

Polynomial moving points and lines on the plane and an olympiad geometry problem solving method

Molnár-Szabó Vilmos
Bsc 2. year
m.sz.vilmos@gmail.com
Supervisor: Frenkel Péter
associate professor

1 Introduction

The *Method of Moving Points* or *Animation* is an olympiad geometry problem solving method, that can be applied to more than half of the geometry problems in my experience, often solving them faster than the intended solution. People have been using it in several contests (including the IMO), even though the theorems in this topic aren't proved precisely. This paper aims to build a precise environment, prove the fundamental and some own theorems of this method and set up the concept of the 'square root' type method, which can solve a broader range of problems than the original method.

This is a quite fresh method, it was introduced in 2019, only proving the weaker versions of the theorems over $\mathbb{R}$ instead of $\mathbb{C}$ (giving only upper bounds for degrees).[3][2]

I proved that the upper bounds for degrees are strict, and for that reason I had to deal with multiplicities, and points at infinity, so I used the standard charts of $\mathbb{CP}^n$ to define local coordinates even at points at infinity. I defined polynomial moving points in a more symmetric way by bivariate homogeneous polynomials. With this definition I could give clear, and short proofs to the fundamental theorems.

When the geometry problem contains an intersection of a line and a conic, the two intersection points' coordinates aren't polynomial, instead a square root of a polynomial appears in them. Since I came across many such problems, I made the square root extension of the method, which is discussed in the second half of my paper. I built up a more general environment to handle these points, and generalized almost every theorem in the first half in this setting.

This method is fairly unknown, and as far as I know these are the only new results until now. I reached out to one of the authors who invented the method, and he also thought about some further ideas, basically on the same path as I did.

Lots of ideas are credited to my supervisor Frenkel Péter.

2 Moving points and lines

2.1 The space we are working in

We work in $\mathbb{CP}^n$ with the standard holomorphic structure. We are especially interested in $\mathbb{CP}^2$, and we consider moving points and lines in this space.

2.2 Moving points

We call a function $P : \mathbb{CP}^1 \rightarrow \mathbb{CP}^n$ a *moving point* in $\mathbb{CP}^n$. We imagine it as if $\mathbb{CP}^1$ was the time, and $P(t)$ was the moving point.

$\mathbb{C}[t, s]_h$ refers to homogeneous bivariate polynomials, and $\mathbb{C}[t, s]_h^d$ refers to homogeneous bivariate polynomials of degree d .

A moving point is *polynomial* if there exist $p_0, \dots, p_n \in \mathbb{C}[t, s]_h^d$ with no common zero except the origin such that

$$P((t : s)) = (p_0(t, s) : \dots : p_n(t, s))$$

with homogeneous coordinates, where d is called the *degree* of the polynomial moving point.

We call $(p_0 : \dots : p_n)$ a polynomial format of P .

Suppose $s \neq 0$ in $(t : s)$, now we can evaluate $(t : s)$ according to the first chart: $\frac{t}{s}$

$$P((t : s)) = P\left(\frac{t}{s}\right) = (p_0(t, s) : \dots : p_n(t, s)) = \left(\frac{p_0(t, s)}{s^d} : \dots : \frac{p_n(t, s)}{s^d}\right) = \left(p_0^*\left(\frac{t}{s}\right) : \dots : p_n^*\left(\frac{t}{s}\right)\right)$$

where $p_0^*, \dots, p_n^* \in \mathbb{C}[t]$ are called the univariate forms of $p_0, \dots, p_n$ respectively.

2.1. Lemma. (The polynomials are unique up to a constant multiplier)

If we have $p_0, \dots, p_n, q_0, \dots, q_n \in \mathbb{C}[t, s]_h$ with no common zeros except the origin, such that

$$P(t) = (p_0(t, s) : \dots : p_n(t, s)) = (q_0(t, s) : \dots : q_n(t, s))$$

then $\exists \lambda \neq 0 \in \mathbb{C} : (p_0, \dots, p_n) = \lambda(q_0, \dots, q_n)$

Proof. Since $(p_0(t) : \dots : p_n(t)) = (q_0(t) : \dots : q_n(t)) \quad \forall t, s \neq \infty$

$$\frac{p_0(t)}{q_0(t)} = \dots = \frac{p_n(t)}{q_n(t)} = \frac{a(t)}{b(t)}$$

where $a, b \in \mathbb{C}[t, s]_h$ are relatively prime polynomials.

$\forall i \quad \frac{p_i}{q_i} = \frac{a}{b} \implies p_i b = q_i a \implies a \mid p_i \text{ and } b \mid q_i \implies a \text{ is a common factor of all } p_i, \text{ but they don't have any, therefore } a \text{ is constant, and so is } b \text{ similarly. Neither } a \text{ nor } b \text{ is constant zero, and therefore } \lambda = \frac{a}{b} \neq 0 \in \mathbb{C} \text{ satisfies } (p_0, \dots, p_n) = \lambda(q_0, \dots, q_n).$ $\square$

2.2. Lemma. A polynomial moving point is holomorphic.

Proof.

Changing from homogeneous coordinates to local coordinates of the charts, and the other way round only use division. Therefore the mapping

$$P(t : s) = (p_0(t, s) : \dots : p_n(t, s))$$

defined by homogeneous coordinates is also holomorphic in local coordinates. $\square$

2.3 Moving lines

We are now in $\mathbb{CP}^2$, however the things mentioned here can be generalised to hyperplanes in $\mathbb{CP}^n$.

A projective line is defined by its homogeneous coordinates, just like points. A polynomial moving line and its degree are defined the same way as for a polynomial moving point.

For example $e(t : s) = (p_0(t, s) : p_1(t, s) : p_2(t, s))$ is a polynomial moving line, where $p_0, p_1, p_2 \in \mathbb{C}[t, s]_d$ with no common zeros except the origin, and d is the *degree* of the polynomial moving point.

3 Distances and areas in the complex plane

Just to clarify some definitions what we will use later:

The distance of two points (p_1, p_2) and (q_1, q_2) is

$$\sqrt{|p_1 - q_1|^2 + |p_2 - q_2|^2}$$

This is a metric.

The distance of a point (p_1, p_2) and a line $Ax + By + C = 0$ is

$$\frac{|Ap_1 + Bp_2 + C|}{\sqrt{|A|^2 + |B|^2}}$$

Twice of the area of the triangle formed by points $P = (p_1, p_2)$, $Q = (q_1, q_2)$ and $R = (r_1, r_2)$ is

$$\begin{vmatrix} p_1 & p_2 & 1 \\ q_1 & q_2 & 1 \\ r_1 & r_2 & 1 \end{vmatrix}$$

3.1. Lemma. $2 \cdot \text{area}(PQR\Delta) = \text{dist}(P, QR) \cdot \text{dist}(QR)$

Since the line through (p_1, p_2) and (q_1, q_2) is $(A, B, C) = (p_1, p_2, 1) \times (q_1, q_2, 1)$,

$$\begin{aligned} & \left| \begin{vmatrix} p_1 & p_2 & 1 \\ q_1 & q_2 & 1 \\ r_1 & r_2 & 1 \end{vmatrix} \right| = |(p_1, p_2, 1) \times (q_1, q_2, 1) \cdot (r_1, r_2, 1)| = |(A, B, C) \cdot (r_1, r_2, 1)| = |Ar_1 + Br_2 + C| = \\ & = \frac{|Ar_1 + Br_2 + C|}{\sqrt{|A|^2 + |B|^2}} \cdot \sqrt{|A|^2 + |B|^2} = \frac{|Ar_1 + Br_2 + C|}{\sqrt{|A|^2 + |B|^2}} \cdot \sqrt{|p_2 - q_2|^2 + |q_1 - p_1|^2} = \text{dist}(P, QR) \cdot \text{dist}(QR) \end{aligned}$$

4 Multiplicities

To state any theorems, we have to define several types of multiplicities. Some multiplicities have an algebraic and a geometric definition, which are equivalent.

$\mathcal{O}_t(f)$ denotes the order of zero of f at t , where f can be one of several things:

- a holomorphic function
- a smooth function from $\mathbb{C}$ to $\mathbb{R}$ (where $f'(x) = \frac{f(x+\Delta) - f(x)}{|\Delta|}$)

- a vector-valued function, in which case we take the minimum of the order of zeros of the coordinate functions, or equivalently the order of zero of the length of the vector.
- a bivariate homogeneous function, in which case for any $t \in \mathbb{CP}$ we fix a nonzero coordinate in the homogeneous coordinate form of t , and take the order of zero of f in the other coordinate.

$\mathcal{O}(f)$ denotes the order of zero of f at a point, where it is clear, which point we are talking about.

$a_1, \dots, \widehat{a_i}, \dots$ means that a_i is omitted from the sequence/sum.

The phrase f is holomorphic at t_0 means that f is holomorphic in a neighborhood of t_0 .

1. point-point multiplicity:

When two moving points, A and B coincide, this can happen with multiplicity. The multiplicity of the coincidence is $\mathcal{O}((A - B))$ at the time of the coincidence. To calculate the derivatives, we use a chart containing the coincidence point.

For example, if at some time A and B are the same, and their derivative vector is the same, they coincide with at least mul. 2.

Proof. Taking two different charts we get the same result:

The transition map is holomorphic with nonsingular Jacobi-matrix, so by the p-p multiplicity lemma 4.1 (see later) we know that the two multiplicities are the same. $\square$

1' geometric definition:

When two moving points, A and B coincide, this can happen with multiplicity. The multiplicity of the coincidence is the order of zero of the distance of A and B . To calculate the distance, we use a chart containing the coincidence point.

