

CLASS NUMBER OF POSITIVE DEFINITE BINARY FORMS OVER EISENSTEIN INTEGERS

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MATHEMATICS BSC 2.

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ABSTRACT. We will use a method developed by Duke, Imamoğlu, and Tóth to treat positive definite binary Hermitian forms with entries in the Eisenstein ring $R = \mathbb{Z} + \mathbb{Z}\omega$, where ω is a primitive third root of unity. Such forms may also be interpreted as 4-dimensional lattices with extra symmetries coming from R . We will construct a smooth partition of unity with respect to the group $SL_2(R)$, which is in fact rational in terms of the entries of the form. We will use this partition of unity to derive analytic class number formulas for positive definite R -integral binary hermitian forms of a fixed discriminant.

1. INTRODUCTION

In this paper, we use the method introduced by [1] to establish an analytic formula for calculating the class number of both positive definite symmetric matrices over the integers, $\mathbb{Z}$, and positive definite Hermitian matrices over the Eisenstein ring, $R = \mathbb{Z} + \mathbb{Z}\omega$, where ω represents a primitive cube root of unity. By employing algebraic techniques, we define a fundamental domain and construct an isomorphic mapping from this domain for symmetric positive definite matrices over $\mathbb{Z}$ to a tetrahedron. Similarly, we create an isomorphism from the fundamental domain of positive definite Hermitian matrices over R to a four-dimensional convex polytope. For both types of positive definite matrices, we define an isomorphism that maps each matrix to a unique point, with these points collectively forming a convex cone. Using volume formulas, we define an equivariant partition of unity and apply Theorem 3.2 to determine the class numbers. We let H_D denote the Hurwitz class number, the number of $SL_2(R)$ -orbits weighted by their stabilizers, and show that

$$H_D = \frac{D^2}{3\pi} \cdot \sum \frac{2a + b_1 + 2d}{d(a+d)(a+b_1+d)(a+b_2+d)(a+b_1-b_2+d)}$$

where the sum is over $\begin{pmatrix} a & b \\ b & d \end{pmatrix}$ positive definite Hermitian matrices over R of determinant D .

2. BACKGROUND AND MOTIVATION

Definition 2.1. Let's define the quadratic forms of n variable in the following way:

$$\mathcal{Q} := \left\{ Q \mid Q : \mathbb{R}^n \rightarrow \mathbb{R}, Q(x_1, x_2, \dots, x_n) = \sum_{1 \leq i \leq j \leq n} \alpha_{ij} x_i x_j, \alpha_{ij} \in \mathbb{R} \right\}$$

Lemma 2.2. $\mathcal{Q}$ is a vector space, further

$$\mathcal{Q} \simeq \mathcal{S}$$

Where $\mathcal{S}$ stands for the symmetric $n \times n$ matrices over $\mathbb{R}$

Proof. Let's define a ϕ function (later will be shown: an isomorphism) that maps from $\mathcal{S}$ to $\mathcal{Q}$:
Satisfying that if $\phi(M) = Q$, then

$$(x_1, x_2, \dots, x_n)^T M (x_1, x_2, \dots, x_n) = Q(x_1, \dots, x_n)$$

Equivalently

$$\sum_{1 \leq i < j \leq n} 2M_{ij}x_i x_j + \sum_{i=1}^n M_{ii}x_i^2 = \sum_{1 \leq i \leq j \leq n} \alpha_{ij}x_i x_j,$$

It is identical to

$$\alpha_{ij} = 2M_{ij} \text{ for } i \neq j$$

and

$$\alpha_{ii} = M_{ii}$$

Where M_{ij} stands for the i -th row's j -th element in M .

It means ϕ is an isomorphism, consequently $\mathcal{Q} \simeq S$

□

Definition 2.3. Let's define the inner product on the vector space $\mathcal{S}$ the following way:

$$\langle A, B \rangle = \text{tr}(AB)$$

We verify the properties of an inner product:

Bilinearity:

$$\langle A + B, C \rangle = \text{tr}((A + B)C) = \text{tr}(AC) + \text{tr}(BC) = \langle A, C \rangle + \langle B, C \rangle$$

$$\langle \lambda A, B \rangle = \text{tr}((\lambda A)B) = \lambda \text{tr}(AB) = \lambda \langle A, B \rangle$$

Symmetry:

$$\langle A, B \rangle = \text{tr}(AB) = \text{tr}(BA) = \langle B, A \rangle$$

Positive-Definiteness:

$$\langle A, A \rangle = \text{tr}(A^2) = \sum_{i,j=1}^n A_{ij}^2 \geq 0$$

and $\langle A, A \rangle = 0$ if and only if $A_{ij} = 0$ for all $1 \leq i, j \leq n$, equivalently if and only if $A = 0$

Thus, $\langle A, B \rangle = \text{tr}(AB)$ is an inner product on S .

Definition 2.4. The following is a group action of $GL_n(\mathbb{R})$ on $\mathcal{S}$:

$$\varphi(g) : \mathcal{S} \rightarrow \mathcal{S}$$

For a $g \in GL_n(\mathbb{R})$, let

$$\varphi(g)X = gXg^T$$

Let's verify that φ is a group action:

$$\varphi : GL_n(\mathbb{R}) \times \mathcal{S} \rightarrow \mathcal{S}$$

Identity property:

$$\varphi(e)X = eXe^T = X$$

for all $X \in \mathcal{S}$, where e stands for the identity element of the group $GL_n(\mathbb{R})$, which is I_n , the $n \times n$ identity matrix.

Associativity:

$$\begin{aligned} \varphi(gh)X &= (gh)X(gh)^T = ghXh^Tg^T = \\ &= g(hXh^T)g^T = \varphi(g)(\varphi(h)X) = \varphi(g)\varphi(h)X \end{aligned}$$

for all $g, h \in GL_n(\mathbb{R})$ and $X \in S$. It means φ is in fact a group action.

Definition 2.5. The orthogonal group on $\mathbb{R}^n$, $O(n)$ is defined as:

$$O(n) := \{g \in GL_n(\mathbb{R}) \mid gg^T = g^T g = I_n\}$$

The elements of $O(n)$ are said to be *orthogonal*.

Lemma 2.6. If g is orthogonal, then $\varphi(g)$ preserves the inner product on $\mathcal{S}$

Proof. We want to show, that $\text{tr}(AB) = \text{tr}((\varphi(g)A)(\varphi(g)B))$

Rewrite it using the definition of $\varphi(g)X$:

$$\text{tr}((\varphi(g)A)(\varphi(g)B)) = \text{tr}(gAg^T gBg^T)$$

If g is orthogonal $g^T g = I_n$, therefore

$$\text{tr}((\varphi(g)A)(\varphi(g)B)) = \text{tr}(gABg^T) = \text{tr}(ABg^T g)$$

Because $\text{tr}(XY) = \text{tr}(YX)$. Using $g^T g = I_n$ again:

$$\text{tr}((\varphi(g)A)(\varphi(g)B)) = \text{tr}(AB)$$

□

Lemma 2.7. $\varphi(g)$ is a linear transformation

$$\mathcal{S} \simeq \mathbb{R}^{\binom{n+1}{2}} \rightarrow \mathbb{R}^{\binom{n+1}{2}}$$

with an $\binom{n+1}{2} \times \binom{n+1}{2}$ transformation matrix M^* , satisfying $\det(M^*) = \det(g)^{n+1}$

Proof. To prove linearity, note that

$$\varphi(g)(X + Y) = g(X + Y)g^T = gXg^T + gYg^T = \varphi(g)X + \varphi(g)Y$$

and

$$\varphi(g)(\lambda X) = g(\lambda X)g^T = \lambda gXg^T = \lambda \varphi(g)X, \quad \forall \lambda \in \mathbb{R}$$

The claim about the determinant will only be used for $n = 2$. Since $\varphi(g)$ is a linear transformation from a vector-space to itself there exists a transformation matrix M^* , satisfying

$$M^*v = \varphi(g)X \quad \forall v \in \mathbb{R}^3$$

Where X is the matrix corresponding to v vector.

