodes. As a result, excluding the border leads to a smaller deviation, which in turn reduces the statistical ambiguity A_s . At the same time, however, the number of nodes is also reduced, which results in a higher A_s . These two effects with the given datasets - in this case wall mosaics with around average 160 cells and 350 nodes - neutralize each other, which results in roughly constant statistical ambiguity per sample.

Since the statistical ambiguity is not sensitive at all to borders we need to investigate the combinatorial ambiguity.

The combinatorial ambiguity is dependent on the four measurable variables $(\bar{n}, \bar{v}, \bar{n}^*, \bar{v}^*)$ and the fundamental equations only. We expect that changes in $\alpha(M)$ between estimations with and without borders will be much more significant, since this estimation method considers the relations between the parameters of the mosaic, in other words the *quality* of the sample.

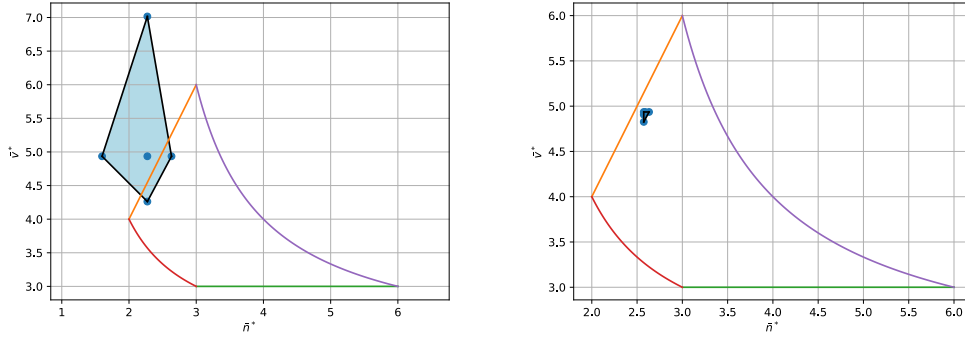


Figure 9: Combinatorial ambiguity of Sample 11 (City wall, Mycenae, Greece) with (left) and without (right) $\alpha(M) = 0.003$ borders displayed on the $(\bar{n}; \bar{v})$ symbolic plane. $\alpha(M) = 1.42$ $\alpha(M)_{interior} = 0.003$

In accordance with our expectation and as displayed on Fig.9 the combinatorial ambiguity is significantly larger with border than without.

Observation 4. *The combinatorial ambiguity is very sensitive to the presence of borders.*

This indicates that the combinatorial ambiguity is a reliable tool for estimating the parameters of wall mosaics, because $\alpha(M)$ is significantly smaller, when only the "good" interior parameters remain, when the *quality* of the parameters improve.

With all the estimated domains displayed on Fig.10, it is more visible that borders indeed increase the combinatorial ambiguity. Without borders, however, the domain of estimations is overlapping with the domain of the measured degrees from $\rho = 1.74$ to $\rho = 2.13$.

Now, I want to re-introduce the definition of *ideal* wall mosaics based on Def.1: we call a wall mosaic ideal if $\alpha(M) = 0$ applies. As displayed on Fig.10 the combinatorial ambiguities of the examined walls are rather small.

The extent of the overlapping of the domains on Fig.11 with the small ambiguities indicate that the parameters of the examined walls are reliable.

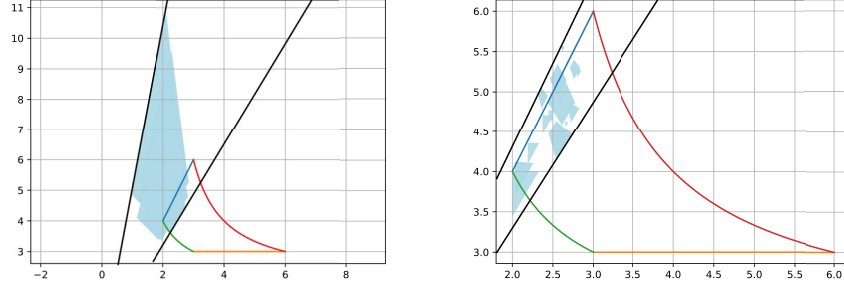


Figure 10: The combinatorial ambiguities of the examined walls measured with (left) and without (right) borders. With borders the covered domain is $\rho = 1.63 - 5.14$ without $\rho = 1.63 - 2.13$ displayed on the $(\bar{n}; \bar{v})$ symbolic plane.

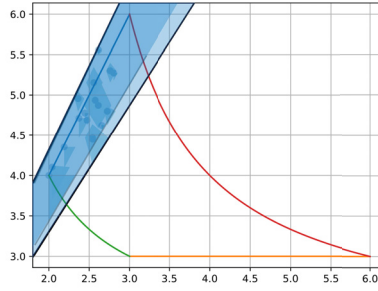


Figure 11: The overlapping domains of the measured degrees $\rho = 1.74 - 2.13$ and the estimations $\rho = 1.63 - 2.13$ displayed on the $(\bar{n}; \bar{v})$ symbolic plane.

Among this set of wall mosaics there is none that fulfills the definition of an ideal wall mosaic. However, with all the points presented above it is secure to claim that wall mosaics - in general - measured without borders can provide resource for further research, as the estimated combinatorial ambiguities are small.

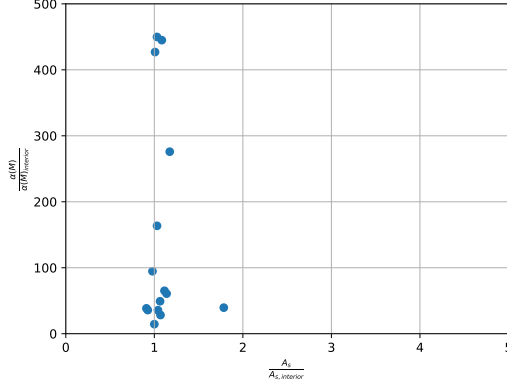


Figure 12: Change in the ambiguities

5 Conclusion

Wall mosaics are finite. Their parameters are filled with errors - compared to of infinite mosaics - caused by the limited size and borders of the sample. We aimed to define the ideal wall mosaics that are finite but have parameters that could be measured on an infinite mosaic.

In order to achieve it, we applied two different estimation methods. First, we calculated the statistical ambiguity of the samples as the area of the confidence-ellipse with the help of confidence intervals - a basic and universal tool of statistical analysis. The statistical ambiguity - when applied to the examined wall samples - showed no sensitivity to borders, so it could not be used in any further investigation.

Second, we newly presented the combinatorial ambiguity that - opposite to the statistical ambiguity - considered the relations between the $\bar{n}, \bar{v}, \bar{n}^*, \bar{v}^*$ parameters of the mosaic. For the introduction of the combinatorial ambiguity new symmetrical fundamental formulae had to be formed, so we revisited the basic definitions of the mean field theory of convex planar mosaics.

We described a formula for calculating $\alpha(M)$ combinatorial ambiguity. We proved that $\alpha(M) = 0$ can only occur if $\bar{n}, \bar{v}, \bar{n}^*, \bar{v}^*$ are parameters that are equal to parameters measured on an infinite sample. It means that in theory there are finite mosaics that fulfill the fundamental formulae - defined for infinite mosaics. Those wall mosaics that the former statement applies to are called *ideal* wall mosaics.

When applied to the wall samples the combinatorial ambiguity showed a significant decrease when the borders were removed opposite to the statistical ambiguity. Therefore, the correct way to estimate the parameters of wall mosaics - as shown on Fig.12 - is the combinatorial ambiguity since it indicates the quality of the sample well.

Among the examined walls there were none with $\alpha(M) = 0$, meaning that the definition of ideal wall mosaic applies to none. However the extent of overlapping of the domains of estimations ($\rho = 1.63 - 2.13$) and of the domains of degrees ($\rho = 1.74 - 2.13$) and the small combinatorial ambiguities indicate that wall mosaics in general measured without borders are close to ideal wall mosaics.

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- [15] Loa P. Traxler Robert J. Sharer. *The Ancient Maya*, page 194. Stanford University Press, 2006.
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- [17] Cultural landscape of sintra. *UNESCO World Heritage Centre*.
- [18] Mark Cartwright. Mycenae. *World History Encyclopedia*, 2009.
- [19] Rhamnous (site). *Perseus Digital Library*.