Proof. (1) and (1') are equivalent definitions:

$$\begin{aligned} \mathcal{O}(\text{dist}(A, B)) &= \mathcal{O}\left(\sqrt{|a_1 - b_1|^2 + |a_2 - b_2|^2}\right) = \frac{1}{2} \min\left(\mathcal{O}(|a_1 - b_1|^2), \mathcal{O}(|a_2 - b_2|^2)\right) = \\ &= \min\left(\mathcal{O}(|a_1 - b_1|), \mathcal{O}(|a_2 - b_2|)\right) = \min\left(\mathcal{O}(a_1 - b_1), \mathcal{O}(a_2 - b_2)\right) = \mathcal{O}((a_1 - b_1, a_2 - b_2)) = \mathcal{O}((A - B)) \end{aligned}$$

$\square$

2. point-line multiplicity:

If a point P lies on hyperplane (line) l at a time t_0 , this can happen with multiplicity. The multiplicity of this is

$$\mathcal{O}_{t_0}((l_1, \dots, l_{n+1}) \cdot (p_1, \dots, p_{n+1}))$$

where $(l_1 : \dots : l_{n+1})$ and $(p_1 : \dots : p_{n+1})$ are polynomial formats of l and p .

2' geometric definition:

If a point P lies on line l , this can happen with multiplicity. The multiplicity of this is the order of zero of the distance between the point and the line. We use the chart containing the place of the event.

Proof. (2) and (2') are equivalent definitions, and also that taking any chart in (2') we get the same result.

The conversion from homogeneous to i-th chart coordinates of a line is putting the i-th coordinate to the last place. Let i be such that p_i is nonzero at the time of the event.

$$|(l_1, \dots, l_{n+1}) \cdot (p_1, \dots, p_{n+1})| = \left| p_i \cdot \left((l_1, \dots, l_{n+1}) \cdot \left(\frac{p_1}{p_i}, \dots, 1, \dots, \frac{p_{n+1}}{p_i} \right) \right) \right| =$$

$$\begin{aligned}
&= \left| p_i \cdot \left(l_1 \frac{p_1}{p_i} + \dots + l_{n+1} \frac{p_{n+1}}{p_i} + l_i \right) \right| = \left| p_i \cdot \sqrt{|l_1|^2 + \dots + |\widehat{l_i}|^2 + \dots + |l_{n+1}|^2} \cdot \left(\frac{|l_1 \frac{p_1}{p_i} + \dots + l_{n+1} \frac{p_{n+1}}{p_i} + l_i|}{\sqrt{|l_1|^2 + \dots + |\widehat{l_i}|^2 + \dots + |l_{n+1}|^2}} \right) \right| = \\
&= \left| p_i \cdot \sqrt{|l_1|^2 + \dots + |\widehat{l_i}|^2 + \dots + |l_{n+1}|^2} \right| \cdot \text{dist}(p, l)
\end{aligned}$$

$p_i \cdot \sqrt{|l_1|^2 + \dots + |\widehat{l_i}|^2 + \dots + |l_{n+1}|^2}$ is a nonzero infinitely differentiable function. Therefore

$$\mathcal{O}((l_1, \dots, l_{n+1}) \cdot (p_1, \dots, p_{n+1})) = \mathcal{O}(\text{dist}(p, l))$$

□

3. line-line multiplicity:

When two moving lines coincide, this can happen with multiplicity. The multiplicity of this is the same as the p-p multiplicity of their poles via any pole-polar transformation.

Proof. This is well-defined:

The composition of two pole-polar transformations is a projective transformation, which preserves p-p multiplicity (proving this later) 4, so calculating the p-p multiplicity via two different pole-polar transformation gives the same answer. □

4. point-conic multiplicity:

When a point P arrives to a fixed conic c , this can happen with multiplicity. The multiplicity of this is the point-line multiplicity of P and the tangent line to c at the arrival point.

5. line-conic multiplicity:

When a moving line l becomes tangent to a fixed conic c , this can happen with multiplicity. The multiplicity of this is the point-line multiplicity of l and the arrival tangency point.

6. 3 point multiplicity:

When three moving points A, B, C become collinear at a time t_0 , this can happen with multiplicity. The multiplicity of this is

$$\mathcal{O}_{t_0}(\det(a|b|c))$$

where a, b and c are polynomial formats of A, B and C .

6'. geometric definition: When three moving points A, B, C become collinear, this can happen with multiplicity. If all three points are on a single chart, the multiplicity is the order of zero of the area of the triangle formed by them according to their chart.

Proof. (6) is equivalent to (6') for any three points on a single chart. We can suppose that they are on the third chart. Their polynomial formats are $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ and $c = (c_1, c_2, c_3)$. We can suppose that the third coordinates are all constant 1, by dividing every coordinate with the third coordinate, it only affects the determinant by a nonzero, holomorphic factor.

$$\mathcal{O}(\det(a|b|c)) = \mathcal{O}(|\det(a|b|c)|) = \mathcal{O}\left(\left|\begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix}\right|\right) =$$

$$\mathcal{O}(\text{area of the triangle formed by } (a_1, a_2) = A, (b_1, b_2) = B \text{ and } (c_1, c_2) = C)$$

□

7. 3 line multiplicity:

When three moving lines e, f, g become concurrent, this can happen with multiplicity. The multiplicity of this is the same as the 3p multiplicity of their poles via any pole-polar transformation.

Proof. This is well-defined:

The composition of two pole-polar transformations is a projective transformation, which preserves 3p multiplicity (proving this later), so calculating the 3l multiplicity with two different pole-polar transformation gives the same answer. $\square$

The next theorem shows us what happens with the multiplicities during a holomorphic transformation.

4.1. Lemma. (p-p multiplicity lemma) Let $p, q : \mathbb{CP} \rightarrow M$ be holomorphic maps and $\phi : M \rightarrow N$ a holomorphic map, where M and N are m and n dimensional complex manifolds respectively. Then for every t_0 where $p(t_0) = q(t_0)$, the p-p multiplicity of $\phi \circ p$ and $\phi \circ q$ at t_0 is at least the p-p multiplicity of p and q at t_0 . Furthermore if ϕ has a Jacobian matrix of rank m at $p(t_0)$, then the two multiplicities are equal.

Proof. Since ϕ is holomorphic, its differential exists, and so there is a neighborhood U around $p(t_0)$ and a constant $c_1 > 0$ such that

$$\forall x, y \in U \quad \text{dist}(\phi(x), \phi(y)) \leq c_1 \cdot \text{dist}(x, y)$$

Then

$$\text{ppmul}(\phi(p(t)), \phi(q(t))) = \mathcal{O}(\text{dist}(\phi(p(t)), \phi(q(t)))) \geq \mathcal{O}(\text{dist}(p(t), q(t))) = \text{ppmul}(p(t), q(t))$$

If ϕ has a Jacobian matrix of rank m at $p(t_0)$, then since it is the maximal rank, the differential of ϕ at t_0 is injective meaning that there is a neighborhood U around $p(t_0)$ and a constants $c_1, c_2 > 0$ such that

$$\forall x, y \in U \quad c_1 \cdot \text{dist}(x, y) \leq \text{dist}(\phi(x), \phi(y)) \leq c_2 \cdot \text{dist}(x, y)$$

Then

$$\text{ppmul}(\phi(p(t)), \phi(q(t))) = \mathcal{O}(\text{dist}(\phi(p(t)), \phi(q(t)))) = \mathcal{O}(\text{dist}(p(t), q(t))) = \text{ppmul}(p(t), q(t))$$

$\square$

4.2. Lemma. All the above mentioned multiplicities don't change via a projective transformation.

Proof.

1. point-point multiplicity:

The $\mathcal{A}$ projective transformation has matrix A . Suppose $\mathcal{A}$ takes a point x , whose i -th homogeneous coordinate is nonzero to $\mathcal{A}(x)$ whose j -th homogeneous coordinate is nonzero. The whole projective transformation consists of three holomorphic mappings: $(x_1, \dots, x_n) \mapsto (x_1 : \dots : x_{i-1} : 1 : x_i : \dots : x_n) \mapsto A \cdot (x_1 : \dots : x_{i-1} : 1 : x_i : \dots : x_n) = (y_1 : \dots : y_{n+1}) \mapsto (\frac{y_1}{y_j}, \dots, \frac{y_j}{y_j}, \dots, \frac{y_{n+1}}{y_j})$. So a projective transformation is holomorphic, and it has a jacobian matrix of rank n .

Now we can apply p-p mul. lemma 4.1, to get that $\text{ppmul}(\mathcal{A}(p(t)), \mathcal{A}(q(t))) = \text{ppmul}(p(t), q(t))$.

2. point-line multiplicity: $\mathcal{A}$ takes a moving point x to Ax , and takes a moving line e to eA^{-1} . (With homogeneous coordinates). If e and x polynomial formats of the moving point and line, then eA^{-1} and Ax are polynomial formats of their images, because multiplying by a nonsingular matrix creates polynomials, and it can't create $(0, 0, 0)$.

$$e \cdot x = (eA^{-1}) \cdot (Ax)$$

So the order of zero of $e \cdot x$ is the order of zero of $(eA^{-1}) \cdot (Ax)$. Therefore the p-l multiplicity of x and e are the same as of their images.

3. line-line multiplicity: We use the pole-polar transformation that takes l to l^T to calculate the l-l multiplicities. A projective transformation, that has matrix A takes l to lA^{-1} . Let e and f be two different moving lines. Then

$$llmul(e, f) = ppmul(e^T, f^T) = ppmul(A^{-1^T} e^T, A^{-1^T} f^T) = llmul(eA^{-1}, fA^{-1})$$

4. point-conic multiplicity:

The image of the tangent line to c at the arrival point is the tangent line to the image of c at the image of the arrival point. Because p-l multiplicity is preserved during projective transformation, p-c multiplicity is preserved too.

