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# **UNCERTAINTY PRINCIPLES IN HARMONIC ANALYSIS**

Research outcome for Scientific Students' Associations Conference

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Budapest, 2024.

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# Chapter 1

## Introduction

The aim of this essay is to discuss an uncertainty principle in harmonic analysis, more precisely an estimation for quaternion valued functions of one variable. Based on the idea of Stein and Shakarchi (cf. [1]) we derive an uncertainty-principle-like estimation.

Our essay is organized as follows: In the next chapter, we give a brief introduction to quaternions. Then we introduce the Schwartz space of quaternion-valued functions giving some important properties, then present a one-dimensional quaternion-valued Fourier-transform, and give some of its' properties. In the last chapter we state and prove our main result: an uncertainty principle concerning this transform.

Budapest, 1 November, 2024

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# Chapter 2

## Introduction to quaternions

Usually the word *quaternions* is attributed to W. R. Hamilton, who introduced their notion in 1843. They can be thought of as an extension of complex numbers. If  $\mathfrak{i} \in \mathbb{C}$  denotes the complex imaginary unit then for the matrices

$$E := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I := \begin{bmatrix} \mathfrak{i} & 0 \\ 0 & -\mathfrak{i} \end{bmatrix}, \quad J := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad K := \begin{bmatrix} 0 & \mathfrak{i} \\ \mathfrak{i} & 0 \end{bmatrix}$$

the following hold (cf. [9]):

$$I^2 = J^2 = K^2 = -E \quad \text{and} \quad IJ = -JI = K, \quad JK = -KJ = I, \quad KI = -IK = J. \quad (2.0.1)$$

It is clear that the set

$$\{\pm E, \pm I, \pm J, \pm K\}$$

is closed with respect to matrix multiplication. The above multiplication properties of enables us to define a noncommutative field with the unit element  $E$  and null element  $\Theta$  where  $\Theta \in \mathbb{R}^{2 \times 2}$  denotes zero matrix.

**Definition.** The set

$$\mathbb{H} := \{aE + bI + cJ + dK \in \mathbb{R}^{2 \times 2} : a, b, c, d \in \mathbb{R}\}$$

is referenced to as quaternions.

The conjugate and the modulus of a quaternion  $Q = aE + bI + cJ + dK \in \mathbb{H}$  is defined by

$$\overline{Q} := Q^* \quad \text{and} \quad |Q| := \sqrt{a^2 + b^2 + c^2 + d^2}.$$

Clearly,

$$QQ^* = |Q|^2 E$$

holds. For sake of brevity we rewrite the above definition in form of

$$\mathbb{H} := \{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} : a, b, c, d \in \mathbb{R}\} \quad (2.0.2)$$

where

$$1 := E, \quad \mathbf{i} := I, \quad \mathbf{j} := J, \quad \mathbf{k} := K.$$

Thus, the length and the conjugate of a quaternion

$$q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H} \quad (2.0.3)$$

are defined by

$$|q| := \sqrt{a^2 + b^2 + c^2 + d^2} \quad \text{and} \quad \bar{q} := a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$$

and clearly

$$q\bar{q} = |q|^2$$

holds. From this, we can also derive the inverse of any quaternion  $0 \neq q \in \mathbb{H}$  as

$$q^{-1} := \frac{\bar{q}}{|q|},$$

and this inverse fulfills the following:

$$q^{-1}q = qq^{-1} = 1 \quad (q \in \mathbb{H}).$$

With help of the real and imaginary part of a quaternion  $q \in \mathbb{H}$

$$\Re(q) := a, \quad \Im(q) := b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$$

we can calculate its conjugate:

$$\bar{q} = \Re(q) - \Im(q).$$

Since quaternions are defined by linear combination of matrices, their exponential is the same as the matrix exponential, which is defined for a square matrix  $M$  as

$$e^M := \sum_{n=0}^{\infty} \frac{M^n}{n!}.$$

Thus, we can calculate the exponential of a matrix  $\xi M$  where  $\xi \in \mathbb{R}$  and  $M \in \{E, I, J, K\}$  as follows (cf. [6], [1]):

- $E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  is a scalar matrix, thus we have

$$e^{\xi E} = e^{\xi} E = \begin{bmatrix} e^{\xi} & 0 \\ 0 & e^{\xi} \end{bmatrix};$$

- $I = \imath \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  is a diagonal matrix which has the consequence that

$$e^{\xi I} = \begin{bmatrix} \cosh(\imath) + \xi \sinh(\imath) & 0 \\ 0 & \cosh(\imath) - \xi \sinh(\imath) \end{bmatrix}$$

holds;

