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Combinatorial Methods in Bootstrap Percolation Problems

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Abstract

The general idea of bootstrap percolation is the following. Given a hypergraph $\mathcal{H}$, each vertex can be either infected or not. Non-infected cells can become infected but not vice versa. Given an initial set of vertices $A \subseteq V(\mathcal{H})$, a vertex becomes infected if it is part of an edge in which it is the only non-infected vertex, and we iterate this process. Restricting ourselves to a special class of hypergraphs, we are interested in the minimum number of initially infected vertices required to infect the entire vertex set. A breakthrough in this problem was achieved by Noga Alon in 1985 with the use of exterior algebra. In 2012, Balogh, Bollobás, Morris, and Riordan established tight bounds in a more general setting with linear algebraic methods. We study the same problem by purely combinatorial means. We prove tight lower bounds in certain cases using elementary methods, and pose an open question to encourage further combinatorial investigation of this problem.

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1 Introduction

We examine the following process. We are given a hypergraph $\mathcal{H}$ and an initial infected vertex set $A_0 \in V(\mathcal{H})$. For all $i \geq 1$, A_i is defined as follows. The vertex set A_i is the union of A_{i-1} and the set of vertices v of $\mathcal{H}$ for which there exists an hyperedge E for which $E \setminus A_{i-1} = \{v\}$ holds [2]. We call such a hyperedge an *infecting set of v* in A_{i-1} , and we also say that E *infects v* .

Notice that (A_i) is a monotone increasing set sequence. We are only interested in this process when $V(\mathcal{H})$ is finite. In this case, there will be a smallest index f for which $A_f = A_{f+1}$. The set A_f is the *full form* of A_0 which we denote by $\overline{A_0}$. If $\overline{A_0} = V(\mathcal{H})$, we say that A_0 *percolates* in $\mathcal{H}$. We also call A_0 a *percolating set* in $\mathcal{H}$.

We call the process described above *percolation by phases*. Another, similar process can be defined. Again, we are given a finite hypergraph $\mathcal{H}$ and an initial infected vertex set $A^{(0)} \in V(\mathcal{H})$. For all $i \geq 1$, $A^{(i)}$ is defined as follows. Pick a vertex v in $V(\mathcal{H}) \setminus A^{(i-1)}$ for which an infecting set exists in $A^{(i-1)}$, if such a vertex exists. Then, we define $A^{(i)}$ as $A^{(i-1)} \cup \{v\}$. This process is called *step-by-step percolation*. At some point during the iteration of this process starting from a set $A^{(0)}$, there will be an index ℓ such that no vertex has an infecting edge in $A^{(\ell)}$. Then, the process terminates. Unlike percolation by phases, this process is generally not deterministic as vertex v above is usually not unique. The set sequence $A^{(i)}$, too, is monotone increasing. It is straightforward to prove that starting from any initial infected vertex set $A^{(0)}$, the terminating set $A^{(\ell)}$ is equal to $\overline{A^{(0)}}$ regardless of the vertices chosen in step-by-step percolation. We make heavy use of this fact in our analysis.

We restrict our attention to a special class of hypergraphs. These hypergraphs can be defined in the following way.

Notation 1.1. We denote the set $\{1, 2, \dots, n\}$ with $[n]$.

Definition 1.2. Fix positive integers d, t, r and $n_1, n_2, \dots, n_d$ such that $d \geq r \geq 1$ and $t \geq 2$. The *hyperrectangle graph* corresponding to these parameters has the vertex set $[n_1] \times [n_2] \times \dots \times [n_d]$. Furthermore, the hyperedges are the Cartesian products $I_1 \times I_2 \times \dots \times I_d$ such that $I_i \subseteq [n_i]$ for all $i \in [d]$, and among the sets I_i , exactly r have size t , and the rest have size 1.

If the value of t and r are clear from context, then by a slight abuse of notation, we also say that a set A percolates in $[n_1] \times [n_2] \times \dots \times [n_d]$ if it percolates in a hyperrectangle graph with parameters d, t, r and $n_1, n_2, \dots, n_d$. Note that the value of d is implied by the vertex set.

Given a hyperrectangle graph and its parameters, we are interested in the minimal size of sets which percolate. We denote this quantity with $m(\underline{n}, d, t, r)$ where $\underline{n}$ denotes the d -tuple $(n_1, n_2, \dots, n_d)$. Noga Alon proved the tight lower bound in the $d = r$ case using exterior algebra in 1985 [1]. In 2012, Balogh, Bollobás, Morris, and Riordan [2] established tight lower (as well as tight upper) bounds for an even more general class of hypergraphs than the one described above using linear algebraic methods.

In this work, we study this problem using only elementary, combinatorial methods. For the sake of self-containment, we include the result of Balogh et al. we prove in special cases. We state the theorem solely for hyperrectangle graphs, their original theorem is more general. The combinatorial proofs in this work were all first presented in my Bachelor's thesis [3].

Theorem 1.3 ([2], Theorem 4.). *Let d, t, r and $n_1, n_2, \dots, n_d$ such that $d \geq r \geq 1$ and $t \geq 2$ be arbitrary positive integers. Then,*

$$m((n_1, n_2, \dots, n_d), d, t, r) = \sum_{s=0}^{r-1} \sum_{I \in \binom{[d]}{s}} (t-1)^{d-s} \prod_{i \in I} (n_i + 1 - t)$$

holds where $\binom{[d]}{s}$ denotes the s -element subsets of $[d]$.