5. line-conic multiplicity:

The image of the tangency point of the moving line on c is the tangency point of the image of the moving line on the image of c . Because p-l multiplicity is preserved during projective transformation, l-c multiplicity is preserved too.

6. 3 point multiplicity:

The three points' polynomial formats are a, b and c . Their images' polynomial formats are Aa, Ab and Ac respectively,

$$\det(a|b|c) = \det(A^{-1} \cdot (Aa|Ab|Ac)) = \det A^{-1} \cdot \det(Aa|Ab|Ac)$$

So the order of zero of $\det(a|b|c)$ and $\det(Aa|Ab|Ac)$ are the same.

7. 3 line multiplicity:

We use the pole-polar transformation that takes l to l^T to calculate the 3l multiplicities. A projective transformation that has matrix A takes l to lA^{-1} . We have three different moving lines, e, f and g .

$$3lmul(e, f, g) = 3pmul(e^T, f^T, g^T) = 3pmul(A^{-1^T} e^T, A^{-1^T} f^T, A^{-1^T} g^T) = 3lmul(eA^{-1}, fA^{-1}, gA^{-1})$$

□

4.3. Lemma. Let $\mathcal{B}$ be a pole-polar transformation, and p and l be a moving point and a moving line. Then the point-line multiplicity of p and l is the same as of $\mathcal{B}(p)$ and $\mathcal{B}(l)$

Proof. The matrix B of $\mathcal{B}$ is nonsingular. $l \cdot p = (lB^{-1})(Bp)$, therefore the p-l multiplicity is preserved. □

5 Degree of statements

Almost every statement that comes up in geometry is true a certain number of times (with multiplicity), or it is always true. This number is called the degree of the statement. If we can show that the statement happens degree+1 times, it implies that the statement is always true!

5.1. Theorem. (Point-line theorem) Let P be a polynomial moving point, and let l be a polynomial moving line. Then the degree of the statement " P is on l " (with point-line multiplicity) is exactly $\deg P + \deg l$.

Proof. $\deg l = a$ and $\deg p = b$.

The p-l multiplicity at t_0 is

$$\mathcal{O}_{t_0}((l_1, \dots, l_{n+1}) \cdot (p_1, \dots, p_{n+1}))$$

where $(l_1 : \dots : l_{n+1})$ and $(p_1 : \dots : p_{n+1})$ are polynomial formats of l and p .

Since the polynomial $l_1 \cdot p_1 + \dots + l_n \cdot p_n$ has exactly $a + b$ zeros with multiplicity, or it is constant zero, p is on l exactly $a + b$ times with p-l multiplicity, or always. □

5.2. Theorem. Let A, B, C be three polynomial moving points. Then the degree of the statement " A, B, C are collinear" (with 3p multiplicity) is exactly $\deg A + \deg B + \deg C$.

Proof. The 3p multiplicity at t_0 is $\mathcal{O}_{t_0}(\det(a|b|c))$ where a, b and c are polynomial formats of A, B and C . Since the polynomial $\det(a|b|c)$ has exactly $a + b + c$ zeros with multiplicity, or it is constant zero, A, B and C are collinear exactly $(\deg A + \deg B + \deg C)$ times (with 3p multiplicity), or always. □

5.3. Theorem. Let e, f, g be three polynomial moving lines. Then the degree of the statement " e, f, g are concurrent" (with 3l multiplicity) is exactly $\deg e + \deg f + \deg g$.

Proof. e, f and g are concurrent with the same 3l multiplicity as their poles are collinear with 3p multiplicity via any pole-polar transformation. Since the degree of their poles are the same, the poles are collinear exactly $(\deg e + \deg f + \deg g)$ times, or always. □

5.4. Theorem. Let A and B be two polynomial moving points on a fixed line. Then the degree of the statement " $A=B$ " (with p-p multiplicity) is exactly $\deg A + \deg B$.

Proof. We send the line to the x-axis with a projective transformation. Let e be the moving line perpendicular to the x-axis, always going through A . This is the line through the fixed ideal point of the y-axis and A , so its degree is $\deg A$ (see L-pp theorem 6.1 later). By the Point-line theorem 5.1 e is on B $\deg A + \deg B$ times with p-l multiplicity. The point-line multiplicity is the distance of e and A , but it is the same as the distance between A and B , which is the point-point multiplicity of A and B , so $A = B$ also $\deg A + \deg B$ times. □

6 L-pp and P-ll

Now let's see the most commonly used theorem in solving geometry problems with this method.

6.1. Theorem. (L-pp theorem) Let P and Q be two polynomial moving points. Then the line PQ is polynomial and its degree is

$$\deg(PQ) = \deg(P) + \deg(Q) - \#(P = Q)$$

where $\#(P = Q)$ denotes the number of times $t \in \mathbb{CP}^1$ when $P = Q$ (with point-point multiplicity).

Proof. For any 3 moving point A, B and C on one chart

$$\text{area}(ABC\Delta) = \text{dist}(C, AB) \cdot \text{dist}(A, B)$$

It follows that for any 3 moving point A, B and C at every t_0

$$3pmul_{t_0}(A, B, C) = plmul_{t_0}(C, AB) + ppmul_{t_0}(A, B)$$

If A, B, C are not on one chart, then we take a projective transformation taking them to one chart, and the multiplicities don't change.

For an arbitrary fixed point F , using Lemma 5.2

$$\deg PQ = \sum_{t \in \mathbb{CP}} plmul_t(PQ, F) = \sum_{t \in \mathbb{CP}} 3pmul_t(P, Q, F) - ppmul_t(P, Q) = \deg P + \deg Q - \#(P = Q)$$

□

6.1 Pole-polar duality of degrees

The following theorem gives us a duality between points and lines in terms of degrees.

6.2. Theorem. Let $\mathcal{C}$ be any nondegenerate conic, and let P and l be a point and line with pole-polar relation via $\mathcal{C}$. Then $\deg(P) = \deg(l)$.

Proof. The polar of a point $\mathbf{P}$ is determined by $\mathbf{l} = A\mathbf{P}$, where A is nonsingular and since $\mathbf{P}$ contains three homogeneous polynomials of the same degree, $\mathbf{l}$ contains their linear combinations, which are also polynomials of the same degree, and they are never all zero. So we get $\deg l = \deg P$. □

6.2 Intersection of two lines

6.3. Theorem. (P-ll theorem) Let e and f be two polynomial moving lines. Then the intersection point $e \cap f$ is polynomial and its degree is

$$\deg(e \cap f) = \deg(e) + \deg(f) - \#(e = f)$$

where $\#(e = f)$ denotes the number of t times when $e = f$ (with line-line multiplicity).

Proof. Let $\mathcal{P}$ be a pole-polar transformation. Applying $\mathcal{P}$ to the whole configuration, we can calculate the degree using the L-pp theorem 6.1, then we can get back the degree of intersection point by applying $\mathcal{P}$ again. That is

$$\begin{aligned} \deg(e \cap f) &= \deg(\mathcal{P}(\overline{\mathcal{P}(e)\mathcal{P}(f)})) = \deg(\overline{\mathcal{P}(e)\mathcal{P}(f)}) = \deg \mathcal{P}(e) + \deg \mathcal{P}(f) - \#(\mathcal{P}(e) = \mathcal{P}(f)) = \\ &= \deg e + \deg f - \#(e = f) \end{aligned}$$

□

6.3 Projective transformation

6.4. Theorem. Let $\mathcal{A}$ be a projective transformation, P a polynomial moving point, and l a polynomial moving line. Then $\deg P = \deg \mathcal{A}(P)$ and $\deg l = \deg \mathcal{A}(l)$

Proof. A projective transformation corresponds to a nonsingular matrix A , so the same proof works as in the proof of Lemma 6.2 with both the point and the line. □

6.5. Corollary. Every isometry, homothety, affine and projective transformation preserves degrees.

7 Proving geometry problems

7.1. Theorem. For every line l , there exists a degree 1 point P , such that the locus of P is l .
For every conic $\mathcal{C}$, there exists a degree 2 point P , such that the locus of P is $\mathcal{C}$.

Proof. Take a degree 1 point, then by the Locus theorem 8.1 (see later) it's moving on a line. Since for every two lines there exists a projective transformation, which sends one to the other, the image of the degree 1 point is moving on our chosen line with degree 1.

Same proof works for conics. □

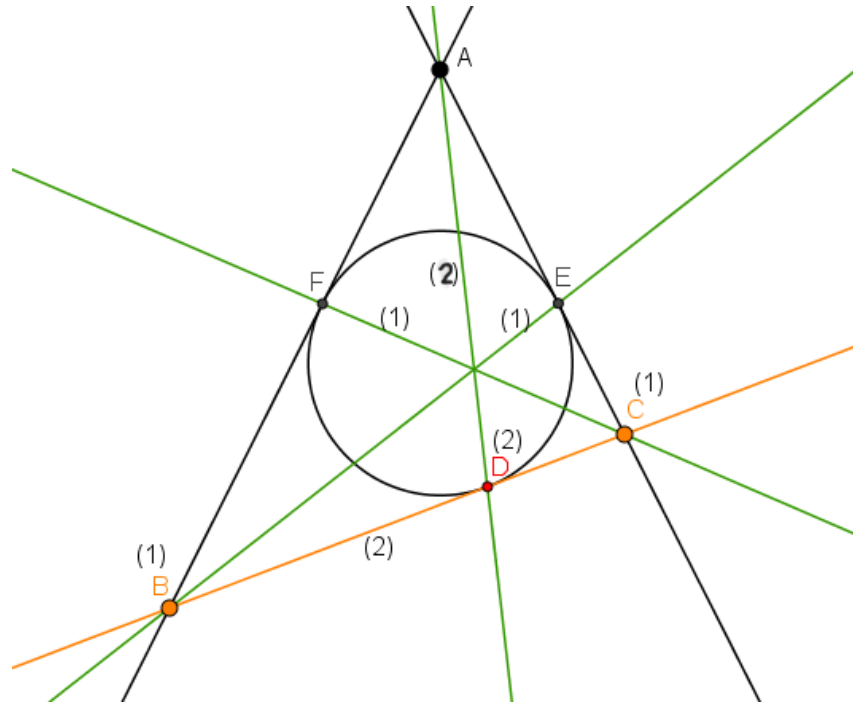
We finally came to the point where we have the necessary tools to use the method.