Denote

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

Note that $\alpha\delta - \beta\gamma \neq 0$. To determine M^* in the standard basis, one need to calculate $\varphi(g)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\varphi(g)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\varphi(g)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Which yields:

$$M^* = \begin{pmatrix} \alpha^2 & 2\alpha\beta & \beta^2 \\ \alpha\gamma & \beta\gamma + \alpha\delta & \beta\delta \\ \gamma^2 & 2\gamma\delta & \delta^2 \end{pmatrix}$$

Therefore

$$\det(\varphi(g)) = \det M^* = \alpha^3\delta^3 - 3\alpha^2\beta\gamma\delta^2 + 3\alpha\beta^2\gamma^2\delta - \beta^3\gamma^3 = (\alpha\delta - \beta\gamma)^3 = \det(g)^3.$$

□

Definition 2.8. Let's use the following notation:

$$\text{PD}_n(\mathbb{R}) := \{Q \in \mathcal{Q} \mid v^T Q v > 0, \quad \forall v \in \mathbb{R}^n \setminus \{0\}\}$$

Where $\text{PD}_n(\mathbb{R})$ stands for $n \times n$ positive definite symmetric matrices over $\mathbb{R}$. As we have previously shown, $\mathcal{Q} \simeq \mathbb{R}^{\binom{n+1}{2}}$, therefore there's an isomorphism ϕ mapping $\text{PD}_n(\mathbb{R})$ into some $\mathcal{P}_n \subset \mathbb{R}^{\binom{n+1}{2}}$.

Lemma 2.9. $\mathcal{P}_n$ is a convex cone.

Proof. Because

$$v^T(Q + P)v = v^T Q v + v^T P v$$

if $Q, P \in PD_n(\mathbb{R})$, then $Q + P \in PD_n(\mathbb{R})$ as well.

$$v^T(\lambda Q)v = \lambda v^T Q v, \forall \lambda \in \mathbb{R}$$

Consequently, if $X \in PD_n(\mathbb{R})$, then $\lambda X \in PD_n(\mathbb{R})$ for all $\lambda > 0$. Equivalently, if $x \in \mathcal{P}_n$, then $\lambda x \in \mathcal{P}_n$ for all $\lambda > 0$, so $\mathcal{P}_n$ is in fact a cone.

If $0 < \lambda < 1$, then for all $X, Y \in PD_n(\mathbb{R})$ $\lambda X, (1 - \lambda)Y \in PD_n(\mathbb{R})$ as well. And it was previously shown, that $\lambda X + (1 - \lambda)Y \in PD_n(\mathbb{R})$ too.

Therefore for all $x, y \in \mathcal{P}_n$ and $0 < \lambda < 1$, $\lambda x + (1 - \lambda)y \in \mathcal{P}$, so $\mathcal{P}_n$ is a convex cone. $\square$

For example $\mathcal{P}_2$ is the infinite cone given by the inequalities

$$a > 0, ac - b^2 > 0.$$

Definition 2.10. We will now restrict the group action, to $SL_n(\mathbb{Z})$, and define an equivalence. We say that $X \sim Y$, if there exists $g \in SL_n(\mathbb{Z})$ such that

$$\varphi(g)X = Y$$

.

Lemma 2.11. The relation $\sim$ defined above is an equivalence relation.

Proof. Reflexive: $\varphi(I_n)X = X$ so $X \sim X$

Symmetric: $X \sim Y \Leftrightarrow$ there exists $g \in SL_n(\mathbb{Z})$ satisfying

$$\varphi(g)X = gXg^T = Y$$

This means that

$$X = g^{-1}Y(g^T)^{-1} = g^{-1}Y(g^{-1})^T$$

Therefore $Y \sim X$

Transitive: If $X \sim Y \Leftrightarrow$ there exists $g \in SL_n(\mathbb{Z})$ satisfying

$$\varphi(g)X = gXg^T = Y$$

and $Y \sim Z \Leftrightarrow$ there exists $h \in SL_n(\mathbb{Z})$ satisfying

$$\varphi(h)Y = hYh^T = Z$$

Then

$$\varphi(hg)X = \varphi(h)(\varphi(g)X) = \varphi(h)Y = Z$$

so $X \sim Z$ $\square$

Lemma 2.12. Every $Q \in \mathcal{P}$ can be written as

$$AA^T = Q$$

For $A \in GL_2(\mathbb{R})$

Proof. Because of the spectral theorem, Q can be diagonalized by an orthogonal matrix:

$$Q = h \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} h^T$$

For $h \in O(2)$ and $\lambda, \mu \in \mathbb{R}^+$ the eigen-values of Q .

Therefore

$$Q = \left(h \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \sqrt{\mu} \end{pmatrix} \right) \left(h \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \sqrt{\mu} \end{pmatrix} \right)^T$$

$\square$

When we observe the positive definite quadratic forms as points of $\mathcal{P}_n \subset \mathbb{R}^{\binom{n+1}{2}}$, there exists a fundamental domain, respect to the above group action. The fundamental domains we will observe are convex polyhedra given as an intersection of half-spaces, by certain inequalities among the entries of the matrix.

We will see below that for $n = 2$ the fundamental domain is given by the following inequalities:

$$\left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \mid 2|b| \leq c \leq a \right\}.$$

The fundamental nature of this will be demonstrated later in 4.5.

Theorem 2.13. *Let $D > 0$ be fixed. The class number of symmetric positive definite matrices with integer entries and determinant D is finite.*

Proof. Using $2|b| \leq c \leq a$:

$$c^2 \leq ac = D + b^2 \leq D + \frac{c^2}{4}$$

So

$$\frac{3}{4}c^2 \leq D$$

There are finitely many options for c and consequently for b as well. Since

$$a = \frac{D - |b|^2}{c}$$

There are finitely many possibility for a as well. □

Our goal is to find formulae for these class numbers.

3. AN IMPORTANT LEMMA

A practical approach to prove Hurwitz-type formulas is simplifying them by constructing equivariant partitions of unity.

Definition 3.1. Assume that Γ is a countable group acting on a space X and ρ is an $X \rightarrow \mathbb{C}$ mapping, such that

$$\sum_{\gamma \in \Gamma} \rho(\gamma x) = 1, \quad \forall x \in X.$$

If this property holds, we will say that ρ induces a Γ -equivariant partition of unity. In the case of $|\Gamma| = \infty$, we require the series to be absolute convergent.

Lemma 3.2. *Assume that Γ acts on X , and $Y \subseteq X$ is a finite union of h orbits*

$$Y = \bigcup_{j=1}^h \Gamma y_j.$$

For all y_j $j \in \{1, 2, \dots, h\}$ denote the stabilizer of y_j the following way:

$$\Gamma_j = \{\gamma \in \Gamma \mid \gamma y_j = y_j\}$$

If $|\Gamma_j| < \infty$ for all $j \in \{1, 2, \dots, h\}$, and ρ induces an equivariant partition of unity then

$$(3.1) \quad \sum_{j=1}^h \frac{1}{|\Gamma_j|} = \sum_{y \in Y} \rho(y).$$

Proof. Given that Y is the union of the orbits Γy_j , the sum may be rearranged as shown below::

$$\sum_{y \in Y} \rho(y) = \sum_{j=1}^h \sum_{y \in \Gamma y_j} \rho(y)$$

Every $y \in \Gamma y_j$ can be written as gy_j for some $g \in \Gamma$. But for all $\gamma \in \Gamma_j$ $g\gamma y_j = gy_j$, therefore:

$$\sum_{j=1}^h \sum_{y \in \Gamma y_j} \rho(y) = \sum_{j=1}^h \frac{1}{|\Gamma_j|} \sum_{\gamma \in \Gamma} \rho(\gamma y_j)$$

Since ρ induces an equivariant partition of unity:

$$\sum_{j=1}^h \frac{1}{|\Gamma_j|} \sum_{\gamma \in \Gamma} \rho(\gamma y_j) = \sum_{j=1}^h \frac{1}{|\Gamma_j|}$$

□

4. EXAMPLE: CONSTRUCTION OF AN EQUIVARIANT PARTITION OF UNITY FOR BINARY QUADRATIC FORMS

In this section, we will find an analytic expression for the class number of the symmetric positive definite 2×2 matrices over $\mathbb{Z}$, with a certain determinant. Firstly, let's observe the matrices over $\mathbb{R}$. A general representor of these matrices is:

$$Q = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

Let's observe the linear transformation

$$\varphi(g) : \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad X \mapsto \varphi(g)X$$

Let M be the set of points, representing reduced symmetric positive definite matrices, such that

$$2|b| \leq c \leq a$$

As introduced in the first section, $\mathcal{P}_2$ is the set of points representing symmetric positive definite matrices in $\mathbb{R}^3$.

Lemma 4.1 (Lagrange reduction, [5]). *$\mathcal{P}_2$ has the following polyhedral decomposition:*

$$\bigcup_{g \in PSL_2(\mathbb{Z})} \varphi(g)M.$$

Here by a polyhedral decomposition, we mean that the polyhedrons $\varphi(g)M$ are either disjoint or meet at a common face.

