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SWINDLERS' GUIDE: THE COMPLEX WAYS OF LOADING DICE

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TUDOMÁNYOS DIÁKKÖRI KONFERENCIA 2024

ANNUAL SCIENTIFIC STUDENT ASSOCIATION'S CONFERENCE 2024

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ABSTRACT. A homogeneous cubic dice falls with equal probabilities $p_i = 1/6$, ($i = 1, 2, \dots, 6$) on its 6 faces. We call this a *fair dice*. If the dice is inhomogeneous then the probabilities p_i may change, but none of them will be zero. We call this a *statistically loaded dice*. However, if the dice is a dodecahedron (with 12 faces) one can have material distributions where some of the probabilities p_i ($i = 1, 2, \dots, 12$) associated with faces become zero. We call this a *deterministically loaded dice*. In this paper I develop a theory which provides information on the number of options we have for deterministic loading.

Beyond loading dice, this theory addresses the following, more general question: If the convex polyhedron P with $f(P)$ faces and $v(P)$ vertices, center of mass o has $S(P)$ stable and $U(P)$ unstable equilibria then we say that P with centroid o is in the equilibrium class (S, U) and we write $(P, o) \in (S, U)$. We denote the set of points $o \in \text{int } P$ for which we have $(P, o) = (i, j)$ by $L_{i,j}$. In general, $L_{i,j}$ is the union of disjoint, convex polyhedra which we call (i, j) -loading zones and the number of which we denote by $n(i, j) \geq 0$. Since we have $f(P) \geq S(P)$, $v(P) \geq U(P)$, we can ask about the total number

$$N(P) = \sum_{i=1}^{f(P)} \sum_{j=1}^{v(P)} n(i, j)$$

of loading zones of P . While we can not provide a general answer, we derive formulae for the planar and also an upper bound for the 3D case.

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1. INTRODUCTION

Our goal is to gain a better understanding on the geometry of the *loading zones* of a convex polyhedron P . Considering inhomogeneous mass distribution, if the center of mass o moves (i.e. P is loaded in various ways) then the respective numbers S, U of stable and unstable balance points may change. In this paper, straying from physical reality, we do not restrict o to be in the interior of P . This is in order to be able to examine all the loading zones either inside or outside the polyhedron. The terms *physical* and *non-physical* will be corresponding to the cases of o being either inside or outside P , respectively.

2. BASIC CONCEPTS

2.1. Physical equilibria.

Definition 1. [1, 2]

Let P be a convex polyhedron and let $o \in \text{int } P$ be a point. We say that P is in equilibrium on a face, edge, or vertex A if there exists $Q \in \text{relint } A$ (the relative interior of A) such that the plane perpendicular to $[o, Q]$ at Q supports P . (Recall that the relative interior is defined in such a way that a singleton's relative interior is itself, though its interior is empty. Thus P may be in equilibrium on a vertex.) We call an equilibrium stable if Q is on the relative interior of a face, unstable if Q is a vertex, and hyperbolic (saddle) otherwise, and we denote their respective numbers by S, U, H .

Now think of a convex polyhedron as set of points at certain distances from a center point. In other words the *radial height* function $r_{P,o} : S^2 \rightarrow \mathbb{R}^+$ can describe the boundary of P . The connection between the equilibria of P and $r_{P,o}$ was investigated by many, among them Arthur Cayley [3] and J.C. Maxwell [4] in the 19th century, and developed into the famous *Morse-theory* [5] associated with Marston Morse in the 20th century.

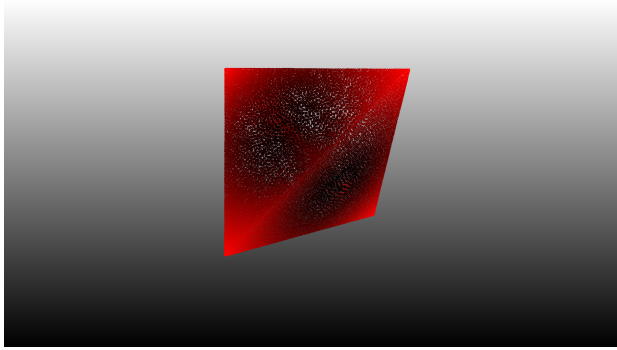
In fact, Maxwell showed in [4] that in Euler's polyhedron formula $f + v - e = 2$ states the connection between the minima, maxima and saddle points of the radial height function, and also that these points represent the number of stable, unstable and saddle-type equilibria, and therefore:

$$(1) \quad S + U - H = 2$$

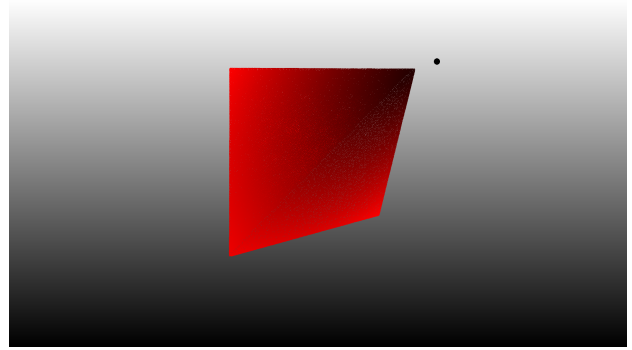
Remark 1. With $o \in \text{int } P$ given, in general cases of a convex polyhedron, no matter how close o is to a vertex, it will be closer to any edge coming from that vertex. And also no matter how close o is to an edge, it will be closer to any faces meeting in that edge. Therefore represent the minima, maxima and saddle-points of $r_{P,o}$ the stable equilibria on faces, the unstable equilibria on vertices, and the saddle-type equilibria on edges, respectively (see Fig. 1a).

2.2. Non-physical equilibria. Since we can apply the radial height function $r_{P,o}$ to $o \notin \text{int } P$ as well we expect (1) to work also for non-physical equilibria. However the statements in Remark 1 does not hold true anymore if $o \notin \text{int } P$, meaning stable, unstable and saddle-type equilibria are not restrained to be on faces, vertices and edges, respectively (see Fig. 1b). Thus a more general definition than Definition 1 has to be used to describe non-physical equilibria.

Definition 2. [4, 6] A three-dimensional weighted convex polyhedron P with center of mass o supported has three sorts of equilibria: stable, unstable, and saddle. These correspond to local minima, maxima, and saddle points of the radial height function $r_{P,o} : S^2 \rightarrow \mathbb{R}^+$ describing the boundary of P as a distance measured from o .



(A) Regular tetrahedron with a physical center of mass (the natural center of mass).



(B) Regular tetrahedron with a non-physical center of mass outside the tetrahedron.