References of Figures

Figures in text

Fig.1 (left) *Digging Up Jericho. Chapter: Mudbricks and Modular Architecture at Tell es-Sultan from the Neolithic to the Bronze Age, Fig.10.* Copyright University of Rome 'La Sapienza' ROSEPAJ

Fig.1 (right) see Wall sample 9

Fig.6 (1) see Wall sample 11

Wall samples

Wall sample 1 *Temple of Athena Nike, Acropolis* by Spyros Kamilalis Worldhistory.org

Wall sample 2 *Bulguksa Temple* by the author

Wall sample 3 *Cusco* by McKay Savage Flickr.com

Wall sample 4 *Polygonal wall, Delphi* by Berthold Werner Worldhistory.org

Wall sample 5 *Hadrian's Wall* by Flickr User Richard 's Saved Photos Flickr.com

Wall sample 6 *Ingapirca* by Roman Korzh (All rights reserved) Flickr.com

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Wall sample 9 *Machu Picchu* by Howard Koons (All rights reserved) Flickr.com

Wall sample 10 *Castelo dos Mouros, Sintra* by Sára Lévy

Wall sample 11 *City wall, Mycenae 1* by Bruce Allardice (All rights reserved) Flickr.com

Wall sample 12 *City wall, Mycenae 2* by Berthold Werner Commons.wikimedia.org

Wall sample 13 *City wall, Mycenae 3* by Flickr User Romanus-too Flickr.com

Wall sample 14 *City Wall, Rhamnous* by George E. Koronaios Commons.wikimedia.org

Wall sample 15 *Sacsayhuaman* by Steve Whittington (All rights reserved) Flickr.com

Wall sample 16 *Tambomchay* by Flickr User C Bourne (All rights reserved) Flickr.com

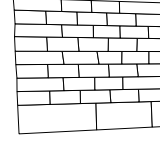
Wall sample 17 *City Wall, Tiryns* by Zdenek Kratochvil Commons.wikimedia.org

Appendices

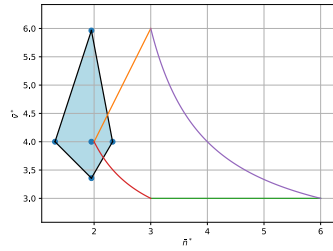
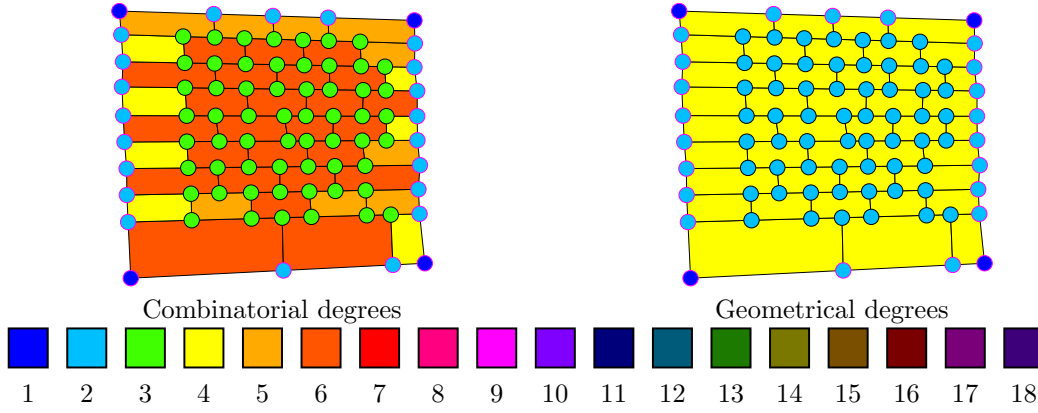
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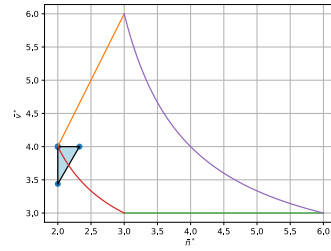
Temple of Athena Nike, Acropolis, Athens Greece



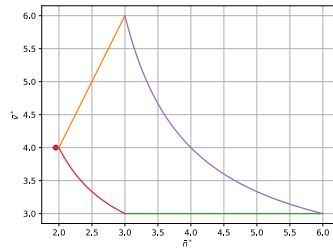
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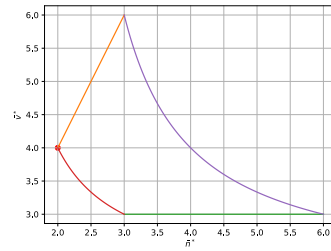
Geometrical uncertainty with borders



Geometrical uncertainty without borders



Statistical uncertainty with borders



Statistical uncertainty without borders

With border	$\bar{n}$	$\bar{v}$	$\bar{n}^*$	$\bar{v}^*$	ρ	ρ^*	$\alpha(M)$	A_s
	2.65	5.44	1.95	4.00	2.05	2.05	1.325	0.000
With no border	$\bar{n}$	$\bar{v}$	$\bar{n}^*$	$\bar{v}^*$	ρ	ρ^*	$\alpha(M)$	A_s
	3.00	5.44	2.00	4.00	1.81	2.00	0.092	0.000

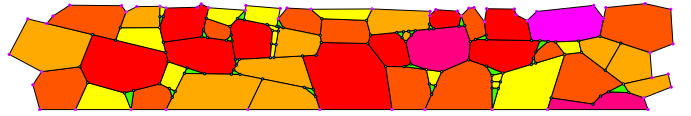
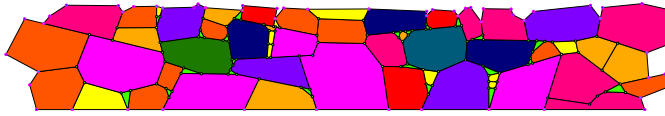
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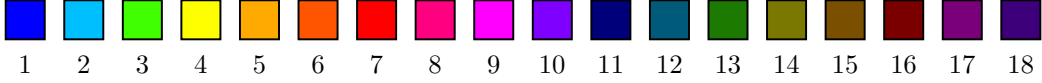
Bulguksa Temple, Gyeongju, South Korea



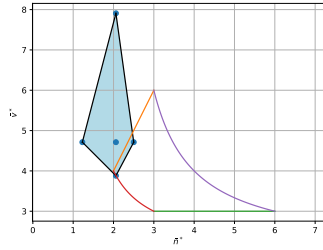
Wall mosaic



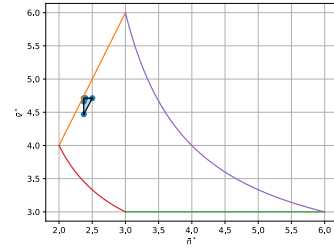
Combinatorial degrees



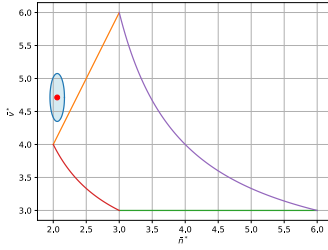
Geometrical degrees



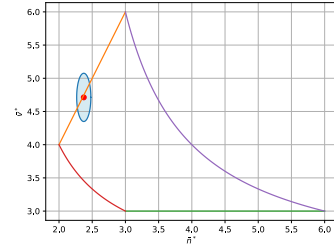
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Geometrical uncertainty without borders



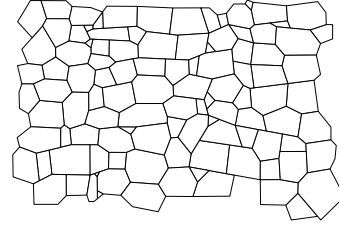
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Statistical uncertainty without borders