2 The Two-Dimensional Case

We visualize the vertex set $[n_1] \times [n_2]$ of a hyperrectangle graph as a two-dimensional array of squares. We index the rows top to bottom in increasing order with the elements of $[n_1]$ and columns left to right in increasing order with the elements of $[n_2]$. If $r = 2$ and $t = 2$, a quadruple of squares or vertices form a hyperedge if and only if they are at the corners of an axis-aligned rectangle, see Figure 1. The white and light gray entries represent non-infected vertices while the dark gray entries represent infected vertices. The dark and light gray entries form an infecting set of the light gray entry. The light gray entry represents the vertex $(2, 5)$.

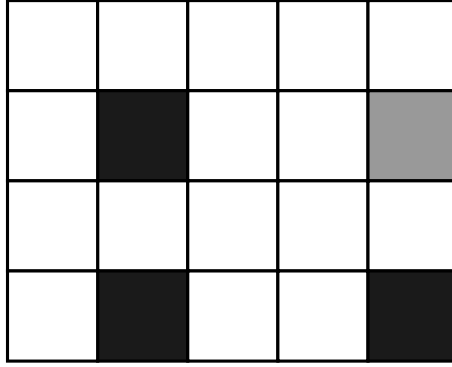


Figure 1: A visualization of an infecting set

Definition 2.1. Fix $k \in [n_1]$, and let S_k be the set of elements of A which have k as their first coordinate. The subset of $[n_2]$ containing the second coordinates of the elements of S_k is the k th row of A . The multiset of the k th rows for all $k \in [n_1]$ is the rows of A . The k th column and the columns of A are defined analogously by switching the roles of $[n_1]$ and $[n_2]$, and of the first and second coordinate. We say that a row (column) is indexed with k if it is the k th row (column).

First, we study the problem when $d = t = r = 2$, and n_1 and n_2 are arbitrary. We present a combinatorial proof of the tight lower bound in this case. This is a special case of Theorem 1.3.

Proposition 2.2. $m((n_1, n_2), 2, 2, 2) = n_1 + n_2 - 1$ for any n_1, n_2 positive integers.

Notice that due to the symmetry in the definition of hyperedges of the hyperrectangle graph, the property of whether a set percolates or not is invariant under row and column permutations. Therefore, one can consider the rows of an initially infected vertex set only to determine if it percolates or not. We apply this observation in the following lemma which is crucial to our elementary proof.

Lemma 2.3. *Let $A \subseteq [n_1] \times [n_2]$ be a percolating set. Let A^- be a set constructed by removing two rows of A and adding their union. Then, the set $A^- \subseteq [n_1 - 1] \times [n_2]$ percolates.*

Proof. Let K be an arbitrary subset of $[n_1] \times [n_2]$. Let $\mathcal{U}(K)$ be the set constructed by removing the row with index n_1 and replacing the row with index $n_1 - 1$ with the union of the rows of K with indices n_1 and $n_1 - 1$. See Figure 2.

Without loss of generality, we can assume that A^- is $\mathcal{U}(A)$. Consider a step-by-step percolation process of A . Let ℓ denote the smallest index for which $A^{(\ell)} = \overline{A}$.

$$A = A^{(0)}, A^{(1)}, A^{(2)}, \dots, A^{(\ell)} = [n_1] \times [n_2]$$

Let K and L be subsets of $[n_1] \times [n_2]$. We show that if L can arise as the union of K and a vertex which has an infecting subset in K , either $\mathcal{U}(L)$ arises the same way from $\mathcal{U}(K)$ or $\mathcal{U}(L) = \mathcal{U}(K)$. From this, the proposition follows, since after removing repeating elements, the sequence

$$A^- = \mathcal{U}(A^{(0)}), \mathcal{U}(A^{(1)}), \mathcal{U}(A^{(2)}), \dots, \mathcal{U}(A^{(\ell)}) = [n_1 - 1] \times [n_2]$$

is a step-by-step percolation process of A^- .

We prove this by case analysis. Let $L = K \cup \{v\}$, where v is the infected vertex with infecting set $E = I_1 \times I_2$.

- If $I_1 \subseteq [n_1 - 2]$, in other words, if E does not intersect the rows of K modified by $\mathcal{U}$, then $\mathcal{U}(L) = \mathcal{U}(K) \cup \{v\}$. Therefore, v is infected by $E^* := E$ in $\mathcal{U}(K)$.
- If $I_1 = \{n_1 - 1, n_1\}$, then $\mathcal{U}(L) = \mathcal{U}(K)$ since the union of rows with indices $(n_1 - 1)$ and n_1 coincide in K and L , and the rest of the rows are identical in K and L .
- Finally, if $|I_1 \cap \{n_1 - 1, n_1\}| = 1$, then we have two subcases.
 - If v is not in the rows indexed $(n_1 - 1)$ or n_1 in K , then $\mathcal{U}(L) = \mathcal{U}(K) \cup \{v\}$. Let $I_1^* = (I_1 \cap [n_1 - 2]) \cup \{n_1 - 1\}$. Then, the vertex v is infected by $E^* := I_1^* \times I_2$ in $\mathcal{U}(K)$.
 - Else, v is in one of the rows indexed $(n_1 - 1)$ or n_1 in K . In this case, let v^* be the vertex is obtained from v by setting its first coordinate to $(n_1 - 1)$. Then $\mathcal{U}(L) = \mathcal{U}(K) \cup \{v^*\}$. If $v^* \in \mathcal{U}(K)$, then $\mathcal{U}(L) = \mathcal{U}(K)$ holds. Otherwise, v^* is infected by the infecting set $E^* := I_1^* \times I_2$ in $\mathcal{U}(K)$ where I_1^* is defined as in the previous subcase.