- $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$  has two complex conjugate eigenvalues:  $\mathfrak{i}$ ,  $-\mathfrak{i}$  from which

$$e^{\xi J} = \cos(\xi)E + \sin(\xi)J = \begin{bmatrix} \cos(\xi) & \sin(\xi) \\ -\sin(\xi) & \cos(\xi) \end{bmatrix}$$

follows;

- for  $K = \begin{bmatrix} a & b \\ c & d \end{bmatrix} := \begin{bmatrix} 0 & \mathfrak{i} \\ \mathfrak{i} & 0 \end{bmatrix}$  we have  $(a - d)^2 + 4bc = 4\mathfrak{i}^2 = -4 \neq 0$ , thus

$$e^{\xi K} = e^{\xi} \begin{bmatrix} \cosh(\mathfrak{i}\xi) & -\mathfrak{i} \sinh(\mathfrak{i}\xi) \\ -\mathfrak{i} \sinh(\mathfrak{i}\xi) & \cosh(\mathfrak{i}\xi) \end{bmatrix}$$

and for a quaternion (2.0.3) with  $v := \sqrt{b^2 + c^2 + d^2} \neq 0$  we have (cf. [8])

$$e^{\xi q} = e^{a\xi} \left( \cos(\sqrt{3}|\xi|) + \frac{\sin(\sqrt{3}|\xi|)}{\sqrt{3}|\xi|} \cdot (b\xi\mathbf{i} + c\xi\mathbf{j} + d\xi\mathbf{k}) \right) \quad (\xi \neq 0).$$

Direct calculation shows that

$$\exp(I + J) = \cos(\sqrt{2})E + \frac{\sin(\sqrt{2})}{2}(I + J),$$

while

$$\exp(I) \cdot \exp(J) = (\cos(1)E + \sin(1)I)(\cos(1)E + \sin(1)J) = \cos^2(1)E + \frac{\sin(2)}{2}(I + J) + \sin^2(1)K$$

which means that

$$\exp(I + J) \neq \exp(I) \cdot \exp(J).$$

This is because  $I$  and  $J$  don't commute:  $[I, J] \neq O$  where

$$[A, B] := AB - BA \quad (A, B \in \mathbb{K}^{d \times d})$$

denotes the so called Lie bracket (cf. [?]).

# Chapter 3

## The quaternionic Schwartz space

In what follows we consider measurable quaternion-valued functions  $f : \mathbb{R} \longrightarrow \mathbb{H}$ . According to (2.0.2) every such a function can be expressed as

$$f(x) = f_1(x) + f_2(x)\mathbf{i} + f_3(x)\mathbf{j} + f_4(x)\mathbf{k} \quad (x \in \mathbb{R}) \quad (3.0.1)$$

where

$$f_n : \mathbb{R} \rightarrow \mathbb{R} \quad (n \in \{1,2,3,4\})$$

are measurable functions. Introducing the quaternion-valued Schwartz space

$$\mathcal{S}(\mathbb{R}, \mathbb{H}) := \left\{ f \in \mathcal{D}^\infty(\mathbb{R}, \mathbb{H}) : \|f\|_{k,l} := \sup_{x \in \mathbb{R}} |x^k f^{(l)}(x)| < \infty \quad (k, l \in \mathbb{N}_0) \right\}$$

and the quaternion-valued  $L^p$  space

$$L^p(\mathbb{R}, \mathbb{H}) := \left\{ f : \mathbb{R} \rightarrow \mathbb{H} : \|f\|_p < \infty \right\}$$

where

$$\|f\|_p := \begin{cases} \|f\|_p := \left( \int_{\mathbb{R}} |f(x)|^p d\lambda_1(x) \right)^{1/p} & (p \in [1, +\infty)), \\ \|f\|_\infty := \text{ess sup}(|f|) & (p = \infty) \end{cases}$$

whereupon  $d\lambda_1(x)$  represents the Lebesgue measure in  $\mathbb{R}$ ;  $L^\infty(\mathbb{R}, \mathbb{H})$  denotes the collection of essentially bounded functions. We can prove the following statement.

**Proposition.** If  $p \in [1, \infty]$  then  $\mathcal{S}(\mathbb{R}, \mathbb{H}) \subset L^p(\mathbb{R}, \mathbb{H})$  holds.