FIGURE 1. Visual representation of the $r_{P,o}$ radial function in physical and non-physical cases. The darker the color the closer is the surface point to the center of mass.

2.3. Loading zones of convex polyhedra.

Definition 3. Let P be a convex polyhedron and let $o \in \mathbb{R}^3$ be a point. By counting stable and unstable relative equilibria of P with respect to o , we associate with o the *equilibrium indices* $i(o), j(o)$. We call a point o *generic* with respect to P if there exists a ball $\mathcal{B}_\epsilon$ of radius ϵ with center o such that $\forall o' \in \mathcal{B}_\epsilon : i(o') = i(o), j(o') = j(o)$. Otherwise, we call o *degenerate* with respect to P . We call o a *physical center of mass* if $o \in \text{int } P$ and we call equilibrium indices associated with physical centers of mass *physical indices*.

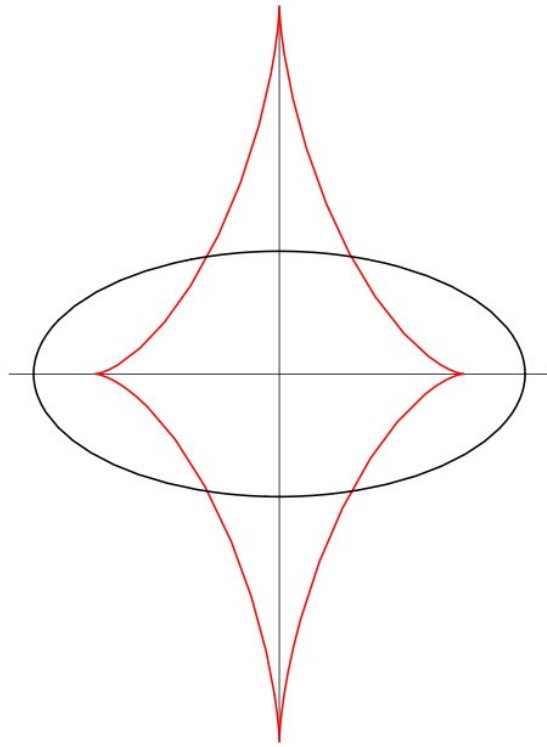
Definition 4. We call the set of generic points $o \in \mathbb{R}^3$ with equilibrium indices $i(o), j(o)$ the (i, j) loading region $R_{i,j}^P$ of P . We denote the number of connected components of $R_{i,j}^P$ by $n_{i,j}^P$. We call the connected components of $R_{i,j}^P$ the (i, j)

loading zones of P and we denote them by $L_{i,j,k}^P, k = 1, 2, \dots, n_{i,j}^P$. We also introduce the total number of loading zones as $N^P = \sum_{i=1}^{f^P} \sum_{j=1}^{v^P} n_{i,j}^P$. We call a loading zone *physical* if it is in the interior of P . We denote the number of physical loading zones by $\tilde{N}^P$. Loading zones may be unbounded (infinite) or compact (finite) domains and for the number of the latter we write $\tilde{N}^P$.

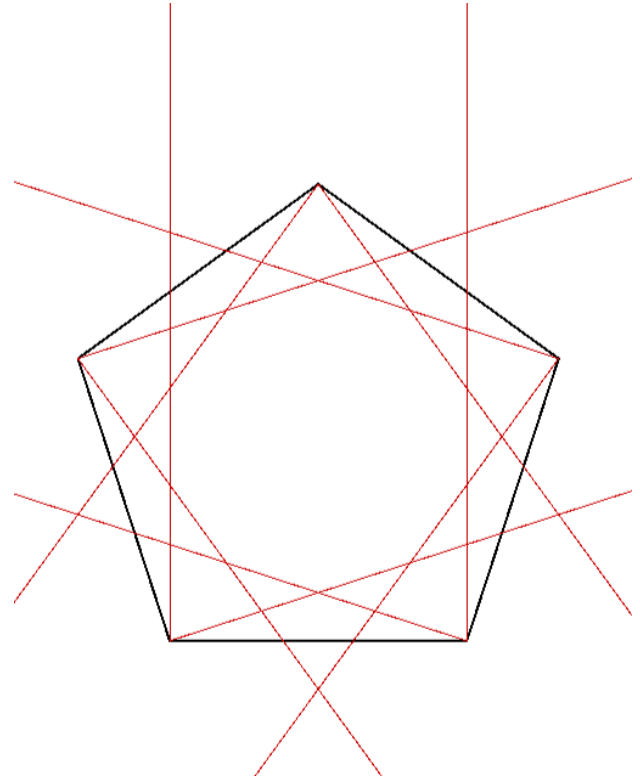
Remark 2. We first observe that the union of all degenerate points with respect to P are equivalent to the union of the two focal surfaces of P , meaning the boundaries of the loading regions and loading zones will be the focal surfaces.

In 2-dimensions the "focal surface" of a curve is a curve, called the *evolute* (see Fig. 2a), which is created by taking the center points of the tangential circle in each point of the curve (the radius of which is defined as the reciprocal of the curvature in the tangential point).

In 3-dimensions the 2 focal surfaces of another surface (which can also be called evolute) are created analogously, by taking the center points of the two circles at every point of the surface tangential to the curvature lines belonging to the principal curvatures. (The radii of the circles are defined as the reciprocals of the two principal curvatures in the tangential point.) [7] The evolutes of polygons and focal surfaces of polyhedra are degenerate.



(A) The evolute of an ellipse



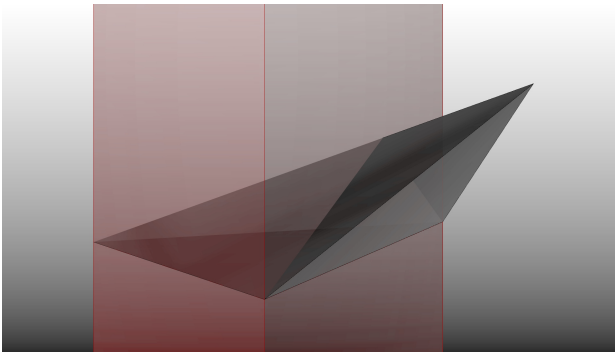
(B) The evolute of a regular pentagon.

FIGURE 2. Evolutes in 2-dimensions.