We have exhausted all cases and the proof is complete. $\square$

Proof of Proposition 2.2. It is easy to check that the set $L := \{(x_1, x_2) \in [n_1] \times [n_2] : x_1 = 1 \text{ or } x_2 = 1\}$ is a percolating set in $[n_1] \times [n_2]$ with $n_1 + n_2 - 1$ elements. It remains to show that $|A| \geq n_1 + n_2 - 1$ for all percolating sets A .

We prove this by induction on n_1 . If $n_1 = 1$ then A can not contain an infecting set. Therefore, it only percolates if and only if $A = \{1\} \times [n_2]$. The size of this set is $n_2 = 1 + n_2 - 1 = n_1 + n_2 - 1$, thus the assertion holds.

Assume that $n_1 \geq 2$ and the proposition holds for all subsets that percolate in $[n_1 - 1] \times [n_2]$. Consider the percolating set $A \subseteq [n_1] \times [n_2]$. Then, A has two rows

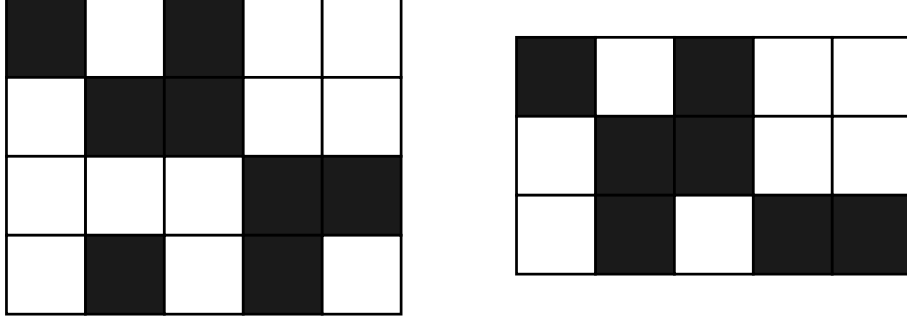


Figure 2: A visualization of a set (left) and its image under operation $\mathcal{U}$ (right)

which are non-disjoint. This is because if $A \neq [n_1] \times [n_2]$, then the rows r_1 and r_2 containing the entries of any infecting set must have a non-empty intersection. Now, consider the set A^- obtained by removing r_1 and r_2 and adding $r_1 \cup r_2$. Due to Lemma 2.3, the set A^- also percolates, and since the rows r_1 and r_2 are non-disjoint, we have $|A| \geq |A^-| + 1$. Then, by the induction hypothesis,

$$|A| \geq |A^-| + 1 \geq (n_1 - 1 + n_2 - 1) + 1 = n_1 + n_2 - 1$$

holds and this completes the proof. $\square$

Next, we show a combinatorial property of bootstrap percolation in the two-dimensional case. We state the following proposition without proof.

Proposition 2.4. *Let $A \subseteq [n_1] \times [n_2]$ be a percolating set with $d = r = 2$ and an arbitrary value of t . If a row (or column) containing exactly $t-1$ elements is removed from A , the resulting set A^- still percolates in $[n_1 - 1] \times [n_2]$ (or in $[n_1] \times [n_2 - 1]$).*

Remark. This proposition yields another proof of the lower bound of Proposition 2.2 via induction on $n_1 + n_2$. Let A be a percolating set. The statement trivially holds in $n_1 = n_2 = 1$. Due to the symmetry of n_1 and n_2 in the definition of hyperedges, one can assume that $n_1 \geq n_2$. If every row of A has at least two elements, the statement holds in this case, as $|A| \geq 2n_1 \geq n_1 + n_2 - 1$. Otherwise, A has at least one row with at most one element. These rows contain at least one element since otherwise A would not percolate since an empty row remains empty throughout the whole percolation process. After the removal of such a row, applying the Proposition 2.4 and the induction hypothesis finishes the proof.

As of now, we are unaware of combinatorial proofs for values of t greater than 2. However, we can prove the tight lower bound under a special condition defined as follows.

Definition 2.5. Let A be a subset of $V(\mathcal{H})$ where $\mathcal{H}$ is an arbitrary hypergraph. The set A *percolates in one phase* if for all vertices in $V(\mathcal{H}) \setminus A$, there exists an infecting set in A .

Note that in the definition of percolation by phases, by setting $A_0 = A$, this is equivalent to $A_1 = V(\mathcal{H})$. We give an elementary proof of Theorem 1.3 in this special case.

Theorem 2.6. *Let $d = r = 2$ and let $t \geq 2$ be arbitrary. Consider an initially infected vertex set $A \subseteq [n_1] \times [n_2]$ with $n_1, n_2 \geq t$. If A percolates in one phase, then its size is at least $(n_1 + n_2) \cdot (t - 1) - (t - 1)^2$.*

Proof. We prove this by induction on n_2 . In the base case, we have $n_2 = t$. It is easy to see that if $A \subseteq [n_1] \times [t]$ percolates in one phase, then A must have at least $t - 1$ rows with t elements, and the rest of the rows of A must have at least $t - 1$ elements. Therefore, A has at least

$$(t - 1) \cdot t + (n_1 - (t - 1)) \cdot (t - 1) = (n_1 + n_2) \cdot (t - 1) - (t - 1)^2$$

elements, proving the assertion.