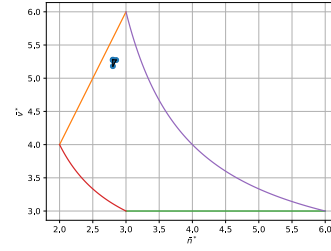
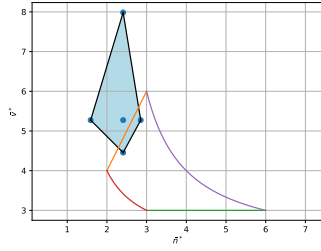
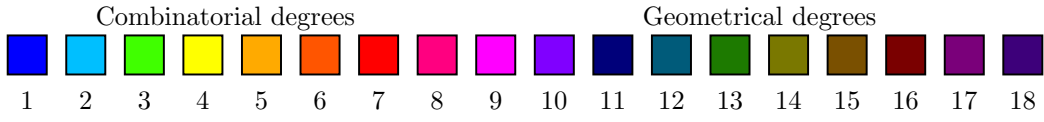
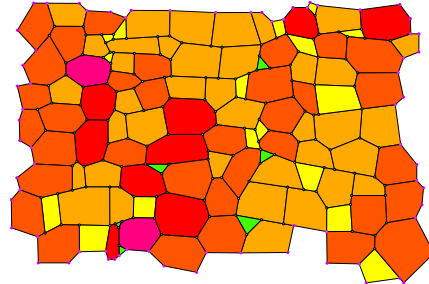
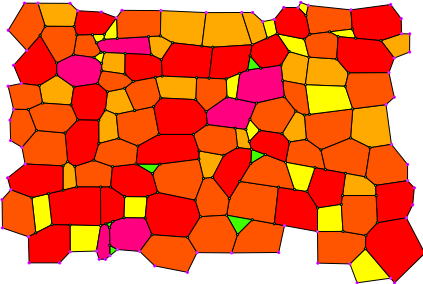
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	2.52	5.77	2.06	4.71	2.29	2.29	2.567	0.125
With no border	$\bar{n}$	$\bar{v}$	$\bar{n}^*$	$\bar{v}^*$	ρ	ρ^*	$\alpha(M)$	A_s
	3.02	5.77	2.37	4.71	1.91	1.99	0.016	0.121

Wall sample 3



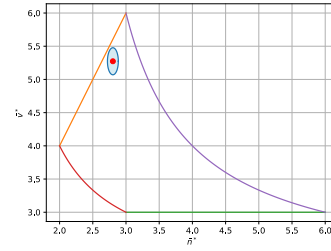
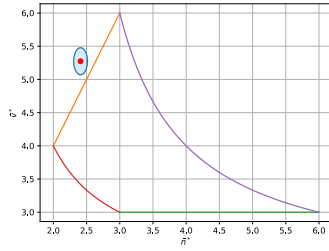
Cusco, Cusco, Peru

Wall mosaic



Geometrical uncertainty with borders

Geometrical uncertainty without borders



Statistical uncertainty with borders

Statistical uncertainty without borders

With border	$\bar{n}$	$\bar{v}$	$\bar{n}^*$	$\bar{v}^*$	ρ	ρ^*	$\alpha(M)$	A_s
	2.60	5.70	2.41	5.27	2.19	2.19	2.226	0.066
With no border	$\bar{n}$	$\bar{v}$	$\bar{n}^*$	$\bar{v}^*$	ρ	ρ^*	$\alpha(M)$	A_s
	3.06	5.70	2.80	5.27	1.86	1.88	0.003	0.052

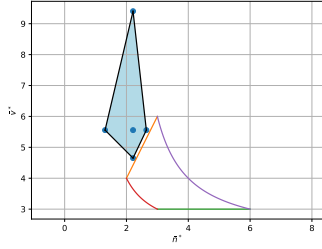
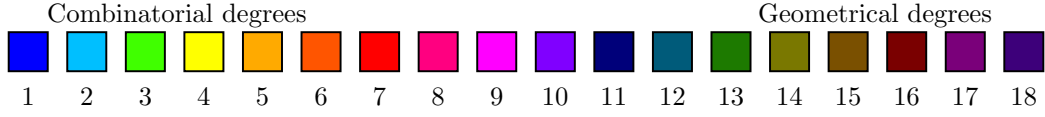
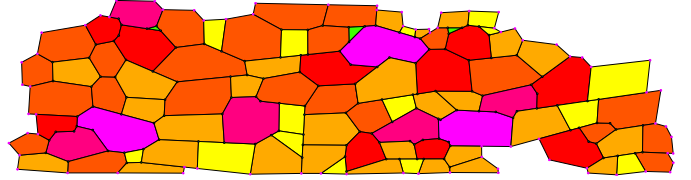
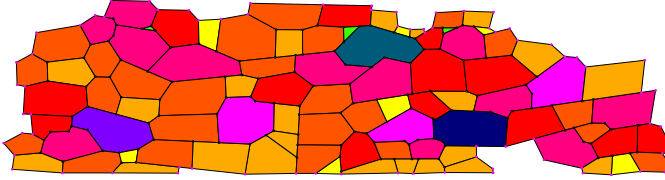
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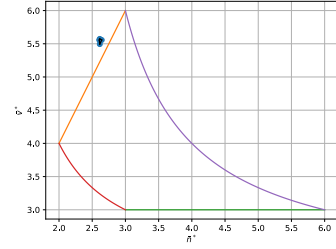
Polygonal Wall, Stoa of the Athenians, Delphi, Greece



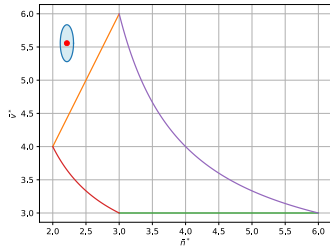
Wall mosaic



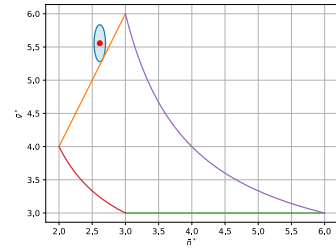
Geometrical uncertainty with borders



Geometrical uncertainty without borders



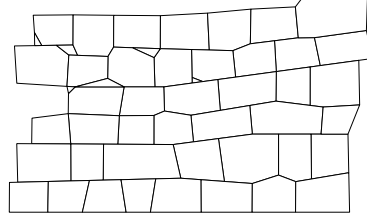
Statistical uncertainty with borders



Statistical uncertainty without borders

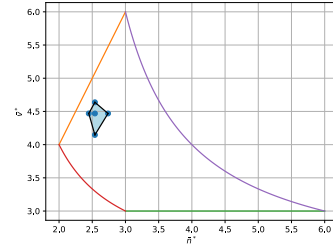
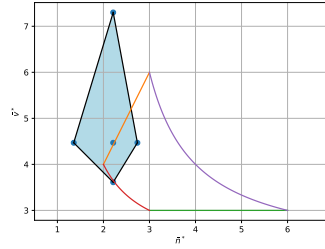
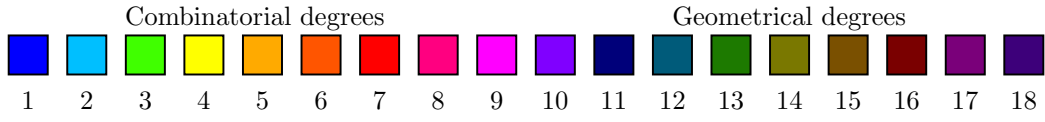
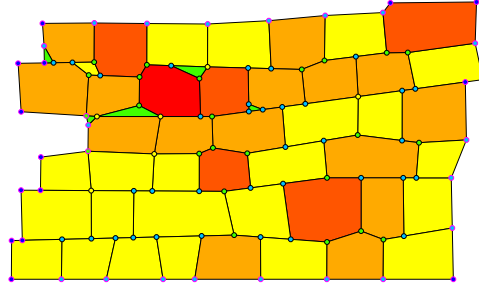
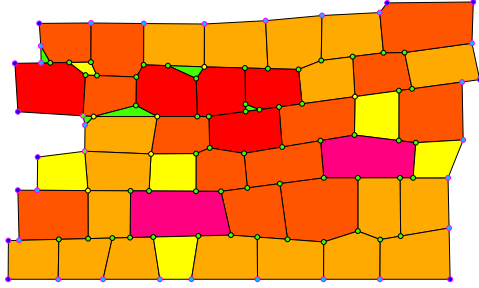
With border	$\bar{n}$	$\bar{v}$	$\bar{n}^*$	$\bar{v}^*$	ρ	ρ^*	$\alpha(M)$	A_s
	2.47	6.20	2.21	5.56	2.51	2.51	3.174	0.087
With no border	$\bar{n}$	$\bar{v}$	$\bar{n}^*$	$\bar{v}^*$	ρ	ρ^*	$\alpha(M)$	A_s
	2.94	6.20	2.61	5.56	2.11	2.13	0.001	0.075