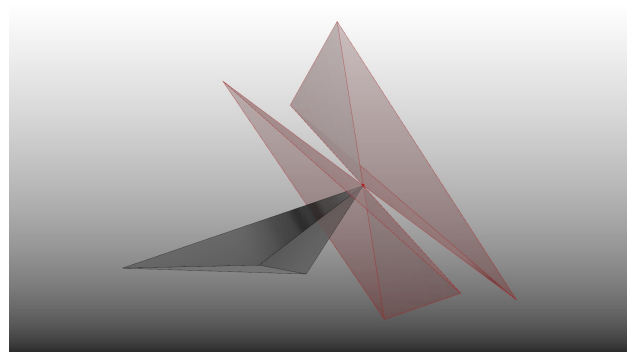
The evolute of a polygon consist of the union of the perpendiculars to the edges in both endpoints of each edge (see Fig. 2b).

The two focal surfaces of P consist of the following:

- (1) the union of surfaces of the infinite right prisms (see Figure 3a) through each face
- (2) the union of the surfaces of infinite pyramids (see Figure 3b) at each vertex, defined by the planes perpendicular to the edges meeting at the vertex



(A) An infinite prism through one face of a tetrahedron is part of its focal surface



(B) An infinite pyramid at the vertex of a tetrahedron is part of its focal surface

FIGURE 3. The two different kinds of focal surfaces of a tetrahedron

Remark 3. Each physical loading zone of a convex polyhedron P is defined as the intersection of a finite number of half spaces (linear inequalities). Since physical loading zones are, by definition, part of the interior of P , so each physical loading zone itself is a convex polyhedron (or the empty set).

Our ultimate goal is to calculate the number of physical cells of the mosaic generated by the focal complex. Due to the complexity of this task, first we aim to provide upper bounds on this number, therefore we make the following simplifications:

- (1) we consider not the actual faces of the prisms and pyramids of the focal complexes, but rather the infinite planes that they are defined by
- (2) we do not consider the geometry of the polyhedra themselves, meaning we do not discuss whether the cells are truly inside or outside a polyhedron

In other words, we *approximate* the loading zone complex of a convex polyhedron to be a 3-dimensional *hyperplane* mosaic. Hyperplane mosaics exist not only in 3, but in all $d \geq 1$ dimensions, and are formed by n linear subspaces of dimension $d - 1$, the so called hyperplanes. We call an arrangement of hyperplanes in d dimensions *generic*, if in d dimensions any collection of d distinct hyperplanes (let $n \geq d$) intersect at exactly one point, and at an intersection point exactly d planes intersect.

The focal complexes of convex polyhedra consist of many parallel planes, sets of planes that have parallel intersection lines, and also sets of planes meeting at one point. Therefore the loading zone complex of a convex polyhedron is highly nongeneric. We first investigate generic hyperplane mosaics in d -dimensions, and then the nongeneric loading zone complex of convex polytopes in 2 and 3-dimensions.

3. FINITE, GENERIC HYPERPLANE MOSAICS IN d -DIMENSIONS

Theorem 1. Let M_n^d be a finite hyperplane mosaic in $\mathbb{R}^d$, composed of n linear subspaces of dimension $d - 1$ which intersect in a generic manner and let us denote the number of closed (bounded) cells by $\tilde{N}_d(n)$ and the number of open (unbounded) cells by $\hat{N}_d(n)$, then we have

$$(2) \quad \tilde{N}_d(n) = \binom{n-1}{d}$$

$$(3) \quad \hat{N}_d(n) = 2 \sum_{i=0}^{d-1} \binom{n-1}{i}$$

Proof. We first note that the total Number $N_d(n)$ of cells can be obtained as

$$(4) \quad N_d(n) = \tilde{N}_d(n) + \hat{N}_d(n)$$

Our next goal is to prove

Lemma 1. $\tilde{N}_d(n) = \binom{n-1}{d}$

Proof. In [8] a recursion is shown for $N_d(n)$. The intersection of the n th hyperplane with the previous $(n - 1)$ hyperplanes is a $(d - 1)$ dimensional hyperplane mosaic with $(n - 1)$ linear subspaces of dimension $d - 2$. This mosaic has $N_{d-1}(n - 1)$ cells. This will be the increment in the number of cells, so we have

$$(5) \quad N_d(n) = N_d(n - 1) + N_{d-1}(n - 1)$$

This is the same formula as for the binomial coefficients: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. Ergo the solutions has to be linear combinations of binomial coefficients. The first d planes create 2^n cells, thus $N_d(d) = 2^d$ (all of them open). The $d+1$ -st cell intersects all cells, except one. Therefore $N_d(d+1) = (2^d) + (2^d - 1)$. Substituting $x = y = 1$ in the binomial formula $(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$ it can be proven that $\sum_{j=0}^d \binom{d}{j} = 2^d$ and $\sum_{j=0}^{d-1} \binom{d}{j} = 2^d - 1$. Since $N_d(d+1) = \sum_{j=0}^d \binom{d+1}{j} = (2^d) + (2^d - 1)$, therefore:

$$(6) \quad N_d(n) := \sum_{j=0}^d \binom{n}{j}$$

and every $N_d(n)$ is reached inductively from $N_d(d)$ and $N_{d-1}(d)$, proving (6) in all cases.

Now calculate $\dot{N}_d(n)$.

If we intersect a hyperplane mosaic with a sphere which contains all intersection points then on the boundary of the sphere we get a $(d-1)$ -dimensional mosaic, each cell of which corresponds to an open cell of the original d -dimensional mosaic.

It will have twice the cells as a hyperplane section. Imagine that you cut the spherical surface into two parts with the n th hyperplane. On both surfaces there will be a $d-1$ dimensional mosaic composed of $n-1$ planes. Therefore the number of all cells are

$$(7) \quad \dot{N}_d(n) = 2N_{d-1}(n-1)$$

Now substitute (7) into (4):

$$\tilde{N}_d(n) = N_d(n) - 2N_{d-1}(n-1)$$

Using (5):

$$\tilde{N}_d(n) = N_d(n-1) - N_{d-1}(n-1)$$

And eventually using (6):

$$(8) \quad \begin{aligned} \tilde{N}_d(n) &= \sum_{j=0}^d \binom{n-1}{j} - \sum_{j=0}^{d-1} \binom{n-1}{j} \\ \tilde{N}_d(n) &= \binom{n-1}{d} \end{aligned}$$

□

Lemma 2. $\dot{N}_d(n) = 2 \sum_{i=0}^{d-1} \binom{n-1}{i}$

Proof. We can expand (7) as:

$$\dot{N}_d(n) + \dot{N}_{d-1}(n) = 2N_{d-1}(n-1) + 2N_{d-2}(n-1)$$

Based on (5) we can rewrite the right hand side to obtain:

$$\dot{N}_d(n) + \dot{N}_{d-1}(n) = 2N_{d-1}(n)$$

which we rearrange as:

$$\dot{N}_d(n) = 2N_{d-1}(n) - \dot{N}_{d-1}(n) = N_{d-1}(n) + \tilde{N}_{d-1}(n) = \dot{N}_{d-1}(n) + 2\tilde{N}_{d-1}(n)$$

