

Upper Bound on the Stick Number of Quad-Knots

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1. Preliminaries

In the present chapter, we summarize the relevant background. In the subsequent chapters, every concept, statement, theorem, and proof is a new result.

Let us define a knot as an embedding of S^1 into $\mathbb{R}^3$. Two embeddings are considered equivalent if there exists an ambient isotopy between them. A knot is then the equivalence class of such embeddings. Non-intersecting collection of knots are called a link.

We refer to the two-dimensional projection of a representative of the link as a link diagram, where every point of the diagram is covered by at most two points of the link. This is essentially a 4-regular graph with additional information: at the crossings, we indicate the over-under relationship between the two intersecting arcs by slightly separating the lower arc at the intersection point. From this, it is clear that the representative can be uniquely reconstructed from the knot diagram.

The Reidemeister theorem [7] states that two diagrams represent the same knot if and only if they can be transformed into each other by a sequence of planar isotopies and the three Reidemeister moves, Figure 1.1.

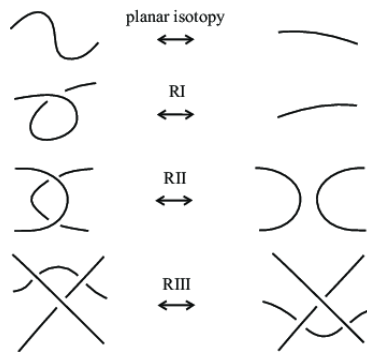


Figure 1.1: The Reidemeister moves

A diagram is called reduced if none of its crossing can be removed by twisting. It is alternating if, given any orientation (for link diagrams, in each component), the traversal at crossings alternately uses over and under arcs. Specifically, at each crossing, the path alternates between going over and under successive crossings. The former case at a crossing

is called an overcrossing, and the latter an undercrossing. A knot is called alternating if it has an alternating diagram.

A unique factorization up to ordering can be defined on knot diagrams (and thus on the knots themselves). Two oriented knots (or links) can be summed by placing them side by side and joining them by straight bars so that orientation is preserved in the sum. This we call the connected sum of the two knots. We define the sum of two diagrams by forming the connected sum along any two arbitrarily chosen arcs, and we refer to the two components in the connected sum as a decomposition. Knots for which, in every decomposition, one of the summands is the unknot are called prime knots.

The crossing number of a knot refers to the number of crossings in its representation with the fewest crossings. It is obviously a knot invariant. Another knot invariant, which we examine in detail in this paper, is the stick number. A stick essentially refers to a line segment. The stick number of a knot indicates the minimum number of sticks required to represent the knot. A lower and upper bound for the stick number is known [3, 6]:

Theorem 1. *For any knot K ,*

$$\frac{1}{2}(7 + \sqrt{8c(K) + 1}) \leq stick(K) \leq \frac{3}{2}(c(K) + 1), \quad (1.1)$$

where $stick(K)$ denotes the stick number, and $c(K)$ denotes the crossing number.

2. Construction

In this section, we define a new family of links, which we will refer to as **quad-links**.

Definition 1. *Given a pair of natural numbers $(n;m)$, take a square grid of n vertical and m horizontal sections on the plane, and then convert the intersections into crossings starting from the top left corner and alternating in both directions in a checkerboard fashion. We call this the **frame** of the square grid. Let's name the initial line segments: the horizontal ones of type- x , and the vertical ones of type- y .*

Remark 1. *Let it be agreed that the upper left intersection is always converted to an over-crossing.*

Note that the diagram is initially derived from line segments, since the endpoints of these line segments will be connected in some way in the pairings discussed below. We refer to these free endpoints simply as **nodes**.

To get a knot from the frame, the nodes must be paired up and connected by arcs.

Definition 2. *The arcs connecting the nodes of the frame are called **outer arcs**, and the arcs belonging to the frame are called **inner arcs**.*

Definition 3. *By completing the frame with the outer arcs, we obtain a link diagram, and the associated link is called a **quad-link**.*

Definition 4. *The nodes between which an outer arc goes are connected to form an **outer pair**. If the set of paired nodes equals the set of nodes, then we speak of a **pairing**.*

The notation is consistent with the representation that, given the nodes on the sides of a rectangle, the frame is inside the rectangle and the outer arcs run in the complement.