**Proof.**

**Step 1.** Assume that  $f \in \mathcal{S}(\mathbb{R}, \mathbb{H})$ . The every  $k, l \in \mathbb{N}_0$  we have

$$\sup_{x \in \mathbb{R}} |x^k f^{(l)}(x)| < \infty,$$

in particular ( $k = 0 = l$ )

$$\sup_{x \in \mathbb{R}} |f(x)| < \infty$$

which means that the essential bound of  $f$  is finite:

$$\text{ess sup}(|f|) < +\infty, \quad \text{i.e.} \quad f \in L^\infty(\mathbb{R}, \mathbb{H}).$$

**Step 2.** Using an idea in [10] we can state that if  $f \in \mathcal{S}(\mathbb{R}, \mathbb{H})$  then  $f$  belongs to  $L^p(\mathbb{R}, \mathbb{H})$ , as well. Indeed,

- $f \in \mathcal{S}(\mathbb{R}, \mathbb{H})$  implies that  $f$  is continuous (and hence measurable) and according to the definition of  $\mathcal{S}(\mathbb{R}, \mathbb{H})$  for sufficient  $C > 0$  the estimation

$$|f(x)| \leq \frac{C}{(1 + |x|)^2} \quad (x \in \mathbb{R})$$

holds, which means that  $f$  has an integrable majorant and therefore  $f \in L^1(\mathbb{R}, \mathbb{H})$ .

- if  $f \in \mathcal{S}(\mathbb{R}, \mathbb{H})$  and  $p \in (1, +\infty)$  then with constants

$$\alpha := \|f\|_\infty \quad \text{and} \quad \beta := \sup \{x^2 \cdot |f(x)| \in \mathbb{R} : x \in \mathbb{R}\}$$

we have

$$\begin{aligned} \|f\|_p^p &= \int_{[-1,1]} |f(x)|^p d\lambda_1(x) + \int_{\mathbb{R} \setminus [-1,1]} |f(x)|^p d\lambda_1(x) \\ &\leq \alpha^p \cdot 2 + \beta^p \cdot \int_{\mathbb{R} \setminus [-1,1]} \frac{1}{|x|^{2p}} d\lambda_1(x) < +\infty. \quad \blacksquare \end{aligned}$$

Now, we can define the one-dimensional quaternion-valued Fourier transform.

**Definition.** The one-dimensional quaternion Fourier transform of a function  $f \in \mathcal{S}(\mathbb{R}, \mathbb{H})$  is defined by the formula

$$\widehat{f}(\xi) := \int_{\mathbb{R}} f(x) \exp(\mathbf{i}\xi x) d\lambda_1(x) \quad (\xi \in \mathbb{R}). \quad (3.0.2)$$

**Remarks.**

1. This definition has the advantage over [4] that we do not pay attention to which Lebesgue space of the index we are working in.
2. Note that even if we replace  $\mathbf{i}$  by  $\mathbf{j}$  or  $\mathbf{k}$  in the above definition, we wouldn't get a significant result, because

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{i} \cdot \mathbf{i} \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix}^2 = - \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix} = -E,$$

$$\mathbf{j} \cdot \mathbf{j} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -E,$$

$$\mathbf{k} \cdot \mathbf{k} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -E,$$

(compare the first formula in (2.0.1)).

3. Using the form (3.0.1) and the additivity of the integral we can write for a function  $f \in \mathcal{S}(\mathbb{R}, \mathbb{H})$  the Fourier transform in the form of

$$\widehat{f}(\xi) = \widehat{f}_1(\xi) + \widehat{f}_2(\xi)\mathbf{i} + \widehat{f}_3(\xi)\mathbf{j} + \widehat{f}_4(\xi)\mathbf{k} \quad (\xi \in \mathbb{R}) \quad (3.0.3)$$

where

$$\widehat{f}_n(\xi) := \int_{\mathbb{R}} f_n(x) e^{i\xi x} d\lambda_1(x) \quad (n \in \{1, 2, 3, 4\}, \xi \in \mathbb{R}).$$

The following theorem states that the Fourier transform interchanges differentiation and multiplication.

**Theorem.** For  $f \in \mathcal{S}(\mathbb{R}, \mathbb{H})$  we have

$$\widehat{f}'(\xi) = -i\xi \widehat{f}(\xi). \quad (3.0.4)$$

**Proof.** From  $f \in \mathcal{S}(\mathbb{R}, \mathbb{H})$  it follows that  $\lim_{\pm\infty} f = 0$ . Thus, integrating by parts (cf. [4]) it is easy to see that

$$\begin{aligned} \int_{\mathbb{R}} f'(x) \exp(i\xi x) d\lambda_1(x) &= [f(x) \exp(i\xi x)]_{-\infty}^{+\infty} - \int_{\mathbb{R}} f(x) \frac{d}{dx} \exp(i\xi x) d\lambda_1(x) \\ &= 0 - i\xi \int_{\mathbb{R}} f(x) \exp(i\xi x) d\lambda_1(x) = -i\xi \widehat{f}(\xi). \quad \blacksquare \end{aligned}$$

Next, we show that derivative of the the Fourier transform is a scalar multiple of the Fourier transform of the position operator (called in the quantum mechanics).

