

FRACTAL ZETA FUNCTIONS FOR p -ADIC AFFINE VARIETIES.

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ABSTRACT. Lapidus and Van Frankenhuysen [10] developed a theory of fractal strings within $\mathbb{R}$, some aspects of which we will explore in Section 2. Their ideas naturally extend to metric spaces which are also topological groups. In this paper, we will focus on the specific case where the group is $(\mathbb{Z}_p, +)$, a Polish group with a metric defined by the p -adic absolute value $|\cdot|_p$. As this structure is not only a group but also a ring, it gives rise to a rich family of spectral strings related to affine varieties. We will introduce a new p -adic zeta function associated to these affine varieties, and examine some of its key properties, such as its rationality, and how the abscissa of convergence aligns with the Minkowski dimension of the variety, which will be defined as an extension of the Minkowski dimension for fractal strings. Additionally, we will present a closed formula for the zeta function as a p -adic integral.

1. INTRODUCTION

Lapidus and Van Frankenhuysen [10] presented a theory of fractal strings set in $\mathbb{R}$ which we will partially discuss in Section 2. The concepts introduced by them most naturally generalize to metric spaces that are topological groups as well. According to a theorem by Birkhoff and Kakutani, a topological group G is metrizable if, and only if, G is Hausdorff, and the identity element has a countable neighbourhood basis. Moreover, if G is metrizable, G gives a compatible metric d , which is left-invariant: $d(xy, xz) = d(y, z)$. A Polish group is a topological group where G as a topological space is Polish¹. It is known that every separable metrizable topological group can be embedded densely to a Polish group. [7]

In this paper we will focus on the special case when the group in question is $(\mathbb{Z}_p, +)$. If we equip $\mathbb{Z}$ with the metric induced by $|\cdot|_p$ (Equation 3.1.), then the Polish group in which it can be densely embedded is $\mathbb{Z}_p$. Since this is not merely a group but a ring, it comes with a large family of spectral strings that arise from affine varieties.

Note that this family only exists over $\mathbb{Z}_p$. Over $\mathbb{R}$ these sets do not give rise to fractal strings, except in dimension one, when they are trivial².

In the present paper, we generalize the one-dimensional real fractal zeta function introduced in [10] to the p -adic setting $(\zeta_{V(I)}(s))$. We will concentrate on affine varieties for which we establish a number of interesting properties. In particular we show

- (1) the rationality of $\zeta_{V(I)}(s)$ (Theorem 3.8.)
- (2) that the abscissa of convergence of $\zeta_{V(I)}(s)$ coincides with the Minkowski dimension generalized from the Minkowski dimension of fractal strings (Theorem 4.8.)
- (3) a closed formula of $\zeta_{V(I)}(s)$ as a p -adic integral (Theorem 5.5.).

¹A topological space is called Polish if it is separable and completely metrizable.

²Or we can say they do not exist even there, since there are only finitely many intervals that have to be left out.

A quick outline of this note is as follows. In Section 2 we review the theory of fractal strings on the line and introduce the fractal zeta function. Then we illustrate the correlation between a fractal string's dimension and its zeta function's abscissa of convergence. After that in Section 3 we extend the definitions presented previously to p -adic affine varieties such as fractal strings, and fractal zeta functions. We then prove the rationality of said functions for the non-singular. In Section 4 we compare different definitions of dimensions on p -adic affine varieties, both algebraic and analytic. We end by discussing the connections between $\zeta_{V(I)}(s)$ and Igusa's local zeta function, which enables us to give a closed formula for the p -adic fractal zeta function.

2. THE GEOMETRIC ZETA FUNCTION OF A FRACTAL STRING

The following definitions and statements, with their proofs, can be found in Lapidus and Van Frankenhuysen [10].

A *fractal string* is a bounded open set (Ω) of the real line, and as such, it consists of countably many disjoint open intervals. Let us denote the complement of Ω with V_Ω . A fractal string may be represented by the lengths of these intervals which form a sequence: $\mathcal{L} = l_1, l_2, l_3, \dots$. Without loss of generality we may assume that $l_1 \geq l_2 \geq \dots > 0$. It is also known that the sum $\sum_{j=1}^{\infty} l_j$ is finite and equal to the Lebesgue measure of Ω .

The generalized Dirichlet-series

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} l_j^s$$

is the geometric zeta function of the fractal string $\mathcal{L}$. Since $\zeta_{\mathcal{L}}(s)$ converges in 1, it gives a holomorphic function for $\operatorname{Re} s > 1^3$.

The Minkowski dimension of the fractal string can be defined as the following:

$$D_{\mathcal{L}} = \inf\{\alpha \geq 0 : \operatorname{vol}(\varepsilon) = O(\varepsilon^{1-\alpha}) \text{ as } \varepsilon \rightarrow 0^+\}$$

(which is the inner Minkowski dimension of $\partial\Omega$), where $\operatorname{vol}(\varepsilon)$ is the volume of the outer ε neighbourhood of V_Ω :

$$\operatorname{vol}(\varepsilon) = \operatorname{vol}_1\{x \in \Omega : d(x, \partial\Omega) < \varepsilon\}$$

Definition 2.1. The abscissa of convergence for the fractal zeta function $\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} l_j^s$ is the following:

$$\sigma_{\mathcal{L}} = \inf\{\alpha \in \mathbb{R} : \sum_{j=1}^{\infty} l_j^s \text{ converges if } \operatorname{Re} s > \alpha\}$$

This is well defined since $\sigma_{\mathcal{L}}$ exists⁴ and $\forall z, s \in \mathbb{C} \ |z^s| = |z|^{\operatorname{Re} s}$, so if for $\alpha \in \mathbb{R}$ $\sum_{j=1}^{\infty} l_j^\alpha$ converges, then based on the Weierstrass M-test and the fact that $\lim_{j \rightarrow \infty} l_j = 0$ the series $\sum_{j=1}^{\infty} l_j^s$ has to converge for any $s \in \mathbb{C}, \operatorname{Re} s > \alpha$. [13]

An interesting fact is that the Minkowski dimension coincides with the abscissa of convergence of a fractal string's zeta function:

Theorem 2.2. [10]

$$D_{\mathcal{L}} = \sigma_{\mathcal{L}}$$

³Recall that for a positive real number a , and $s \in \mathbb{C}$, $a^s = e^{s \log a}$.

⁴Since $\lim_{j \rightarrow \infty} l_j = 0$, for any $\alpha \in \mathbb{R}_{\leq 0}$ there exists j_α , that if $j > j_\alpha$, then $l_j^\alpha \geq 1$, which means that $\sum_{j=1}^{\infty} l_j^\alpha = \infty$, thus $\sigma_{\mathcal{L}} \geq 0$.