A geometry problem usually takes the following format: There are some initial objects, from which we construct further objects, and then we have to prove a statement about these objects.

We first take any configuration of the initial objects, and we want to prove the statement for this configuration. We fix some of the initial objects. Then we 'animate' one of the initial objects, this will be the 'initial point'. We choose a locus (line or conic) to the initial point, and animate it to move like a degree 1 polynomial point if the locus is a line, or a degree 2 polynomial point if the locus is a conic. Then the rest of the objects are uniquely determined by the initial objects, so they are moving along, and in almost every case they will be polynomial. With some rules we can calculate the other objects' degrees. We then calculate the degree of the statement by the already known degrees, and find degree+1 special cases, where the statement is true. We can choose simpler or degenerate configurations for special cases. This proves that at every state of the initial point the statement is true, specially at the place from where we 'started' the animation.

Here is a practice problem with solution.

7.2. Exercise. (Gergonne-point) In triangle ABC the tangency points of the incircle are D , E and F on sides BC , CA and AB respectively. Prove that AD , BE and CF are concurrent.



Proof. We use the incircle-animation, which comes up very often in problem solving, and is very useful. We start with a general configuration of triangle ABC .

Let's fix A, E, F and the incircle, and start animating point D with degree 2 on the incircle.

Since side BC is the polar of D via the incircle, BC has degree 2.

B is the intersection of the fixed side AB , and the degree 2 side BC . We can use the P-II theorem 6.3 to calculate the degree of B , that is

$$\deg(B) = \deg(AB) + \deg(BC) - \#(AB = BC) = 0 + 2 - 1 = 1$$

AB coincides with BC exactly once (with p-p mul 1), when $D = F$.

Similarly C is also of degree 1.

To calculate the degree of EB , we use the L-pp theorem 6.1. That is

$$\deg(EB) = \deg(B) + \deg(E) - \#(B = E) = 1 + 0 - 0 = 1$$

B and E never coincide.

Similarly FC is also of degree 1.

AD is of degree $0 + 2 - 0 = 2$.

The statement that BE, FC, AD are concurrent, is of degree $\deg BE + \deg FC + \deg AD = 1 + 1 + 2 = 4$.

Then it is sufficient to find 5 special cases where the statement is true.

1. $AB = AC$

Now ABC is isosceles, and the statement is true for every isosceles triangle by symmetry.

2. $BA = BC$

Another isosceles case.

3. $CA = CB$

And another.

4. $D = E$

This is a degenerate case. $C = E = D$ because of the degeneration. Therefore AD , BE and CF all go through this point, so they are concurrent.

5. $D = F$

The symmetric pair of the previous case.

Now we proved that the statement is true for every position of D . Therefore it is true for the original position, which was a general case, so the statement is true for every triangle ABC . $\square$

8 Locus of a polynomial moving point

8.1. Theorem. (Locus-theorem) H is a degree n polynomial moving point. Then the locus of H is a degree k algebraic curve, where $k \mid n$. For every smooth point A of the curve, H passes through A $\frac{n}{k}$ times (with p-p mul.). For every order d singularity B , H passes through B $\frac{n}{k} \cdot d$ times (with p-p mul.)

8.2. Example.

A degree 1 point is always moving on a fixed line, and takes all its positions once. (with p-p mul.)

A degree 2 point is either moving on a fixed conic, and takes all its positions once, or it is moving on a fixed line, and takes all its positions twice. (with p-p mul.)

Proof. Let $P, Q, R \in \mathbb{C}[t]$ be the univariate forms of the polynomial format of H . Then $H(t) = (P(t) : Q(t) : R(t))$ for every $t \neq \infty$, res_t denotes resultant in variable t .

$\text{res}_t(xR(t) - P(t), yR(t) - Q(t))$ is an x, y -variate polynomial, where for every (x, y) satisfying $\text{res}_t(xR(t) - P(t), yR(t) - Q(t)) = 0$ there exists some $t \neq \infty$ for which $xR(t) - P(t) = yR(t) - Q(t) = 0$, therefore $x = \frac{P(t)}{R(t)}, y = \frac{Q(t)}{R(t)}$. This means that at every finite (x, y) point of the $\text{res}_t(xR(t) - P(t), yR(t) - Q(t)) = 0$ affine algebraic curve H passes through at least one time.

This isn't true for one (x, y) , because here we are using a parametric resultant (x and y are the parameters of the polynomials), and the usual resultant uses polynomials with nonzero main coefficient, while this parametric resultant allows this. When only one polynomial has zero main coefficient, the parametric and the usual resultant differ only by a nonzero multiplier, so they are both zero or both nonzero. The error happens when both polynomials have zero main coefficient, but in this case this is true for only one (x, y) , which is actually $H(\infty)$, so H goes through this point of the curve also.

Now we have an affine algebraic curve c , and H goes through every point of c . But since H is continuous on the projective plane, it takes the ideal points of the projective extension of c also.

If we substitute $P(t)$, $Q(t)$ and $R(t)$ in x, y, z respectively in the homogenous form of c , we get a t -polynomial, which has infinite zeros, therefore it is constant zero. This means that H is always on c . (even at $t=\infty$, as we saw before)

For the same reason c is irreducible.

Let $f : c \rightarrow \mathbb{N}$ be the function indicating how many times H passes through a certain point on c . (with p-p mul.)

We want to show that f is constant, except the singularities, where the value is d times bigger, if d is the degree of the singularity.

In case c is a line, $f(P) = \text{number of } H \text{ passing through a fixed line going through } P$, which is n by the Point-line theorem 5.1, and thus we proved the statement. From now on we suppose c is not a line.

f is upper-bounded by n , because if H is passing through a fixed point P $n + 1$ times, the coordinates of $H - P$ are degree n rational functions with $n + 1$ zeros, so they are constant zero, and so the 'moving' point is always the same as the fixed point.

A point P is *problematic* if it's singularity or $x'(t) = 0$, where $H(t_0) = P$ and $x(t)$ is the x coordinate-function of H at t_0 according to the charts containing t_0 and P .

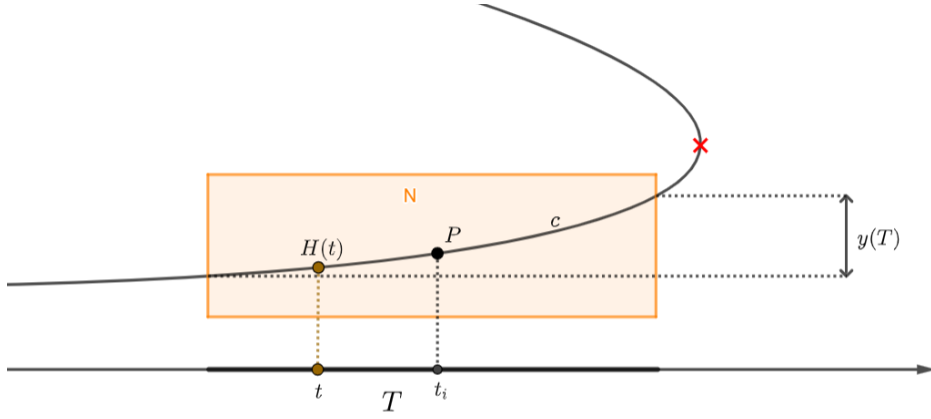
There are only finitely many problematic points, because there are finitely many singularities, and $x'(t) = 0$ finitely many times, or always. In the latter case c is a line.

Lemma 1: f is locally constant in any nonproblematic point.

Let P be a point on c , that is not problematic. Let $t_1, \dots, t_m$ be the times when $H(t_i) = P$ (m times at total). $x(t)$ and $y(t)$, are the euclidean coordinates of H according to the charts containing t_i and P .

Since P is a smooth point, and $\frac{\partial F}{\partial y} \neq 0$ at P (otherwise $x'(t) = 0$), we can apply holomorphic implicit function theorem, and so we have a neighborhood N in which the curve looks like a holomorphic $y(x)$ graph.

By holomorphic inverse function theorem, there exists a neighborhood T of t_i , where $x(T)$ is a neighborhood of $x(t_i)$, and $x(t)$ is a bijection between T and $x(T)$ ($x'(t) \neq 0$). In addition we can suppose $x(T) \times y(T) \subset N$, and $x(T) \times y(T)$ doesn't contain problematic points.



For every $x \in x(T)$, this x value will be held by H for exactly one t . In $(x(T) \times y(T)) \cap c$, every x value is held by only one point (it is in N), and therefore that one point will be held by H for exactly one t . The conclusion is that H goes through every point exactly once in $(x(T) \times y(T)) \cap c$, which is a neighborhood of $H(t_i)$ on c . The point-point multiplicity is 1 everywhere, since $x'(t) \neq 0 \implies (x(t), y(t))' \neq 0$.

At every t_i , H goes through P with p-p mul. 1 ($x'(t) \neq 0$). So $f(P) = m$. But for every i , there is a neighborhood U_i of P , where H goes through every point of U_i once within a neighborhood T_i of t_i . We can suppose that the T_i -s don't intersect. In $U = \bigcup_{i=1}^m U_i$, every point is taken by H exactly m different times within $\bigcup_{i=1}^m T_i$, with mul. 1, so we are left to show that outside of $\bigcup T_i$, $H(t) \neq P$.