We substitute (2):

$$\dot{N}_d(n) = \dot{N}_{d-1}(n) + 2 \binom{n-1}{d-1}$$

Now we consider the initial condition $\dot{N}_1(n) \equiv 2$ to arrive at the summation:

$$\dot{N}_d(n) = 2 + 2 \sum_{i=2}^d \binom{n-1}{i-1} = 2 + 2 \sum_{i=1}^{d-1} \binom{n-1}{i}$$

Since $\binom{n}{0} = 1$

$$(9) \quad \dot{N}_d(n) = 2 \sum_{i=0}^{d-1} \binom{n-1}{i}$$

□

This way we proved Theorem 1. ■

4. BOUNDS OF THE LOADING ZONES OF CONVEX POLYTOPES IN 2 AND 3-DIMENSIONS

Before discussing the geometric bounds on the number of loading zones we also show the number of combinatorial possibilities.

4.1. Combinatorial bounds on the number N^P of loading zones in general polygons.

Lemma 3. Let P be a convex polygon with e^P edges. Then we have $N^P \leq N(e^P) = \sum_{i=0}^{e^P} \binom{e^P}{i} 2^{e^P-i}$.

Proof. To identify a loading zone, we have to consider the following linear inequalities. First, we pick a value $i \in \{0, 1, 2, \dots, e^P\}$, then we pick i edges from, among the e^P edges with which the i (stable) equilibria are associated. We can perform this in $\binom{e^P}{i}$ different ways. Now we regard one of these choices. To assure that the number of equilibria is *exactly* i , we have to prohibit centers of mass which admit equilibria on any other (non-selected) edge. We have $e^P - i$ such edges. Centers of mass producing equilibrium on an edge lie within a strip bordered by two parallel lines, so we have two alternative ways to prohibit equilibria on any edge. This accounts for the factor 2^{e^P-i} . $\square$

Lemma 4. $N(e^P) = 3^{e^P}$

Proof. We use a simple variant of the binomial formula $(x+1)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k}$. By substituting $x = 2$, we get $3^n = (2+1)^n = \sum_{k=0}^n \binom{n}{k} 2^{n-k}$ $\square$

4.2. Combinatorial bounds on the number N^P of loading zones in general polyhedra.

Lemma 5. Let P be a convex polyhedron with F faces and V vertices. Let us assume that the degree of each face and each vertex is constant and these values are f, v , respectively. Then we have $N^P \leq N(F, V, f, v) = \sum_{i=0}^F \sum_{j=0}^V \binom{F}{i} \binom{V}{j} f^{F-i} v^{V-j}$.

Proof. This is analogous to the proof of Lemma 3. Now we pick i faces and j vertices, accounting for the factor $\binom{F}{i} \binom{V}{j}$. Then we add to each system $(F-i) + (V-j)$ linear inequalities, each of these prohibiting additional stable and unstable equilibria. On each non-equilibrium face we can pick f such inequalities and on each non-equilibrium vertex we can pick v such inequalities. This accounts for the factor $f^{F-i} v^{V-j}$. $\square$

Example. We first compute the estimate for the tetrahedron.

$$\begin{aligned}
N(4, 4, 3, 3) &= \binom{4}{0} \binom{4}{0} 3^4 3^4 + \binom{4}{0} \binom{4}{1} 3^4 3^3 + \binom{4}{0} \binom{4}{2} 3^4 3^2 + \binom{4}{0} \binom{4}{3} 3^4 3^1 + \binom{4}{0} \binom{4}{4} 3^4 3^0 + \\
&+ \binom{4}{1} \binom{4}{0} 3^3 3^4 + \binom{4}{1} \binom{4}{1} 3^3 3^3 + \binom{4}{1} \binom{4}{2} 3^3 3^2 + \binom{4}{1} \binom{4}{3} 3^3 3^1 + \binom{4}{1} \binom{4}{4} 3^3 3^0 + \\
&+ \binom{4}{2} \binom{4}{0} 3^2 3^4 + \binom{4}{2} \binom{4}{1} 3^2 3^3 + \binom{4}{2} \binom{4}{2} 3^2 3^2 + \binom{4}{2} \binom{4}{3} 3^2 3^1 + \binom{4}{2} \binom{4}{4} 3^2 3^0 + \\
&+ \binom{4}{3} \binom{4}{0} 3^1 3^4 + \binom{4}{3} \binom{4}{1} 3^1 3^3 + \binom{4}{3} \binom{4}{2} 3^1 3^2 + \binom{4}{3} \binom{4}{3} 3^1 3^1 + \binom{4}{3} \binom{4}{4} 3^1 3^0 + \\
&+ \binom{4}{4} \binom{4}{0} 3^0 3^4 + \binom{4}{4} \binom{4}{1} 3^0 3^3 + \binom{4}{4} \binom{4}{2} 3^0 3^2 + \binom{4}{4} \binom{4}{3} 3^0 3^1 + \binom{4}{4} \binom{4}{4} 3^0 3^0 = \\
&= 1 * 1 * 81 * 81 + 1 * 4 * 81 * 27 + 1 * 6 * 81 * 9 + 1 * 4 * 81 * 1 + 1 * 1 * 81 * 1 + \\
&+ 4 * 1 * 27 * 81 + 4 * 4 * 27 * 27 + 4 * 6 * 27 * 9 + 4 * 4 * 27 * 1 + 4 * 1 * 27 * 1 + \\
&+ 6 * 1 * 9 * 81 + 6 * 4 * 9 * 27 + 6 * 6 * 9 * 9 + 6 * 4 * 9 * 1 + 6 * 1 * 9 * 1 + \\
&+ 4 * 1 * 3 * 81 + 4 * 4 * 3 * 27 + 4 * 6 * 3 * 9 + 4 * 4 * 3 * 3 + 4 * 1 * 3 * 1 + \\
&1 * 1 * 1 * 81 + 1 * 4 * 1 * 27 + 1 * 6 * 1 * 9 + 1 * 4 * 1 * 3 + 1 * 1 * 1 * 1 = 63592.
\end{aligned}$$

4.3. Geometric bounds on the number N^P of loading zones in general polygons. The focal complex of a polygon is defined by perpendiculars at the endpoints of the edges. At each vertex 2 lines pass, so the loading zones of a polygon is a mosaic composed of n pairs of parallel lines.

We first discuss finite, generic hyperplane mosaics in the plane which are produced by the intersection of n straight lines. We assume that 3 or more lines can not intersect at one point.