We will illustrate this theorem through an example:

Example 2.3. The Cantor string is the complement of the ternary Cantor set in $[0,1]$ (thus $l_1 = \frac{1}{3}, l_2 = l_3 = \frac{1}{9}; l_4 = l_5 = l_6 = l_7 = \frac{1}{27}, \dots$).

$$V_C(\varepsilon) = \sum_{j: l_j < 2\varepsilon} l_j + \sum_{j: l_j \geq 2\varepsilon} 2\varepsilon = 2\varepsilon \cdot (2^n - 1) + \sum_{k=n}^{\infty} 2^k 3^{-k-1} = 2\varepsilon \cdot 2^n + \left(\frac{2}{3}\right)^n - 2\varepsilon$$

if $0 < \varepsilon \leq \frac{1}{2}$ and $3^{-n} \geq 2\varepsilon > 3^{-n-1}$, because the discs covering the endpoints of the interval overlap if $l_j < 2\varepsilon$. To find the Minkowski dimension we may assume that $2\varepsilon = 3^{-n}$, because any other ε is between two of this form and thus through the squeeze theorem we get the same result:

$$V_C(\varepsilon) = 2\varepsilon \cdot 2^n + \left(\frac{2}{3}\right)^n - 2\varepsilon = 3 \cdot \left(\frac{2}{3}\right)^n - \frac{2}{3^n} = \frac{2}{3^n} (3 \cdot 2^{n-1} - 1)$$

So to get the Minkowski dimension we take the $\log_\varepsilon$ of the tube of the fractal:

$$\frac{\log 2 - n \log 3 + \log(3 \cdot 2^{n-1} - 1)}{-n \log 3} \sim \frac{n \log 2 - (n-1) \log 3}{-n \log 3} \sim 1 - \frac{\log 2}{\log 3}$$

which makes $D_C = \log_3 2$.

The zeta function of the Cantor string is:

$$\zeta_C(s) = \sum_{k=1}^{\infty} 2^{k-1} 3^{-ks} = \frac{1}{2} \sum_{k=1}^{\infty} 2^k 3^{-ks} < \infty \iff \left|\frac{2}{3^s}\right| < 1 \iff \frac{2}{3^{\operatorname{Re} s}} \iff \operatorname{Re} s > \log_3 2$$

so the abscissa of convergence is $\log_3 2$.

In general, the geometric zeta function may not have an analytic continuation to all of $\mathbb{C}$, but can have a meromorphic extension on a so called window's ([10]) neighbourhood, which helps us define said window's complex dimension. Complex dimensions can be used to derive information about the geometry of a fractal. The definition of a fractal can be generalized to higher dimensions, for example, by introducing fractal sprays. The geometric zeta function then can give us a formula for the spectral zeta function, helping us find the spectrum which tells us about the geometry of the fractal. [10]

3. p -ADIC FRACTAL STRINGS ASSOCIATED TO AFFINE VARIETIES

3.1. Background. Let us introduce a new absolute value on the integers and then extend it to the rational numbers⁵ [9]. Let's fix a prime p and define for $a \in \mathbb{Z}$

$$(3.1) \quad |a|_p = p^{-k} \iff (p^k | a) \wedge (p^{k+1} \nmid a)$$

Since for any $a, b \in \mathbb{Z} : |ab|_p = |a|_p |b|_p$:

$$\left|\frac{a}{b}\right|_p = \frac{|a|_p}{|b|_p}$$

is well defined on $\mathbb{Q}$. The completion of this metric space gives us the p -adic numbers: $\mathbb{Q}_p$ [8]. We use $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ as the notation for the unit ball in $\mathbb{Q}_p$. It is also the closure of $\mathbb{Z}$ in $\mathbb{Q}_p$.

⁵A theorem of Ostrowski shows that the possible absolute values on $\mathbb{Q}$ are equivalent to either the usual $|\cdot|$ or one of the $|\cdot|_p$ -s introduced here. [9]

It will be convenient to use another realization of $\mathbb{Z}_p$. $\mathbb{Z}_p$ is isomorphic to the set that can be defined as $\{\sum_{k=0}^{\infty} a_k p^k : a_k \in \{0, 1, \dots, p-1\} \forall k\}$ (formal sums) or the set of infinite sequences $x_1, x_2, \dots$ where $\forall k \in \mathbb{N} \ x_{k+1} \equiv x_k \pmod{p^k}$. For any $x, y \in \mathbb{Z}_p$:

$$|x - y|_p \leq p^{-k} \iff p^k |x - y| \iff x \equiv y \pmod{p^k}$$

So $\{\sum_{k=0}^{\infty} a_k p^k : a_k \in \{0, 1, \dots, p-1\} \forall k, a_0 = b_0, \dots, a_n = b_n\}$ is the representation of a ball in $\mathbb{Z}_p$ with p^{-n} radius.

Proposition 3.1. *$\mathbb{Z}_p$ is homeomorphic to the Cantor set.*

Proof. It follows from Brouwer's theorem [2] that every non-empty, perfect, compact, totally disconnected metric space is homeomorphic to the Cantor set, thus it is sufficient to prove that $\mathbb{Z}_p$ has these properties:

- (1) *perfect*: If $x \in \mathbb{Z}_p$ and $|x|_p = p^{-k}, k \in \mathbb{N}$, then $B_{p^{-k}}(x) \subset \mathbb{Z}_p$.
- (2) *compact*: It is enough to show that $\mathbb{Z}_p$ is sequential compact.
Let $(x_n)_{n \in \mathbb{N}} \subset \mathbb{Z}_p$ be a sequence. Define $a_0 \in \{0, \dots, p-1\}$ such that $x_n \equiv a_0 \pmod{p}$ for infinitely many x_n -s, and let x_{n_0} be one of them. Recursively define $a_k \in \{0, \dots, p-1\}$ such that $x_n \equiv \sum_{i=0}^k a_i p^i \pmod{p^{k+1}}$ for infinitely many x_n -s, and let x_{n_k} be one of them.
Then $\lim_{k \rightarrow \infty} x_{n_k} = x = \sum_{i=0}^{\infty} a_i p^i$, since $|x - x_{n_k}|_p \leq p^{-(k+1)}$.
- (3) *totally disconnected*: It is enough to see that $\forall x \in \mathbb{Z}_p$ has a neighbourhood base consisting of clopen subsets. Since $x \mapsto x + y$ is a homeomorphism in $\mathbb{Z}_p$ for any $y \in \mathbb{Z}_p$, we may assume that $x = 0$. $\{B_{p^{-k}}(0) | \forall k \in \mathbb{N}\}$ is a neighbourhood base of 0 with clopen subsets.