Now, let A be a set that percolates in one phase in $[n_1] \times [n_2]$ where $n_2 > t$. The first column of A must contain at least $t - 1$ elements, since otherwise no infecting set could exist for the non-infected vertices in the first column. Now, consider the set A^- obtained from A by removing its first column. We have two cases in the induction step.

First case. Assume that A^- percolates in one phase in $[n_1] \times ([n_2] \setminus \{1\})$. Then, $|A| \geq |A^-| + (t - 1)$, and the induction hypothesis yields

$$|A| \geq |A^-| + (t - 1) \geq (n_1 + n_2 - 1) \cdot (t - 1) - (t - 1)^2 + (t - 1) = (n_1 + n_2) \cdot (t - 1) - (t - 1)^2.$$

This verifies the theorem in this case.

Second case. Assume that A^- does not percolate in one phase. Our aim is to construct a set A^* for which the following requirements hold.

- (I) The size of A^* is the same as the size of A .
- (II) The set A^* has at least $t - 1$ elements in its first column.
- (III) By removing the first column of A^* , the resulting set $(A^*)^-$ percolates in one phase in $[n_1] \times ([n_2] \setminus \{1\})$.

Requirements (II) and (III) ensure that the calculations of the first case can be applied to A^* , which give the desired lower bound for $|A|$ as well since $|A| = |A^*|$ by requirement (I).

Since A^- does not percolate in one phase, there exists a non-infected vertex $v \in [n_1] \times ([n_2] \setminus \{1\})$ which only has an infecting set $E_0 = I_1 \times I_2$ with $1 \in I_1$. We may assume without loss of generality that $v = (t, t)$ and that one of its infecting sets is $E_0 = [t] \times [t]$.

We construct A^* from A row by row in the following way. Let $r_k \subseteq [n_2]$ denote the k th row of A in the sense of Definition 2.1. If $1 \in r_k$ and $|r_k \cap [t]| < t$, then let the k th row of A^* be $r_k \setminus \{1\} \cup \{\min([t] \setminus r_k)\}$. Less formally and more visually, this means that in this case, we construct the k th row of A^* from the k th row of A by moving the first element of the row to the leftmost empty place in that row. If these two conditions do not hold for a row r_k of A , we let the k th row of A^* be r_k , in other words, we leave the row unchanged. See Figure 3 for an example of this construction with $t = 2$.

Now, we verify that the requirements (I), (II), and (III) hold for A^* . Since A^* is constructed in such a way that each of its rows have the same number of elements as the row of the same index in A , requirement (I) is satisfied. By the construction of A^* , the edge E_0 defined above guarantees that $[t - 1] \times \{1\} \subseteq A^*$, implying requirement (II).

Finally, we prove that requirement (III) holds. Let $v \notin (A^*)^-$ denote an arbitrary non-infected vertex in $[n_1] \times ([n_2] \setminus \{1\})$. Since $A^- \subseteq (A^*)^-$ by the construction of A^* , if v has an infecting set in A^- , then v also has an infecting set in $(A^*)^-$. Thus, we can assume that v has no infecting set in A^- , and we need to verify that v has an infecting set in $(A^*)^-$. Since A percolates in one phase, v has an infecting set $E = I_1 \times I_2$ in A . As vertex v does not have an infecting set in A^- , $1 \in I_2$ must hold. Since E is an infecting set of a vertex outside the first column, E must have t elements in the first column of A . Denote the index of the column v belongs to by $k_v \geq 2$. Let $k^* = \min([t] \setminus (I_2 \setminus \{k_v\}))$ which is well-defined since $|I_2 \setminus \{k_v\}| = t - 1$. The minimality of k^* implies that $I_1 \times [k^* - 1] \subseteq A$, therefore $I_1 \times \{k^*\} \subseteq A^*$ by the construction of A^* . Thus, since $v \notin A^*$, the vertex v can not be in the column indexed k^* , i.e. $k^* \neq k_v$. Therefore, $k^* \notin I_2$, that is, E avoids the k^* th column of $[n_1] \times [n_2]$. Now, consider the hyperedge $E^* := I_1 \times (I_2 \setminus \{1\} \cup \{k^*\})$. Since $I_1 \times \{k^*\} \subseteq A^*$, $A^- \subseteq (A^*)^-$, and $E \setminus \{v\} \subseteq A$, the set E^* is an infecting set of v . The choice of v was arbitrary, therefore it follows that $(A^*)^-$ percolates in one phase.

With this, all requirements are verified, therefore the calculations in the first case can indeed be applied to A^* . This finishes the proof. $\square$

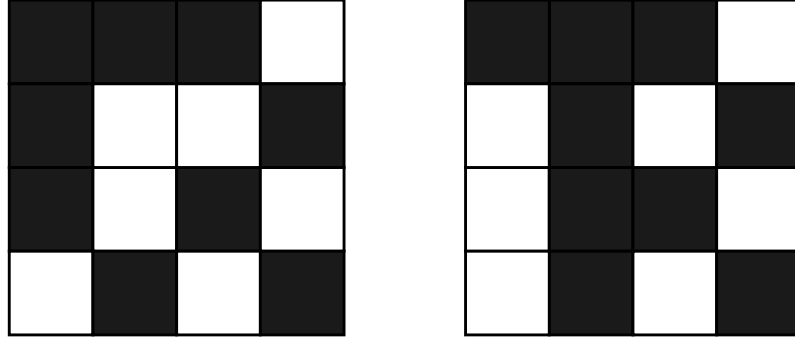


Figure 3: An example of a set A percolating in one phase with $t = 2$ (left) and its corresponding set A^* (right)

3 A Proof for General Number of Dimensions

In this section, we generalize the proof provided for Proposition 2.2 to any number of dimensions. We do this by generalizing the definitions and the lemma introduced in the previous section. Then, we provide an elementary proof of the following proposition.