4.3.1. *Finite, generic hyperplane mosaics in 2-dimensions.* We use the knowledge of Theorem 1.

Lemma 6. *Let M_n be a generic hyperplane mosaic in the plane, consisting of n straight lines. Then, for the number of (closed and open) cells of M_n we have*

$$(10) \quad N_2(n) = \frac{1}{2}(n^2 + n + 2).$$

Proof. By substituting $d = 2$ into (6), we get (10). □

Lemma 7. *Let M_n be a generic hyperplane mosaic in the plane, consisting of n straight lines. Then, for the number of closed cells of M_n we have*

$$(11) \quad \bar{N}_2(n) = \frac{1}{2}(n^2 - 3n + 2).$$

Proof. By substituting $d = 2$ into (2) we get (11). □

4.3.2. *General formula for arbitrary degeneracies in 2 dimensions.* The n lines composing the mosaic M_n may be in generic relative positions, and in that case the number of closed cells is given by formula (11). However, there may be degenerated relative positions: two lines may be parallel or three lines may meet at one point. We call these codimension 1 degeneracies. If we have $k + 1$ parallel lines, or $k + 2$ lines intersecting at one point, then we have a codimension k degeneracy. Below we give a general formula to compute the number of closed cells if the number and codimension of degeneracies is known. The formula are related to the following

Observation 1. *Let M_n be a planar hyperplane mosaic, composed of n lines and let us assume that the relative location of these lines has one degeneracy of codimension k . We can resolve this degeneracy by perturbing the equations of the lines, using k independent parameters. Resolving the degeneracy in this manner results in a generic mosaic $\bar{M}_n$ and the number of closed cells of the generic mosaic will be greater or equal compared to the number of closed cells of the degenerate mosaic, i.e. by resolving new closed cells are created (or no closed cells are created). Our fundamental observation is that in 2-dimensions, the increase in the number of closed cells solely depends on the codimension of the degeneracy and does not depend on any other feature of M_n .*

Lemma 8. *Let M_n be a hyperplane mosaic composed of n straight lines. Assume that in the relative locations of these n lines we have k_i degeneracies of codimension i , $i = 1, 2, \dots, s$. Then, for the number of closed cells in M_n we can write*

$$(12) \quad \bar{N}(M_n) = \bar{N}_2(n) - \sum_{i=1}^s k_i \bar{N}_2(i + 2).$$

Proof. First consider $m + 2$ lines meeting at one point somewhere in M_N . Now move m of the $m + 2$ lines such that new cells are only created in M_N within a small ϵ radius of the original intersection point. In this area we can observe a generic mosaic composed of $m + 2$ straight lines (see Fig. 4 upper drawing). This way the number of the new cells created by resolving this $i = m$ codimensional degeneracy can be calculated using formula (11). Secondly, consider a set of $m + 1$ parallel lines in M_N . By rotating the $m + 1$ st line with an ϵ angle so it intersects with the other m lines "almost in infinity", thus closing m originally open cells in M_N (see Fig. 4 bottom drawing). By rotating the lines one-by-one (with each ϵ being different) we get that the amount of generated closed cells can be calculated using the formula $Sn = m + (m - 1) + (m - 2) + \dots + 3 + 2 + 1 = \frac{m(m+1)}{2} = \bar{N}_2(m + 2)$. We see that by removing any $i = m$ codimensional degeneracy we add $\bar{N}_2(m + 2)$ closed cells to the mosaic. The difference of the number of closed cells of the generic mosaic and the sum of the generated closed cells by resolving the degeneracies gives the amount of closed cells in the original M_N mosaic, which is formally written in (12). □

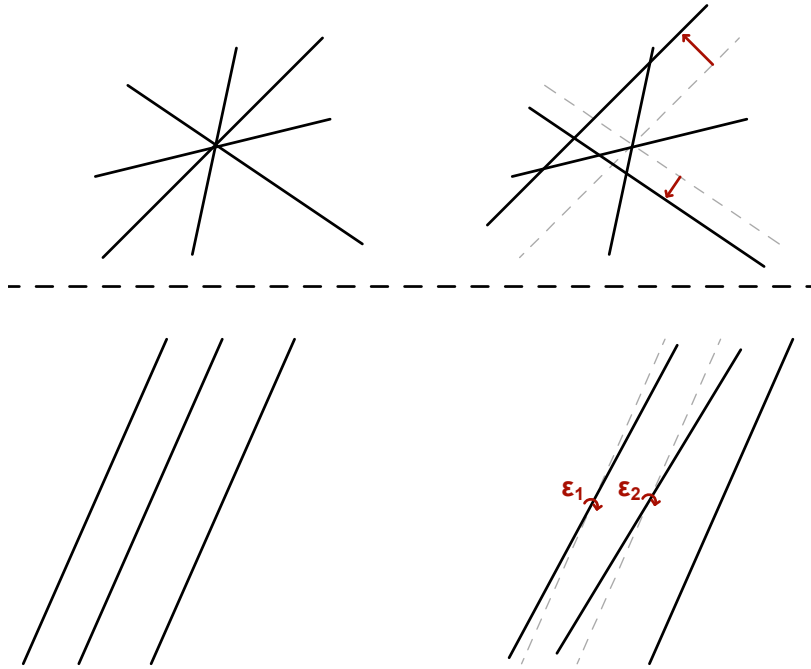


FIGURE 4. Resolving degeneracies of codimension 2

Our next aim is to prove

Lemma 9. *Let P be a polygon with n vertices which has no pair of parallel edges. Then for the number of closed cells we have*

$$(13) \quad \tilde{N}^P = 2n^2 - 4n + 1$$

Proof. The loading zones of a polygon are defined by $k_i = n$ pairs of parallel lines of codimension $i = s = 1$. Substituting into (12) gives $\tilde{N}_2(2n) - n\tilde{N}_2(3)$, which using (11) yields (13). $\square$

Lemma 10. *Let P be a polygon with n vertices which has l pairs of parallel edges. Then for the number of closed cells we have*

$$(14) \quad \tilde{N}^P(l) = \tilde{N}^P - 4l$$

Proof. The loading zones of such polygon are defined by $(n - 2l)$ pairs of parallel lines of codimension 1, and l pairs of parallel lines of codimension 3. Substituting into (12) gives $\tilde{N}_2(2n) - (n - 2l)\tilde{N}_2(3) - l\tilde{N}_2(5)$. Using (11) and (13) yields (14). $\square$

Definition 5. We say that 2 edges of a polygon are *in front of* each other if the perpendicular line pair through their endpoints coincide.