□

By Haar's theorem [4] on every locally compact topological group there exists a left-translation-invariant measure called left Haar measure. On a commutative topological group, this is right-translation-invariant as well and thus is simply called Haar measure. We define the Haar measure of $\mathbb{Z}_p$ to be 1: $\lambda(\mathbb{Z}_p) = 1$.

To determine the Haar measure on $\mathbb{Z}_p$, note that for any $k \in \mathbb{N}$ $\mathbb{Z}_p$ is the disjoint union of p^k balls of radius p^{-k} all of whom are translates of each other. Moreover any ball of radius p^{-k} is one of these balls and hence,

$$\lambda(B_{p^{-k}}(x)) = p^{-k}$$

for any x . Finally any ball agrees with a ball of radius p^{-k} for some k : if $p^{-(k+1)} < r \leq p^{-k}$ then $B_r(x) = B_{p^{-k}}(x)$ and so, the Haar measure of a ball with radius r has to be p^{-k} for the unique k such that $p^{-(k+1)} < r \leq p^{-k}$.

We make the product space $\mathbb{Z}_p^n$ a metric space induced by the maximum norm:

$$|(x_1, \dots, x_n)^T|_p = \max\{|x_1|_p, \dots, |x_n|_p\}$$

This also has a Haar measure and applying the same as before, the normalized Haar measure in $\mathbb{Z}_p^n$ is

$$\forall x \in \mathbb{Z}_p^n, p^{-(k+1)} < r \leq p^{-k} : \lambda(B_r(x)) = p^{-nk}.$$

3.2. Introducing the p -adic fractal zeta function.

Definition 3.2. A p -adic fractal string (V) is the union of countably many disjoint open balls.

In every bounded metric space, we can call a bounded open set a fractal string if it is the disjoint union of countably many open balls. Because of the structure of $\mathbb{Z}_p$, all bounded open sets are fractal strings.

In the definition of such a decomposition of a p -adic fractal string, the radii of the balls are not unique, hence some care is needed when defining the fractal zeta function. Open balls in $\mathbb{Z}_p$ have neither a well defined center nor a well defined radius. It can happen that $B_r(x) = B_s(y)$ without $x = y$ or $r = s$. However this can only happen if $p^{-(k+1)} < r, s \leq p^{-k}$ and $|x - y|_p < p^{-k}$ [3]. The fact that any open ball of radius r is the union of p balls of radii r/p further complicates this issue. Thus a p -adic fractal string is not uniquely defined, we can give the same set as the disjoint union of countably many open balls in numerous ways.

Definition 3.3. Given an open set U of $\mathbb{Z}_p$ we will say that a ball $B_r(x) \subset U$ is maximal (with respect to U) if for any balls $B_s(y) \subset U$ if $B_r(x) \subset B_s(y)$ then $B_r(x) = B_s(y)$. Given a maximal ball in U $B_r(x)$ we define its radius to be p^{-k} with $p^{-(k+1)} < r \leq p^{-k}$.

We will require that in the collection that gives our fractal string no finite unions give another ball of $\mathbb{Z}_p$, thus, each ball present in the representation of the fractal string as a union of countably many disjoint open balls is maximal with respect to the fractal string.

To introduce the p -adic zeta function of varieties we will first prove that the complement of affine varieties are fractal strings by Definition 3.2., the union of countably many disjoint maximal balls.

Definition 3.4. Let $f_1, \dots, f_k$ be a polynomials of n -variables with integer coefficients and I the ideal generated by them. Then

$$V(I) = \{x \in \mathbb{Z}_p^n : f(x) = 0, \forall f \in I\}$$

is the variety of $f_1, \dots, f_k$.

Let us introduce the following notation: if $x = \sum_{k=0}^{\infty} a_k p^k$, then $[x]_m = \sum_{k=0}^{m-1} a_k p^k$. Let $N(p^k)$ denote the number of solutions $\mod p^k$ and set $N(p^0) = 1$.

Proposition 3.5. $\mathbb{Z}_p^n \setminus V(I)$ is a fractal string and for each $k \in \mathbb{Z}^+$ there are $p^n N(p^{k-1}) - N(p^k)$ maximal balls of radius p^{-k} in the complement of $V(I)$.

Lemma 3.6. Let $f \in \mathbb{Z}_p[x_1, \dots, x_n]$ and $x \in \mathbb{Z}_p^n$. Then $f(x) = 0$ if and only if $f([x]_k) \equiv 0 \mod p^k$ for all $k \in \mathbb{Z}^+$.

Proof. The map $x \mapsto x \mod p^k$ is a ring homomorphism and $x \equiv [x]_k \mod p^k$. Thus if $\forall k f([x]_k) \equiv 0 \mod p^k$, then $\forall k f(x) \equiv 0 \mod p^k$, which means that for any positive integer k $|f(x)|_p < p^{-k}$, thus $f(x) = 0$.

If $f(x) = 0$, then $f(x) \equiv 0 \mod p^k$ for any $k \in \mathbb{Z}^+$ and since $x \equiv [x]_k \mod p^k$ we get that $\forall k f([x]_k) \equiv 0 \mod p^k$. $\square$

Using Lemma 3.6. we can now prove Proposition 3.5.

Proof. If $[x]_k$ is no longer a solution for $f([x]_k) = 0$ (where as $[x]_{k-1}$ is) then the p^{-k} ball around $[x]_k$ is in the complement of $V(f)$, because any point of $\mathbb{Z}_p^n$ can only differ from $[x]_k$ till the k th digit in their formal series representation. The complement is the disjoint union of such balls.

If $I = (f_1, \dots, f_m)$ then $V(I) = \bigcap_{i=1}^m V(f_i)$ and in $\mathbb{Z}_p^n$ the finite intersection of the union of countably many disjoint balls is countably many disjoint balls.

All maximal balls in the complement of $V(I)$ of radius p^{-k} come from solutions mod $p^{-(k-1)}$, thus there are $p^n N(p^{k-1}) - N(p^k)$ of them. $\square$

We generalize the definition presented in Section 2. by defining the p -adic fractal zeta function of the p -adic fractal string V . This is similar to the one-dimensional case, except the radii and the multiplicities are not unique. However we can assign to V the unique sequence $\{k_j\}_{j=1}^\infty$ if $V = \bigcup_j^* \bigcup_{i=1}^{*k_j} B_{p^{-j}}(x_{ji})$ and for each x_{ji} that's the maximum radius ball around it that's in the complement: $\forall 1 \leq i \leq k_j : B_{p^{-(j-1)}}(x_{ji}) \not\subset V$.