Wall sample 5



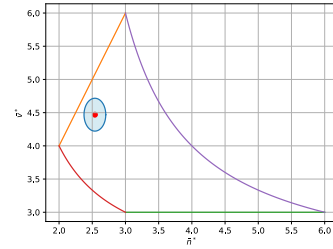
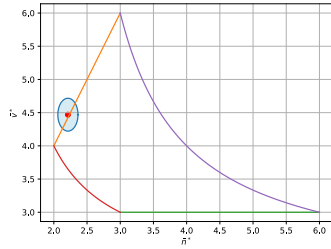
Hadrian's Wall, Northumberland, England

Wall mosaic



Geometrical uncertainty with borders

Geometrical uncertainty without borders



Statistical uncertainty with borders

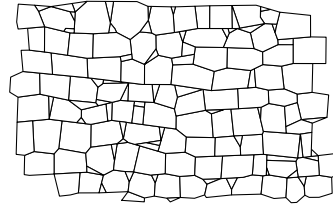
Statistical uncertainty without borders

With border	$\bar{n}$	$\bar{v}$	$\bar{n}^*$	$\bar{v}^*$	ρ	ρ^*	$\alpha(M)$	A_s
	2.61	5.27	2.21	4.47	2.02	2.02	2.551	0.118
With no border	$\bar{n}$	$\bar{v}$	$\bar{n}^*$	$\bar{v}^*$	ρ	ρ^*	$\alpha(M)$	A_s
	3.10	5.27	2.54	4.47	1.70	1.76	0.071	0.127

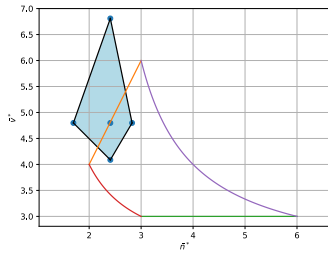
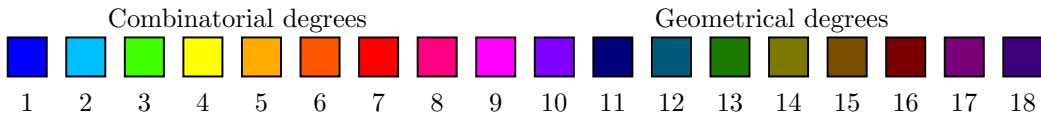
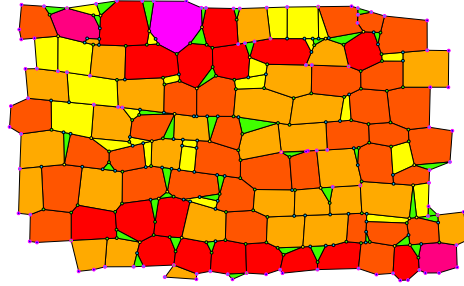
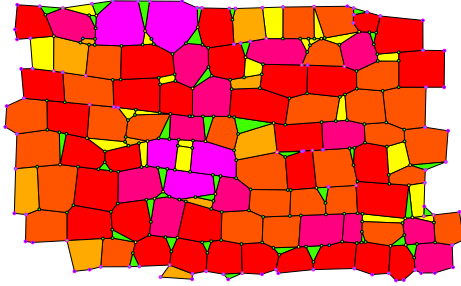
Wall sample 6



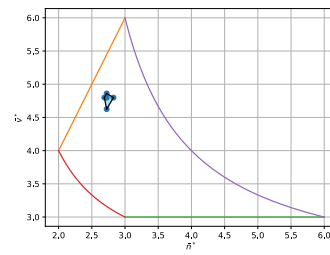
Ingapirca, Cañar Province, Ecuador



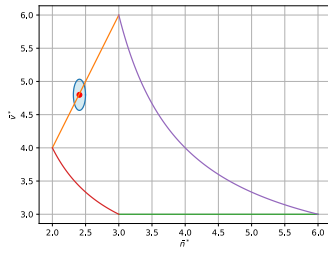
Wall mosaic



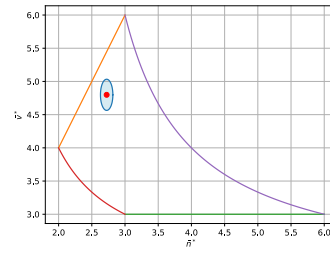
Geometrical uncertainty with borders



Geometrical uncertainty without borders



Statistical uncertainty with borders



Statistical uncertainty without borders

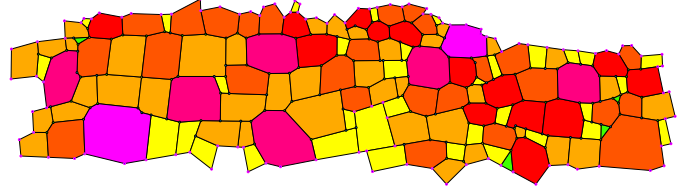
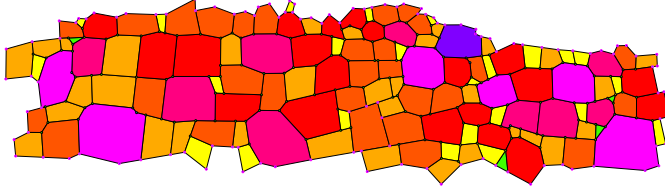
With border	$\bar{n}$	$\bar{v}$	$\bar{n}^*$	$\bar{v}^*$	ρ	ρ^*	$\alpha(M)$	A_s
	2.71	5.40	2.41	4.80	1.99	1.99	1.537	0.067
With no border	$\bar{n}$	$\bar{v}$	$\bar{n}^*$	$\bar{v}^*$	ρ	ρ^*	$\alpha(M)$	A_s
	3.12	5.40	2.72	4.80	1.73	1.76	0.016	0.069

Wall sample 7

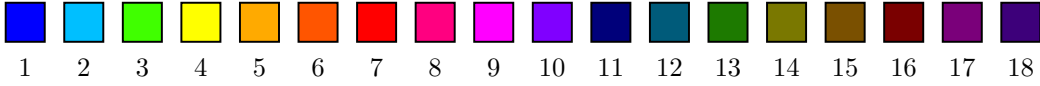


Huaca de los monos, Kusilluchayuq, Cusco, Peru

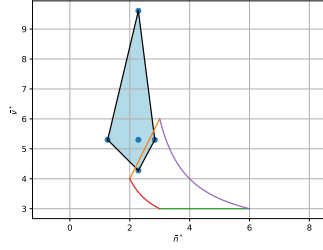
Wall mosaic



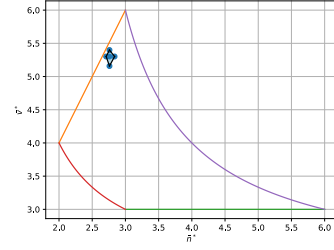
Combinatorial degrees



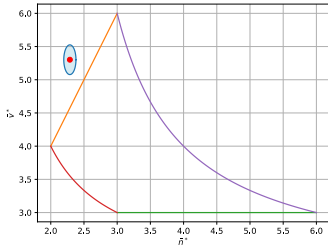
Geometrical degrees



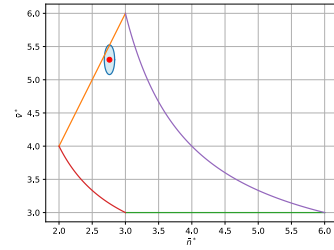
Geometrical uncertainty with borders



Geometrical uncertainty without borders



Statistical uncertainty with borders



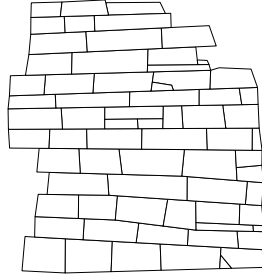
Statistical uncertainty without borders

With border	$\bar{n}$	$\bar{v}$	$\bar{n}^*$	$\bar{v}^*$	ρ	ρ^*	$\alpha(M)$	A_s
	2.48	5.74	2.29	5.30	2.32	2.32	4.201	0.064
With no border	$\bar{n}$	$\bar{v}$	$\bar{n}^*$	$\bar{v}^*$	ρ	ρ^*	$\alpha(M)$	A_s
	3.02	5.74	2.76	5.30	1.90	1.92	0.015	0.055