Theorem 2. *Let P be a polygon with n vertices which has l pairs of parallel edges (not in front of each other) and m edges in front of each other. Then for the number of closed cells we have*

$$(15) \quad \tilde{N}^P(l, m) = \tilde{N}^P(l) + 2m^2 - 4m(n - 1)$$

Proof. The loading zones of such polygon are defined by $(2n - 2m)$ lines, of which $(n - m - 2l)$ are parallel lines of codimension 1, and l are pairs of parallel lines of codimension 3. Substituting into (12) gives

$$\tilde{N}_2(2n - 2m) - (n - m - 2l)\tilde{N}_2(3) - l\tilde{N}_2(5)$$

Using (11) and (14) yields (15) $\square$

4.4. Geometric bounds on the number N^P of loading zones in general polyhedra. First we investigate generic hyperplane mosaics in 3D. We use the knowledge of Theorem 1.

Lemma 11. *Let M_n be a generic hyperplane mosaic in 3D, consisting of n planes. Then, for the number of (bounded and unbounded) cells of M_n we have*

$$(16) \quad N_3(n) = \frac{1}{6}(n^3 + 5n + 6).$$

Proof. We substitute $d = 3$ into (6) to get (16). $\square$

Next we establish the number of closed cells.

Lemma 12. *Let M_n be a generic hyperplane mosaic in 3D, consisting of n planes. Then, for the number of bounded cells of M_n we have*

$$(17) \quad \bar{N}_3(n) = \frac{1}{6}(n^3 - 6n^2 + 11n - 6).$$

Proof. We substitute $d = 3$ into (2) to get (17). □

4.4.1. *Arbitrary degeneracies in 3-dimensions.* Observation 1 is *not* valid in 3-dimensions: the number of new closed cells created by resolving a degeneracy will not only depend on the codimension of the latter but may also depend on the geometry of the original mosaic. This implies that there is no general formula for 3-dimensional mosaics, or at least not without introducing something else than the codimensions of degeneracies.

5. LOADING "DICE"

We would like to show the various ways of loading both 2 and 3 dimensional dice. Of course no one plays with 2-dimensional polygons any game, although it would not be impossible. We can imagine a 2-dimensional dice as a long perpendicular prism, with a base of any regular polygon. If the height of the prism is much longer than the edge lengths of the base, statistically the dice will almost always land on the side faces (or we can simply say we only assign values to the side faces.)

To show the connection between the loading zones and the equilibria of a polygon or polyhedron, we are going to introduce the term *equilibrium matrix*.

Definition 6. [9] Let P be a convex polyhedron with f faces and v vertices. Then an $f \times v$ size M_P matrix is the equilibrium matrix of P , and the $m_p(i, j)$ elements of M_P are defined by the following rule: let $m_p(i, j) = 1$ if there exist a polyhedron in the $(i, j)^E$ equilibrium class, and otherwise $m_p(i, j) = 0$. We denote the sum of the elements μ_P .

We note that in 2-dimensions the equilibrium matrix of an n sided convex polygon (considered as an $f = 1$ faced polyhedron) is a binary vector of length n .

The μ_P sum of the elements of the equilibrium matrix shows how many different $m_p(i, j)$ class can be achieved, and $m_p(i, j) = 1$ elements show (i, j) equilibrium classes can be achieved. In other words which possible rolling outcomes can we get rid of with a proper loading.

However the information in the matrix does not include exactly *which i faces (and j vertices) have a chance to be rolled (and which not)*. For further information it is inevitable to construct the loading zones themselves.

Each $L_{i,j}^P$ physical loading zone of a polyhedron has a corresponding $m_p(i, j) = 1$ element in the equilibrium matrix (as a non-injective surjection).

5.1. 2-dimensional "dice" - exact numbers for regular polygons. In [9] it was shown that the equilibrium matrices

of triangles can only be $M_T = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \vee \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and for each there exist a triangle. It was also shown that the equilibrium

matrices of quadrangles can only be $M_Q = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \vee \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \vee \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \vee \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \vee \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ and for each there exist a quadrangle.

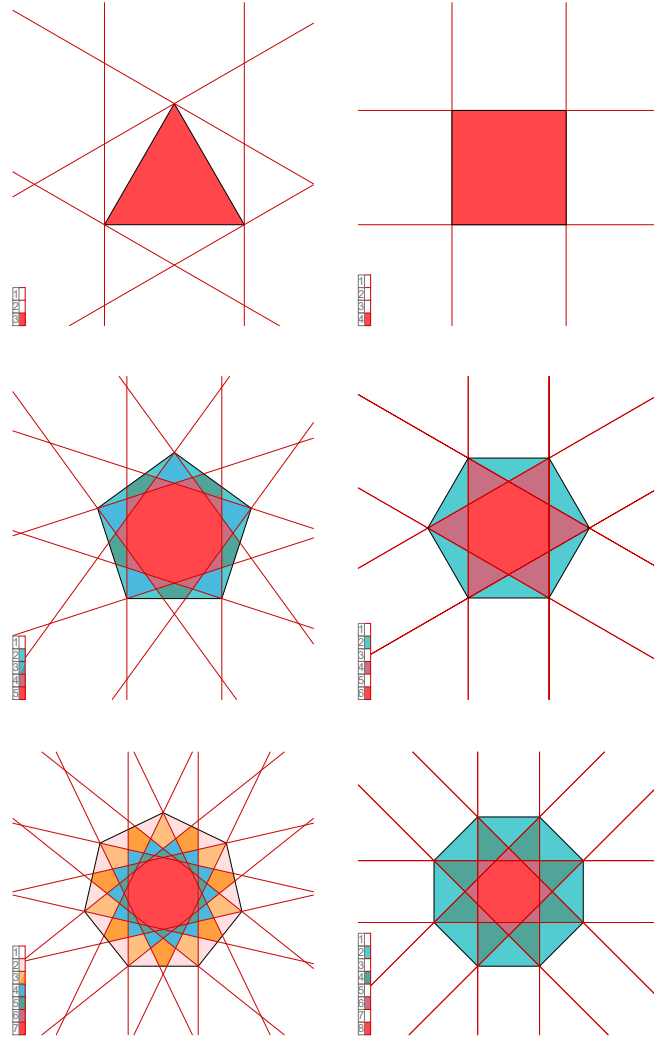


FIGURE 5. The construction of the loading zones of the first 6 regular polygons, along with their M_P colored equilibrium vectors. The coloring shows the number of equilibria belonging to each loading zone.

The main difference between regular polygons, in terms of closed cells of their loading zones, is determined by the parity of the number of edges. A regular polygon with an *odd* number of edges has no parallel edges ($m = l = 0$). Therefore the number of closed cells can be calculated using (13), as for irregular polygons. However an n -sided regular polygon with an *even* number of edges has n pairs of edges in front of each other, therefore we can use (15) with $l = 0$ and $m = n/2$.