Definition 3.7. Assume that $V = \bigcup_j^* \bigcup_{i=1}^{*k_j} B_{p^{-j}}(x_{ji})$ is a decomposition into maximal balls as above. Then let

$$\zeta_V(s) = \sum_{j=1}^{\infty} k_j p^{-js}$$

the fractal zeta function of the p -adic fractal string V .

In Proposition 3.5. we proved that if $V = \mathbb{Z}_p^n \setminus V(I)$, then $k_j = p^n N(p^{j-1}) - N(p^j)$. And so

$$\zeta_{V(I)}(s) = \sum_{k=1}^{\infty} (p^n N(p^{k-1}) - N(p^k)) p^{-ks}$$

is the p -adic fractal zeta function of the variety of I .

3.3. Properties of the p -adic zeta function. Before going on to the most striking property of $\zeta_{V(I)}(s)$ and present the proof, we will now calculate the zeta function of an example.

Theorem 3.8. If $I \triangleleft \mathbb{Z}[x_1, \dots, x_n]$ is an ideal, then the series $\zeta_{V(I)}$ can be expressed as a rational function of $T = p^{-s}$.

Example 3.9. $f(x, y) = x^2 + y^2 - 1$ we are looking for $([x]_k, [y]_k)$ pairs that satisfy the following knowing that $([x]_{k-1}, [y]_{k-1})$ satisfied the congruence mod p^{k-1} :

$$\begin{aligned} [x]_k^2 + [y]_k^2 &\equiv 1 \pmod{p^k} \\ ([x]_{k-1} + a_{k-1}p^{k-1})^2 + ([y]_{k-1} + b_{k-1}p^{k-1})^2 &\equiv 1 \pmod{p^k} \\ [x]_{k-1}^2 + a_{k-1}[x]_{k-1}p^{k-1} + [y]_{k-1}^2 + b_{k-1}[y]_{k-1}p^{k-1} &\equiv 1 \pmod{p^k} \end{aligned}$$

Since $[x]_{k-1}^2 + [y]_{k-1}^2 \equiv 1 \pmod{p^{k-1}}$, we have $cp^{k-1} \equiv [x]_{k-1}^2 + [y]_{k-1}^2 - 1 \pmod{p^k}$ for some $c \in \{0, \dots, p-1\}$, which gives

$$c + a_{k-1}[x]_{k-1} + b_{k-1}[y]_{k-1} \equiv 0 \pmod{p}$$

Since we reached a linear equality which gives us p solutions we can say that

$$N(p^k) = pN(p^{k-1}) = p^{k-1}N(p)$$

which means that for each k there are $p^2 N(p^{k-1}) - N(p^k)$ number of balls with p^{-k} radii. The geometric zeta function would be:

$$\begin{aligned} \sum_{k=1}^{\infty} (p^2 N(p^{k-1}) - N(p^k)) p^{-ks} &= N(p) \sum_{k=1}^{\infty} (p^k - p^{k-1}) p^{-ks} = \\ &= \frac{N(p)}{p} (p-1) \sum_{k=1}^{\infty} p^{k(1-s)} = N(p)(p-1) \frac{1}{1-p^{1-s}} p^{-s} = N(p)(p-1) \frac{T}{1-pT} \end{aligned}$$

which is a rational function of $T = p^{-s}$.

Definition 3.10. Let K be a field and $f \in K[x_1, \dots, x_n]$. The variety $V(f)$ is called non-singular if $\{x = (x_1, \dots, x_n) \mid f(x) = 0, \frac{\partial f}{\partial x_1}(x) = 0, \dots, \frac{\partial f}{\partial x_n}(x) = 0\} = \emptyset$.

The variety $V(I)$ is called non-singular over K if the vector space $\{(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x)) \mid f \in I\}$ has the same dimension $\forall x \in V(I)$ over K .

When this maximum dimension is m we say that dimension of $V(I)$ is $n - m$.

This paper will only give a self-contained proof of Theorem 3.8. when $V(I)$ is non-singular over $\mathbb{Z}/p\mathbb{Z}$ ⁶. In Section 5 we consider the general case, including singular varieties and show, using a hard theorem of Igusa that relies on Hironaka's resolution of singularities, that the fractal zeta function is rational even for singular varieties.

Our previous method (Example 3.9.) fails in singular cases since the number of solutions mod p^k aren't distributed as evenly as for non-singular varieties. The number of solutions around a non-singular point mod p^k can be given as a geometric sequence, but around a singular point it varies and has to be calculated separately. The difficulties are well illustrated via the following example, which is the simplest singularity:

Example 3.11. $f(x, y) = xy$

$xy \equiv 0 \pmod{p}$ if and only if at least one of them are dividable by p .

$xy \equiv [x]_2[y]_2 \equiv [x]_1[y]_1 + p(b_1[x]_1 + a_1[y]_1) \pmod{p^2}$

If $[x]_1 \not\equiv 0 \pmod{p}$ or $[y]_1 \not\equiv 0 \pmod{p}$, thus (x, y) is not in the p^{-2} neighbourhood of the singularity, then this gives p solutions. If $[x]_1 \equiv [y]_1 \equiv 0 \pmod{p}$, then it yields p^2 solutions.

Similarly for each m the p^{-m} neighbourhood of the singularity behaves differently and depending on the parity of m it brings 1 or p^2 results which complicates our previous method.

Luckily this can easily be resolved in the following way. Let us use the following notation:

$$M_k(p^l) = |\{([x]_k, [y]_k) \mid xy \equiv 0 \pmod{p^k}, \gcd([x]_k, p^k) = p^l\}|$$

Since

$$\{([x]_k, [y]_k) \mid xy \equiv 0 \pmod{p^k}\} = \cup_{l=0}^{*k} \{([x]_k, [y]_k) \mid xy \equiv 0 \pmod{p^k}, \gcd([x]_k, p^k) = p^l\}$$

it is enough to find $M_k(p^l)$ to get $N(p^k)$, because

$$N(p^k) = \sum_{l=0}^k M_k(p^l).$$

⁶If a variety is non-singular over $\mathbb{Z}/p\mathbb{Z}$, then it follows that it is non-singular over $\mathbb{Q}_p$, but the converse does not hold.