Proof. Using the following steps, we can reduce any positive definite matrix to one in M .

Step1:

$$\varphi \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} a' & b+nc \\ b+nc & c \end{pmatrix}$$

Where $2|b'| = 2|b + nc| \leq c' = c$ for a suitable n .

Step2: If $c > a$ use

$$\varphi \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} c & -b \\ -b & a \end{pmatrix}$$

If we do these steps iteratively since the absolute value of each entry decreases after a finite number of steps the reduced matrix satisfies $2|b| \leq c \leq a$.

Therefore

$$\mathcal{P}_2 \subset \bigcup_{g \in PSL_2(\mathbb{Z})} \varphi(g)M$$

A direct calculation (see e.g. [5]), shows that if $x, y \in \text{int}M$ and $\varphi(g)x = \varphi(h)y$ then $g = h$ and $x = y$. □

For $y \in \mathcal{P}_2$ let's introduce the following half-space:

$$F_y = \{x \in \mathcal{P}_2 \mid \langle x, y \rangle < 1\}$$

Lemma 4.2. *The following holds for all $y \in \mathcal{P}_2$:*

$$\text{vol}(\mathcal{P}_2 \cap F_y) = \sum_{g \in PSL_2(\mathbb{Z})} \text{vol}(\varphi(g)M \cap F_y)$$

Proof. Since $\mathcal{P}_2$ has the above polyhedral decomposition, the equality holds. □

Lemma 4.3. *For all $y \in \mathcal{P}_2$, $g \in PSL_2(\mathbb{R})$ and $S \subset \mathbb{R}^3$ measurable set (it will be only used for M and $\mathcal{P}_2$):*

$$\text{vol}(\varphi(g)S \cap F_y) = \text{vol}(S \cap F_{\varphi(g^T)y})$$

Proof. Because

$$\varphi(g)S = \{\varphi(g)z \mid z \in S\}$$

we can get the following set equality:

$$F_y \cap \varphi(g)S = \{x \in \varphi(g)S \mid \langle x, y \rangle < 1\} = \{\varphi(g)z \mid z \in S, \langle \varphi(g)z, y \rangle < 1\}$$

Due to the determinant volume scaling theorem, for all measurable set $H \subset \mathbb{R}^3$

$$\text{vol}(\varphi(g)H) = \det(\varphi(g)) \text{vol}(H) = \text{vol}(H)$$

Holds for all $g \in PSL_2(\mathbb{R})$, because $\det(\varphi(g)) = \det(g)^3 = 1$.

Therefore

$$\text{vol}(\varphi(g)S \cap F_y) = \text{vol}(\{z \in S \mid \langle \varphi(g)z, y \rangle < 1\})$$

Recalling the definition of inner product:

$$\langle \varphi(g)z, y \rangle = \text{tr}(gzg^T y) = \text{tr}(zg^T yg)$$

The second equality holds, because $\text{tr}(ab) = \text{tr}(ba)$. Note that $\varphi(g^T)y = g^T yg$. Rewriting the inner product:

$$\langle \varphi(g)z, y \rangle = \langle z, \varphi(g^T)y \rangle$$

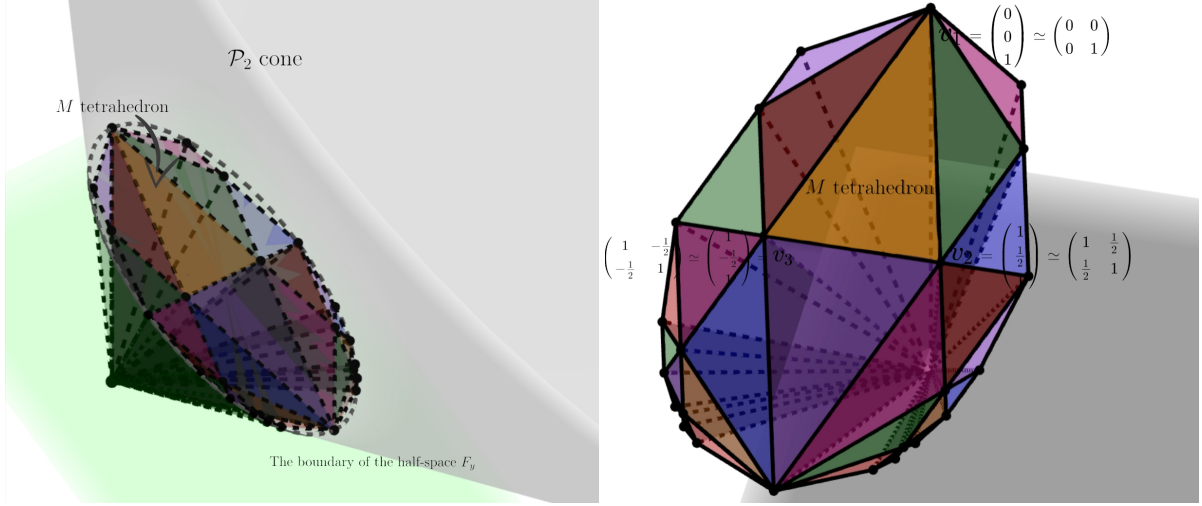
Therefore

$$\text{vol}(\varphi(g)S \cap F_y) = \text{vol}(\{z \in S \mid \langle z, \varphi(g^T)y \rangle < 1\}) = \text{vol}(S \cap F_{\varphi(g^T)y})$$

□

Note that since $g \in PSL_2(\mathbb{Z}) \Leftrightarrow g^T \in PSL_2(\mathbb{Z})$ the previous two lemmas show that

$$\text{vol}(\mathcal{P}_2 \cap F_y) = \sum_{g \in PSL_2(\mathbb{Z})} \text{vol}(M \cap F_{\varphi(g)y}).$$



On the left hand side $\text{vol}(\mathcal{P}_2 \cap F_y)$ only depends on the determinant of y . In fact we have the following.

Lemma 4.4.

$$\text{vol}(\mathcal{P}_2 \cap F_y) = \frac{\pi}{12 \det(y)^{\frac{3}{2}}}$$

Proof. Because of the spectral theorem every $y \in \mathcal{P}_2$ can be written as

$$h \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} h^T$$

Where h is an orthogonal matrix over $\mathbb{R}$ of determinant 1, and λ and μ are positive real numbers.

Since $\det h = 1$, we can apply 4.3, yielding:

$$\text{vol}(\mathcal{P}_2 \cap F_y) = \text{vol}(\mathcal{P}_2 \cap F_{\varphi(h^T)y}) = \text{vol}(\{z \in \mathcal{P}_2 | \langle z, \varphi(h^T)y \rangle < 1\})$$

Note, that:

$$\varphi(h^T)y = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

Reintroducing it into the original equality.:

$$\text{vol}(\mathcal{P}_2 \cap F_y) = \text{vol}(\{(a, b, c) | \lambda a + \mu c < 1, a > 0, ac - b^2 > 0\})$$

Let's use the substitution:

$$(x, y, z) = (\lambda a, \sqrt{\lambda \mu} b, \mu c)$$

$(a, b, c) \mapsto (x, y, z)$ is an affine transformation, so the following holds for the volume:

$$\begin{aligned} & \text{vol}(\{(a, b, c) | \lambda a + \mu c < 1, a > 0, ac - b^2 > 0\}) = \\ &= \frac{1}{\lambda^{\frac{3}{2}} \mu^{\frac{3}{2}}} \text{vol}(\{(x, y, z) | x + y < 1, x > 0, xz - y^2 > 0\}) \end{aligned}$$

This can be readily computed using the following integral:

$$\int_0^1 \int_0^{1-x} \int_{-\sqrt{xz}}^{\sqrt{xz}} 1 \, dy \, dz \, dx = \frac{\pi}{12}$$

We know that $\det(y) = \lambda\mu$, therefore:

$$\text{vol}(\mathcal{P}_2 \cap F_y) = \frac{\pi}{12 \det(y)^{\frac{3}{2}}}$$

□

Lemma 4.5.

$$\text{vol}(M \cap F_y) = \frac{1}{6c(a+b+c)(a-b+c)}$$

Where

$$y = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

Proof. Define the vectors

$$v_1 = (0, 0, 1), v_2 = \left(1, \frac{1}{2}, 1\right), v_3 = \left(1, -\frac{1}{2}, 1\right)$$

It's easy to see they define M . By define I mean that

$$M = \{\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 \mid \lambda_i \in \mathbb{R}^+\}$$

Therefore $M \cap F_y$ is a tetrahedron spanned by $\alpha_1 v_1, \alpha_2 v_2, \alpha_3 v_3$, where $\alpha_i \in \mathbb{R}^+$ such that $\langle \alpha_i v_i, y \rangle = 1$ By span I mean that

$$M \cap F_y = \{\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \mid \alpha_i \in \mathbb{R}^+, \alpha_1 + \alpha_2 + \alpha_3 \leq 1\}$$

Let's solve these equations:

$$\text{tr} \left(\begin{pmatrix} 0 & 0 \\ 0 & \alpha_1 \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) = \alpha_1 c = 1$$

$$\frac{1}{2} \text{tr} \left(\begin{pmatrix} 2\alpha_i & \pm\alpha_i \\ \pm\alpha_i & 2\alpha_i \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) = \alpha_i (a \pm b + c) = 1$$

Where $i = 2, 3$ and there's a $+$, when $i = 2$ and a $-$, when $i = 3$.