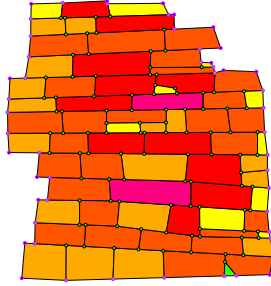
Wall sample 8



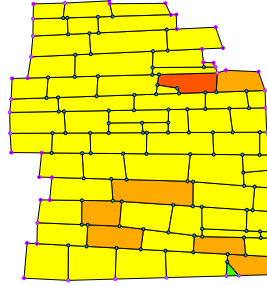
Lubaantun, Toledo District, Belize



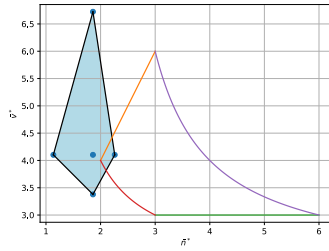
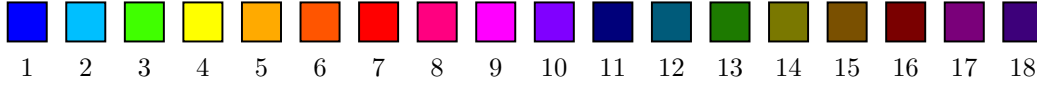
Wall mosaic



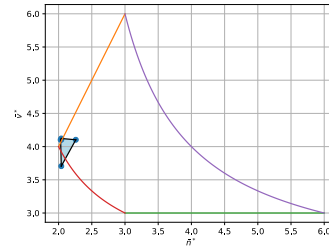
Combinatorial degrees



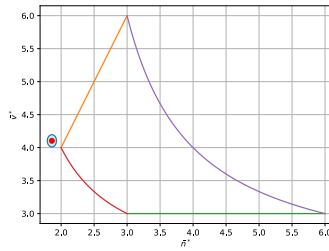
Geometrical degrees



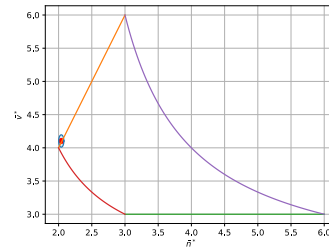
Geometrical uncertainty with borders



Geometrical uncertainty without borders



Statistical uncertainty with borders



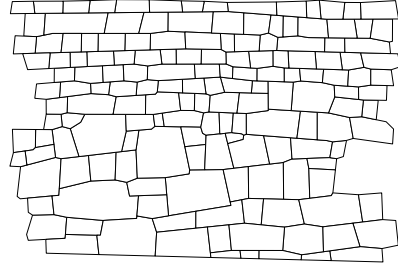
Statistical uncertainty without borders

With border	$\bar{n}$	$\bar{v}$	$\bar{n}^*$	$\bar{v}^*$	ρ	ρ^*	$\alpha(M)$	A_s
	2.55	5.63	1.86	4.10	2.21	2.21	1.880	0.020
With no border	$\bar{n}$	$\bar{v}$	$\bar{n}^*$	$\bar{v}^*$	ρ	ρ^*	$\alpha(M)$	A_s
	2.99	5.63	2.04	4.10	1.88	2.01	0.048	0.011

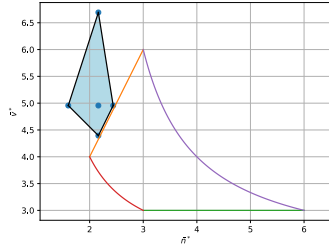
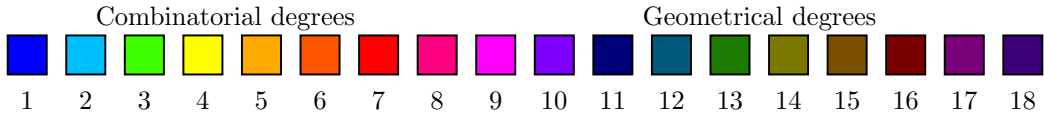
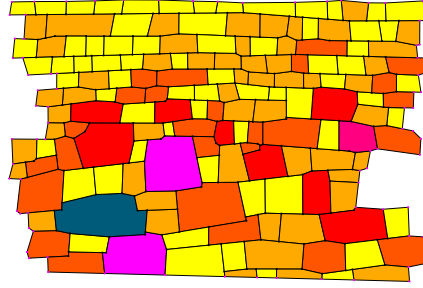
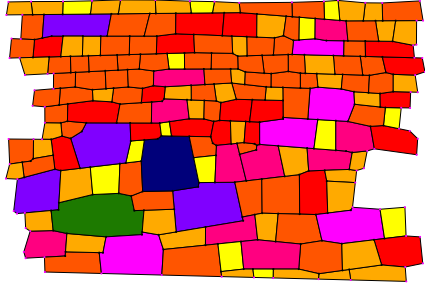
Wall sample 9



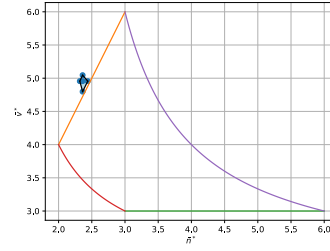
Machu Picchu, Cusco, Peru



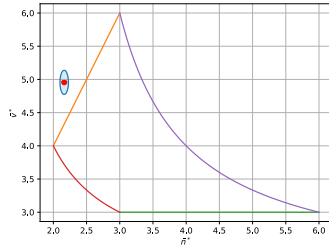
Wall mosaic



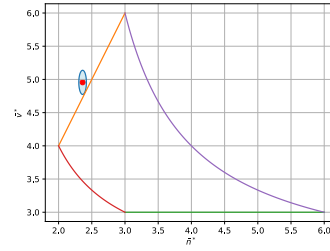
Geometrical uncertainty with borders



Geometrical uncertainty without borders



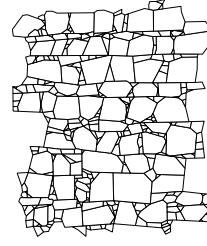
Statistical uncertainty with borders



Statistical uncertainty without borders

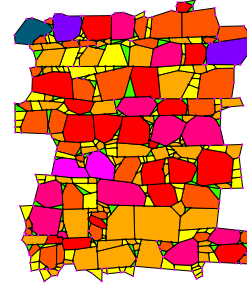
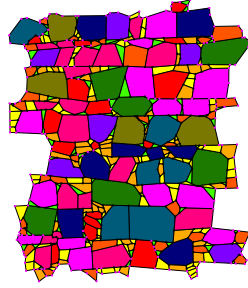
With border	$\bar{n}$	$\bar{v}$	$\bar{n}^*$	$\bar{v}^*$	ρ	ρ^*	$\alpha(M)$	A_s
	2.65	6.06	2.16	4.96	2.29	2.29	0.962	0.036
With no border	$\bar{n}$	$\bar{v}$	$\bar{n}^*$	$\bar{v}^*$	ρ	ρ^*	$\alpha(M)$	A_s
	2.94	6.06	2.36	4.96	2.07	2.10	0.015	0.033

Wall sample 10



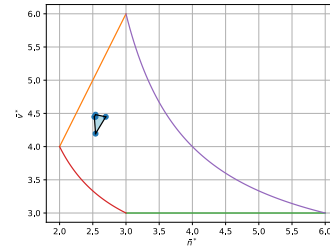
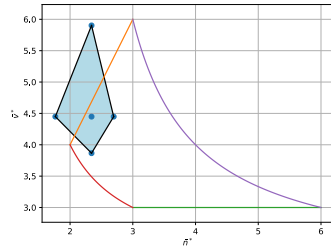
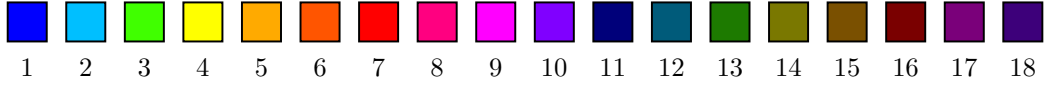
Castelo dos Mouros, Sintra, Portugal