**Theorem.** For  $f \in \mathcal{S}(\mathbb{R}, \mathbb{H})$  we have

$$(\widehat{f})'(\xi) = i\widehat{f}_*(\xi) \quad \text{where} \quad f_*(x) := xf(x) \quad (x \in \mathbb{R}). \quad (3.0.5)$$

**Proof.** Integrating by parts we have

$$\begin{aligned} (\widehat{f})'(\xi) &= \frac{d}{d\xi} \int_{\mathbb{R}} f(x) \exp(i\xi x) d\lambda_1(x) = \int_{\mathbb{R}} f(x) ix \exp(i\xi x) d\lambda_1(x) \\ &= i \int_{\mathbb{R}} xf(x) \exp(i\xi x) d\lambda_1(x) = i \int_{\mathbb{R}} f_*(x) \exp(i\xi x) d\lambda_1(x) \\ &= i\widehat{f}_*(\xi). \quad \blacksquare \end{aligned}$$

The last two results enable us to show that the Gaussian

$$f(x) := c \cdot \exp(-\pi x^2) \quad (c \in \mathbb{H}, x \in \mathbb{R})$$

is a fixed point of the quaternioan Fourier transform (3.0.2). Indeed,  $F(0) = c$  and for the derivative of the Function

$$F(\xi) := \widehat{f}(\xi) \quad (\xi \in \mathbb{R})$$

we have (cf. (3.0.2))

$$\begin{aligned} F'(\xi) &= \int_{\mathbb{R}} \frac{d}{d\xi} (c \cdot \exp(-\pi x^2) \cdot \exp(i\xi x)) \, d\lambda_1(x) = \int_{\mathbb{R}} c \cdot \exp(-\pi x^2) (ix) \exp(i\xi x) \, d\lambda_1(x) \\ &= -\frac{i}{2\pi} \cdot \int_{\mathbb{R}} c \cdot (-2\pi x) \exp(-\pi x^2) \exp(i\xi x) \, d\lambda_1(x) = -\frac{i}{2\pi} \cdot \int_{\mathbb{R}} f'(x) \exp(i\xi x) \, d\lambda_1(x) \\ &= -\frac{i^2}{2\pi} \widehat{f'}(\xi) = \frac{\xi}{2\pi} \cdot F(\xi). \end{aligned}$$

This means that  $F$  fulfills the initial value problem

$$F(0) = c, \quad F'(\xi) = \frac{\xi}{2\pi} \cdot F(\xi) \quad (\xi \in \mathbb{R}),$$

whose solution is clearly the function

$$F(\xi) = c \exp\left(-\frac{\xi^2}{4\pi}\right) \quad (\xi \in \mathbb{R}, c \in \mathbb{H}).$$

**Theorem.** The dilation formula has in case of (3.0.2) the form

$$\widehat{F}(\xi) = \widehat{f}(\xi) e^{-i\xi y} \quad (\xi, y \in \mathbb{R})$$

where  $f \in \mathcal{S}(\mathbb{R}, \mathbb{H})$  and for given  $x \in \mathbb{R}$  the function  $F$  is defined by

$$F(y) := f(x + y) \quad (y \in \mathbb{R}).$$

**Proof.** By substitution method we have

$$\begin{aligned} \widehat{F}(\xi) &= \int_{\mathbb{R}} F(y) \exp(i\xi y) \, d\lambda_1(y) = \int_{\mathbb{R}} f(x + y) \exp(i\xi y) \, d\lambda_1(y) \\ &= \int_{\mathbb{R}} f(u) \exp(i\xi(u - x)) \, d\lambda_1(u) = \int_{\mathbb{R}} f(u) \exp(i\xi u) \cdot \exp(-i\xi x) \, d\lambda_1(u) \\ &= \exp(-i\xi x) \int_{\mathbb{R}} f(u) \exp(i\xi u) \, d\lambda_1(u) = \widehat{f}(\xi) \exp(-i\xi x). \quad \blacksquare \end{aligned}$$

Similarly as in [5] we can prove that for functions  $f, g \in \mathcal{S}(\mathbb{R}, \mathbb{H})$  the

$$\langle f, g \rangle = \frac{1}{2\pi} \langle \widehat{f}, \widehat{g} \rangle$$

holds where

$$\langle f, g \rangle := \int_{\mathbb{R}} f(x) \overline{g(x)} d\lambda_1(x).$$

From this follows the famous Plancherel's formula

$$\int_{\mathbb{R}} |f(x)|^2 d\lambda_1(x) = \frac{1}{2\pi} \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 d\lambda_1(\xi). \quad (3.0.6)$$

Similarly as in the simple case we can formulate and prove the Fourier inversion formula (cf. [11]).