Based on the construction of the first 6 regular polygons (see Figure 5) we make the following conclusions on the $\tilde{N}^P$ number of physical cells and M_P equilibrium matrices of regular polygons:

$$(1) \text{ For } n = 3: \tilde{N}^P = 1 < \tilde{N}^P = 7; M_P = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; \mu_P = 1$$

$$(2) \text{ } n = 4: \tilde{N}^P = \tilde{N}^P = 1; M_P = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}; \mu_P = 1$$

$$\begin{aligned}
 (3) \text{ For } n = 2k; k \in N^+, k > 2: \tilde{N}^P = \tilde{N}^P + n; M_P &= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ \dots \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}; \mu_P = n/2 \\
 (4) \text{ For } n = 2k + 1, k \in N^+, k > 1: \tilde{N}^P = \tilde{N}^P; M_P &= \begin{pmatrix} 0 \\ 1 \\ 1 \\ \dots \\ 1 \\ 1 \end{pmatrix}; \mu_P = n - 1
 \end{aligned}$$

Table 1 summarizes the results:

TABLE 1. The $\tilde{N}_P$ number of bounded and $\tilde{N}^P$ physical cells of the first 18 regular polygons and the μ_P sum of the elements of their equilibrium matrices

$n = 2k + 1$	m	$\tilde{N}^P$	$\tilde{N}^P$	μ_P	$n = 2k$	m	$\tilde{N}^P$	$\tilde{N}^P$	μ_P
3	0	7	1	1	4	2	1	1	1
5	0	31	31	4	6	3	7	13	3
7	0	71	71	6	8	4	17	25	4
9	0	127	127	8	10	5	31	41	5
11	0	199	199	10	12	6	49	61	6
13	0	287	287	12	14	7	71	85	7
15	0	391	391	14	16	8	97	113	8
17	0	511	511	16	18	9	127	145	9
19	0	647	647	18	20	10	161	181	10

As you can see the equilateral triangle and the square dice can only be loaded statistically, but any other 2-dimensional dice can be loaded deterministically, however with certain geometrical limitations that one has to consider when trying to load a dice to his preference. First of all, whether the dice-to-be-loaded has odd or even number of sides.

5.2. 3-dimensional "dice" - upper bounds. Most conventionally, 4, 6, 8, 12 or 20 sided dice are used for gaming, and not suprisingly these come in the form of the 5 platonic solids (see Fig. 6). Other than these, the most common 10 sided dice has the form pentagonal trapezohedron (see Fig. 7).

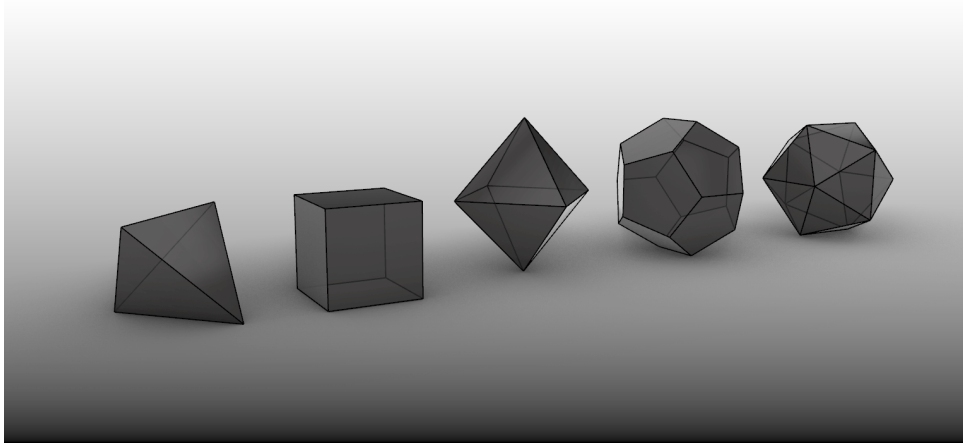


FIGURE 6. The five platonic solids, from left to right: tetrahedron, hexahedron, octahedron, dodecahedron and icosahedron

5.2.1. *Platonic solids.* Below we provide combinatorial upper bounds on the number of cells of platonic solids based on geometrical considerations.

- (1) **Tetrahedron.** The dihedral angles of a regular tetrahedron are less than 90 degrees. Also, all the angles between the edges meeting at a vertex are less than 90 degrees (exactly 60), therefore none of the focal surfaces intersect the tetrahedron. This means the tetrahedron is *omni-omnistic* (carrying equilibria on all faces and all vertices) for arbitrary center of mass: $\tilde{N}_P = 1$.
- (2) **Hexahedron - Cube.** The dihedral angles of a regular cube are exactly 90 degrees. Also, the angles between the edges are exactly 90 degrees, therefore none of the focal surfaces intersect the cube. This means the cube is *omni-omnistic* for arbitrary center of mass: $\tilde{N}_P = 1$.
- (3) **Octahedron.** The dihedral angles of the octahedron are greater than 90 degrees, so the prisms of the focal surfaces intersect the octahedron, therefore it can be loaded so there is no stable equilibria on all faces. All the angles between the edges meeting at a vertex are either exactly 90 degrees (the angles not on the faces) or 60 degrees (the angles on the faces), meaning the octahedron is *omni-unstable* (carrying unstable equilibria on all vertices) for arbitrary center of mass. We can calculate an upper bound using the equation in Lemma 5. Considering the invariability in numbers of unstable equilibria we have the following: $N^P \leq N(F, f) = \sum_{i=0}^F \binom{F}{i} f^{F-i}$.

$$\begin{aligned}
 N(8, 3) &= \binom{8}{0} 3^8 + \binom{8}{1} 3^7 + \binom{8}{2} 3^6 + \binom{8}{3} 3^5 + \binom{8}{4} 3^4 + \binom{8}{5} 3^3 + \binom{8}{6} 3^2 + \binom{8}{7} 3^1 + \binom{8}{8} 3^0 = \\
 &= 1 * 6561 + 8 * 2187 + 28 * 729 + 56 * 243 + 70 * 81 + 56 * 27 + 28 * 9 + 8 * 3 + 1 * 1 = \\
 &= 6561 + 17496 + 20412 + 13608 + 5670 + 1512 + 252 + 24 + 1 \\
 &= 65536
 \end{aligned}$$

This seems to be a really crude upper bound. Considering that only the $3 * 8$ planes belonging to the faces play part in creation of the physical cells, we can also say an upper bound using Lemma 4:

$$\tilde{N}_P^{Octahedron} \leq N_d(24) = 2325$$

- (4) **Dodecahedron and Icosahedron.** Both in the case of the dodecahedron and icosahedron all 120 planes of the focal surface intersect the solid body, therefore the best upper bound we can give is:

$$\tilde{N}_P^{Dodecahedron} = \tilde{N}_P^{Icosahedron} \leq N_d(120) = 288101$$