Therefore

$$\alpha_1 = \frac{1}{c}, \alpha_2 = \frac{1}{a+b+c}, \alpha_3 = \frac{1}{a-b+c}$$

Using the determinant tetrahedron volume formula:

$$\text{vol}(M \cap F_y) = \frac{1}{6} \left| \det \begin{pmatrix} \alpha_1 v_1 \\ \alpha_2 v_2 \\ \alpha_3 v_3 \end{pmatrix} \right| = \frac{|\alpha_1 \alpha_2 \alpha_3|}{6}$$

Since

$$a + c \geq 2\sqrt{ac} \geq 2|b| > |b|$$

holds, all of the above values for α_i are positive.

□

Theorem 4.6. For $y = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ let

$$\rho(y) = \frac{2 \det(y)^{\frac{3}{2}}}{\pi} \frac{1}{c(a+b+c)(a-b+c)}.$$

Then ρ induces a partition of unity as in 3.2.

Proof. As we saw in 4.2:

$$\text{vol}(\mathcal{P}_2 \cap F_y) = \sum_{g \in PSL_2(\mathbb{Z})} \text{vol}(\varphi(g)M \cap F_y)$$

Because 4.3:

$$\text{vol}(\mathcal{P}_2 \cap F_y) = \sum_{g \in PSL_2(\mathbb{Z})} \text{vol}(M \cap F_{\varphi(g)y})$$

And finally, because of 4.4 and 4.5:

$$\sum_{g \in PSL_2(\mathbb{Z})} \text{vol}(M \cap F_{\varphi(g)y}) = \frac{\pi}{12 \det(y)^{\frac{3}{2}}} \sum_{g \in PSL_2(\mathbb{Z})} \rho(\varphi(g)y) = \frac{\pi}{12 \det(y)^{\frac{3}{2}}}$$

Therefore

$$\sum_{g \in PSL_2(\mathbb{Z})} \rho(\varphi(g)y) = 1$$

□

4.1. Application to class numbers. Fix a determinant $D > 0$. To apply Lemma 3.2 we need to show that positive definite integral binary quadratic forms of determinant D have finitely many $\Gamma = PSL_2(\mathbb{Z})$ -orbits and that each representative have finite stabilizer group.

That for a fix $D = \det(y)$ the class number is finite over $PSL_2(\mathbb{Z})$, as established in Theorem 2.13 We denote this class number by h .

We introduce the following notation:

$$M_D = \{y \in M \mid \det(y) = D, y \in M_2(\mathbb{Z}) \cap \mathcal{S}\}$$

Can be expressed as:

$$M_D = \{y_j \mid j = 1, \dots, h\}$$

Also

$$\mathcal{P}_D = \bigcup^* \Gamma y_j$$

Where $\mathcal{P}_D$ stands for the subset of $\mathcal{P}_2$ with D determinant and integer entries. This follows from the fundamental nature of M .

Therefore, applying 3.2, we can conclude that:

$$\sum_{j=1}^h \frac{1}{|\Gamma_j|} = \sum_{y \in \mathcal{P}_D} \rho(y)$$

We now make a remark about the stabilizer groups Γ_j . As we have previously shown in 2.12, y_j can be written as

$$y_j = a_j a_j^T$$

for $a_j \in GL_2(\mathbb{R})$

Lemma 4.7.

$$\Gamma_j = PSL_2(\mathbb{Z}) \cap a_j O(2) a_j^{-1}$$

is a finite group.

Proof.

$$\Gamma_j = \{g \in PSL_2(\mathbb{Z}) \mid g y_j g^T = y_j\}$$

We know that

$$g y_j g^T = (g a_j)(g a_j)^T = a_j a_j^T$$

Equivalently,

$$(a_j^{-1} g a_j)(a_j^{-1} g a_j)^T = I_2$$

So it means

$$(a_j^{-1}ga_j) \in O(2)$$

It's the same as

$$g \in a_j O(2) a_j^{-1}$$

Since the intersection of a compact set $(a_j O(2) a_j^{-1})$ and a discrete set $(SL_2(\mathbb{Z}))$ is finite the results is proved. $\square$

Denote $o = a_j^{-1}ga_j \in O(2)$, and let $k(x)$ represent the characteristic polynomial of g . Since $g \in PSL_2(\mathbb{Z})$, k 's coefficients are integers.

$$k(x) = x^2 - \text{tr}(g)x + \det(g)$$

We know that

$$\text{tr}(g) = \text{tr}(a_j^{-1}oa_j) = \text{tr}(oa_j a_j^{-1}) = \text{tr}(o)$$

Since o is orthogonal, $o_{11}^2 + o_{12}^2 = 1 = o_{21}^2 + o_{22}^2$, which implies $|\text{tr}(o)| \leq 2$.

If $\text{tr}(o) = \pm 2$, then $o = \pm I_2$, which leads to $g = \pm I_2$. Since only $I_2 \in PSL_2(\mathbb{Z})$, this gives us one possibility for g for each $y \in \mathcal{P}_2$.

This reduces the problem to the case where the characteristic polynomial of g is either $x^2 + 1$, $x^2 + x + 1$ or $x^2 - x + 1$.

In each case g can be written as

$$g = sj s^{-1}$$

because of the Jordan Canonical form Theorem, where

$$j = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}$$

with $k(x) = (x - x_1)(x - x_2)$, for $x_1, x_2 \in \mathbb{C}$

If we are able to find a g' matrix over real numbers such that $k(g') = 0$, then g can be expressed as:

$$g = sg' s^{-1}$$

Where $s^{-1} \in GL_2(\mathbb{R})$.

Recalling the original equation:

$$(sg' s^{-1})y(sg' s^{-1})^T = y$$

Equivalently

$$g' \left(s^{-1}y (s^{-1})^T \right) g'^T = s^{-1}y (s^{-1})^T$$

At this point, it becomes straightforward to compute $s^{-1}y (s^{-1})^T$, which in turn allows us to determine s and subsequently reveals the only possible g in each case.

Case 1, g is the root of $x^2 + 1$:

$$g' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Therefore

$$s^{-1}y (s^{-1})^T = \alpha I_2$$

So

$$y = (\sqrt{\alpha}s)(\sqrt{\alpha}s)^T$$

And

$$g = (\sqrt{\alpha}s) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (\sqrt{\alpha}s)^{-1}$$

By expressing both y and g in the aforementioned manner leads to the conclusion that g is over integers if and only if $\frac{y}{\sqrt{\det(y)}}$ is over integers.

If $\sqrt{\det(y)}$ isn't rational, $g \notin PSL_2(\mathbb{Z})$. Therefore if $\det(y)$ isn't a perfect square, this case yields no solutions.

The Case 2-3 when g is the root of $x^2 \pm x + 1$ is treated similarly, such stablizers can only arise when $\det y = 3n^2$.

Therefore for $D \neq n^2, 3n^2$: $|\Gamma_j| = 1$ so in this case

$$h = \frac{2D^{\frac{3}{2}}}{\pi} \sum_{\substack{ac-b^2=D, \\ a>0}} \frac{1}{c(a+b+c)(a-b+c)}$$

5. THE GEOMETRY OF BINARY HERMITIAN FORMS

5.1. Hermitian forms. Use the following notation:

$$\mathcal{Q} := \left\{ X = \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} \mid a, d \in \mathbb{R}, b \in \mathbb{C} \right\}$$

Similarly to the positive definite symmetric quadratic forms, the elements of $\mathcal{Q}$ can be represented as a vector, but now in $\mathbb{R}^4$.

For instance

$$X \mapsto (a, d, b_1, b_2)$$

is a bijection, where $b = b_1 + b_2i$.