If $0 \leq l < k$ and $\gcd([x]_k, p^k) = p^l$, then $p^{k-l} | y$, so

$$M_k(p^l) = (p^{k-l+1} - p^{k-l})p^l = (p-1)p^k.$$

Now $M_k(p^k) = p^k$, since $[x]_k$ has to be 0 and $[y]_k$ can be anything. Thus

$$N(p^k) = \sum_{l=0}^k M_k(p^l) = k(p-1)p^k + p^k.$$

Just like in the previous example, the p -adic fractal zeta function can be written as

$$\begin{aligned} \zeta_{V(f)}(s) &= \sum_{k=1}^{\infty} (p^2 N(p^{k-1}) - N(p^k)) p^{-ks} = \\ &= \sum_{k=1}^{\infty} ((k-1)(p-1)p^{k+1} + p^{k+1} - k(p-1)p^k - p^k) p^{-ks} = \\ &= (p-1)^2 \sum_{k=1}^{\infty} (k-1)(p^{1-s})^k = p^{2(1-s)}(p-1)^2 \sum_{k=1}^{\infty} (k-1)(p^{1-s})^{k-2}. \end{aligned}$$

Since

$$\sum_{k=1}^{\infty} (k-1)T^{k-2} = \left(\sum_{k=1}^{\infty} T^{k-1} \right)' = \left(\frac{1}{1-T} \right)' = \frac{1}{(1-T)^2}$$

we have

$$\zeta_V(f)(s) = p^2(p-1)^2 \frac{T^2}{1-2pT+p^2T^2}$$

a rational function in $T = p^{-s}$.

3.4. Proof of rationality of the zeta function. Now we turn to building up our proof for Theorem 3.8. We begin by introducing a lemma.

Lemma 3.12. *If $f \in \mathbb{Z}[x_1, \dots, x_n]$ and its variety is non-singular over $\mathbb{Z}/p\mathbb{Z}$ then the number of solutions $\bmod p^k$ satisfies*

$$N(p^k) = p^{(n-1)(k-1)} N(p)$$

Proof. Because of Lemma 3.6. we may assume that a given $[x]_{k-1}$ was a solution $\bmod p^{k-1}$ and we are interested in how many $[x]_k$ solutions $\bmod p^k$ are such that $[x]_k \equiv [x]_{k-1} \bmod p^{k-1}$. For every i between 1 and n if $x_i = \sum_{j=0}^{\infty} b_{i,j} p^j$ then $[x_i]_k = [x_i]_{k-1} + b_{i,k-1} p^{k-1}$. Let us use the following notations:

$$\begin{aligned} \mathbf{x} &= (x_1, \dots, x_n) \\ \forall k : \mathbf{b}_k &= (b_{1,k}, \dots, b_{n,k}) \\ \nabla f(\mathbf{x}) &= \left(\frac{\partial f}{\partial x_1}(\mathbf{x}), \dots, \frac{\partial f}{\partial x_n}(\mathbf{x}) \right) \\ \forall k : \langle \nabla f(\mathbf{x}), \mathbf{b}_k \rangle &= \sum_{i=1}^n b_{i,k} \frac{\partial f}{\partial x_i}(\mathbf{x}) \end{aligned}$$

Then we get

$$f([x]_k) \equiv f([x]_{k-1} + \mathbf{b}_{k-1} p^{k-1}) \bmod p^k$$

Notice that for any $x \in \mathbb{Z}_p$

$$[x]_k^d \equiv ([x]_{k-1} + bp^{k-1})^d \equiv [x]_{k-1}^d + d[x]_{k-1}^{d-1}bp^{k-1} \pmod{p^k}$$

because any component which would give a higher degree would have a constant multiplier that is divisible by p^k .

And so

$$\begin{aligned} [x_1]_k^{d_1} \dots [x_n]_k^{d_n} &\equiv [x_1]_{k-1}^{d_1} \dots [x_n]_{k-1}^{d_n} + \left(\sum_{i=1}^n d_i b_{i,k} [x_i]_{k-1}^{d_i-1} \prod_{i \neq j} [x_j]_{k-1}^{d_j} \right) p^{k-1} \pmod{p^k} \\ &\equiv [x_1]_{k-1}^{d_1} \dots [x_n]_{k-1}^{d_n} + \langle \nabla([x_1]_{k-1}^{d_1} \dots [x_n]_{k-1}^{d_n}), \mathbf{b}_{\mathbf{k}-1} \rangle p^{k-1} \pmod{p^k} \end{aligned}$$

Thus

$$f([\mathbf{x}]_k) \equiv f([\mathbf{x}]_{k-1}) + \langle \nabla f(\mathbf{x}), \mathbf{b}_{\mathbf{k}-1} \rangle p^{k-1} \pmod{p^k}$$

Just like in the example the components which aren't dependent on $b_{i,k-1}$ give

$$f([\mathbf{x}]_{k-1}) \equiv 0 \pmod{p^{k-1}}$$

which means there is a $c = c([\mathbf{x}]_{k-1}) \in \{0, \dots, p-1\}$ such that $f([\mathbf{x}]_{k-1}) \equiv cp^{k-1} \pmod{p^k}$.

Thus $f([\mathbf{x}]_k) \equiv 0 \pmod{p^k}$ if and only if

$$c + \langle \nabla f([\mathbf{x}]_{k-1}), \mathbf{b}_{\mathbf{k}-1} \rangle \equiv c + \langle \nabla f([\mathbf{x}]_1), \mathbf{b}_{\mathbf{k}-1} \rangle \equiv 0 \pmod{p}.$$

Therefore

$$N(p^k) = \sum_{\substack{[\mathbf{x}]_{k-1} \pmod{p^{k-1}} \\ f([\mathbf{x}]_{k-1}) \equiv 0}} |\{\mathbf{b} \in (\mathbb{Z}/p\mathbb{Z})^n : \langle \nabla f([\mathbf{x}]_1), \mathbf{b} \rangle \equiv -c([\mathbf{x}]_{k-1}) \pmod{p}\}|.$$

Since f is non-singular, $\nabla f \neq 0$, and so:

$$\begin{aligned} N(p^k) &= \sum_{\substack{[\mathbf{x}]_{k-1} \pmod{p^{k-1}} \\ f([\mathbf{x}]_{k-1}) \equiv 0}} |\{\mathbf{b} \in (\mathbb{Z}/p\mathbb{Z})^n : \langle \nabla f([\mathbf{x}]_{k-1}), \mathbf{b} \rangle \equiv -c([\mathbf{x}]_{k-1}) \pmod{p}\}| = \\ &= |\{\mathbf{b} \in (\mathbb{Z}/p\mathbb{Z})^n : \langle \nabla f([\mathbf{x}]_{k-1}), \mathbf{b} \rangle \equiv 0 \pmod{p}\}| N(p^{k-1}) = p^{n-1} N(p^{k-1}). \end{aligned}$$