**Theorem (Multiplication formula).** If  $f, g \in \mathcal{S}(\mathbb{R}, \mathbb{H})$  then

$$\int_{\mathbb{R}} f(x) \widehat{g}(x) d\lambda_1(x) = \int_{\mathbb{R}} \widehat{f}(x) g(x) d\lambda_1(x) \quad (3.0.7)$$

**Proof.** Using the Fubini's theorem (cf. [3]) we have for functions  $f, g \in \mathcal{S}(\mathbb{R}, \mathbb{H})$ :

$$\begin{aligned} \int_{\mathbb{R}} f(x) \widehat{g}(x) d\lambda_1(x) &= \int_{\mathbb{R}} f(x) \left( \int_{\mathbb{R}} g(y) \exp(\mathbf{i}xy) d\lambda_1(y) \right) d\lambda_1(x) \\ &= \int_{\mathbb{R}} g(y) \left( \int_{\mathbb{R}} f(x) \exp(\mathbf{i}xy) d\lambda_1(x) \right) d\lambda_1(y) \\ &= \int_{\mathbb{R}} g(y) \widehat{f}(y) d\lambda_1(y). \quad \blacksquare \end{aligned}$$

**Theorem (Fourier inversion).** If  $f \in \mathcal{S}(\mathbb{R}, \mathbb{H})$  then

$$f(x) = \mathcal{F}^{-1}(\widehat{f})(x) := \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\xi) e^{-\mathbf{i}\xi x} d\xi \quad (x \in \mathbb{R}). \quad (3.0.8)$$

**Proof.**

**Step 1.** By the above remark resp. by the representation (3.0.3) we can restrict ourself to the case when  $f$  is a function in  $\mathcal{S}(\mathbb{R}, \mathbb{R})$ . We show that

$$f(0) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\xi) d\lambda_1(\xi).$$

Indeed, if we for a given  $\delta > 0$  define

$$G_\delta(x) := \frac{1}{2\pi} e^{-\pi\delta x^2} \quad \text{and} \quad K_\delta(x) := \frac{1}{\sqrt{\delta}} e^{-\pi x^2/\delta} \quad (x \in \mathbb{R})$$

then (cf. [2])  $\widehat{G_\delta} = K_\delta$  and by multiplication formula we get

$$\int_{\mathbb{R}} f(x) K_\delta(x) d\lambda_1(x) = \int_{\mathbb{R}} \widehat{f}(\xi) G_\delta(\xi) d\lambda_1(\xi).$$

It can be shown (cf. [11], [2]) that

$$\lim_{\delta \rightarrow 0} \int_{\mathbb{R}} f(x) K_\delta(x) d\lambda_1(x) = f(0) \quad \text{and} \quad \lim_{\delta \rightarrow 0} G_\delta = \frac{1}{2\pi}.$$

**Step 2.** For arbitrary given  $x \in \mathbb{R}$  let

$$F(y) := f(x + y) \quad (y \in \mathbb{R}),$$

then

$$f(x) = F(0) = \frac{1}{2\pi} \int_{\mathbb{R}} F(\xi) d\lambda_1(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\xi) e^{-i\xi x} d\lambda_1(\xi). \quad \blacksquare$$

**Theorem (Donoho-Stark).** Let  $f \in \mathcal{S}(\mathbb{R}, \mathbb{H})$ ,  $\|f\|_2 = 1$  and  $\varepsilon_1, \varepsilon_2$  such nonnegative numbers that  $\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2} < 1$ . If sets  $T, \Omega \subset \mathbb{R}$  are chosen such that

$$\int_T |f(x)|^2 d\lambda_1(x) \geq 1 - \varepsilon_1 \quad \text{and} \quad \int_\Omega |\widehat{f}(\xi)|^2 d\lambda_1(\xi) \geq 1 - \varepsilon_2,$$

then the following holds:

$$\lambda_1(T) \cdot \lambda_1(\Omega) \geq (1 - \sqrt{\varepsilon_1} - \sqrt{\varepsilon_2})^2.$$

**Proof.** (Compare [12]). If one of  $T$  or  $\Omega$  is unbounded then the proof is trivial. Otherwise, let us define the following orthogonal projections:

$$P_T f := \chi_T f \quad (f \in \mathcal{S}(\mathbb{R}, \mathbb{H}))$$

and

$$Q_\Omega f(x) := \mathcal{F}^{-1} \left( \chi_\Omega \widehat{f} \right) (x) = \int_\Omega \widehat{f}(\xi) e^{ix\xi} d\lambda_1(\xi) \quad (f \in \mathcal{S}(\mathbb{R}, \mathbb{H}), x \in \mathbb{R}).$$