Definition 5.1. Let's define the inner product similarly. On the vector space $\mathcal{Q}$:

$$\langle A, B \rangle = \text{tr}(AB)$$

We verify the properties of an inner product:

Bilinearity:

$$\langle A + B, C \rangle = \text{tr}((A + B)C) = \text{tr}(AC) + \text{tr}(BC) = \langle A, C \rangle + \langle B, C \rangle$$

$$\langle \lambda A, B \rangle = \text{tr}((\lambda A)B) = \lambda \text{tr}(AB) = \lambda \langle A, B \rangle$$

Symmetry:

$$\langle A, B \rangle = \text{tr}(AB) = \text{tr}(BA) = \langle B, A \rangle$$

Positive-Definiteness:

$$\langle A, A \rangle = \text{tr}(A^2) = a^2 + d^2 + 2|b|^2 \geq 0$$

and $\langle A, A \rangle = 0$ if and only if $a = |b| = d = 0$, therefore $b = 0$, so $A = 0$.

Thus, $\langle A, B \rangle = \text{tr}(AB)$ is an inner product on S .

Definition 5.2. Let's define the group action on $\mathcal{Q}$. Let's say $GL_2(\mathbb{C})$ acts on $\mathcal{Q}$ the following way:

$$\varphi(g)X = gX\bar{g}^T$$

And

$$\varphi(g) : X \mapsto \varphi(g)X$$

Proposition 5.3. $\varphi(g)$ is a $\mathcal{Q} \rightarrow \mathcal{Q}$ linear mapping, with a transformation matrix M , such that

$$\det(M) = \det(g)^4$$

Proof. We need to verify that $\varphi(g)X \in \mathcal{Q}$, equivalently:

$$\left(\overline{gX\bar{g}^T}\right)^T = gX\bar{g}^T$$

It can be shown in the following way:

$$\left(\overline{gX\bar{g}^T}\right)^T = (\bar{g}\bar{X}g^T)^T = g\bar{X}^T\bar{g}^T$$

Since $\bar{X}^T = X$, the to be proven is true.

To prove linearity, note that

$$\varphi(g)(X + Y) = g(X + Y)\bar{g}^T = gX\bar{g}^T + gY\bar{g}^T = \varphi(g)X + \varphi(g)Y$$

and

$$\varphi(g)(\lambda X) = g(\lambda X)\bar{g}^T = \lambda gX\bar{g}^T = \lambda \varphi(g)X, \quad \forall \lambda \in \mathbb{R}$$

Since $\varphi(g)$ is a linear mapping from $\mathcal{Q} \simeq \mathbb{R}^4$ to itself, so there exists a 4×4 transformation matrix, M , such that

$$Mv = \varphi(g)X$$

whenever v and X are corresponding. We will show that $\det(M) = \det(g)^4$.

Since Cartan decomposition g can be written in the following form:

$$g = k_1 D k_2$$

Where k_1, k_2 are unitary matrices over $\mathbb{C}$, and D is a diagonal matrix over $\mathbb{R}$, containing g 's two eigenvalues λ and μ .

Since φ is a group action (and consequently associative):

$$\varphi(g) = \varphi(k_1)\varphi(D)\varphi(k_2)$$

Therefore

$$\det(\varphi(g)) = \det(\varphi(k_1)) \det(\varphi(k_2)) \det(\varphi(D))$$

Since k_i is unitary $\det(\varphi(k_i))$ is 1, therefore

$$\det(\varphi(g)) = \det(\varphi(D))$$

Where

$$D = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

$$\varphi(D) \begin{pmatrix} x & y \\ \bar{y} & z \end{pmatrix} = \begin{pmatrix} \lambda^2 x & \lambda \mu y \\ \lambda \mu \bar{y} & \mu^2 z \end{pmatrix}$$

Writing $\varphi(D)$'s M' transformation matrix in the identity basis:

$$M' = \begin{pmatrix} \lambda^2 & 0 & 0 & 0 \\ 0 & \lambda \mu & 0 & 0 \\ 0 & 0 & \lambda \mu & 0 \\ 0 & 0 & 0 & \mu^2 \end{pmatrix}$$

Therefore

$$\det(M') = \det(D)^4$$

Consequently

$$\det(\varphi(g)) = \det(M) = \det(g)^4$$

□

Definition 5.4. Let $\mathcal{S}$ be the set of 2×2 positive definite Hermitian forms:

$$\mathcal{S} = \{X \in \mathcal{Q} \mid \bar{v}^T X v > 0 \quad \forall v \in \mathbb{C}^2 \setminus \{0\}\}$$

Definition 5.5. We define $\mathcal{P}^* \subset \mathbb{R}^4$ as

$$\mathcal{P}^* = \{(a, b_1, b_2, d) \mid a + d > 0, ad - b_1^2 - b_2^2 > 0\}$$

Lemma 5.6. *The following property holds:*

$$\mathcal{S} \simeq \mathcal{P}^*$$

Proof. A matrix

$$X = \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix}$$

is positive definite if and only if its eigenvalues are positive real numbers.

They are roots of the characteristic polynomial:

$$k(X) = \lambda^2 - (a + d)\lambda + (ad - |b|^2)$$

Therefore both $a + d$ and $ad - |b|^2$ must be positive, since they are the sum and product of the roots, respectively.

If the discriminant of k :

$$(a + d)^2 - 4(ad - |b|^2)$$

is non-negative as well, both roots are positive real numbers. $\Delta(k) > 0$ holds, since:

$$(a + d)^2 - 4(ad - |b|^2) = (a - d)^2 + 4|b|^2$$

□

Lemma 5.7. $\mathcal{P}^*$ is a convex cone in $\mathbb{R}^4$

Proof. Since 5.6 $\mathcal{P}^* \simeq \mathcal{S}$, so it is sufficient to see that if $X \in \mathcal{S}$, then $\lambda X \in \mathcal{S}$ as well, for $\lambda \in \mathbb{R}^+$.

$$\bar{v}^T (\lambda X) v = \lambda \bar{v}^T X v$$

for all $v \in \mathbb{C}^2$, therefore $X \in \mathcal{S} \Leftrightarrow \lambda X \in \mathcal{S}$ for $\lambda \in \mathbb{R}^+$.

Again because of 5.6 it is enough to show that if

$$X, Y \in \mathcal{S}, \text{ then } \lambda X + (1 - \lambda)Y \in \mathcal{S}$$

for $0 < \lambda < 1$. We already saw that λX and $(1 - \lambda)Y$ are positive definite, so we want to show that if $X, Y \in \mathcal{S}$, then $X + Y \in \mathcal{S}$.

$$\bar{v}^T (X + Y) v = \bar{v}^T X v + \bar{v}^T Y v > 0$$

for all $v \in \mathbb{C}^2 \setminus \{0\}$, consequently $\mathcal{P}^*$ is a convex cone.

□

Lemma 5.8. *Every $X \in \mathcal{S} \simeq \mathcal{P}^*$ can be decomposed the following way:*

$$X = A \bar{A}^T$$

Where $A \in GL_2(\mathbb{C})$.

Proof. Due to the spectral theorem, a X Hermitian matrix can be diagonalized by a unitary matrix:

$$X = U J \bar{U}^T$$

We also know that X 's all eigenvalues are positive real numbers, because it's positive definite, so J can be written as: $J = \begin{pmatrix} \lambda^2 & 0 \\ 0 & \mu^2 \end{pmatrix}$, where λ^2, μ^2 are the the eigenvalues of X . Then

$$A = U \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

satisfies the above criteria. □

5.2. Some volume calculations. For a vector $y \in \mathcal{Q}$ let:

$$F_y := \{x \in \mathcal{Q} \mid \langle x, y \rangle < 1\}$$

Lemma 5.9. *For every $y \in \mathcal{P}^*$, $g \in PSL_2(R)$ and $S \subset \mathbb{R}^4$ measurable set the following holds:*

$$\text{vol}(\varphi(g)S \cap F_y) = \text{vol}(S \cap F_{\varphi(\bar{g}^T)y})$$

Proof. Because

$$\varphi(g)S = \{\varphi(g)z \mid z \in S\}$$

the following set equality can be established:

$$F_y \cap \varphi(g)S = \{x \in \varphi(g)S \mid \langle x, y \rangle < 1\} = \{\varphi(g)z \mid z \in S, \langle \varphi(g)z, y \rangle < 1\}$$

Due to the determinant volume scaling theorem, for all measurable $H \subset \mathbb{R}^4$:

$$\text{vol}(gH) = \det(\varphi(g)) \text{vol}(H) = \text{vol}(H)$$

holds for all $g \in PSL_2(R)$, because $\det(\varphi(g)) = \det(g)^4 = 1$.