There exists a U^* neighborhood of P , so that $t \notin \bigcup T_i \implies H(t) \notin U^*$. Otherwise there exists a t_n sequence of times outside of $\bigcup T_i$, so that $H(t_n) \rightarrow P$. But since $t_n \in \mathbb{CP}$, there exists a convergent subsequence of t_n converging to some t_0 . Since $H(t)$ is continuous, $H(t_0) = P$, but since $\bigcup T_i$ is open $t_0 \notin \bigcup T_i$, which is a contradiction.

Therefore H goes through every point exactly m times in $U \cap U^*$, which means that f is locally constant at P .

With this we proved lemma 1.

c without the problematic points is path-connected, because c without the singular points is path-connected [1], and every smooth point has a neighborhood on c , that is homeomorphic to an open set in $\mathbb{C}$, so omitting finitely many points doesn't affect path-connectedness.

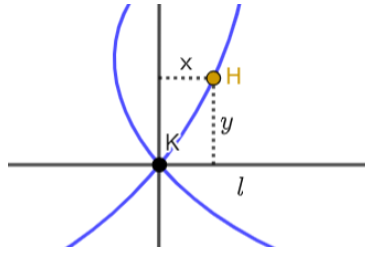
Let c' be c without the problematic points.

f is locally constant on c' , which is path-connected, so f is constant on all c' .

Lemma 2: A line l is intersecting c at K . l is not tangent to c at K . Then $H(t) = K$ with the same p-p multiplicity as $H(t) \in l$ with p-l multiplicity+.

Using a projective transformation we can suppose that K is the Origin and l is the x-axis. This doesn't change the multiplicities. $\lim(\overline{HK})$ is tangent to c , but l is not, so $\lim(HK) \neq x\text{-axis}$, or equivalently $\lim \frac{y(t)}{x(t)} \neq 0$, where x and y are the coordinate functions of H .

$$ppmul(H, K) = \min(\mathcal{O}(x), \mathcal{O}(y)) = \mathcal{O}(y) = \mathcal{O}(|y|) = \text{dist}(H, x\text{-axis}) = plmul(H, l)$$



Let k be the degree of c .

Now let l be a line not tangent to c , so that no problematic points are on l . Then l intersects c at k different non-problematic points K_i . The Point-line theorem 5.1 states that H goes through l a total of n times with point-line mul. By Lemma 2, we know that

$$n = \sum_t plmul(H, l) = \sum_{i=1}^k \sum_t ppmul(H, K_i) = \sum_{i=1}^k f(K_i)$$

Since f is constant on c' , $f = \frac{n}{k}$. So we proved the theorem for the non-problematic points.

Let P be a problematic point, which is an order d singularity (we include smooth points too with $d = 1$). Let l be a line through P , that is not tangent to c , and doesn't go through problematic points except P . l intersects c at $k - d$ different points not counting P . So with similar calculation as above, $f(P) = n - (k - d)\frac{n}{k} = d \cdot \frac{n}{k}$. With this we proved the theorem for problematic points too.

□

8.1 Locus of a polynomial moving line

8.3. Lemma. A degree 1 line always has a fixed point around which it rotates, and takes all its positions once. (with l-l mul.)

Proof. Take a pole-polar transformation, apply the fact that a degree 1 point is moving on a fixed line, and do the inverse of the pole-polar, to get back the original figure, where the degree 1 line always lies on the pole of the fixed line, which is a fixed point. Also the pole-polar equivalent of a point-point multiplicity is a line-line multiplicity. □

8.4. Lemma. A degree 2 line is either always tangent to a conic, and takes all its positions once, or it has a fixed point around which it rotates, and takes all its positions twice. (with l-l mul.)

Proof. The pole of the line is a degree 2 point, so there are two cases. First, the pole is moving on a conic, taking all its positions once with p-p mul. This means that the line is always tangent to the dual of this conic, implying the first scenario of the statement. Secondly, the pole is moving on a line taking all its positions twice with p-p mul, implying the second scenario of the statement. $\square$

9 Mászó-theorem

While solving a geometry problem, calculating the degree of a line through two points takes a lot of time, because we have to check all the coincidences of the two points. However when both points move on the same conic, Mászó-theorem gives an explicit degree of the line through them. And also the degree is quite low!

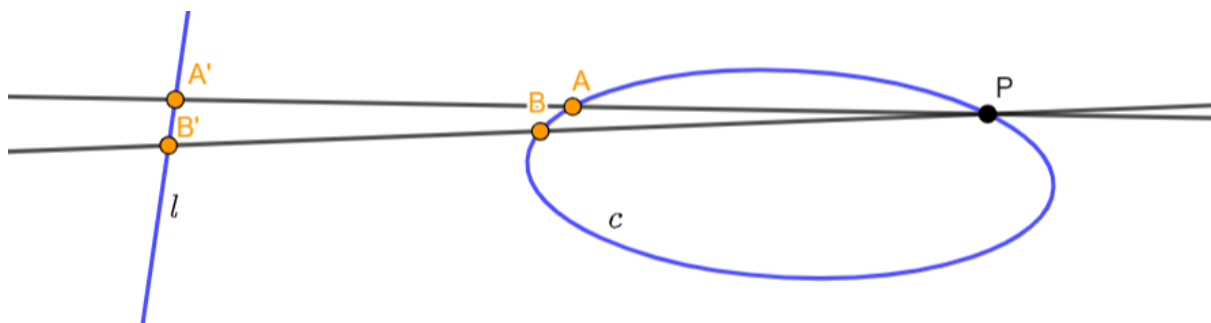
9.1. Lemma. If a polynomial point moves on a conic, it has even degree.

Proof. The Locus-theorem 8.1 states that if a degree n point moves on a degree k conic, then $k \mid n$. $k = 2$ gives us the proof. $\square$

9.2. Theorem. (Mászó-theorem) Two polynomial points A and B move on the same conic. Then

$$\deg AB = \frac{\deg A + \deg B}{2}$$

Proof. $\deg A = a$, $\deg B = b$. Let P be a fixed point on the conic $\mathcal{C}$, and l a line not passing through P . According to the Locus-theorem 8.1, since A moves on $\mathcal{C}$, it passes through P exactly $\frac{a}{2}$ times (with p-p mul.) This means that the line AP has degree $0 + a - \frac{a}{2} = \frac{a}{2}$. Let $A' = AP \cap l$, B' similarly. $\deg A' = 0 + \frac{a}{2} - 0 = \frac{a}{2}$. Similarly $\deg B' = \frac{b}{2}$. A' and B' are both moving on l , so applying Lemma 5.4 they coincide exactly $\frac{a}{2} + \frac{b}{2}$ times (with p-p multiplicity).



Let C be a degree 2 moving point on the conic. We construct C' similarly as before, so $\deg C' = 1$. C' as a function of time is a biholomorphism between l and $\mathbb{CP}$, because the Locus-theorem states that $C'(t)$ goes through at every point of l exactly once with nonzero derivative. We can apply the Inverse function theorem to get that C'^{-1} exists, and is holomorphic. For the same reason, C as a function of time is a biholomorphism between $\mathcal{C}$ and $\mathbb{CP}$. The $\phi : C(t) \mapsto C'(t)$ mapping is a biholomorphism, because $\phi = C' \circ C^{-1}$.

We can now apply the p-p multiplicity lemma 4.1, to conclude that the p-p mul. of A and B is the same as of A' and B' .

This means that A and B coincide exactly the same number of times as A' and B' do, which is $\frac{a+b}{2}$.

We know that $\deg AB = a + b - \#(A = B) = a + b - \frac{a+b}{2} = \frac{a+b}{2}$.

□

We use the next theorem in a geometry problem when we construct the second intersection point of a circle and a line, such that the first intersection point is already given.

9.3. Theorem. (Inverse of Mászó-theorem) Let $\mathcal{C}$ be a fixed conic, A a polynomial moving point on it, and l a polynomial line always going through A . The second intersection of $\mathcal{C}$ and l is B . Then B is polynomial, and its degree satisfies

$$\deg l = \frac{\deg A + \deg B}{2}$$

Proof. With a projective transformation we can send any conic into a circle, so we can suppose $\mathcal{C}$ is a circle with center O . l has a polynomial ideal point (intersection with the ideal line, which is fixed). A 90° rotation sends this ideal point into the perpendicular direction's ideal point, so connecting this with O gives us a polynomial line, which is the perpendicular bisector of A and B . A polynomial point's reflection across a polynomial line is also polynomial, and therefore B is polynomial.

We can now use Mászó-theorem to conclude that indeed

$$\deg l = \frac{\deg A + \deg B}{2}$$

□

9.4. Theorem. Let A and B be two polynomial moving points on a fixed conic. Then the degree of the statement " $A=B$ " is $\frac{\deg A + \deg B}{2}$ (with point-point multiplicity).

Proof. Mászó-theorem: $\deg AB = \frac{\deg A + \deg B}{2}$

L-pp theorem: $\deg AB = \deg A + \deg B - \#(A = B) \implies \#(A = B) = \frac{\deg A + \deg B}{2}$

We use p-p multiplicity to calculate $\#(A = B)$.

□

10 Square root moving points

It naturally comes up in geometry problems to calculate the degree of the intersection of a polynomial moving line and a fixed conic. However unlike all previous cases, these intersection points are not necessarily polynomial. They are so-called 'square root points'. Their homogeneous coordinates are in the following set:

Let $\mathbb{C}[t, s, \sqrt{H}]_h = \{a + b\sqrt{H} \mid a, b \in \mathbb{C}[t, s]_h, \deg a = \deg b + r\}$, where $H \in \mathbb{C}[t, s]_h$, such that H is not a square of any polynomial, and $\deg H = 2r$ for an $r \in \mathbb{N}$.