From Plancherel's formula (3.0.6) it follows that

$$\|Q_\Omega f\|_2 = \|\chi_\Omega \widehat{f}\|_2 \leq \|\widehat{f}\|_2 = \frac{1}{\sqrt{2\pi}} \|f\|_2, \quad (3.0.9)$$

and from the conditions imposed on  $f$  we also have

$$\|f - P_T f\|_2^2 = \int_{T^c} |f(x)|^2 d\lambda_1(x) = \int_{\mathbb{R}} |f(x)|^2 d\lambda_1(x) - \int_T |f(x)|^2 d\lambda_1(x) \leq \varepsilon_1.$$

We can obtain a similar bound for  $Q$ :

$$\|f - Q_\Omega f\|_2^2 = \|\widehat{f} - \widehat{Q_\Omega f}\|_2^2 = \|\widehat{f} - \chi_\Omega \widehat{f}\|_2^2 = \|\chi_{\Omega^c} \widehat{f}\|_2^2 \leq \frac{1}{\sqrt{2\pi}} \cdot \varepsilon_2 \leq \varepsilon_2.$$

Using (3.0.9) and a triangle-inequality we can obtain the following estimation:

$$\|f - Q_\Omega P_T f\|_2 \leq \|f - Q_\Omega f\|_2 + \|Q_\Omega(f - P_T f)\|_2 \leq \sqrt{\varepsilon_2} + \|f - P_T f\|_2 \leq \sqrt{\varepsilon_2} + \sqrt{\varepsilon_1},$$

and with another triangle inequality we can obtain the following inequality:

$$\|Q_\Omega P_T f\|_2 \geq \|f\|_2 - \|f - Q_\Omega P_T f\|_2 \geq 1 - \sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}. \quad (3.0.10)$$

From the definition of the projection operators we get for every  $x \in \mathbb{R}$

$$\begin{aligned} Q_\Omega P_T f(x) &= \mathcal{F}^{-1} \left( \chi_\Omega \left( \widehat{P_T f} \right) \right) (x) = \frac{1}{2\pi} \int_\Omega \left( \int_T f(t) e^{it\xi} d\lambda_1(t) \right) e^{-ix\xi} d\lambda_1(\xi) \\ &= \frac{1}{2\pi} \int_T \int_\Omega f(t) e^{i\xi(x-t)} d\lambda_1(\xi) d\lambda_1(t), \end{aligned}$$

where the order of integration can be swapped, since  $T$  and  $\Omega$  are bounded sets, and so

$$\int_\Omega \left( \int_T |f(t)| d\lambda_1(t) \right) d\lambda_1(\xi) \leq \lambda_1(\Omega) \left( \int_\mathbb{R} |f(t)|^2 d\lambda_1(t) \right)^{\frac{1}{2}} \lambda_1(T)^{\frac{1}{2}} < \infty.$$

Hence, by defining

$$K(x, t) := \frac{1}{2\pi} \chi_T(t) \int_\Omega e^{i\xi(x-t)} d\lambda_1(\xi) = \frac{1}{2\pi} \chi_T(t) T_t \left( \mathcal{F}^{-1} \chi_\Omega \right) (x) \quad ((x, t) \in \mathbb{R}^2)$$

we can write

$$Q_\Omega P_T f(x) = \int_\mathbb{R} K(x, t) f(t) dt \quad (f \in \mathcal{S}(\mathbb{R}, \mathbb{H}), x \in \mathbb{R}).$$

Using Hölder's inequality we can express the norm of  $Q_\Omega P_T f$  as

$$\begin{aligned} \|Q_\Omega P_T f\|_2^2 &= \int_\mathbb{R} \left| \int_\mathbb{R} K(x, t) f(t) d\lambda_1(t) \right|^2 d\lambda_1(x) \leq \int_\mathbb{R} \int_\mathbb{R} \|K(x, t)\|_2^2 d\lambda_1(t) \int_\mathbb{R} |f(t)|^2 d\lambda_1(t) d\lambda_1(x) \\ &= \|f\|_2^2 \cdot \int_\mathbb{R} \int_\mathbb{R} |K(x, t)|^2 d\lambda_1(t) d\lambda_1(x) \quad (f \in \mathcal{S}(\mathbb{R}, \mathbb{H})). \end{aligned}$$