Remark 2. *A node can belong to only one pair*

3. Pairings

For the moment, let's look at the case where we are trying to obtain a knot from the pairing. Note that if we additionally tailor the pairing in such a way to extend the given alternating structure, then we get a diagram with a minimum number of intersections by the *Tait theorem* [5], which says that every reduced, alternating diagram contains the minimal possible number of crossings. Our goal in this chapter is to prove the following statement:

No matter how the outer arcs are drawn, the resulting intersections can be assigned crossing types in such a way that the diagram alternates.

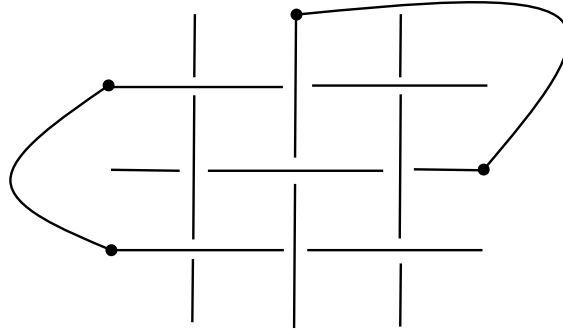


Figure 3.1: The frame of a square grid with two outer pairs

Definition 5. *The nodes originally belonging to the same line segment are called **internal pair**.*

Definition 6. *A **traversal** is defined as the path that starts from the left node of the top horizontal line segment and alternately moves to the outer and inner pairs of nodes.*

Proposition 1. *Each traversal belonging to a pairing is a cycle.*

Proof. We need to prove that during the traversal, we always return to the starting node, and before that, each node is touched at most once.

Let us group the nodes associated with the traversal according to whether we reached them from their outer pair or their inner pair. Let the former be called inner nodes and the latter outer nodes. Assume indirectly that there is a node that is touched more than once, and it is not the starting node. Let this be the first such node in the traversal. If this node was an inner point on both its first and second touches, then we must have reached it both times from its inner pair, meaning we would have touched the inner pair twice earlier. Similarly, if it was an outer node both times, we would have touched the outer pair twice earlier, so this is a contradiction.

If it was an inner node the first time and an outer node the second time, we would travel from it to its outer pair and then return from the outer pair, meaning the outer pair would be touched twice earlier. Conversely, if it was an outer node the first time and an inner node the second, we would travel from it to its inner pair and then return from the inner pair, thus it contradicts again the assumption that the arc we chose was the first, that we touched twice.

Thus, we have shown that in the traversal, each node is touched at most once before returning to the starting node. Since we only have finitely many nodes, we must eventually return to the starting node. $\square$

If all vertices are touched in a single cycle, the diagram is a *knot*; otherwise, it is a *link*, and the cycles correspond to components of the link.

There is another important characteristic of the matching that we will not permit for now. If we aim to obtain a reduced diagram, we cannot pair corners. That is, the left endpoint of the top horizontal segment cannot be paired with the upper endpoint of the leftmost vertical segment, nor can the right endpoint of the top horizontal segment be paired with the upper endpoint of the rightmost vertical segment. The same applies to the bottom horizontal endpoints, which cannot be paired with the lower endpoints of the nearest vertical segments. This pairing would result in a *Type I Reidemeister move* at the corners.

Let us examine what pairings of nodes yield an alternation. To do this, we first need to establish a few conventions. We will number the nodes starting from the upper left horizontal endpoint and proceeding counterclockwise. Thus, the pairings can be correspondingly mapped to closed intervals over $\mathbb{Z}$, where the two endpoints of each interval correspond to the integers assigned to the paired endpoints.

It immediately follows that under this correspondence, two outer arcs cross each other if and only if the assigned intervals have a non-empty intersection, without inclusion. Therefore, it does not matter how we traverse graphically the outer arc between the two endpoints.