Theorem 3.1. *Let $d \geq 2$ and $n_1, n_2, \dots, n_d$ be arbitrary positive integers. Then,*

$$m((n_1, n_2, \dots, n_d), d, 2, 2) = \sum_{i=1}^d n_i - (d - 1).$$

Definition 3.2. Let $A \subseteq [n_1] \times [n_2] \times \dots \times [n_d]$ and let $k \in [d]$. Define the following mapping:

$$\Pi_k : \mathbb{Z}^d \rightarrow \mathbb{Z}^{d-1}, (x_1, x_2, \dots, x_d) \mapsto (x_1, x_2, \dots, x_{k-1}, x_{k+1}, \dots, x_d).$$

Let A_k^m denote the set of elements of A which have m as their k th coordinate. Then, the multiset

$$S_k = \{\Pi_k(A_k^m) : m \in [n_k]\}_{\text{multi}}$$

is the slices of A along the k th coordinate. We refer to an element of this set as a slice of A along the k th coordinate.

Remark. The rows and columns of a two-dimensional set are the slices of this set according to their first and second coordinate respectively. Similarly to the two-dimensional case, whether or a set percolates or not remains invariant under permutation of the slices along any coordinate.

Lemma 3.3. *Let $A \subseteq [n_1] \times [n_2] \times \dots \times [n_d]$ be a percolating set. Let A^- be a set constructed by removing two slices along the same coordinate of A and adding their union as a new slice. Then, A^- percolates.*

Sketch of proof. We take a very similar approach to proving this as in the proof of Lemma 2.3. Fix $k \in [d]$. Let $\mathcal{U}$ denote the operation that maps a set $K \subseteq [n_1] \times [n_2] \times \dots \times [n_d]$ to the set constructed by removing the slices of K along the k th coordinate indexed n_k and $n_k - 1$, and adding their union as a slice along the k th coordinate with index $n_k - 1$. We may assume without loss of generality that $A^- = \mathcal{U}(A)$. We want to show the following. Apply the operation $\mathcal{U}$ to each element of a step-by-step percolation process of A . After removing the repeating elements, we obtain a step-by-step percolation process of A^- .

First, we examine what happens when the vertex infected within a set in a step-by-step percolation process can be infected using an infecting set entirely contained by a slice along the k th coordinate. In this case, the same infecting set is present in the image of the set with respect to $\mathcal{U}$, or the vertex infected by it is already in the image, possibly by changing the k th coordinate to $n_k - 1$ in both cases. If a vertex can not be infected with an infecting set entirely contained in a slice along the k th coordinate, observe the following. The infecting set is contained within a subset (a “two-dimensional plane”) of $[n_1] \times [n_2] \times \dots \times [n_d]$ in which only two coordinates take on more than one value, one of which is the k th coordinate. By omitting the rest of coordinates, we return to the two-dimensional case. Here, Lemma 2.3 finishes the proof if the k th coordinate takes on the role of the first coordinate. $\square$

Definition 3.4. The *edgesum* of the product $[n_1] \times [n_2] \times \dots \times [n_d]$ is $\sum_{i=1}^d n_i$.

Proof of Theorem 3.1. In [2], a percolating set of size $\sum_{i=1}^d n_i - (d - 1)$ is presented in the hyperrectangle graph $[n_1] \times [n_2] \times \dots \times [n_d]$, for $t = r = 2$. (This is the set $L_{n_1, \dots, n_d}^{d, 2, 2}$ defined in Notation 4.1.) It remains to show that if A is a percolating set in the hyperrectangle graph $[n_1] \times [n_2] \times \dots \times [n_d]$, then $|A| \geq \sum_{i=1}^d n_i - (d - 1)$.

The proof is by induction on the edgesum of the vertex set of the hyperrectangle graph. In the base case, $n_i = 1$ for all $i \in [d]$, thus the edgesum is d . In this case, a set percolates if and only if it has one element, therefore the proposition holds.

Assume that the proposition holds if the edgesum is $N - 1$. Now, let A be a percolating set with edgesum $N \geq d + 1$. If $A \neq [n_1] \times [n_2] \times \dots \times [n_d]$ – otherwise, the theorem holds, – A has to contain an infecting set E . Let k be a coordinate in which two elements of the infecting set differ. Then, the slices along the k th coordinate which contain the four elements of E are non-disjoint.

According to Lemma 3.3, the set A^- obtained from A by replacing these two slices with their union still percolates. Since these slices are not disjoint, $|A| \geq |A^-| + 1$ holds. The induction hypothesis yields that the size of A is at least $N - 1 - (d - 1) + 1 = N - (d - 1)$, and this finishes the proof. $\square$

4 Open Questions

We conclude by posing a few open questions relating to the combinatorial investigation of bootstrap percolation.