10.1. Lemma. $\mathbb{C}[t, s, \sqrt{H}]_h$ contains the homogeneous elements of a graded ring with index set $\{r, r+1, \dots\}$, such that an element $a + b\sqrt{H}$ has degree $\deg a$.

Proof.

$$(a + b\sqrt{H}) + (c + d\sqrt{H}) = (a + c) + (b + d)\sqrt{H}$$

Therefore adding two element with the same degree creates a third element with that degree.

$$(a + b\sqrt{H})(c + d\sqrt{H}) = (ac + bdH) + (bc + ad)\sqrt{H}$$

$\deg(ac) = \deg(bdH) = \deg(bc + ac) + r$, therefore multiplying two elements of degree p and q creates an element with degree $p + q$. $\square$

Let $\mathbb{C}[t, s, \sqrt{H}]_h^d$ denote the set of degree d elements in $\mathbb{C}[t, s, \sqrt{H}]_h$.

10.2. Definition. $a + b\sqrt{H}$ and $a - b\sqrt{H}$ are conjugate pairs, the conjugate of $p \in \mathbb{C}[t, s, \sqrt{H}]_h$ is denoted $\bar{p}$.

10.3. Lemma. This conjugation is an automorphism of $\mathbb{C}[t, s, \sqrt{H}]_h$.

Proof. It is an automorphism of $\mathbb{C}[t, s, \sqrt{H}]$ (allowing non-homogeneous elements), because $\sqrt{H}$ and $-\sqrt{H}$ have the same minimal polynomial: $x^2 - H = 0$, so there exists an automorphism fixing $\mathbb{C}[t]$, and taking $\sqrt{H}$ to $-\sqrt{H}$. That must be the above defined conjugation. Now $\mathbb{C}[t, s, \sqrt{H}]_h$ is closed to conjugation, so the conjugation is also an automorphism of this subset. $\square$

10.4. Remark. $\sqrt{H}$ isn't well defined, only $\pm\sqrt{H}$ as a 2-valued function. The elements of $\mathbb{C}[t, s, \sqrt{H}]_h$ are also not well defined, but from now on we only use expressions that are symmetric with respect to conjugation at the actual method, and these are well defined, and we only use the expressions that are not symmetric formally.

Conjugate pair square root valued functions also have multiplicity analogous to the polynomial version.

$$\mathcal{O}_t((f, \bar{f})) = \frac{\mathcal{O}_t(f \cdot \bar{f})}{2}$$

$f \cdot \bar{f}$ is symmetric to conjugation, so it is necessarily a polynomial. Square root multiplicity is always a half-integer.

10.5. Definition. An ordered pair of moving points (P_1, P_2) is a *square root point pair* if there exist $p_0, \dots, p_n \in \mathbb{C}[t, s, \sqrt{H}]_h^d$ such that

$$(P_1(t : s), P_2(t : s)) = ((p_0(t, s) : \dots : p_n(t, s)), (\bar{p}_0(t, s) : \dots : \bar{p}_n(t, s)))$$

with homogeneous coordinates.

The *degree* of the square root point pair is

$$d - \frac{\#(|(p_0, \dots, p_n)| = 0 \text{ or } |(\bar{p}_0, \dots, \bar{p}_n)| = 0)}{2}$$

where $\#(|(p_0, \dots, p_n)| = 0 \text{ or } |(\bar{p}_0, \dots, \bar{p}_n)| = 0)$ is taken with multiplicity.

When $(p_0, \dots, p_n) = 0$, P_1 is defined as $\frac{(p_0, \dots, p_n)}{|(p_0, \dots, p_n)|}$, P_2 similarly.

We call $(p_0 : \dots : p_n)$ a polynomial format of (P_1, P_2) .

10.6. Lemma. Degree is well defined, and always a positive half-integer.

Proof. The square root point-line theorem states that a square root point pair passes through a fixed line exactly 2 times the degree of the point pair, so the degree must only depend on the moving point pair as a function and not on its polynomial format. It follows that degree is a positive half-integer too. $\square$

We can define moving line pairs the same way as moving point pairs.

10.7. Remark. A polynomial moving point/line of degree n paired with itself form a square root point/line pair of degree n .

10.1 Intersection of a moving line and a fixed conic

10.8. Theorem. The two (not necessarily different) moving intersection points of a polynomial moving line of degree n and a fixed conic either form a square root point pair of degree n or they are both polynomial.

Proof. We want to prove that every coordinate is in $\mathbb{C}[t, s, \sqrt{H}]_h^d$ for some H and d , and also that P and Q are conjugate pairs, and also that it is degree n . A projective transformation keeps this square root form, the conjugate relation and the degree, because it only uses ring operations. So we can suppose that the conic is the origin-centered, radius 1 circle. Let $l : (A : B : C)$ be a degree n polynomial moving line. Then $Ax + By + C = 0$ is the equation of the line, and $x^2 + y^2 = 1$ is the equation of the circle.

From the line equation $y = \frac{-C-Ax}{B}$

Substituting this into the circle equation gives us

$$x^2 + \left(\frac{-C-Ax}{B} \right)^2 = 1 \iff (A^2 + B^2)x^2 + (2AC)x + (C^2 - B^2) = 0$$

$$x_{1,2} = \frac{-AC \pm \sqrt{A^2C^2 - (C^2 - B^2)(A^2 + B^2)}}{A^2 + B^2} = \frac{-AC \pm B\sqrt{A^2 + B^2 - C^2}}{A^2 + B^2}$$

Doing the same with y :

$$y_{1,2} = \frac{-BC \pm A\sqrt{A^2 + B^2 - C^2}}{A^2 + B^2}$$

In homogeneous coordinates:

$$P = (-AC + B\sqrt{A^2 + B^2 - C^2} : -BC - A\sqrt{A^2 + B^2 - C^2} : A^2 + B^2)$$

$$Q = (-AC - B\sqrt{A^2 + B^2 - C^2} : -BC + A\sqrt{A^2 + B^2 - C^2} : A^2 + B^2)$$

Taking $H = A^2 + B^2 - C^2$, if H the square of some polynomial, then both P and Q are polynomial points, and if H is not the square of some polynomial, then the coordinates are indeed in $\mathbb{C}[t, s, \sqrt{H}]_h^{2n}$ with a conjugate relation.

The degree of a point pair $(p_0, p_1, p_2), (\overline{p_0}, \overline{p_1}, \overline{p_2})$ is the half the degree of the polynomial point $(p_0\overline{p_0}, p_1\overline{p_1}, p_2\overline{p_2})$. In this case that is

$$\begin{aligned} & ((AC)^2 - B^2(A^2 + B^2 - C^2) : (BC)^2 - A^2(A^2 + B^2 - C^2) : (A^2 + B^2)^2) = \\ & = ((A^2 + B^2)(C^2 - A^2) : (A^2 + B^2)(C^2 - B^2) : (A^2 + B^2)^2) = \end{aligned}$$

$$= (C^2 - A^2 : C^2 - B^2 : A^2 + B^2)$$

This is never all zero, otherwise (A, B, C) would be all zero, therefore this is a degree $2n$ point, so (P, Q) is of degree n . $\square$

10.9. Theorem. (Polynomial intersection theorem) Let l be a moving line, and c be a fixed conic. If at every time, when l becomes tangent to c , this happens with an even l - c multiplicity, then the intersections of l and c are two polynomial points.

Proof.

Lemma 1: Let f be a holomorphic function, which is not constant zero. Then the following are equivalent:

1. f has even order of zero at t_0
 2. There exists a holomorphic g , such that $g^2 = f$ at a neighborhood of t_0
1. $\Leftarrow$ 2. If g has order of zero a , $f = g^2$ has order of zero $2a$.
1. $\Rightarrow$ 2. $f = t^{2a} \cdot h$, where h is a holomorphic function, that is not zero at t_0 . Since t_0 is not a limit point of zeros of f , there exists a neighborhood, where h has no zeros. Here exists a holomorphic j , such that $j^2 = h$. Then $g = t^a \cdot j$ is a holomorphic function, such that $g^2 = f$ at this neighborhood.

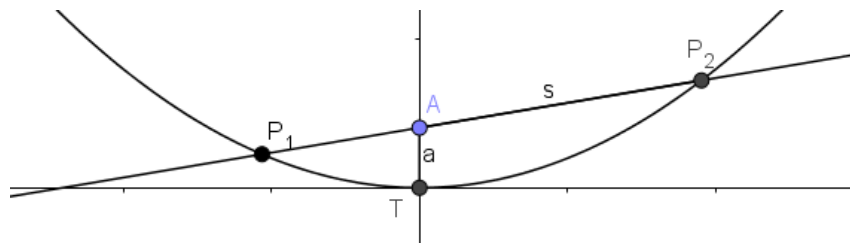
Lemma 2: If l is tangent to c at some t_0 , and the tangency has even l - c multiplicity, then the two intersection points are holomorphic at t_0 .

Let the tangency point of l on c at t_0 be T .

With a projective transformation send T to the Origin, and c to the $x^2 = y$ parabola.

Projective transformations don't change multiplicities and holomorphicity.

Let A be the intersection of the y -axis and l . At a time near t_0 , A is of distance a from the x -axis, and l has slope s .



Since l is the x -axis at t_0 , s converges to 0, so $plmul(l, T) = plmul(A, x\text{-axis})$ (the distances have ratio converging to 1). Therefore a also has even order of zero.