Definition 7. *Let us call an interval **even** if its endpoints have opposite parity and **odd** if they have the same parity.*

Remark 3. *This terminology is consistent with whether there is an even or odd number of points (integers) between them, i.e., in the open interval $(n; m)$.*

Proposition 2. *An odd interval must have an intersection with another interval.*

Proof. An odd number of interior points cannot be paired within the interval (i.e., under the arc). □

With the same reasoning we get the following, stronger statement.

Lemma 1. *It is necessary for an odd interval to have an odd number of crossings, and for an even interval to have an even number (including zero) of crossings.*

Proof. For both parity types of intervals, it is necessary that the number of interior points paired with another interior point is even. □

We introduce one more convention: the intersection points on an arc are ordered based on the left nodes of the corresponding intervals; that is, if the arc $[a; b]$ is intersected by the arcs $[c; d]$ and $[e; f]$, where $c < e$, then following the counterclockwise orientation introduced at the nodes, the intersection with the arc $[c; d]$ comes first, followed by the intersection with the arc $[e; f]$.

Remark 4. *This does not impose any restriction, as if the arcs $[c; d]$ and $[e; f]$ also intersect, it does not matter whether this happens before or after their intersections with $[a; b]$, since this would simply reverse all three crossings, which still results in an alternation.*

Theorem 2. *The necessary and sufficient condition for an alternating pairing is that for the intervals corresponding to the pairings, an even interval must have an even number, allowing zero, intersections, while an odd interval must have an odd number of intersections.*

Proof. The necessity was already justified above. Here, we only prove the sufficiency.

Let's start drawing the arcs according to the order of the endpoints, meaning first we draw the arc represented by the interval $[1; a]$, then the one corresponding to $[2; b]$ as long as $2 \neq a$, and so on. The only issue that can arise is if, when drawing an arc, it needs to cross another arc in such a way that no matter how we choose the crossing, it disrupts the alternation, because one of the arcs would have two consecutive crossings of the same type (either over or under). This means that both arcs exit the same type of crossing right before their common intersection.

So, assume indirectly that such a problem occurs with the intersecting arcs $[a; c]$ and $[b; d]$. In this case, we have $a < b < c < d$. Let's examine the interval $[a; b]$. The arcs that intersect exactly one of the two arcs $[a; c]$ or $[b; d]$ before their common intersection will have exactly one of their nodes within the $[a; b]$ interval.

If the interval $[a; b]$ is even, then the nodes a and b come from crossings of opposite types, and the interval can have zero or an even number of crossings. If there are no crossings, the arcs still exit from crossings of opposite types, just as they do when the interval is even, because both arcs will have the same parity in the number of crossings, meaning that right before their common intersection, the arcs $[a; c]$ and $[b; d]$ exit from crossings of opposite types. Thus a crossing can be assigned to their intersection that continues the alternation.

The same reasoning holds when the interval $[a; b]$ is odd. In this case, a and b exit from crossings of the same type, and the interval must have an odd number of crossings. This means that the parity of the number of crossings on the arcs differs, so right before their common intersection, the arcs exit from crossings of opposite types. So again, a crossing can be assigned to their intersection, which alternates with the previous crossings.

We have shown that if we draw the arcs according to the ordering of their nodes and place the intersections on them, then when drawing the next arc, no issues can arise, provided the conditions of the statement are met. With this, we have inductively proven the sufficiency of the claim as well. $\square$

In fact, we have proven the desired statement, which we introduced at the beginning of the chapter.

Corollary 3. *No matter how the external arcs are drawn, the resulting intersections can be assigned crossing types in such a way that the diagram alternates.*

Proof. Let us assume indirectly that there exists a pairing for which the corresponding intersections cannot be assigned crossing types in a way that makes the diagram alternate. This would mean that, starting to assign crossings to the intersection points, we encounter the same issue as before: no matter how we choose the crossing type, it breaks the alternation by resulting in two consecutive crossings of the same type (either over or under) on one of the arcs. We have already proven that this is not possible if the crossings are assigned according to the order of the nodes. $\square$

4. Stick representation

We study quad-knots. In this chapter, we provide a stick representation that will be useful for the estimation in the next chapter.