Consider  $K$ 's norm:

$$\begin{aligned} \int_\mathbb{R} |K(x, t)|^2 d\lambda_1(x) &= \frac{1}{2\pi} \chi_T(t) \cdot \|T_t (\mathcal{F}^{-1} \chi_\Omega)\|_2^2 = \frac{1}{2\pi} \chi_T(t) \cdot \|\mathcal{F}^{-1} \chi_\Omega\|_2^2 \\ &= \frac{1}{2\pi} \chi_T(t) \cdot 2\pi \cdot \|\chi_\Omega\|_2^2 = \lambda_1(\Omega) \cdot \chi_T(t) \quad (t \in \mathbb{R}), \end{aligned}$$

and hence we have

$$\int_\mathbb{R} \int_\mathbb{R} |K(x, t)|^2 d\lambda_1(x) d\lambda_1(t) = \lambda_1(\Omega) \cdot \lambda_1(T).$$

From this and (3.0.10) we obtain the desired result:

$$1 - \sqrt{\varepsilon_1} - \sqrt{\varepsilon_2} \leq \|Q_\Omega P_T f\|_2 \leq \int_\mathbb{R} \int_\mathbb{R} |K(x, t)|^2 d\lambda_1(x) d\lambda_1(t) \leq \sqrt{\lambda_1(T) \cdot \lambda_1(\Omega)}. \quad \blacksquare$$

# Chapter 4

## Uncertainty principle for the one-dimensional quaternionic Fourier-transform

Introducing the operators (cf. [11])

$$\begin{aligned} H(f) &:= -f'' + (f_*)_* \quad (f \in \mathcal{S}(\mathbb{R}, \mathbb{H})), \\ A_t(f) &:= f' + tf_* \quad (f \in \mathcal{S}(\mathbb{R}, \mathbb{H}), t \in \mathbb{R}), \\ C_t(f) &:= -f' + tf_* \quad (f \in \mathcal{S}(\mathbb{R}, \mathbb{H}), t \in \mathbb{R}) \end{aligned}$$

we can prove the following statements using an idea and some elaboration of it in the classical case ([11], [7]).

**Proposition.** For any number  $t \in \mathbb{R}$  and functions  $f, g \in \mathcal{S}(\mathbb{R}, \mathbb{H})$  we have

1.  $\langle A_t f, g \rangle = \langle f, A_t^* g \rangle$ , that is  $A_t$  and  $C_t$  are Hermitian adjoints;
2.  $\langle A_t^* A_t f, f \rangle \geq 0$ ;
3.  $A_1 C_1 = H - I$ .

**Proof.**

**Step 1.** Using the method of integration by parts, the left hand side  $\langle A_t f, g \rangle$  is equal to

$$\begin{aligned} &= \int_{\mathbb{R}} (f'(x) + txf(x)) \overline{g(x)} d\lambda_1(x) = \int_{\mathbb{R}} f'(x) \overline{g(x)} d\lambda_1(x) + \int_{\mathbb{R}} txf(x) \overline{g(x)} d\lambda_1(x) \\ &= 0 - \int_{\mathbb{R}} f(x) \overline{g'(x)} d\lambda_1(x) + \int_{\mathbb{R}} txf(x) \overline{g(x)} d\lambda_1(x), \end{aligned}$$

and we can rewrite the right hand side  $\langle f, A_t^* g \rangle$  in the form

$$\begin{aligned} &= \int_{\mathbb{R}} f(x) \overline{(-g'(x) + txg(x))} d\lambda_1(x) = \int_{\mathbb{R}} -f(x) \overline{g'(x)} d\lambda_1(x) + \int_{\mathbb{R}} f(x) \overline{txg(x)} d\lambda_1(x) \\ &= - \int_{\mathbb{R}} f(x) \overline{g'(x)} d\lambda_1(x) + \int_{\mathbb{R}} tx f(x) \overline{g(x)} d\lambda_1(x). \end{aligned}$$

This means that

$$\langle A_t f, g \rangle = - \int_{\mathbb{R}} f(x) \overline{g'(x)} d\lambda_1(x) + \int_{\mathbb{R}} tx f(x) \overline{g(x)} d\lambda_1(x) = \langle f, A_t^* g \rangle.$$

**Step 2.** By using the adjoint property of the two operators and the nonnegativity of the norm, we get

$$\langle A_t^* A_t f, f \rangle = \langle A_t f, A_t f \rangle = \|A_t f\|_{L_2}^2 \geq 0.$$

**Step 3.** Using the method of integration by parts again, we have for any  $f \in \mathcal{S}(\mathbb{R}, \mathbb{H})$

$$\begin{aligned} C_1 A_1(f) &= -(f' + f_*)' + (f' + f_*)_* = -f'' - (f_*)' + f'_* + (f_*)_* \\ &= -f'' - f - f'_* + f'_* + (f_*)_* = -f'' - f + (f_*)_* = \\ &= -f'' + (f_*)_* - f = H(f) - I(f). \end{aligned}$$

The operator  $H$  called Hermite operator (is a quantum analogues of the harmonic oscillator), furthermore  $A_1$  and  $C_1$  are known in the quantum mechanics as annihilation operator and creation operator, respectively.