Using induction on k we get:

$$N(p^k) = p^{(n-1)(k-1)} N(p)$$

□

Lemma 3.13. *If $V(I)$ is non-singular over $\mathbb{Z}/p\mathbb{Z}$ and of dimension $n-m$, then the number of solutions $\pmod{p^k}$ is the following:*

$$N(p^k) = p^{(n-m)(k-1)} N(p)$$

Proof. Let $f_1, \dots, f_m \in I$ be polynomials so that in $[\mathbf{x}]_1$ their gradients span the vector space $\{(\frac{\partial f}{\partial x_1}(\mathbf{x}), \dots, \frac{\partial f}{\partial x_n}(\mathbf{x})) | f \in I\}$, thus, they are linearly independent. Since $V(I)$ is non-singular and with algebraic dimension $n-m$, there exist such polynomials in I .

Like in our previous proof for each f_i we get a c_i so that

$$c_i p^{k-1} \equiv f_i([\mathbf{x}]_{k-1}) \pmod{p^k}$$

and so we are looking for the number of $\mathbf{b} \in (\mathbb{Z}/p\mathbb{Z})^n$ which satisfy

$$\forall i : \langle \nabla f_i([\mathbf{x}]_{k-1}), \mathbf{b} \rangle \equiv \langle \nabla f_i([\mathbf{x}]_1), \mathbf{b} \rangle \equiv -c_i \pmod{p}$$

So we are looking for the number of solutions $\bmod p$ in $(\mathbb{Z}/p\mathbb{Z})^n$ for the equation $J\mathbf{x} = -\mathbf{c}$, where $J_{ij} = \frac{\partial f_i}{\partial x_j}([\mathbf{x}]_1)$ is the element in the i th row in the j th column, J is the Jacobian matrix of $F = (f_1, \dots, f_m)^T$.

By the definition of $f_1, \dots, f_m$ the rank of J is m . Therefore, since $\dim(\text{im } J) = m$,

$$J : (\mathbb{Z}/p\mathbb{Z})^n \rightarrow (\mathbb{Z}/p\mathbb{Z})^m$$

is surjective, and $\dim(\ker J) = n - m$ through the dimension theorem for vector spaces. Therefore, there are p^{n-m} solutions for $J\mathbf{x} = \mathbf{y}$ for any $\mathbf{y} \in (\mathbb{Z}/p\mathbb{Z})^m$. Thus

$$N(p^k) = |\{\mathbf{b} \in (\mathbb{Z}/p\mathbb{Z})^n : J\mathbf{b} \equiv -\mathbf{c} \pmod{p}\}| N(p^{k-1}) = p^{n-m} N(p^{k-1}) = p^{(n-m)(k-1)} N(p).$$

□

We will now prove Theorem 3.8. in the non-singular case.

Proof. We may assume that $V(I)$ has dimension $n - m$. Through Proposition 3.5. there are $p^n N(p^{k-1}) - N(p^k)$ maximal balls of radius p^{-k} in the complement of $V(I)$. Using Lemma 3.13. we get:

$$\begin{aligned} \zeta_{V(I)}(s) &= \sum_{k=1}^{\infty} (p^n N(p^{k-1}) - N(p^k)) p^{-ks} = N(p) \sum_{k=1}^{\infty} (p^{(n-m)(k-2)+n} - p^{(n-m)(k-1)}) p^{-ks} = \\ &= N(p) (p^{-(n-2m)} - p^{-(n-m)}) \sum_{k=1}^{\infty} (p^{n-m-s})^k = N(p) (p^{-(n-m)} - p^{-n}) \frac{p^{n-s}}{1 - p^{n-m-s}} = \\ &= N(p) (p^m - 1) \frac{T}{1 - p^{n-m} T} \end{aligned}$$

□

Our results can be expressed in the terms of the generating function of the sequence $N(p^k)$. This is called the Poincaré series of $V(I)$ [11, 12]:

Definition 3.14.

$$P_{V(I)}(T) = \sum_{k=0}^{\infty} N(p^k) p^{-nk} T^k$$

We then have the following connection:

Proposition 3.15.

$$\zeta_{V(I)}(s) = (p^n T - 1) P_{V(I)}(p^n T) + 1$$

Note that by Lemma 3.13. The Poincaré series of a non-singular variety $V(I)$ is also a rational function.

4. DIMENSION CONCEPTS FOR p -ADIC VARIETIES AND FRACTAL ZETA FUNCTIONS

The dimension of an algebraic variety can be defined in many ways. One may take the dimension to be the maximum dimension of the tangent spaces at the variety's non-singular points like we did in Definition 3.10. Any ideal $I = (f_1, \dots, f_m)$ generated by m independent polynomials over a field will have at least $n - m$ as its variety's dimension in the affine space $\mathbb{A}^n$.

Since we are in a space equipped with a metric, this gives us the opportunity to introduce and observe other concepts for the dimension of the variety. Recall from Section 2 that for one-dimensional real fractal strings, the abscissa of convergence of a fractal string's zeta function coincides with its Minkowski dimension. We will

now prove that this still holds for a non-singular p -adic variety's zeta function after defining its Minkowski dimension. The proof will consist of proving for each of them separately that they agree with the algebraic dimension.

As before let $I \triangleleft \mathbb{Z}[x_1, \dots, x_n]$.

Definition 4.1. The abscissa of convergence for the p -adic fractal zeta function $\zeta_{V(I)}(s) = \sum_{j=1}^{\infty} k_j p^{-js}$ is the following:

$$\sigma_{V(I)} = \inf \{ \alpha \in \mathbb{R} : \sum_{j=1}^{\infty} k_j p^{-js} \text{ converges if } \operatorname{Re} s > \alpha \}$$

Proposition 4.2. *The abscissa of convergence of the p -adic zeta function of $V(I)$ when non-singular over $\mathbb{Z}/p\mathbb{Z}$ and with algebraic dimension $n - m$, is $n - m$.*

Proof.

$$\begin{aligned} \zeta_{V(I)}(s) &= N(p)(p^{-(n-2m)} - p^{-(n-m)}) \sum_{k=1}^{\infty} (p^{n-m-s})^k < \infty \iff \\ \iff |p^{n-m-s}| < 1 &\iff \operatorname{Re}(n - m - s) < 0 \iff n - m < \operatorname{Re} s \end{aligned}$$

□

The Minkowski dimension, also called the box dimension of a set in a metric space measures how well it can be covered by balls. We may use a different approach and define the Minkowski dimension using the set's Minkowski content. In the non-singular case both ways lead to the same result, the algebraic dimension.