Naturally, with any alternating matching where the traversal does not split into multiple cycles and no corners are connected, we obtain a reduced knot with a minimal number of crossings.

In $\mathbb{R}^3$, consider the plane $z = 0$. Arrange the segments in a regular manner, that is, place the x-type segments inside a rectangle, leaving equal space between neighboring segments, and align them parallel to each other and to the x-axis. Similarly, for the y-type segments, align them parallel to each other and to the y-axis. These m y-type segments define $m - 1$ bands, each bounded by two neighboring segments.

Next, we rotate the x-type segments around an axis parallel to the y-axis, located within one of the bands defined by the y-type segments — say, the first band — such that we rotate consecutive segments in alternating directions by an angle smaller than ϵ . After this rotations we still refer to these segments as x-type segments.

Definition 8. *We define the surface spanned by two adjacent x-type segments as the surface obtained by connecting points on the two segments that share the same x-coordinate with a line segment. This is the surface spanned by the two segments. The surface spanned by n x-type segments is the union of the surfaces spanned by each pair of adjacent segments.*

Let us denote the surface spanned by the n x-type segments after the above rotations as F , see Figure 4.1. Notice that the extreme points of F will be exactly the x-type segments, except for a single point on each, which will be the saddle points of F .

Thus, we have represented the x-type segments with a single stick, leaving the y-type segments to be addressed. If we view F as the union of the surfaces spanned by the $n - 1$ pairs of neighboring x-type segments, the alternating pattern of the frame essentially means that the neighboring y-type segments need to pierce these subsurfaces of F from

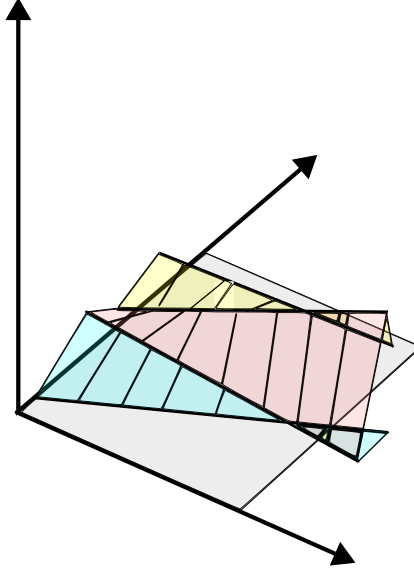


Figure 4.1: Stretched surface by the x-type line segments

opposite directions. Given that F has the aforementioned property of the locus of extreme points, one of the two neighboring y-type segments can be represented with a single stick (corresponding to its position in the plane $z = 0$). The other y-type segment is then represented by a broken line consisting of $n - 1$ segments, which zigzags above and below the x-type segments, avoiding them. An exception to this is the leftmost y-type segment and the one adjacent to it, as the saddle points of F are located between them. Therefore, both of these segments can be represented with a single stick, see Figure 4.

Turning to the outer arcs, let us assign them a directed graph where the outer arcs are the vertices. An edge exists between two vertices if the corresponding outer arcs intersect, and the edge should point towards the vertex representing the upper arc. We call this the **intersection graph** of the knot. For now, we only consider such quad-knots whose intersection graphs don't contain cycles.

In this case, a topological ordering of the graph edges provides an ordering of the heights of the intersecting outer arcs of the knot diagram. If, for each endpoint of every extreme stick in the frame, we draw a ray at an angle of α , as Figure 4 shows, then each

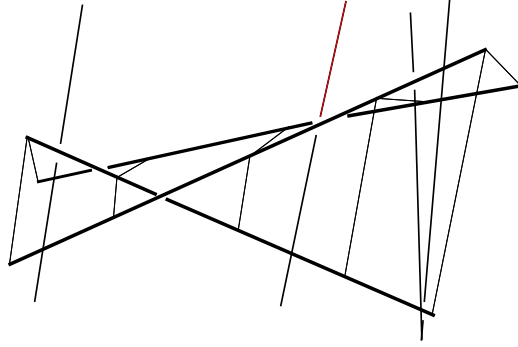


Figure 4.2: Adjacent y-type segments around the saddle point

outer arc corresponds to a pair of such rays. Assign heights bijectively to these pairs of rays such that the heights preserve the order induced by the topological arrangement. By drawing the segment for the outer arc between the two rays at their corresponding height, we obtain a representation that preserves the relational structure of the outer arcs. With this, the stick representation of the quad-knot is complete.