Notation 4.1. Let the values of the parameters d, t, r and $n_1, n_2, \dots, n_d$ be fixed. Let $L_{n_1, \dots, n_d}^{d, t, r}$ denote the subset of elements of $[n_1] \times [n_2] \times \dots \times [n_d]$ which has at most $r - 1$ coordinates with value more than $t - 1$.

In [2], it is proved that $L_{n_1, \dots, n_d}^{d, t, r}$ percolates, and that

$$|L_{n_1, \dots, n_d}^{d, t, r}| = m((n_1, n_2, \dots, n_d), d, t, r)$$

holds for any values of d, t, r and $n_1, n_2, \dots, n_d$. Therefore, not only does $L_{n_1, \dots, n_d}^{d, t, r}$ percolate in the hyperrectangle graph with vertex set $[n_1] \times [n_2] \times \dots \times [n_d]$ and parameters d, t, r but it does so with the least possible number of initially infected vertices.

Definition 4.2. Let $\mathcal{H}$ be a hyperrectangle graph with arbitrary d, t , and r parameters, and let A be an arbitrary subset of $V(\mathcal{H})$. Let E be a hyperedge of the hypergraph which is an infecting set of some vertex v . In other words, let $E \in E(\mathcal{H})$ such that $|E \cap A| = t^r - 1$ and $E \setminus A = \{v\}$. Now, let $w \in E \setminus \{v\}$ arbitrary. The *shift operation* on A with respect to E and w maps A to $A \cup \{v\} \setminus \{w\}$. If w is the vertex of E with the maximal coordinate sum, this operation is a *maximal shift operation*. We say that a set A can be *transformed into A^* with a sequence of (maximal) shift operations* if there is a sequence of sets of which A is the first element, and every other element can be obtained by a (maximal) shift operation on the previous element, and this sequence contains A^* .

Definition 4.3. Let $\mathcal{H}$ be a hyperrectangle graph and let $A \subseteq V(\mathcal{H})$ be arbitrary. The infecting set $E \in E(\mathcal{H})$ is in *standard position* if the vertex $v \in E \setminus A$ has the maximal coordinate sum among the vertices in E .

Note that a maximal shift operation can not be performed on a hyperedge if and only if it is in standard form. An example of a maximal shift operation with $d = r = 2$ and $t = 3$ can be seen on Figure 4. The dark gray entries are the infected vertices which along with the light gray entry form an infecting set of the light gray entry.

Let $\mathcal{H}$ be a hyperrectangle graph, let $A \subseteq V(\mathcal{H})$ be arbitrary. Let $E \in E(\mathcal{H})$ be an arbitrary infecting set with $v \in E \setminus A$ and $w \in E \setminus \{v\}$. Let A^* be the image of a shift operation on A with respect to E and w . Observe a step-by-step percolation process of A and A^* . First, let E infect v in A , and let E and w in A^* . Notice that we get the same set after this step. Therefore, $\bar{A} = \bar{A}^*$ holds. Consequently, transforming a percolating set using a sequence of shift operations yields a percolating set of the same size.

Since $L_{n_1, \dots, n_d}^{d, t, r}$ is percolating set with the minimal possible number of infected vertices, we raise the following question.

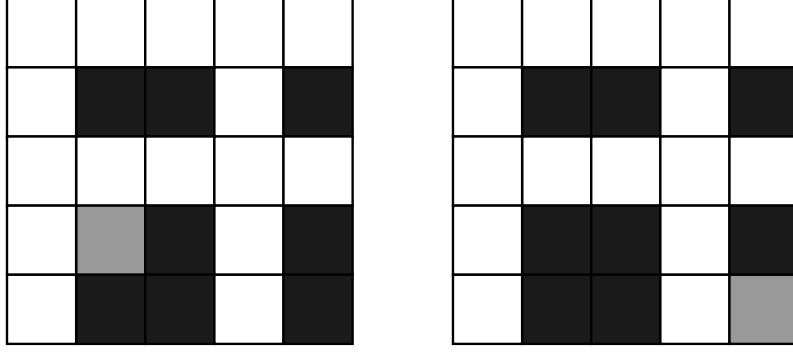


Figure 4: An infecting set (left) and the result of a maximal shift operation performed with respect to it (right)

Open Question 4.4. *For what values of d, t , and r does it hold that all percolating sets $A \subseteq [n_1] \times [n_2] \times \dots \times [n_d]$ can be transformed into A' using a sequence of shift operations such that $L_{n_1, \dots, n_d}^{d, t, r} \subseteq A'$? Can this be done using a sequence of maximal shift operations only?*

Note that verifying this for specific values of d, t , and r yields a tight lower bound for $m(\underline{n}, d, t, r)$ with these parameters. We also raise the following question which relaxes the assumptions posed in the previous question.

Open Question 4.5. *Assume that $d = r$. For what values of d and t does there exist a sequence of shift operations for any percolating set $A \subseteq [n_1] \times [n_2] \times \dots \times [n_d]$ which transforms A into a set which percolates in one phase?*

Note that an affirmative answer in the case of $d = 2$ would give a proof for the tight lower bound using Theorem 2.6 for any value of t . This was the main motivation behind Theorem 2.6. We finish this work by answering both questions in Open Question 4.4 in the affirmative in the case of $d = t = r = 2$.