Now, we can use the method outlined in Problem 23 formulated in [11].

**Proposition.** Suppose that the function  $f \in \mathcal{S}(\mathbb{R}, \mathbb{H})$  is normed, that is  $\|f\|^2 := \langle f, f \rangle = 1$  holds. Then

$$\left( \int_{\mathbb{R}} x^2 |f(x)|^2 d\lambda_1(x) \right) \cdot \left( \int_{\mathbb{R}} |f'(x)|^2 d\lambda_1(x) \right) \geq \frac{1}{4}.$$

**Proof.** We start by substituting for and rewriting the inequality from the previous proposition.

$$0 \leq \langle A_t^* A_t f, f \rangle = \int_{\mathbb{R}} t^2 x^2 f(x) \overline{f(x)} d\lambda_1(x) - \int_{\mathbb{R}} t f(x) \overline{f'(x)} d\lambda_1(x) - \int_{\mathbb{R}} f''(x) \overline{f(x)} d\lambda_1(x).$$

We know that for any quaternion multiplied by its' conjugate is its' length squared and therefore

$$\begin{aligned}
0 &\leq \langle A_t^* A_t f; f \rangle \\
&= \int_{\mathbb{R}} t^2 x^2 f(x) \overline{f(x)} d\lambda_1(x) - \int_{\mathbb{R}} t f(x) \overline{f(x)} d\lambda_1(x) - \int_{\mathbb{R}} f''(x) \overline{f(x)} d\lambda_1(x) \\
&= t^2 \cdot \int_{\mathbb{R}} x^2 f(x) \overline{f(x)} d\lambda_1(x) - t \cdot \langle f, f \rangle - \int_{\mathbb{R}} f''(x) \overline{f(x)} d\lambda_1(x) \\
&= t^2 \cdot \int_{\mathbb{R}} x^2 f(x) \overline{f(x)} d\lambda_1(x) - t \cdot 1 - \int_{\mathbb{R}} f''(x) \overline{f(x)} d\lambda_1(x)
\end{aligned}$$

and we have obtained a quadratic polynomial in  $t$ , whose principal coefficient is nonnegative (since  $x^2 |f(x)|^2$  is nonnegative). This can only be nonnegative at every point if its' determinant is nonpositive, that is,

$$1 - 4 \left( \int_{\mathbb{R}} x^2 |f(x)|^2 d\lambda_1(x) \right) \cdot \left( \int_{\mathbb{R}} |f'(x)|^2 d\lambda_1(x) \right) \leq 0,$$

and by rewriting this we obtain the inequality stated in the theorem:

$$\left( \int_{\mathbb{R}} x^2 |f(x)|^2 d\lambda_1(x) \right) \cdot \left( \int_{\mathbb{R}} |f'(x)|^2 d\lambda_1(x) \right) \geq \frac{1}{4}. \quad \blacksquare$$

**Remark.** Using Plancherel's formula (3.0.6) and the identity that connects a function's the quaternionic Fourier transform to its' derivative, we can write the above inequality as

$$\begin{aligned}
\frac{1}{4} &\leq \left( \int_{\mathbb{R}} x^2 |f(x)|^2 d\lambda_1(x) \right) \cdot \left( \int_{\mathbb{R}} |f'(x)|^2 d\lambda_1(x) \right) \\
&= \left( \int_{\mathbb{R}} x^2 |f(x)|^2 d\lambda_1(x) \right) \cdot \left( \frac{1}{2\pi} \int_{\mathbb{R}} |\widehat{f}'(\xi)|^2 d\lambda_1(\xi) \right) = \\
&= \left( \int_{\mathbb{R}} x^2 |f(x)|^2 d\lambda_1(x) \right) \cdot \left( \frac{1}{2\pi} \int_{\mathbb{R}} |\widehat{f}(\xi) \cdot \xi \cdot (-\mathbf{i})|^2 d\lambda_1(\xi) \right),
\end{aligned}$$

and so we can state our main result which is the following uncertainty principle:

**Proposition.** Suppose that  $f \in \mathcal{S}(\mathbb{R}, \mathbb{H})$  and  $\|f\|_2^2 = 1$ . Then

$$\left( \int_{\mathbb{R}} x^2 |f(x)|^2 d\lambda_1(x) \right) \cdot \left( \int_{\mathbb{R}} |\widehat{f}(\xi) \cdot \xi \cdot (-\mathbf{i})|^2 d\lambda_1(\xi) \right) \geq \frac{\pi}{2}.$$

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