We can calculate the x coordinates of the intersections:

$$y = x^2 = a + sx$$

$$x_{1,2} = \frac{s \pm \sqrt{s^2 + 4a}}{2}$$

$s^2 + 4a$ is a holomorphic function, and since both s^2 and $4a$ have even order of zero, $s^2 + 4a$ has so. From Lemma 1, there exist two holomorphic functions $\pm g$, such that $(\pm g)^2 = s^2 + 4a$. Therefore the x -coordinates are holomorphic functions at t_0 . $y_{1,2} = x_{1,2}^2$, meaning that the y -coordinates are also holomorphic at t_0 , so the intersection points are holomorphic at t_0 . We proved Lemma 2.

The homogeneous coordinates of the intersections of a polynomial line l , and a fixed conic c are in $\mathbb{C}\left[t, s, \sqrt{H}\right]_h$. Take a case when $H(t, s) = 0$, $P_1 = P_2$ and l is tangent to c .

Suppose $(t : s)$ is on the first chart of $\mathbb{CP}$ (similarly works if it is on the second chart). In a neighbourhood of $(t : s)$ the intersection points can be written in the form

$$P_{1,2} = \left(a \pm b\sqrt{h}, c \pm d\sqrt{h}\right)$$

where $a, b, c, d \in \mathbb{C}(t)$, $h \in \mathbb{C}[t]$, $h(t) = H(t, 1)$, and the coordinates are meant according to the chart of T .

Now l is tangent to c , so from the condition and Lemma 2 P_1 and P_2 are holomorphic at $t_0 = \frac{t}{s}$. This means that one of their coordinates, for example $a + b\sqrt{h}$ is holomorphic at t_0 , and with some operations we get that $\pm\sqrt{b^2h}$ are holomorphic at t_0 . From Lemma 1, b^2h has an even order of zero at t_0 , and thus h has so. This means that H has even order of zero at $(t : s)$. We conclude that every zero of H (except the origin) is of even order, and so $H = G^2$, where $G \in \mathbb{C}[t, s]_h$.

So the intersections are polynomial points. □

10.10. Theorem. Let P be a moving point, and c be a fixed conic. If at every time, when P lies on c , this happens with an even multiplicity, then the tangents from P to c are two polynomial lines.

Proof. The tangency points of the tangent lines from P are X and Y . XY is the polar of P , and by Lemma 4.3 $plmul(XY, T) = plmul(P, t)$, where T is the arrival point of P on c , and t is the tangent line to c at T . We know that both multiplicities are even. Then by the previous theorem, X and Y are polynomial, and the tangent lines are so. □

10.2 Constructing further square root points

10.11. Lemma. Every geometric construction that uses only field operations constructs only square root point/line pairs from square root point/line pairs.

Proof. Such transformations can be expressed with homogeneous coordinates using multiplication, and addition. Therefore all coordinates stay in the set $\mathbb{C}\left[t, s, \sqrt{H}\right]$ (allowing non-homogeneous elements). But the new point/line is well-defined in terms of time, so choosing $(\lambda t, \lambda s)$ instead of (t, s) must change all three coordinates by the same multiplier, so they are homogeneous by the same degree, thus they are in $\mathbb{C}\left[t, s, \sqrt{H}\right]_h^d$ for some d . □

10.12. Corollary. projective transformation, pole-polar, line through two points, intersection of two lines, affine combination of some points, inversion about any conic all construct only square root point/lines pairs from square root point/lines pairs. (Basically any construction step that comes up in olympiad geometry)

10.13. Definition. When a geometric construction creates a square root point/line pair from polynomial points/lines, it is called a *bipartition*. The product pair is called *biparted pair*.

10.14. Example. The following are biparted pairs:

1. The intersections of a polynomial moving line and a fixed conic
2. The two tangents from a polynomial moving point to a fixed conic

3. The two angle bisectors of two polynomial moving lines
4. The two intersections of a fixed line and moving circle with polynomial center and radius
5. The complex distance of two polynomial moving points:

$$cdist(P, Q) = \pm \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2}$$

10.15. Definition. Two square root points/lines are *geometrically conjugate* if they can be constructed from a biparted pair using only field operations in a logical symmetric way, that is if we swap the role of the two points/lines of the biparted pair, the constructed points/lines also swap.

10.16. Example. Let A be a polynomial moving point somewhere, and P and Q are the intersections of a moving line and a fixed conic. Then lines AP and AQ are geometrically conjugate, because if we swap the roles of P and Q , AP and AQ also swap.

10.17. Lemma. Two geometrically conjugate points/lines also form a square root point/line pair.

Proof. The biparted points/lines are both geometrically and algebraically conjugate. From them we construct geometrically conjugate points/lines only using ring operations on their homogeneous coordinates. Therefore the coordinates of these points/lines will be algebraically conjugate, since algebraic conjugation is an automorphism. $\square$

10.18. Lemma. A square root point/line that can be constructed from a biparted pair using only field operations, and that is logically symmetric to the biparted pair is a polynomial point/line.

10.19. Example. The midpoint of the intersections of a moving line and a fixed conic is polynomial.

Proof. The point/line is its own geometric conjugate, so the coordinates of the point/line are also their own conjugates, so the coefficient of $\sqrt{H}$ is zero in them, and only polynomials remain. $\square$

10.3 Square root multiplicities

We can also define all types of multiplicities mentioned in the polynomial part for square root point pairs. For example, the p-l multiplicity of a point pair (P_1, P_2) and (l_1, l_2) is

$$plmul_t((P_1, P_2), (l_1, l_2)) = \frac{\mathcal{O}(dist(P_1, l_1) \cdot dist(P_2, l_2))}{2}$$

or equivalently the algebraic definition:

$$plmul_t((P_1, P_2), (l_1, l_2)) = \frac{\mathcal{O}((p_1 \cdot l_1) \cdot (p_2 \cdot l_2))}{2} - \frac{\mathcal{O}(p_1 = 0) + \mathcal{O}(p_2 = 0) + \mathcal{O}(l_1 = 0) + \mathcal{O}(l_2 = 0)}{2}$$

where p_1, p_2, l_1, l_2 vectors are the polynomial formats of P_1, P_2, L_1, L_2 respectively.

The algebraic and the geometric definitions are equivalent, because

$$\mathcal{O}(dist(P_1, l_1)) = \mathcal{O}\left(\frac{p_1 \cdot l_1}{|p_1| \cdot |l_1|}\right)$$

Similarly we can define every multiplicity analogously.

Multiplicity is always a half-integer.

10.4 Square root degree of statements

We can define degree for statements about square root point/line pairs.

10.20. Definition. The degree of a statement about point/line pairs $(A_1, B_2), \dots, (A_k, B_k)$ with a certain multiplicity is

$$\sum_{t \in \mathbb{CP}} \text{mul}_t((A_1, B_2), \dots, (A_k, B_k))$$

The degree of statements can be used in the following way:

10.21. Theorem. A degree n statement about point/line pairs $(A_1, B_2), \dots, (A_k, B_k)$ is true for either $A_1, \dots, A_k$ or $B_1, \dots, B_k$ exactly $2n$ times with multiplicity.

Proof.

$$2n = 2 \cdot \sum_{t \in \mathbb{CP}} \text{mul}_t((A_1, B_2), \dots, (A_k, B_k)) = \sum_{t \in \mathbb{CP}} \text{mul}_t(A_1, \dots, A_k) + \sum_{t \in \mathbb{CP}} \text{mul}_t(B_1, \dots, B_k)$$

□

10.22. Theorem. (Square root point-line theorem) Let (P_1, P_2) be a square root point pair, and let (l_1, l_2) be a square root line pair. Then the degree of the statement " P_1 is on l_1 or P_2 is on l_2 " (with sqrt point-line multiplicity) is exactly $\deg(P_1, P_2) + \deg(l_1, l_2)$.

Proof. Let p_1, p_2, l_1, l_2 vectors be the polynomial formats of P_1, P_2, L_1, L_2 respectively.

$$\begin{aligned} & \sum_{t \in \mathbb{CP}} \text{plmul}_t((P_1, P_2), (l_1, l_2)) = \\ &= \sum_{t \in \mathbb{CP}} \frac{\mathcal{O}((p_1 \cdot l_1) \cdot (p_2 \cdot l_2))}{2} - \sum_{t \in \mathbb{CP}} \frac{\mathcal{O}(p_1 = 0) + \mathcal{O}(p_2 = 0) + \mathcal{O}(l_1 = 0) + \mathcal{O}(l_2 = 0)}{2} = \\ &= \deg p_1 + \deg l_1 - \frac{\#(p_1 = 0 \text{ or } p_2 = 0)}{2} - \frac{\#(l_1 = 0 \text{ or } l_2 = 0)}{2} = \deg(P_1, P_2) + \deg(l_1, l_2) \end{aligned}$$

□

And analogously every degree of statement type theory works the same way as in the polynomial case.

10.5 Square root versions of further theorems

In fact every geometric construction is calculated the same way as in the polynomial case.

For example

10.23. Theorem. Let $\mathcal{A}$ be a projective transformation, (P_1, P_2) a square root point pair, and (l_1, l_2) a square root line pair. Then $\deg(P_1, P_2) = \deg(\mathcal{A}(P_1), \mathcal{A}(P_2))$ and $\deg(l_1, l_2) = \deg(\mathcal{A}(l_1), \mathcal{A}(l_2))$

Proof. Let A be the nonsingular matrix of $\mathcal{A}$. Let p be the polynomial format of (P_1, P_2) . Then Ap consists of the linear combinations of the elements of p , so they are of the same degree as the elements of p . And also $\mathcal{O}(|p|) = \mathcal{O}(|Ap|)$. So

$$\deg(P_1, P_2) = \deg p - \mathcal{O}(|p|) = \deg Ap - \mathcal{O}(|Ap|) = \deg(\mathcal{A}(P_1), \mathcal{A}(P_2))$$

Similarly with (l_1, l_2) . □

An other example:

10.24. Theorem. (Sqrt L-pp theorem) Let (P_1, P_2) and (Q_1, Q_2) be two square root point pairs. Then the degree of square root line pair (P_1Q_1, P_2Q_2) is

$$\deg(P_1Q_1, P_2Q_2) = \deg(P_1, P_2) + \deg(Q_1, Q_2) - \#((P_1, P_2) = (Q_1, Q_2))$$

where $\#((P_1, P_2) = (Q_1, Q_2))$ denotes half the number of times $t \in \mathbb{CP}^1$ when $P_1 = Q_1$ or $P_2 = Q_2$ (with square root point-point multiplicity).