Therefore

$$\text{vol}(\varphi(g)S \cap F_y) = \text{vol}(\{z \in S \mid \langle \varphi(g)z, y \rangle < 1\})$$

Recalling the definition of inner product:

$$\langle \varphi(g)z, y \rangle = \text{tr}(gz\bar{g}^T y) = \text{tr}(z\bar{g}^T yg)$$

The second equality holds, because $\text{tr}(ab) = \text{tr}(ba)$. Note that $\varphi(\bar{g}^T)y = \bar{g}^T yg$. Rewriting the inner products:

$$\langle \varphi(g)z, y \rangle = \langle z, \varphi(\bar{g}^T)y \rangle$$

Therefore

$$\text{vol}(\varphi(g)S \cap F_y) = \text{vol}(\{z \in S \mid \langle z, \varphi(\bar{g}^T)y \rangle < 1\}) = \text{vol}(S \cap F_{\varphi(\bar{g}^T)y})$$

□

Lemma 5.10. *For all $y \in \mathcal{P}^*$:*

$$\text{vol}(\mathcal{P}^* \cap F_y) = \text{vol}(\mathcal{P}^* \cap F_{\varphi(\bar{g}^T)y})$$

Where vol stands for the 4-dimensional Lebesgue measure.

Since 5.9:

$$\text{vol}(\varphi(g)\mathcal{P}^* \cap F_y) = \text{vol}(\mathcal{P}^* \cap F_{\varphi(\bar{g}^T)y})$$

$\mathcal{P}^* = \varphi(g)\mathcal{P}^*$, because $x = \varphi(g)(\varphi(g^{-1})x)$.

Lemma 5.11. *For all $y \in \mathcal{P}^*$:*

$$\text{vol}(\mathcal{P}^* \cap F_y) = \frac{\pi}{48 \det(y)^2}$$

Proof. Using spectral theorem for every $y \in \mathcal{P}^*$ there exists a h unitary matrix, such that:

$$\varphi(h) \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} = y.$$

Since $|\det(h)| = 1$, there exists a matrix r in form of $r = \begin{pmatrix} e^{-i\varphi} & 0 \\ 0 & 1 \end{pmatrix}$ such that $\det(hr) = 1$, and

$$\varphi(r) \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

. Therefore:

$$\varphi(hr) \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} = y$$

Additionally,

$$\varphi \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix}$$

For

$$c_1 = \frac{\sqrt{\mu}}{\sqrt[4]{\mu\lambda}}, \quad c_2 = \frac{\sqrt{\lambda}}{\sqrt[4]{\mu\lambda}}, \quad \text{and } \gamma = \sqrt{\lambda\mu}$$

Therefore there exists a matrix g , with 1 determinant, such that:

$$\varphi(g) \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix} = y, \quad \text{equivalently } \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix} = \varphi(\bar{g}^T)y.$$

Because of 5.10:

$$\text{vol}(\mathcal{P}^* \cap F_y) = \text{vol}\left(\mathcal{P}^* \cap F_{\varphi(\bar{g}^T)y}\right) = \text{vol}(\{z \in \mathcal{P}^* | \langle z, \varphi(\bar{g}^T)y \rangle < 1\})$$

Since

$$f(\bar{g}^T)y = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix}$$

We can calculate the volume:

$$\text{vol}(\{z \in \mathcal{P}^* | \langle z, \varphi(\bar{g}^T)y \rangle < 1\}) = \frac{1}{2} \text{vol}(\{(a, b_1, b_2, d) \in \mathbb{R}^4 | ad - b_1^2 - b_2^2 > 0, 1 > \gamma a + \gamma d > 0\})$$

Since the volume of the parallelepiped formed by the vectors $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$ is 2.

Using substitution:

$$x = \gamma a, y = \gamma b_1, z = \gamma b_2, w = \gamma d$$

The above volume is

$$\frac{1}{2\gamma^4} \text{vol}(\{(x, y, z, w) \in \mathbb{R}^4 | xw - y^2 - z^2 > 0, 1 > x + w > 0\})$$

It can be calculated using the integral:

$$\frac{1}{2} \int_{x=0}^1 \int_{w=0}^{1-x} \int_{y=-\sqrt{xw}}^{\sqrt{xw}} \int_{z=-\sqrt{xw-y^2}}^{\sqrt{xw-y^2}} 1 \, dz \, dy \, dw \, dx = \frac{\pi}{48}$$

Therefore

$$\text{vol}(\mathcal{P}^* \cap F_y) = \frac{\pi}{48\gamma^4} = \frac{\pi}{48 \det(y)^2}$$

□

6. CONSTRUCTION OF A PARTITION OF UNITY

6.1. Hermitian forms and the Eisenstein ring. Let

$$R = \left\{ a + b\omega \mid a, b \in \mathbb{Z}, \omega = \frac{-1 + \sqrt{3}i}{2} \right\}$$

be the ring of Eisenstein integers. (It is commutative and a Euclidean domain.)

Definition 6.1. Let

$$PSL_2(R) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in R, ad - bc = 1 \right\} / \{\pm I_2\}$$

Let

$$\mathcal{Q}^* := \left\{ X = \begin{pmatrix} a & b \\ b & d \end{pmatrix} \mid a, d \in \mathbb{Z}, b \in R \right\}.$$

The group $PSL_2(R)$ acts on binary hermitian forms via φ . $\mathcal{Q}^*$ is an analogue of the positive definite integral binary forms over $\mathbb{Z}$. Our goal is to derive a class number formula for these forms when the determinant is fixed. We start with determining a polyhedral decomposition of $\mathcal{Q}$ with respect to the group $PSL_2(R)$.

Definition 6.2. We define $\mathcal{P} \subset \mathbb{R}^4$ the following way:

$$\mathcal{P} = \{(a, b_1, b_2, d) \mid a + d > 0, ad - b_1^2 - b_2^2 + b_1b_2 > 0\}$$

Lemma 6.3. $\mathcal{P} \simeq \mathcal{P}^*$, moreover $P = TP^*$, for

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{2}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Proof. $\mathcal{P} \simeq \mathcal{S}$, because

$$\phi : \begin{pmatrix} a & b \\ b & d \end{pmatrix} \mapsto (a, b_1, b_2, d)^T$$

Is an isomorphism where $b = b_1 + b_2\omega$ for $b_1, b_2 \in \mathbb{R}$. Therefore it is also equivalent to $\mathcal{P}^* \simeq \mathcal{S}$, where the corresponding vector to the above matrix is $(a, b'_1, b'_2, d)^T$, where $b'_1 + b'_2i = b$ for $b'_1, b'_2 \in \mathbb{R}$.

Since $b_1 + b_2\omega = b'_1 + b'_2i$, $b_1 = b'_1 + \frac{b'_2}{\sqrt{3}}$, and $b_2 = \frac{2b'_2}{\sqrt{3}}$. This implicates the above transition matrix. $\square$

Let's use the notations in the figure.

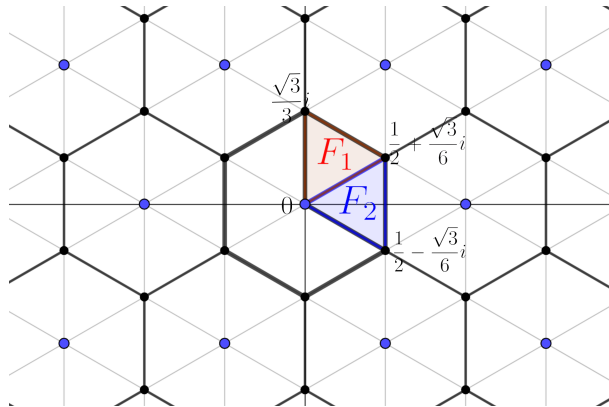


FIGURE 2. F and the Eisenstein integers

The blue points represent the Eisenstein integers the axes are the real and the imaginary axis, and the hexagons are the Dirichlet-Voronoi cell of each Eisenstein integer.

We let $F = F_1 \cup F_2$. Explicit descriptions of the planar domains F_1, F_2 will be given in the next section.

Definition 6.4. Let M be the following domain:

$$M = \left\{ \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} \mid 0 \leq a \leq d, \text{ and } \frac{b}{a} \in F \right\}.$$

Clearly $M = M_1 \cup^* M_2$, where M_1 and M_2 are defined the following way:

$$M_i = \left\{ \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} \mid 0 \leq a \leq d, \text{ and } \frac{b}{a} \in F_i \right\}$$

Theorem 6.5. M is a fundamental domain for the action of $SL_2(R)$ on $\mathbb{Q}$.