Proposition 4.6. *Let $A \subseteq [n_1] \times [n_2]$ be a percolating set. Then, A can be transformed using a sequence of maximal shift transformations into a set which has n_2 elements in its first row and n_1 elements in its first column.*

Proof. Let A be a percolating set. Perform a sequence of maximal shift operations of A until it is no longer possible to do so. Since the sum of the coordinates in the set strictly decreases sequence after performing a maximal shift operation, this sequence is finite for any percolating set. Let A^* the last element of this sequence.

We now define a relation over the rows of A^* . Let r_1 and r_2 be rows of A^* , and let $r_1 \sim r_2$ if and only if they are non-disjoint. Since all rows of A^* are non-empty, the relation $\sim$ inherits reflexivity and symmetry from set intersection. Over the rows of A^* , this relation is also transitive. Let r_i, r_j , and r_k be i th, j th, and k th rows of A^* respectively, and let $r_i \sim r_j$ and $r_j \sim r_k$. Since all infecting sets are in standard position in A^* , $\min r_i = \min r_j = \min(r_i \cap r_j)$ holds. For analogous reasons, $\min r_j = \min r_k = \min(r_j \cap r_k)$ also holds. Therefore, $\min r_i = \min r_k$, thus r_i and r_k are not disjoint, proving that $\sim$ is indeed transitive over the rows A^* .

Let $C_1, C_2, \dots, C_s$ denote the equivalence classes of the rows of A^* with respect to $\sim$, and for $i = 1, 2, \dots, s$, let $\hat{r}_i \subseteq [n_2]$ be the element of C_i with the lowest row index. Notice that $\hat{r}_i$ contains the union of all other rows in its equivalence class.

Otherwise, there would exist an infecting set in A^* not in standard position. The rows $\widehat{r}_1, \widehat{r}_2, \dots, \widehat{r}_s$ are pairwise disjoint since they are in pairwise different equivalence classes. From the above, it follows easily that

$$\overline{A^*} = \bigcup_{i=1}^s (C_i^{\text{ind}} \times \widehat{r}_i) \quad (1)$$

where $C_i^{\text{ind}} \subseteq [n_1]$ is the set of row indices of the rows in C_i .

So the set A^* percolates if and only if there is only equivalence class with $C_1^{\text{ind}} = [n_1]$ and $\widehat{r}_1 = [n_2]$. Therefore, the first row of A^* contains n_2 elements. Since all elements of this equivalence class share the same minimum and $\min \widehat{r}_1 = \min [n_2] = 1$, every other row must contain the element 1. Therefore, the first column of A^* contains n_1 elements. This completes the proof. $\square$

Remark. The above proof shows that the full form of a set $A \subseteq [n_1] \times [n_2]$ always has the form

$$(I_1 \times J_1) \cup (I_2 \times J_2) \cup \dots \cup (I_s \times J_s)$$

for some nonnegative integer s and pairwise disjoint sets $I_1, I_2, \dots, I_s \subseteq [n_1]$ and pairwise disjoint sets $J_1, J_2, \dots, J_s \subseteq [n_2]$, c.f. formula (1).

Összegzés

Ebben a dolgozatban a következő folyamatot vizsgáljuk. Adott egy $\mathcal{H}$ hipergráf és egy $A_0 \subseteq V(\mathcal{H})$ halmaz. Az A_0 halmazt *kezdetben gyepes mezők halmazának*, a $V(\mathcal{H})$ halmazra nézett komplementerét *nem gyepes mezőknek* nevezzük. Az A_i ($i \geq 1$) halmazt a következőféleképpen definiáljuk rekurzívan. Az A_i halmaz az A_{i-1} és azon v csúcsok halmazának uniója, amelyekhez létezik olyan E hiperél, hogy $E \setminus A_{i-1} = \{v\}$ teljesül. Egy ilyen, a v csúcshoz tartozó E hiperél *v gyepesítő halmazának* nevezzük az A_{i-1} halmazban.

Ezt a folyamatot véges hipergráfok esetében vizsgáljuk. Ebben az esetben létezik egy olyan legkisebb f küszöbindex, amelyre $A_f = A_{f+1}$ teljesül. Az A_f halmaz az A_0 halmaz *teljes alakja*, és amennyiben $A_f = V(\mathcal{H})$, akkor azt mondjuk, hogy A_0 *perkolál* a $\mathcal{H}$ hipergráfban.

A dolgozatban a következő folyamatot is felhasználjuk. Az előzőhöz hasonlóan adott egy $\mathcal{H}$ hipergráf és egy $A \subseteq V(\mathcal{H})$ halmaz. A *lépésenkénti gyepesedési folyamatban* a halmazzsorozat első eleme az A halmaz. A sorozat egy új elemét úgy kapjuk meg az előzőből, hogy egy tetszőleges gyepesítő halmazhoz tartozó nem gyepes mezőt hozzávesszük a gyephalmazhoz, amennyiben gyepesítő halmaz létezik. Könnyen megmutatható, hogy amikor egy lépésenkénti gyepesedési folyamat elakad (azaz amikor nincsen gyepesítő halmaz az aktuális gyephalmazban), akkor mindig A teljes alakját kapjuk a választott gyepesítő halmazok megválasztásától függetlenül. Ezt a tényt erősen kihasználjuk a bizonyításainkban.