Proof. For any three point pairs (A_1, A_2) , (B_1, B_2) and (C_1, C_2)

$$3pmul_{t_0}((A_1, A_2), (B_1, B_2), (C_1, C_2)) = plmul_{t_0}((C_1, C_2), (A_1B_1, A_2B_2)) + ppmul_{t_0}((A_1, A_2), (B_1, B_2))$$

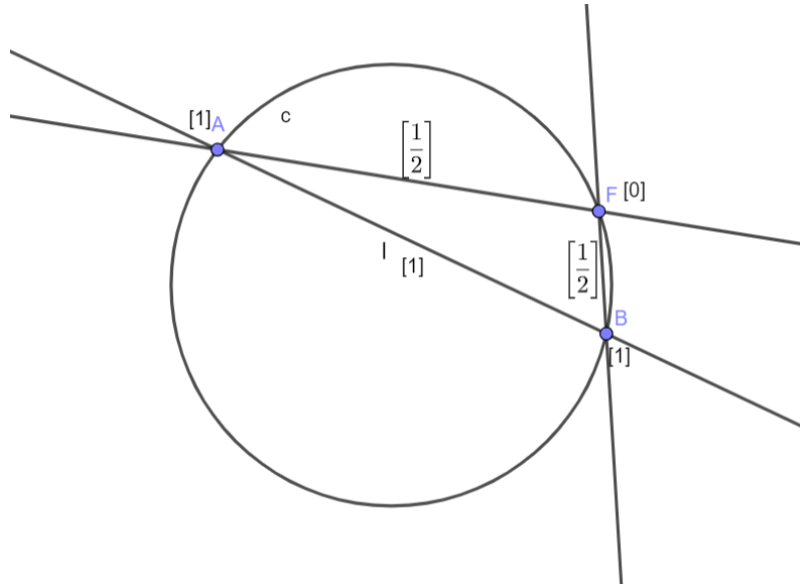
That's because for all types of multiplicities

$$mul((A_1, A_2), (B_1, B_2), (C_1, C_2)) = \frac{mul(A_1, B_1, C_1) + mul(A_2, B_2, C_2)}{2}$$

and for the ordinary form of these multiplicities the above equation holds.

The rest of the proof goes the same way as the original proof of the L-pp theorem 6.1. □

10.25. Example. The intersections of a fixed circle c and a degree 1 polynomial moving line l are A and B . F is a fixed point on c . The lines FA and FB form a square root line pair of degree $0 + 1 - \frac{1}{2} = \frac{1}{2}$. $F = A$ or B happens one time, when line l passes through F . Here exactly one of A and B coincides with F , so on average A and B coincide with F $\frac{1}{2}$ times.



10.26. Conjecture. (Square root Locus theorem) A degree $\frac{n}{2}$ square root point P and $\bar{P}$ are always moving on a shared degree k algebraic curve, where $k \mid n$, furthermore for every smooth point A of the curve, P and $\bar{P}$ together pass through A $\frac{n}{k}$ times (with p-p mul.). For every order d singularity B , P and $\bar{P}$ together pass through B $\frac{n}{k} \cdot d$ times (with p-p mul.)

Square root Mászó-theorem can also be proven analogously, however in the proof we use Square root Locus theorem, what is only a conjecture. But it can be easily proved going around Locus-theorem.

10.27. Lemma. If a square root point pair moves on a conic, it has integer degree.

10.28. Theorem. (Sqrt Mászó-theorem) Two square root point pairs (A_1, A_2) and (B_1, B_2) move on the same conic. Then

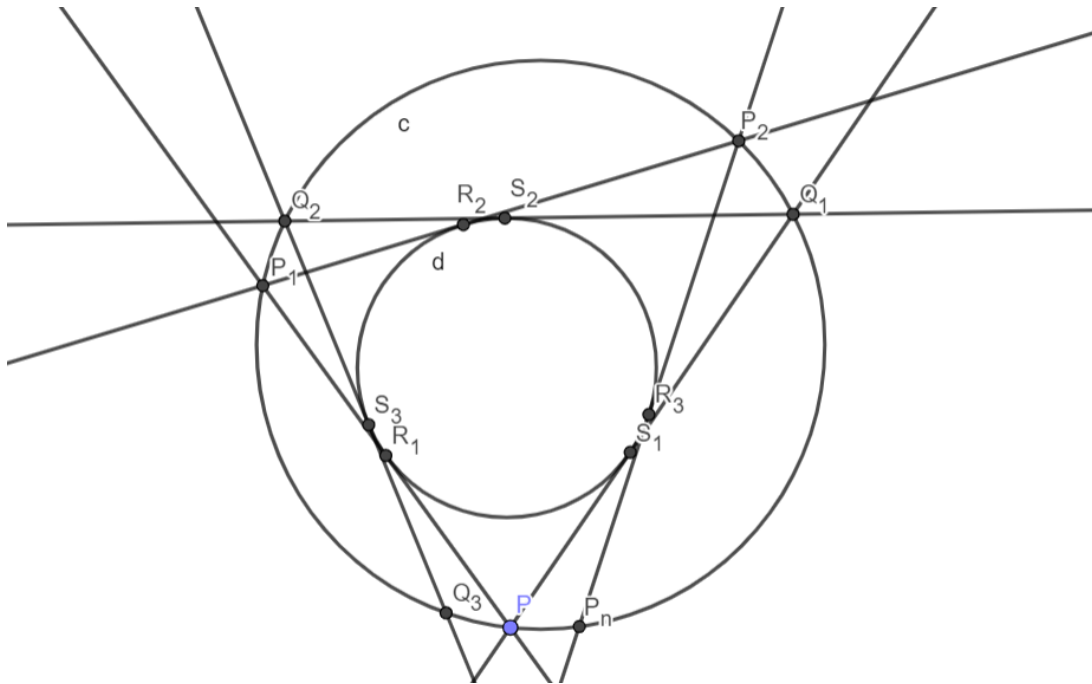
$$\deg(A_1B_1, A_2B_2) = \frac{\deg(A_1, A_2) + \deg(B_1, B_2)}{2}$$

10.29. Theorem. (Inverse of Sqrt Mászó-theorem) Let $\mathcal{C}$ be a fixed conic, (A_1, A_2) a square root point pair on it, and (l_1, l_2) a square root line pair such that l_1 and l_2 always goes through A_1 and A_2 respectively. The second intersection of $\mathcal{C}$ and l_1 and l_2 is B_1 and B_2 respectively. Then (B_1, B_2) form a square root point pair, and its degree satisfies

$$\deg(l_1, l_2) = \frac{\deg(A_1, A_2) + \deg(B_1, B_2)}{2}$$

11 Example problem for square root animation

11.1. Exercise. (Poncelet-porism) Let C and D be two conics, such that there exists a polygon with n sides inscribed in C and circumscribed around D . Prove that for every point P of C there exists such n -gon having P as a vertex.



Proof. Define distinct points $P_0, \dots, P_n \in C$ such that $P_{i-1}P_i$ is tangent to D for every $i = 1, \dots, n$, $P_0 = P$, and P_0P_1 is at the 'left' side of D .

The statement is equivalent to $P_0 = P_n$

We can similarly define $Q_0, \dots, Q_n$, but Q_0Q_1 is at the 'right' side of D .

Let $R_1, \dots, R_n$ be the tangency points of $P_0P_1, \dots, P_{n-1}P_n$ on D . Similarly we define S_i to be the tangency point of $Q_{i-1}Q_i$ on D .

Now we fix C , D , and animate P so that it is a degree 2 polynomial moving point on C .

R_1S_1 is the polar of P to D , so it is also of degree 2.

A polynomial moving line of degree 2 intersects D at a square root point pair of degree 2.

The polar of (R_1, S_1) is (P_0P_1, Q_0Q_1) , so it is also degree 2.

By the inverse of Sqrt Mászó-tétel $\frac{\deg(P_1, Q_1)+2}{2} = 2$, so $\deg(P_1, Q_1) = 2$.

And so on, we iterate sqrt pole-polar theorems and sqrt inverse Mászó-theorems to get that every square root point/line pair on the figure is of degree 2.

Now (P_n, Q_n) is of degree 2, and P is on degree 2.

So the degree of the statement " $P = P_n$ or $P = Q_n$ " is $\frac{2+2}{2} = 2$.

This means that " $P = P_n$ or $P = Q_n$ " happens more than 4 times, it is always true, and when they are both true, that counts as 2.

Luckily $P = P_n \iff P = Q_n$. So it is sufficient to find 3 cases when both statements are true.

The condition of the porism states that there exists a good n -gon, so when P is at the vertices of the n -gon we instantly have n cases when both $P = P_n$ and $P = Q_n$.

$n \geq 3$, so we are finished. □

References

- [1] Matt Kerr. *Lecture Notes Algebraic Geometry III/IV*.
- [2] Anant Mudgal Zack Chroman, Gopal K. Goel. The method of animation. 2019.
- [3] Vladyslav Zveryk. The method of moving points. 2019.