5.2.2. *Pentagonal Trapezohedron.* Such polyhedron can be constructed from 10 kite facets, so all dihedral angles are the same, approximately 116,56 degrees. It has 12 vertices, on 2 of which there are only acute angles between the edges therefore they must carry unstable equilibria. The faces give $4 * 10$ planes and the vertices $10 * 3$ planes that can construct physical cells. Therefore the upper bound is:

$$\tilde{N}_P^{PTrapezohedron} \leq N_d(70) = 57726$$

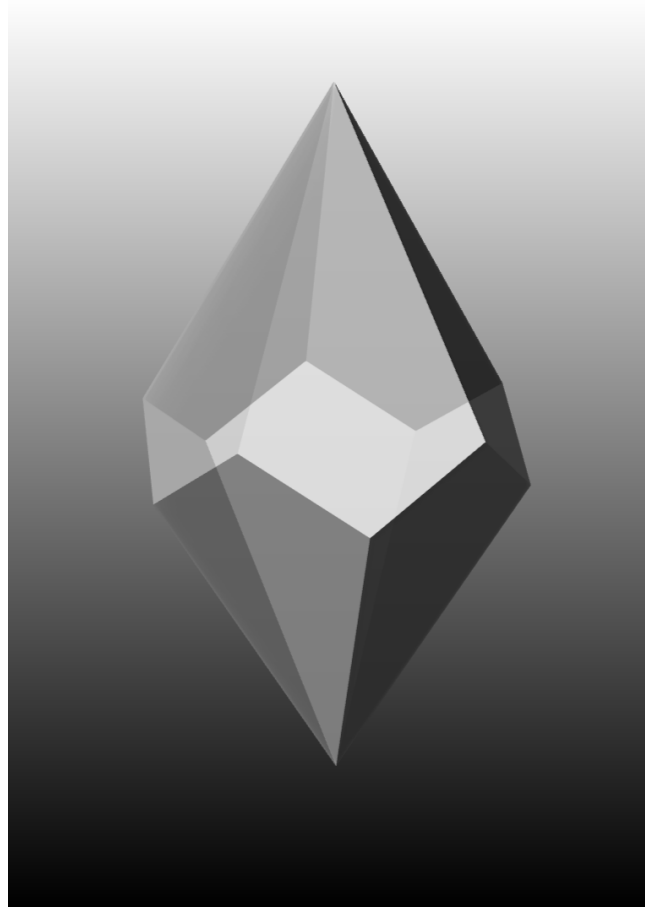


FIGURE 7. The pentagonal trapezohedron

5.2.3. *General tetrahedron.* For a general tetrahedron, by considering that the focal complex consist of $n = 2 * 3 * 4 = 24$ planes:

$$\tilde{N}^{tetrahedron} \leq N_3(24) = 2325$$

.

6. ENDNOTE

In this paper we showed the basic concepts of calculating the number of cells created by the focal surfaces of polygons and polyhedra in order to give upper bounds on the number of physical loading zones. In 2-dimensions we could give an exact formula for the number of closed cells of the focal complex, however we could only get exact numbers for the physical cells for regular polygons. The 3-dimensional case is even more complex, we could only give crude upper bounds, and at the moment cannot handle degeneracies.

In the future a numeric approach of finding the number of physical loading zones for polyhedra should be considered, with the same approach as the combinatorial bounds were calculated. Each plane of the focal surface and the boundary of the polyhedron can be handled by an adequate system of linear inequalities where logical connections can also make the initial approximations unnecessary, therefore allowing the calculation of exact the number of closed cells.

7. TABLE OF FIGURES

Number of figure	Source
1	Rhino 8 - Grasshopper - self made
2a	https://mathworld.wolfram.com/EllipseEvolute.html
2b	Rhino 8 - GH - ArchiCad 27 - self made
3	Rhino 8 - Grasshopper - self made
4	ArchiCad27 - self made
5	Rhino 8 - GH - ArchiCad 27 - self made
6	Rhino 8 - self made
7	Rhino 8 - self made

8. ACKNOWLEDGEMENTS

Gergő Almádi kindly acknowledges support from the NKFIH Hungarian Research Fund grant 134199 and NKFIH grant TKP2021 BME-NVA, carried out at Budapest University of Technology and Economics. G.A. expresses his grateful appreciation of the gift representing the Albrecht Science Fellowship. G.A.'s research has been supported by the program EKOP-24-2 by KIM.

This paper is a result of the ongoing cooperation with Robert J. MacG. Dawson from Saint Mary's University in Halifax, so hereby we express our gratitude toward him and thank his thoughts, work and support that also helped to form this paper.

REFERENCES

- [1] Gábor Domokos et al. "Balancing polyhedra". In: *arXiv preprint arXiv:1810.05382* (2018).
- [2] Péter L Várkonyi and Gábor Domokos. "Mono-monostatic bodies: the answer to Arnold's question". In: *Math. Intelligencer* 28.4 (2006), pp. 34–38.
- [3] A. Cayley. "XL. On contour and slope lines". In: *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* 18.120 (1859), pp. 264–268. DOI: 10.1080/14786445908642760. eprint: <https://doi.org/10.1080/14786445908642760>.
- [4] JC Maxwell. "On hills and dales. Volume XL". In: *The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science* (1870).
- [5] John Willard Milnor. *Morse theory*. 51. Princeton university press, 1963.
- [6] Gergő Almádi et al. "On Equilibria of Tetrahedra". In: *The Mathematical Intelligencer* (Sept. 2023). ISSN: 1866-7414. DOI: 10.1007/s00283-023-10294-2. URL: <https://doi.org/10.1007/s00283-023-10294-2>.
- [7] David Hilbert and Stephan Cohn-Vossen. *Anschauliche geometrie*. Springer-Verlag, 2011.
- [8] Jakob Steiner. "Einige Gesetze über die Theilung der Ebene und des Raumes". In: *Jacob Steiner's Gesammelte Werke: Herausgegeben auf Veranlassung der königlich preussischen Akademie der Wissenschaften*. Ed. by Karl-Editor Weierstrass. Cambridge Library Collection - Mathematics. Cambridge University Press, 2013, pp. 77–94.
- [9] Almádi Gergő. "Inhomogén politópok mechanikai komplexitása - avagy van-e egy tetraédernek lelke?" In: *BME ÉPK, Tudományos Diákköri Konferencia, Geometria Szekció* (2022).