Wall mosaic



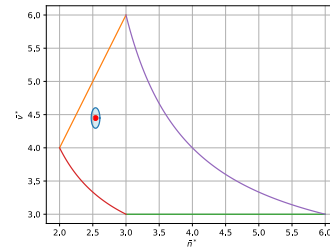
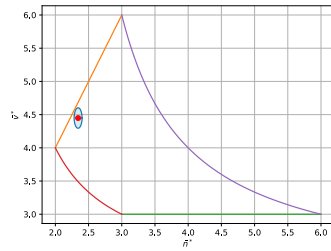
Combinatorial degrees

Geometrical degrees



Geometrical uncertainty with borders

Geometrical uncertainty without borders



Statistical uncertainty with borders

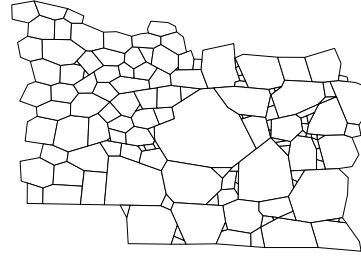
Statistical uncertainty without borders

With border	$\bar{n}$	$\bar{v}$	$\bar{n}^*$	$\bar{v}^*$	ρ	ρ^*	$\alpha(M)$	A_s
	2.79	5.30	2.34	4.45	1.90	1.90	0.946	0.029
With no border	$\bar{n}$	$\bar{v}$	$\bar{n}^*$	$\bar{v}^*$	ρ	ρ^*	$\alpha(M)$	A_s
	3.13	5.30	2.54	4.45	1.69	1.75	0.025	0.032

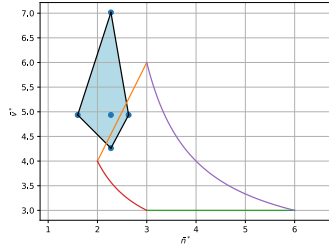
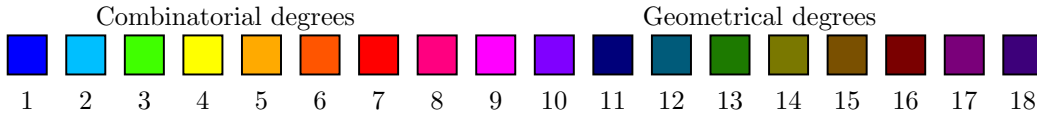
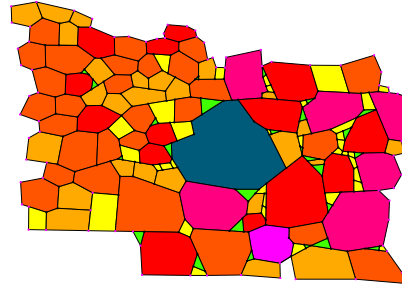
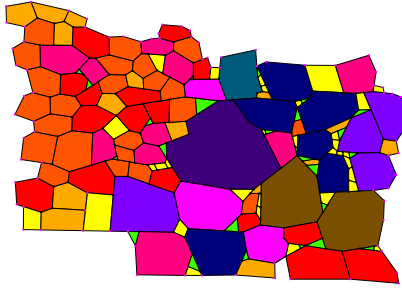
Wall sample 11



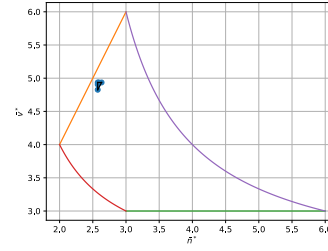
City Wall, Mycenae, Greece



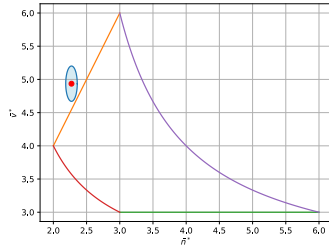
Wall mosaic



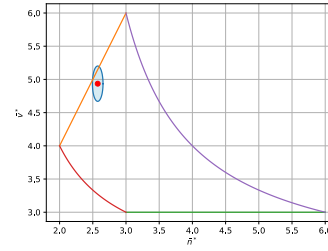
Geometrical uncertainty with borders



Geometrical uncertainty without borders



Statistical uncertainty with borders



Statistical uncertainty without borders

With border	$\bar{n}$	$\bar{v}$	$\bar{n}^*$	$\bar{v}^*$	ρ	ρ^*	$\alpha(M)$	A_s
	2.65	5.75	2.27	4.94	2.17	2.17	1.422	0.072
With no border	$\bar{n}$	$\bar{v}$	$\bar{n}^*$	$\bar{v}^*$	ρ	ρ^*	$\alpha(M)$	A_s
	3.05	5.75	2.57	4.94	1.89	1.92	0.003	0.067

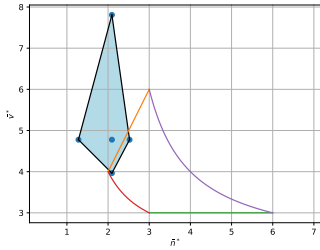
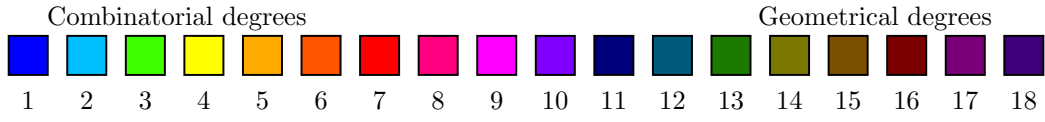
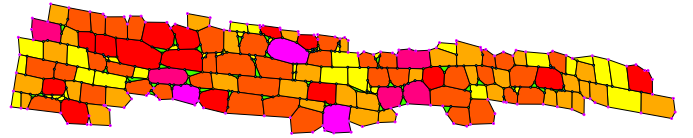
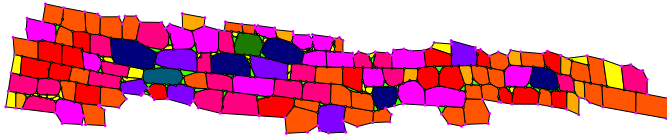
Wall sample 12



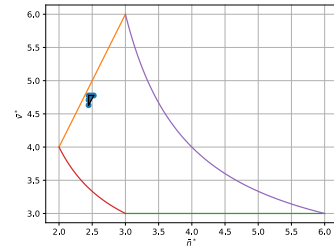
City Wall, Mycenae, Greece



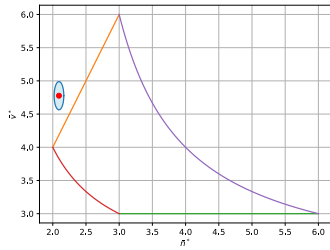
Wall mosaic



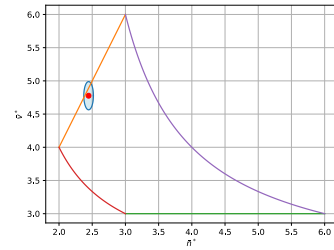
Geometrical uncertainty with borders



Geometrical uncertainty without borders



Statistical uncertainty with borders



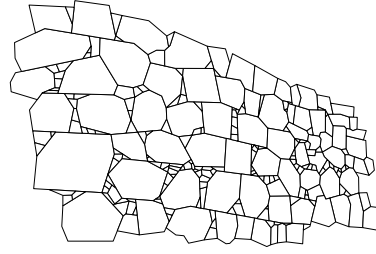
Statistical uncertainty without borders

With border	$\bar{n}$	$\bar{v}$	$\bar{n}^*$	$\bar{v}^*$	ρ	ρ^*	$\alpha(M)$	A_s
	2.54	5.79	2.09	4.78	2.28	2.28	2.388	0.049
With no border	$\bar{n}$	$\bar{v}$	$\bar{n}^*$	$\bar{v}^*$	ρ	ρ^*	$\alpha(M)$	A_s
	3.04	5.79	2.45	4.78	1.90	1.95	0.006	0.048