This means that $\mathcal{P}$ has the following polyhedral decomposition:

$$\mathcal{P} = \bigcup_{g \in PSL_2(R)} \varphi(g)\mathcal{M} = \bigcup_{g \in PSL_2(R)} (\varphi(g)\mathcal{M}_1 \cup \varphi(g)\mathcal{M}_2)$$

Proof. We propose a method to transform a fixed positive definite Hermitian form, H , into a reduced form, $H' \in M$. Let a be the smallest number that can be expressed in the form $(\bar{z}_1 \ z_2) H \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, where $z_1, z_2 \in R$. Consequently, the greatest common divisor of z_1 and z_2 in the Eisenstein ring is 1. Therefore there exists $w_1, w_2 \in R$, such that $\det \begin{pmatrix} z_1 & w_1 \\ z_2 & w_2 \end{pmatrix} = 1$. Denote $g = \begin{pmatrix} z_1 & w_1 \\ z_2 & w_2 \end{pmatrix}$

$$\varphi(g)H = \begin{pmatrix} a & \beta \\ \bar{\beta} & \gamma \end{pmatrix}$$

We will now follow similar steps as in the real case to obtain $\frac{b}{a} \in F$.

$$\varphi \begin{pmatrix} 1 & 0 \\ x_1 - x_2\omega & 1 \end{pmatrix} \begin{pmatrix} a & \beta \\ \bar{\beta} & \gamma \end{pmatrix} = \begin{pmatrix} a & \beta + (x_1 + x_2\omega)a \\ \bar{\beta} + (x_1 + x_2\omega)\bar{a} & \gamma' \end{pmatrix}$$

Since the above illustrated hexagonal Dirichlet-Voronoi cells partition $\mathbb{C}$, $\frac{\beta}{a}$ is in one of them.

If it's in the Dirichlet-Voronoi cell of $-x_1 - x_2\omega$, then $\frac{\beta + (x_1 + x_2\omega)a}{a} = \frac{\beta}{a} + x_1 + x_2\omega$ is inside 0's Dirichlet Voronoi cell. So it's either in F or in the image of F rotated by 120° or 240° . Denote $b' = \beta + (x_1 + x_2\omega)a$ and $d' = \gamma'$

$$\varphi \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix} \begin{pmatrix} a & b' \\ \bar{b}' & d' \end{pmatrix} = \begin{pmatrix} a & \omega b' \\ \omega^2 \bar{b}' & d' \end{pmatrix}$$

This rotates $\frac{b'}{a}$ by 120° , so after at most two applications, a form

$$H' = \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix}$$

can be achieved, where $\frac{b}{a} \in F$.

Since a 's minimality, $\begin{pmatrix} 0 & 1 \end{pmatrix} H' \begin{pmatrix} 0 \\ 1 \end{pmatrix} = d \geq a$.

The final step, that $\varphi(g)x = \varphi(h)y$ for $x, y \in \text{int}(M)$ if and only if $g = h$ and $x = y$ can be found in [2]. $\square$

6.2. Explicit description of the polyhedron M . Using the description via matrices the condition $x \in F_1$ or F_2 can be expressed as a set of inequalities. The same holds for the property that a matrix is positive definite. For example we will use the fact that for such matrices $a, d > 0$.

$\frac{b}{a} \in F_1$ if and only if

$$\begin{aligned} 0 &\leq \Re(b) \\ \Im\left(\frac{b}{a}\right) &\leq -\frac{\sqrt{3}}{3}\Re\left(\frac{b}{a}\right) + \frac{\sqrt{3}}{3} \end{aligned}$$

$$\Im\left(\frac{b}{a}\right) \geq \frac{\sqrt{3}}{3} \Re\left(\frac{b}{a}\right)$$

It will be convenient to use a different description, via a change of basis. Use the following correspondence between $\mathcal{Q}$ and $\mathbb{R}^4$:

$$\begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} \mapsto (a, b_1, b_2, d)$$

Where $b = b_1 + b_2\omega$, where $b_1, b_2 \in \mathbb{R}$.

Define the set of points corresponding to matrices in M_i as $\mathcal{M}_i$ and to matrices in M as $\mathcal{M}$.

Rewriting the above inequalities:

$$\mathcal{M}_1 = \left\{ (a, b_1, b_2, d) \mid b_2 \leq 2b_1, b_1 \leq 2b_2, b_1 + b_2 \leq a, d \leq a \right\}$$

Since it's defined as the intersection of 4 hyper-half-spaces, it's a(n infinite) simplex. Therefore, we may notice that $\mathcal{M}_1$ can be expressed the following way:

$$\mathcal{M}_1 = \{ \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 \mid \alpha_i \in \mathbb{R}^+ \}$$

For

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}^T, v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}^T, v_3 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 3 \end{pmatrix}^T \text{ and } v_4 = \begin{pmatrix} 3 \\ 1 \\ 2 \\ 3 \end{pmatrix}^T$$

We can do similar calculations for $\mathcal{M}_2$ and get:

$$\mathcal{M}_2 = \{ \alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3 + \alpha_4 w_4 \mid \alpha_i \in \mathbb{R}^+ \}$$

For

$$v_1 = w_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}^T, v_2 = w_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}^T, v_3 = w_3 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 3 \end{pmatrix}^T \text{ and } w_4 = \begin{pmatrix} 3 \\ 1 \\ -1 \\ 3 \end{pmatrix}^T$$

So the fundamental domain, $\mathcal{M}$ can be written as $\mathcal{M} = \mathcal{M}_1 \cup^* \mathcal{M}_2$, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are 4-dimensional infinite simplexes, with a common facet.

6.3. Construction of a partition of unity. Recall that by Theorem 6.5 the following lemma holds:

Lemma 6.6. *For all $y \in \mathcal{P}$:*

$$\text{vol}(\mathcal{P} \cap F_y) = \sum_{g \in SL_2(\mathbb{C})} \text{vol}(\varphi(g)\mathcal{M} \cap F_y) = \sum_{g \in SL_2(\mathbb{C})} (\text{vol}(\varphi(g)\mathcal{M}_1 \cap F_y) + \text{vol}(\varphi(g)\mathcal{M}_2 \cap F_y))$$

Where vol stands for the 4-dimensional Lebesgue measure.

If T is a simplex:

$$T := \left\{ v \mid v = \sum_{i=1}^4 \alpha_i v_i, \alpha_i \geq 0, \sum_{i=1}^4 \alpha_i \leq 1 \right\}$$

For given, linearly independent 4 dimensional vectors v_1, v_2, v_3, v_4 .

And T_∞ is the following infinite simplex:

$$T_\infty := \left\{ v \mid v = \sum_{i=1}^4 \alpha_i v_i, \alpha_i \geq 0 \right\}$$

Now we present a lemma:

Lemma 6.7. *For every 4 dimensional y vector which isn't orthogonal to either of the vectors v_i :*

$$\text{vol}(T_\infty \cap F_y) = \frac{\text{vol}(T)}{|\text{tr}(v_1 y) \text{tr}(v_2 y) \text{tr}(v_3 y) \text{tr}(v_4 y)|}$$

Proof. Let S_y be the hyper-plane defining F_y hyper-half-space:

$$S_y := \{v \mid \langle v, y \rangle = 1\}$$

Denote the intersection of the lines λv_i and the hyper-plane S_y as follows:

$$\alpha_i v_i \in S_y \Leftrightarrow \langle \alpha_i v_i, y \rangle = 1$$

Since $\alpha_i \in \mathbb{R}$ scalar, and the inner-product is linear:

$$\frac{1}{\langle v_i, y \rangle} = \frac{\langle \alpha_i v_i, y \rangle}{\langle v_i, y \rangle} = \frac{\alpha_i \langle v_i, y \rangle}{\langle v_i, y \rangle} = \alpha_i$$

Recalling the definition of the inner product:

$$\alpha_i = \frac{1}{\text{tr}(v_i y)}$$

Because of the simplex volume formula, and the fact that the identity basis's vectors forms a parallelepiped with volume 2:

$$\text{vol}(T) = \frac{1}{48} \left| \det \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \right|$$

And

$$\text{vol}(T_\infty \cap F_y) = \frac{1}{48} \left| \det \begin{pmatrix} \alpha_1 v_1 \\ \alpha_2 v_2 \\ \alpha_3 v_3 \\ \alpha_4 v_4 \end{pmatrix} \right| = |\alpha_1 \alpha_2 \alpha_3 \alpha_4| \text{vol}(T)$$