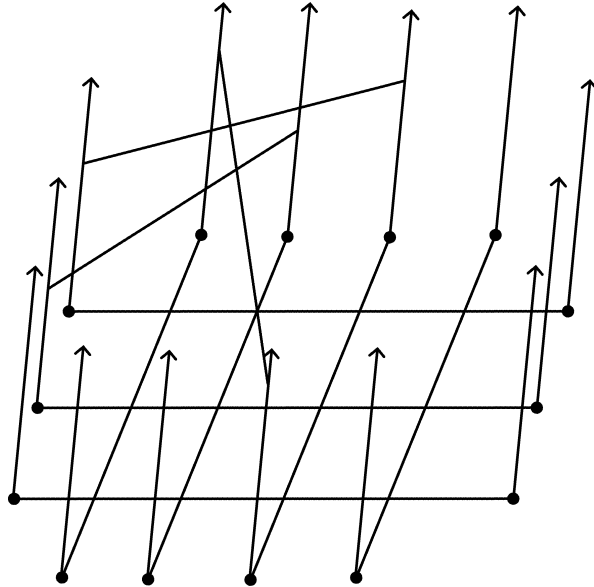


Figure 4.3: The rays of angle α on the extreme sticks

5. New upper bound

In this chapter, we provide an upper bound for the stick number of the quad-knots we introduced, and then we show that this is an improvement over the currently known upper bound, Theorem 1.

Theorem 4. *For a quad-knot K with parameters (n, m) , and acyclic intersection graph, $stick(K) \leq 3n + 2m + \frac{m}{2}n$.*

Proof. We use the stick representation that we introduced in 4. Since the roles of the x-type and y-type segments are symmetric, we can assume $n \geq m$. As it is evident, we represent the type of segment that occurs more frequently with a single stick. In the frame, we use n sticks for the x-type segments, and at least $\lceil \frac{m}{2} \rceil$ y-type segments can also be represented with a single stick. For the remaining segments, $\lfloor \frac{m}{2} \rfloor (n - 1)$ sticks are needed in total. Then, we add $2(n + m)$ sticks for the rays at angle α , and for the $n + m$ outer arcs, one stick each. In total, this gives at most:

$$3n + 2m + \frac{m}{2}n.$$

□

Now, let us examine how the obtained bound compares to the previously known upper bound. The quad-knots do not belong to the subfamilies of torus knots or two-bridge knots (for which specific upper bounds are known [1, 4]); therefore, the best-known bound applicable to them is Theorem 1. In the case of a quad-knot with parameters (n, m) , there are already nm crossings in the frame, which gives us a lower bound for the crossing number. The known upper bound is

$$\frac{3}{2}(c(K) + 1)$$

which in our case is at least

$$\frac{3}{2}(nm + 1).$$

The inequality

$$3n + 2m + \frac{m}{2}n \leq \frac{3}{2}(nm + 1)$$

$$0 \leq (n - 2)(m - 3) - 5$$

holds for $m \geq 4$ and $n \geq 7$. Thus, the bound given in Theorem 4 is stronger than the previous upper bound when $\max(n, m) \geq 7$ and $\min(n, m) \geq 4$. That is, with the exception of a few small examples, always.

Remark 5. *We can think of quad-knots as generalizations of well-known special knot families such as weaving knots, or turk's head knots [2]. Generalizing our new result, and understanding if it can yield better bounds is subject to work in progress.*

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6. Appendix

Saját eredmények:

1. Introduction of a new knot family, the so-called quad-knots, including their definition and the construction of their knot diagrams.
2. Introduction of the concept of pairing within the knot family, and through this, examination of the conditions for alternation in quad-knots.
3. Construction of the stick representation for quad-knots.
4. Derivation of a sharper upper bound, compared to the general upper bound, for a subset of quad-knots.