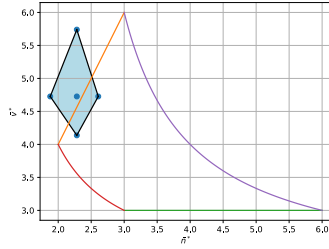
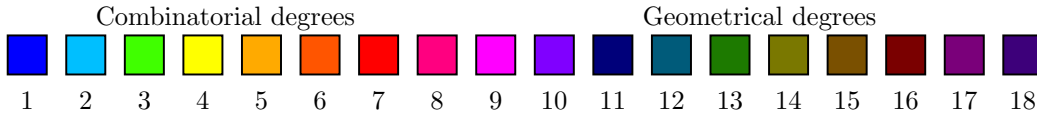
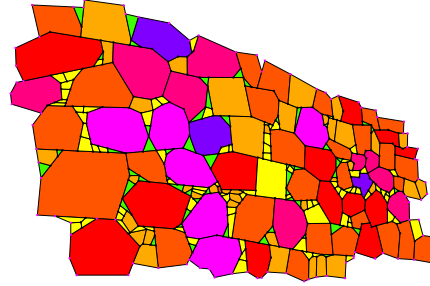
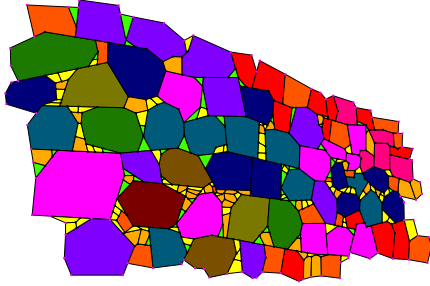
Wall sample 13



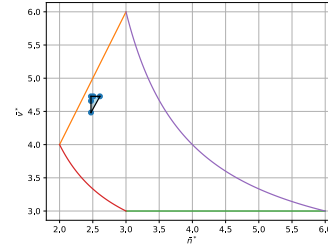
City Wall, Mycenae, Greece



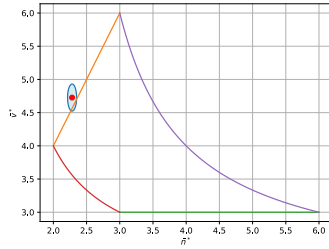
Wall mosaic



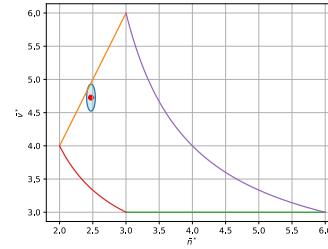
Geometrical uncertainty with borders



Geometrical uncertainty without borders



Statistical uncertainty with borders



Statistical uncertainty without borders

With border	$\bar{n}$	$\bar{v}$	$\bar{n}^*$	$\bar{v}^*$	ρ	ρ^*	$\alpha(M)$	A_s
	2.80	5.74	2.28	4.69	2.05	2.05	0.472	0.043
With no border	$\bar{n}$	$\bar{v}$	$\bar{n}^*$	$\bar{v}^*$	ρ	ρ^*	$\alpha(M)$	A_s
	3.06	5.74	2.47	4.69	1.88	1.90	0.001	0.042

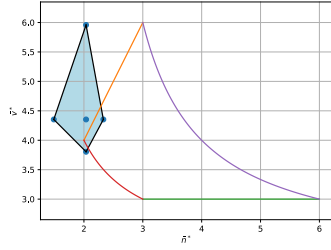
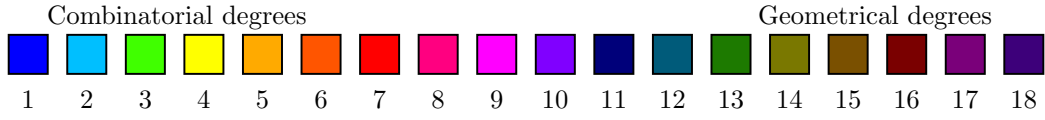
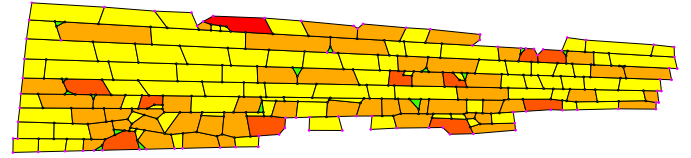
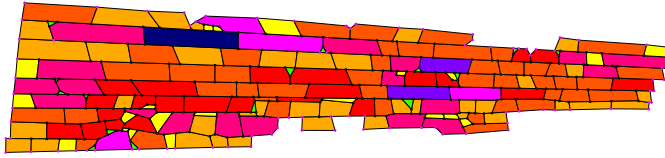
Wall sample 14



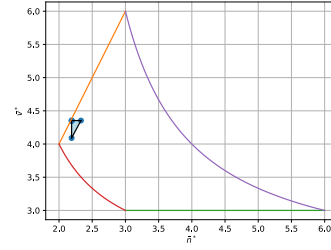
City Wall, Rhamnous, Greece



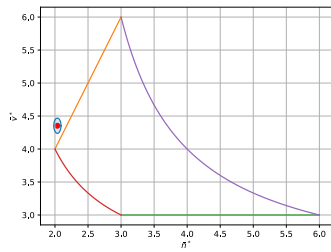
Wall mosaic



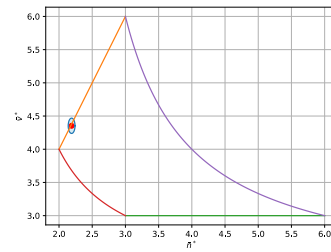
Geometrical uncertainty with borders



Geometrical uncertainty without borders



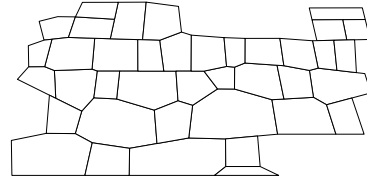
Statistical uncertainty with borders



Statistical uncertainty without borders

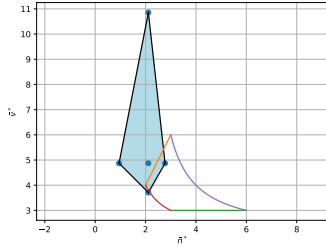
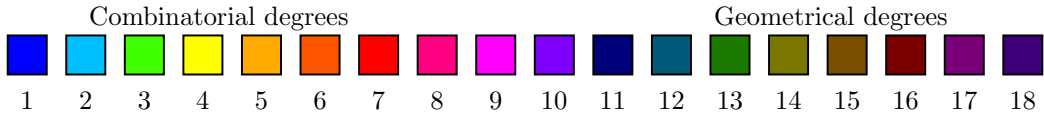
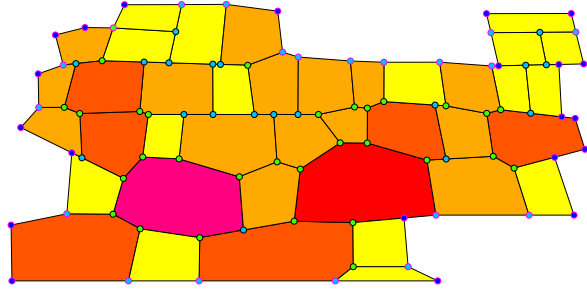
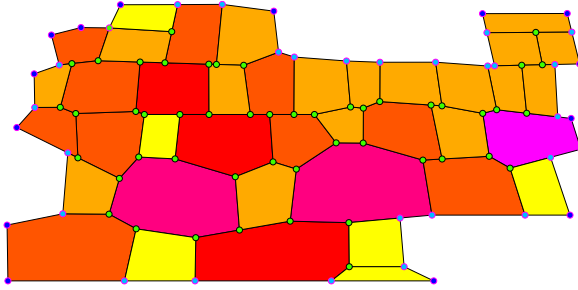
With border	$\bar{n}$	$\bar{v}$	$\bar{n}^*$	$\bar{v}^*$	ρ	ρ^*	$\alpha(M)$	A_s
	2.68	5.74	2.04	4.35	2.14	2.14	0.908	0.020
With no border	$\bar{n}$	$\bar{v}$	$\bar{n}^*$	$\bar{v}^*$	ρ	ρ^*	$\alpha(M)$	A_s
	3.01	5.74	2.19	4.35	1.91	1.99	0.018	0.019