Definition 4.3. The volume of the ε tube of the variety is

$$\operatorname{vol}(\varepsilon) = \lambda(\{x \in \mathbb{Z}_p^n \setminus V(I) \mid \exists y \in V(I) : |x - y|_p < \varepsilon\}).$$

Similarly to fractal strings, we can define the variety's Minkowski dimension using the tube's volume:

Definition 4.4. The Minkowski dimension of the variety $V(I)$ is

$$D_{V(I)} = \inf \{ \alpha \geq 0 : \operatorname{vol}(\varepsilon) = O(\varepsilon^{n-\alpha}) \text{ as } \varepsilon \rightarrow 0^+ \}.$$

Notice that for plane curves and surfaces in $\mathbb{R}^3$ this gives the expected results: If γ is a smooth, simple, closed plane curve, then [14]

$$\operatorname{vol}(\varepsilon) = 2\operatorname{Length}(\gamma)\varepsilon = O(\varepsilon) = O(\varepsilon^{2-1})$$

thus, the dimension is 1 as expected.

If Σ is an oriented closed surface in $\mathbb{R}^3$, then Weyl's tube formula for a surface in $\mathbb{R}^3$ gives [14]

$$\operatorname{vol}(\varepsilon) = 2\operatorname{Area}(\Sigma)\varepsilon + \frac{4\pi}{3}\chi(\Sigma)\varepsilon^3 = O(\varepsilon) = O(\varepsilon^{3-2})$$

where $\chi(\Sigma)$ is the Euler characteristic of the surface, which makes the dimension to be 2.

Proposition 4.5. *The Minkowski dimension of the variety $V(I)$ when non-singular over $\mathbb{Z}/p\mathbb{Z}$ and with algebraic dimension $n - m$, is $n - m$.*

Proof. It is sufficient to show for $\varepsilon = p^{-l}$. If $x \in Z_p^n \setminus V(I)$ and $\forall y \in V(I) : |x - y|_p \geq p^{-k}$, then $B_{p^{-k}}(x) \subset \mathbb{Z}_p^n \setminus V(I)$. So to find $\text{vol}(p^{-l})$, we need to add the measure of the balls with radii at most $p^{-(l+1)}$ in the representation of the fractal string:

$$\text{vol}(p^{-l}) = \sum_{k=l+1}^{\infty} (p^n N(p^{k-1}) - N(p^k)) p^{-nk}$$

From Lemma 3.13. we find that in this case $N(p^k) = p^{(n-m)(k-1)} N(p)$, and so

$$\begin{aligned} \text{vol}(p^{-l}) &= N(p) \sum_{k=l+1}^{\infty} (p^{(n-m)(k-2)+n} - p^{(n-m)(k-1)}) p^{-nk} = \\ &= N(p) (p^{-(n-2m)} - p^{-(n-m)}) \sum_{k=l+1}^{\infty} p^{-mk} = \\ &= N(p) (p^{-(n-2m)} - p^{-(n-m)}) \cdot \frac{p^{-ml}}{p^m - 1} = \frac{N(p)}{p^{(n-m)}} p^{-ml} \end{aligned}$$

Thus

$$\log_{p^{-l}}(\text{vol}(p^{-l})) = \frac{\log(N(p)) - (lm + n - m) \log(p)}{-l \log(p)}$$

And so

$$\lim_{l \rightarrow \infty} \log_{p^{-l}}(\text{vol}(p^{-l})) = m$$

□

Or we may use the general definition for a set's Minkowski dimension:

Definition 4.6. Let $N(V(I), \varepsilon)$ denote the minimal number of sets with diameter at most ε that cover $V(I)$. The box dimension of the variety $V(I)$ is the limit

$$\dim(V(I)) = \lim_{\varepsilon \rightarrow 0} \frac{\log N(V(I), \varepsilon)}{\log \frac{1}{\varepsilon}}$$

when it exists. [1]

Proposition 4.7. *The box dimension of the variety $V(I)$ when non-singular over $\mathbb{Z}/p\mathbb{Z}$, where I is with algebraic dimension $n - m$, is $n - m$.*

Proof. It is sufficient to show for $\varepsilon = p^{-k}$, because $N(V(I), \varepsilon) = N(V(I), p^{-k})$ for some k . However $N(V(I), p^{-k}) = N(p^k)$, because for each $x \in V(I)$ the balls with radius p^{-k} around the solutions of $\text{mod } p^k$ give a minimal cover.

$$\dim(V(I)) = \lim_{\varepsilon \rightarrow 0} \frac{\log(N(V(I), \varepsilon))}{\log \frac{1}{\varepsilon}} = \lim_{k \rightarrow \infty} \frac{\log(N(V(I), p^{-k}))}{k \log(p)} = \lim_{k \rightarrow \infty} \frac{\log(N(p^k))}{k \log(p)}$$

But since $V(I)$ is non-singular we know that $N(p^k) = p^{(n-m)(k-1)} N(p)$, thus,

$$\dim V(I) = \lim_{k \rightarrow \infty} \frac{\log(p^{(n-m)(k-1)} N(p))}{k \log(p)} = \lim_{k \rightarrow \infty} \left((n-m) \frac{k-1}{k} + \frac{\log(N(p))}{k \log(p)} \right) = n - m$$

□

Thus, we have proved the following:

Theorem 4.8. *The Minkowski dimension of a non-singular variety over $\mathbb{Z}/p\mathbb{Z}$ aligns with the abscissa of convergence of its p -adic zeta function.*

5. CONNECTION TO THE IGUSA LOCAL ZETA FUNCTION

The case of a singular variety's fractal zeta function can be dealt with if we consider its connections to the Poincaré series (Definition 3.14.). This research was initiated by Jun-Ichi Igusa, in a more generalized context regarding the so called Igusa local zeta function [6]:

$$Z_{K,\phi}(s) = \int_{K^n} \phi(x) |f(x)|_K^s |dx|_K$$

for $s \in \mathbb{C}$ with $\operatorname{Re} s > 0$ where K is a local field, $|\cdot|_K$ is an absolute value on K , $f \in K[x_1, \dots, x_n]$ is a polynomial, $|dx|_K$ a Haar measure compatible with that absolute value and $\phi \in \mathcal{S}(K^n)$ is a compactly supported locally constant function⁷.