Substituting the above results into α_i gives the expression we aimed to prove. $\square$

Proposition 6.8.

$$\text{vol}(\mathcal{M} \cap F_y) = \frac{1}{144} \cdot \left(\frac{1}{d(a+d)(a+b_1+d)} \right) \left(\frac{1}{a+b_2+d} + \frac{1}{a+b_1-b_2+d} \right)$$

Proof. Due to 6.7 it is sufficient to compute $\text{tr}(vy)$, where v is one of the spanning vector of M_1 or M_2 :

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 2+\omega \\ 2+\bar{\omega} & 3 \end{pmatrix}, \text{ and either } \begin{pmatrix} 3 & 1+2\omega \\ 1+2\bar{\omega} & 3 \end{pmatrix} \text{ or } \begin{pmatrix} 3 & 1-\omega \\ 1-\bar{\omega} & 3 \end{pmatrix}$$

These can be calculated easily:

$$\text{tr} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b_1+b+2\omega \\ b_1+b_2\bar{\omega} & d \end{pmatrix} = d$$

$$\text{tr} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b_1+b+2\omega \\ b_1+b_2\bar{\omega} & d \end{pmatrix} = a+d$$

$$\text{tr} \begin{pmatrix} 3 & 2+\omega \\ 2+\bar{\omega} & 3 \end{pmatrix} \begin{pmatrix} a & b_1+b+2\omega \\ b_1+b_2\bar{\omega} & d \end{pmatrix} = 3(a+b_1+d)$$

$$\text{tr} \begin{pmatrix} 3 & 1+2\omega \\ 1+2\bar{\omega} & 3 \end{pmatrix} \begin{pmatrix} a & b_1+b+2\omega \\ b_1+b_2\bar{\omega} & d \end{pmatrix} = 3(a+b_2+d)$$

$$\text{tr} \begin{pmatrix} 3 & 1-\omega \\ 1-\bar{\omega} & 3 \end{pmatrix} \begin{pmatrix} a & b_1+b+2\omega \\ b_1+b_2\bar{\omega} & d \end{pmatrix} = 3(a+b_1-b_2+d)$$

By substituting these values, and $\text{vol}(\mathcal{M}_1) = \frac{1}{48} \left| \det \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \right| = \frac{1}{16}$:

$$\text{vol}(\mathcal{M}_1 \cap F_y) = \frac{1}{144 \cdot d(a+d)(a+b_1+d)(a+b_2+d)}$$

And for $\mathcal{M}_2$:

$$\text{vol}(\mathcal{M}_2 \cap F_y) = \frac{1}{144 \cdot d(a+d)(a+b_1+d)(a+b_1-b_2+d)}$$

Adding these up, because $\mathcal{M} = \mathcal{M}_1 \cup^* \mathcal{M}_2$ we get the desired result. $\square$

Theorem 6.9. For $y = \begin{pmatrix} a & b_1+b_2\omega \\ b_1+b_2\bar{\omega} & d \end{pmatrix}$ let

$$\rho(y) := \frac{\det(y)^2}{3\pi} \cdot \frac{2a+b_1+2d}{d(a+d)(a+b_1+d)(a+b_2+d)(a+b_1-b_2+d)}$$

Then ρ induces an equivariant partition of unity as in 3.2

Proof. As we saw in 6.6:

$$\text{vol}(\mathcal{P} \cap F_y) = \sum_{g \in PSL_2(R)} \text{vol}(\varphi(g)\mathcal{M} \cap F_y)$$

Using 5.9:

$$\text{vol}(\mathcal{P} \cap F_y) = \sum_{g \in PSL_2(R)} \text{vol}(\mathcal{M} \cap F_{\varphi(g)y})$$

Since 5.11

$$\frac{\pi}{48} \det(y)^2 = \sum_{g \in PSL_2(R)} \text{vol}(\mathcal{M} \cap F_{\varphi(g)y})$$

According to the lemma 6.8, we have:

$$\text{vol}(\mathcal{M} \cap F_{\varphi(g)y}) = \rho(\varphi(g)y) \cdot \frac{48 \det(\varphi(g)y)^2}{\pi}$$

Since $\det(y) = \det(\varphi(g)y)$, ρ induces a $PSL_2(R)$ -equivariant partition. $\square$

6.4. Summary: the class number formula. Let's fix $D = \det(y)$. To apply Lemma 3.2 we need to show that the number of $PSL_2(R)$ orbits on $\mathbb{Q}_D^* = \{y \in \mathbb{Q}^* \mid \det(y) = D\}$ is finite, that each y has a finite stabilizer. We start with the finiteness of the class number.

Theorem 6.10. For a fixed determinant, D , the class number is finite.

Proof. Since $\frac{b}{a} \in F$:

$$\sqrt{3}|b| \leq a$$

And as it was previously stated:

$$a \leq d$$

Analogously to the real case:

$$a^2 \leq ad = D + |b|^2 \leq D + \frac{a^2}{3}$$

Consequently,

$$\frac{2}{3}a^2 \leq D$$

So there are finitely many options for a and there are finitely many options for b as well, using the fact that the set of points representing an Eisenstein integer forms a nowhere dense set in $\mathbb{C}$. $\square$

Now, I turn to the proof of the finiteness of the stabilizer.

Lemma 6.11.

$$\Gamma_j = a_j U(2) a_j^{-1} \cap PSL_2(R)$$

Where $y_j = a_j \overline{a_j}^T$, such decomposition exist due to 5.8. And $U(2)$ stands for the group of unitary matrices over $\mathbb{C}$.

Proof. We can do the exact same thing as in 4.7

$$g \in \Gamma_j \Leftrightarrow (ga_j) \overline{(ga_j)}^T = a_j \overline{a_j}^T$$

Multiplying by a_j^{-1} the right and by $(\overline{a_j}^T)^{-1}$ the left:

$$(a_j^{-1} ga_j) \overline{(a_j^{-1} ga_j)}^T = I_2$$

Equivalently,

$$a_j^{-1} ga_j \in U(2)$$

So

$$g \in a_j U(2) a_j^{-1}$$

And since $g \in PSL_2(R)$, it's equivalent with

$$g \in a_j U(2) a_j^{-1} \cap PSL_2(R)$$

$\square$

Since Γ_j is the intersection of a compact and a discrete set, $|\Gamma_j|$ is finite.

When $PSL_2(R)$ acts on $\mathcal{P}$, with $\rho(y) = C \text{vol}(\mathcal{M} \cap F_y)$, since 6.9, ρ induces an $PSL_2(R)$ -equivariant partition of unity. Let's use the following notations:

$$\mathcal{M}_i^{(D)} = \{y \in \mathcal{M}_i \mid \det(y) = D, z \in \mathcal{Q}^*\}$$

$$\mathcal{M}_D = \{y \in \mathcal{M} \mid \det(y) = D, z \in \mathcal{Q}^*\} = \mathcal{M}_1^{(D)} \cup^* \mathcal{M}_2^{(D)}$$

$$\mathcal{P}_D = \{y \in P \mid \det(y) = D, z \in \mathcal{Q}^*\}$$

Since $\mathcal{M}_D$ is a fundamental domain, $\mathcal{M}_D$ can be written as:

$$\mathcal{M}_D = \{y_1, y_2, \dots, y_h\}$$

And

$$\mathcal{P}_D = \bigcup^* \Gamma y_j$$

Definition 6.12. The Hurwitz class number of positive definite binary hermitian forms over the Eisenstein ring is

$$H_D = \sum_{j=1}^h \frac{1}{|\Gamma_j|}$$

where Γ_j is the stabilizer group of y_j :

$$\Gamma_j = \{g \in PSL_2(R) \mid \varphi(g)y_j = y_j\}$$

It was shown by Hurwitz that this is a more natural constant than the number of orbits h . Among others this (and not h) appears in many analytic number theory results. Using 3.2 and our Theorem 6.9 we have the following corollary

Theorem 6.13. *Let $D > 0$ be given and let H_D be the Hurwitz class number of $\mathcal{Q}_D^*$, the set of positive definite binary Hermitian forms over the Eisenstein ring R of discriminant D . Then*

$$H_D = \frac{D^2}{3\pi} \cdot \sum \frac{2a + b_1 + 2d}{d(a+d)(a+b_1+d)(a+b_2+d)(a+b_1-b_2+d)}$$

where the sum is over $\begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} \in \mathcal{Q}_D^*$.

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