Wall sample 15

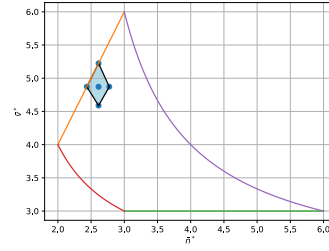


Sacsayhuaman, Cusco, Peru

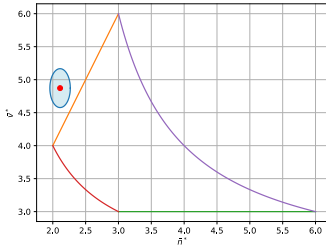
Wall mosaic



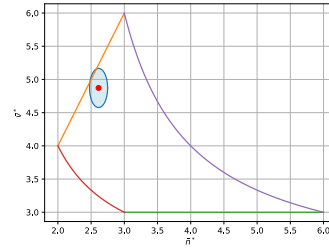
Geometrical uncertainty with borders



Geometrical uncertainty without borders



Statistical uncertainty with borders



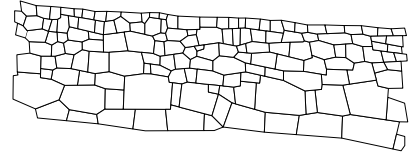
Statistical uncertainty without borders

With border	$\bar{n}$	$\bar{v}$	$\bar{n}^*$	$\bar{v}^*$	ρ	ρ^*	$\alpha(M)$	A_s
	2.39	5.51	2.11	4.87	2.31	2.31	6.529	0.144
With no border	$\bar{n}$	$\bar{v}$	$\bar{n}^*$	$\bar{v}^*$	ρ	ρ^*	$\alpha(M)$	A_s
	3.00	5.51	2.61	4.87	1.84	1.86	0.107	0.126

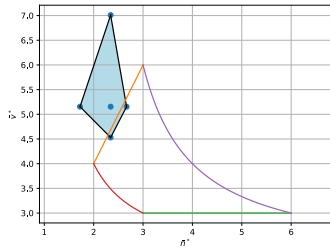
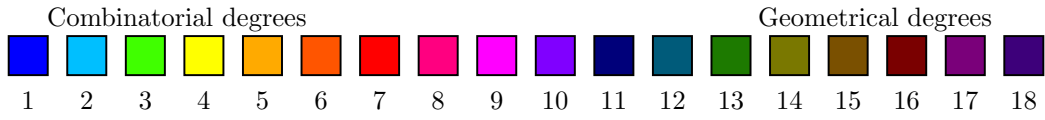
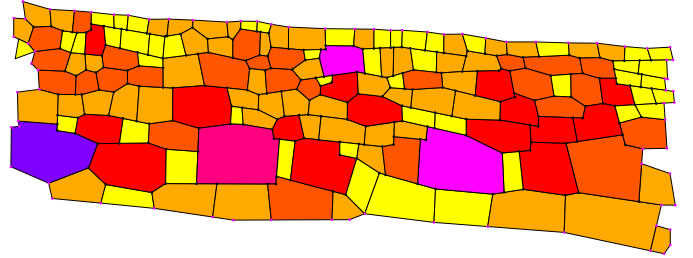
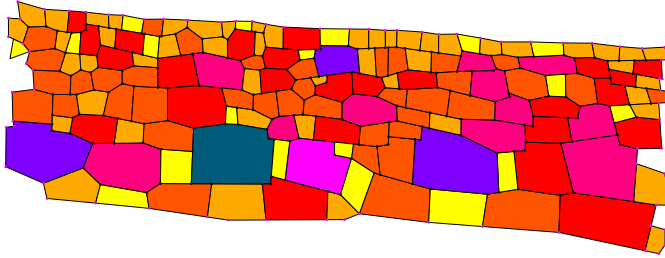
Wall sample 16



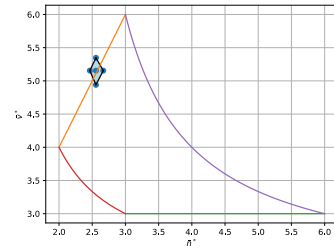
Tambomchay, Cusco, Peru



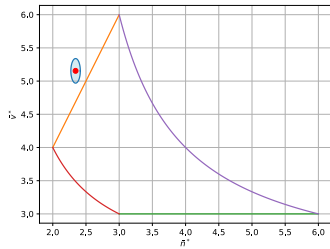
Wall mosaic



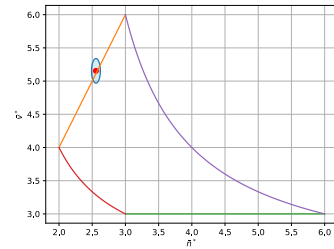
Geometrical uncertainty with borders



Geometrical uncertainty without borders



Statistical uncertainty with borders



Statistical uncertainty without borders

With border	$\bar{n}$	$\bar{v}$	$\bar{n}^*$	$\bar{v}^*$	ρ	ρ^*	$\alpha(M)$	A_s
	2.67	5.87	2.34	5.15	2.20	2.20	1.169	0.041
With no border	$\bar{n}$	$\bar{v}$	$\bar{n}^*$	$\bar{v}^*$	ρ	ρ^*	$\alpha(M)$	A_s
	2.96	5.87	2.56	5.15	1.98	2.02	0.041	0.039

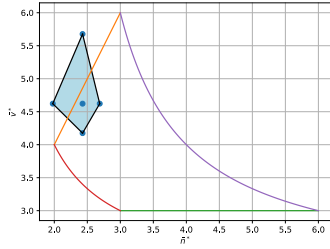
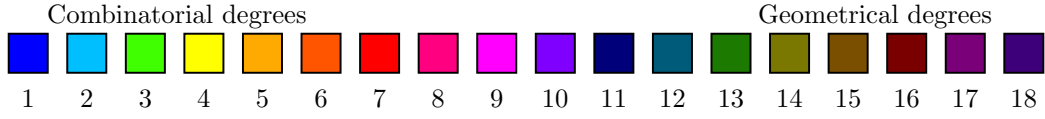
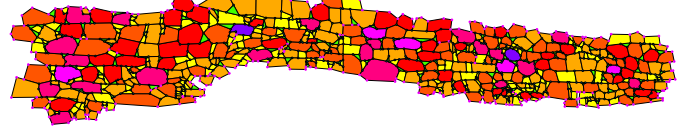
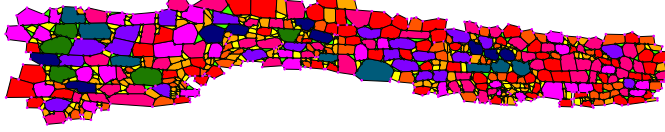
Wall sample 17



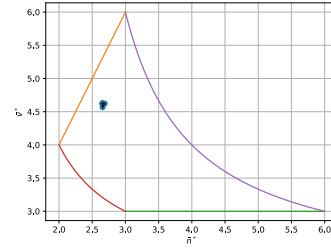
City Wall, Tiryns, Greece



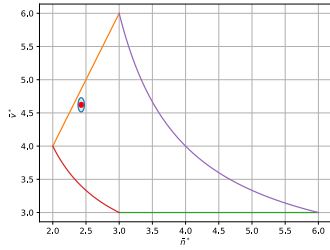
Wall mosaic



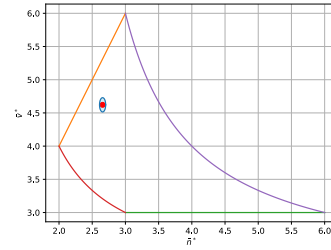
Geometrical uncertainty with borders



Geometrical uncertainty without borders



Statistical uncertainty with borders



Statistical uncertainty without borders

With border	$\bar{n}$	$\bar{v}$	$\bar{n}^*$	$\bar{v}^*$	ρ	ρ^*	$\alpha(M)$	A_s
	2.86	5.44	2.43	4.62	1.90	1.90	0.534	0.016
With no border	$\bar{n}$	$\bar{v}$	$\bar{n}^*$	$\bar{v}^*$	ρ	ρ^*	$\alpha(M)$	A_s
	3.15	5.44	2.66	4.62	1.73	1.74	0.001	0.016