If $K = \mathbb{Q}_p$ and $\phi = \chi_{\mathbb{Z}_p^n}$ the characteristic function of $\mathbb{Z}_p^n$ then we get the Igusa local zeta function for p -adic numbers:

$$Z_{V(f)}(s) = \int_{\mathbb{Z}_p^n} |f(x)|_p^s d\lambda(x)$$

A survey of the main results can be found in [11, 12].

Theorem 5.1. $Z_{V(f)}(s)$ is a rational function in $T = p^{-s}$.

Igusa [6] showed this in 1974 using Hironaka's theorem [5] about the resolution of singularities.

It is also known that the above mentioned Poincaré series can be given as a rational function of the Igusa local zeta function. We will present a proof for this as seen below.

Proposition 5.2. [6]

$$Z_{V(f)}(s) = \frac{(p^{-s} - 1)P_{V(f)}(p^{-s}) + 1}{p^{-s}}$$

Proof.

$$\begin{aligned} Z_{V(f)}(s) &= \int_{\mathbb{Z}_p^n} |f(x)|_p^s d\lambda(x) = \sum_{k=0}^{\infty} \int_{\{x \in \mathbb{Z}_p^n \mid |f(x)|_p = p^{-k}\}} p^{-ks} d\lambda(x) = \\ &= \sum_{k=0}^{\infty} p^{-ks} \int_{\{x \in \mathbb{Z}_p^n \mid |f(x)|_p = p^{-k}\}} d\lambda(x) = \sum_{k=0}^{\infty} p^{-ks} \lambda(\{x \in \mathbb{Z}_p^n \mid |f(x)|_p = p^{-k}\}) = \\ &= \sum_{k=0}^{\infty} p^{-ks} (\lambda(\{x \in \mathbb{Z}_p^n \mid |f(x)|_p \leq p^{-k}\}) - \lambda(\{x \in \mathbb{Z}_p^n \mid |f(x)|_p \leq p^{-(k+1)}\})) \end{aligned}$$

Now observe that:

$$\lambda(\{x \in \mathbb{Z}_p^n \mid |f(x)|_p \leq p^{-k}\}) = \lambda(\{x \in \mathbb{Z}_p^n \mid f([x]_k) \equiv 0 \pmod{p^k}\}) = p^{-nk} N(p^k)$$

⁷A so called Schwartz-function on K^n .

Which means that:

$$\begin{aligned}
Z_{V(f)}(s) &= \sum_{k=0}^{\infty} p^{-ks} (\lambda(\{x \in \mathbb{Z}_p^n \mid |f(x)|_p \leq p^{-k}\}) - \lambda(\{x \in \mathbb{Z}_p^n \mid |f(x)|_p \leq p^{-(k+1)}\})) = \\
&= \sum_{k=0}^{\infty} p^{-ks} (p^{-nk} N(p^k) - p^{-n(k+1)} N(p^{k+1})) = \\
&= \sum_{k=0}^{\infty} p^{-ks} p^{-nk} N(p^k) - \frac{1}{p^{-s}} \sum_{k=0}^{\infty} p^{-ks} p^{-nk} N(p^k) = \\
&= P_{V(f)}(p^{-s}) - \frac{1}{p^{-s}} P_{V(f)}(p^{-s}) = \frac{(p^{-s} - 1) P_{V(f)}(p^{-s}) + 1}{p^{-s}}
\end{aligned}$$

□

Based on Proposition 5.2. we may deduce the following corollary.

Corollary 5.3.

$$P_{V(f)}(T) = \frac{1 - T \cdot Z_{V(f)}(s)}{1 - T}$$

Notice that the Igusa local zeta function is only defined on varieties given by a principal ideal, but all varieties can be given in such a form:

Proposition 5.4. *If $I \triangleleft \mathbb{Z}_p[x_1, \dots, x_n]$, then $\exists f \in \mathbb{Z}_p[x_1, \dots, x_n]$, so that $V(I) = V(f)$.*

Here $V(I)$ refers to the set of points in $\mathbb{Z}_p$ as in Definition 3.4. and not the the abstract algebraic variety in the sense of scheme theory.

Proof. Let I be the ideal generated by $f_1, \dots, f_k \in \mathbb{Z}_p[x_1, \dots, x_n]$, then it is sufficient to find a $g \in \mathbb{Z}_p[x_1, \dots, x_k]$ that's only root is $(0, \dots, 0) \in \mathbb{Z}_p^k$ since then $f = g(f_1, \dots, f_k)$ satisfies the conditions given.

It is enough to show that there is an $h \in \mathbb{Z}_p[x_1, x_2]$ that on $\mathbb{Z}_p^2$ vanishes only at the point $(0, 0)$. Given such an h , the polynomial

$$g(x_1, \dots, x_k) = h(x_1, h(x_2, \dots h(x_{k-2}, h(x_{k-1}, x_k)) \dots))$$

is a polynomial with integer coefficients whose only root is $(0, \dots, 0) \in \mathbb{Z}_p^k$.

Let $h(x, y) = x^2 - py^2$. We will show that $h(x, y)$ meets the above conditions, i.e. if $x, y \in \mathbb{Z}_p$ then $h(x, y) = 0$ if and only x, y are both 0. It is enough to see that $y = 0$, because $h(x, y) = 0$ if and only if $x^2 = py^2$, thus if $y = 0$, then $x^2 = 0$, and so $x = 0$. To prove this, we will show that $\forall k \ p^k | y$ using induction on k :

For the base case assume first that $k = 1$: since $x^2 = py^2$, $p | x^2$, thus $p | x$, meaning $p^2 | py^2$, which shows, that $p | y$.

Now assume that $p^k | y$: again since $x^2 = py^2$, $p^{2k+1} | x^2$, meaning $p^{k+1} | x$, which shows, that $p^{2k+2} | py^2$, and so $p^{k+1} | y$. □

Since the Igusa local zeta function can be given as a rational function in T , and the Poincaré series is a rational function of the Igusa local zeta function, and our fractal zeta function is a linear function of the Poincaré series, we can see that it also can be given as a rational function of T . The relationships between the three functions suggest the following theorem.

Theorem 5.5. *The p -adic zeta function can be given in a closed formula as a p -adic integral.*

Proof. From Proposition 3.15. and 5.2. we get

$$\zeta_{V(f)}(s) = p^{-(n-s)} Z_{V(f)}(-(n-s)) = p^{-(n-s)} \int_{\mathbb{Z}_p^n} |f(x)|_p^{-(n-s)} d\lambda(x)$$

so we have given our fractal zeta function in a closed formula. $\square$

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