Vizsgálatunk tárgyát az alábbi gráfok képezik. Jelölje $[n]$ az $\{1, 2, \dots, n\}$ halmazt. Ekkor a hipertéglatestgráfokat a következő módon definiáljuk. Rögzítsük a d, r, t és $n_1, n_2, \dots, n_d$ pozitív egész paramétereket úgy, hogy $d \geq r \geq 1$ és $t \geq 2$. Egy d -dimenziós $\mathcal{H}$ *hipertéglatestgráf* csúcshalmaza a $[n_1] \times [n_2] \times \dots \times [n_d]$ Descartes-szorzat. A hipergráf élhalmazában pontosan azok az $I_1 \times I_2 \times \dots \times I_d$ Descartes-szorzatok szerepelnek, amelyekben $I_i \subseteq [n_i]$ minden $i \in [d]$ esetén, és az I_i halmazok közül pontosan r darab elemszáma t , a többi egyelemű.

Ezen hipergráfok esetében szeretnénk meghatározni azt, hogy egy halmaznak legalább hány eleműnek kell lennie ahhoz, hogy perkoláljon. Erre elemi kombinatorikus módszer nem ismert. Noga Alon 1985-ben éles alsó korlátot bizonyított $d = r$ esetben Grassmann-algebra segítségével [1]. Balogh, Bollobás, Morris és Riordan 2012-ben lineáris algebrai módszerekkel határozta meg az éles korlátot a hipertéglatestgráfoknál általánosabb gráfosztályra [2].

Ebben a dolgozatban ezt a kérdést elemi kombinatorikai eszközökkel vizsgáljuk, összevetve őket a [2] cikkben szereplő eredményekkel. Először kombinatorikusan igazoljuk az éles alsó korlátot abban az esetben, ha $d = t = r = 2$, amit az alábbi módon teszünk meg.

Egy kétdimenziós hipertéglatestgráf csúcshalmazának egy $A \subseteq [n_1] \times [n_2]$ részhalmazának k -adik során a

$$\{j \in [n_2] : (k, j) \in A\}$$

halmazt értjük. Első körben igazoljuk a következő lemmát. Amennyiben egy A perkoláló halmazból elveszünk két sort, és azt azok uniójával helyettesítjük, akkor az így kapott halmaz szintúgy perkolál. Ezt követően ezt felhasználva egy teljes indukciós bizonyítást adunk arra alapozva, hogy egy perkoláló halmaznak mindig van két nem diszjunkt sora.

Emellett kitekintünk a síkon az általános t paraméter esetére a d és r paraméterek értékeit megőrizve. Nem ismerünk olyan kombinatorikus bizonyítást $t \geq 3$ esetén, amely teszőleges perkoláló halmaz esetében igazolja az éles alsó korlátot. Mindazonáltal tettünk lépést ez irányba, a következő az egyik fő eredményünk. Azt mondjuk, hogy egy A halmaz *egy fázisban perkolál*, ha bármely nem gyepes mezőjéhez tartozik egy gyepesítő halmaz az A halmazban. Azt igazoltuk elemi módszerekkel, hogy ha egy $A \subseteq [n_1] \times [n_2]$ halmaz egy fázisban perkolál, akkor A elemszámára teljesül az éles alsó korlát. A bizonyítás teljes indukcióval történik, amely két alesetre bomlik. Az első esetben az indukciós hipotézis közvetlenül alkalmazható. A másik esetben a kezdeti gyephalmaz „jobbra tömörítésével” azt egy olyan kezdeti gyephalmazzá alakítjuk át, amely visszavezeti azt az első esetre.

Ezt követően a dolgozat másik fő eredményét mutatjuk be. Ez az, hogy kombinatorikus módszerekkel igazolható az éles alsó korlát $t = r = 2$ esetben d teszőleges értékére. Ez a bizonyítás a $d = t = r = 2$ esetben vázolt bizonyítás általánosítása. A sorok és oszlopok magasabb dimenziós általánosítását vezetjük be, amit *szeleteknek* nevezünk. Ez után a két dimenzióban felhasznált lemma segítségével ezen lemma magasabb dimenziós, szeletekre vonatkozó megfelelőjének bizonyítását vázoljuk. A kétdimenziós esethez hasonlóan a bizonyítás az általánosított lemmát alkalmazza, és az előzőekhez hasonlóan teljes indukcióval történik.

Az utolsó fejezetben több nyitott kérdést teszünk fel, amelyet ígéretesnek tartunk a kérdéskör további kombinatorikai vizsgálatában. Bevezetjük a shift operációt, amely egy olyan operáció a kezdetben gyepes csúcsok halmazain, amely megőrzi a teljes alakot. A shift operációt a következő módon definiáljuk.

Legyen $\mathcal{H}$ egy hipertéglatestgráf teszőleges d, t és r paraméterekkel, és legyen A a $\mathcal{H}$ hipertéglatestgráf csúchalmazának teszőleges részhalmaza. Legyen E egy olyan hiperél, amely egy v csúcs gyepesítő halmaza az A halmazban, azaz legyen $E \in E(\mathcal{H})$ olyan, hogy $|E \cap A| = t^r - 1$ és $E \setminus A = \{v\}$. Legyen $w \in E \setminus \{v\}$ teszőleges. Az A halmazon az E és w szerinti *shift operáció* az A halmazhoz a $A \cup \{v\} \setminus \{w\}$ halmazt rendeli.

Emellett még három másik fogalmat vezetünk be a nyitott kérdéseink kimondásához. Ha w az E hiperél legnagyobb koordinátaösszegű csúcsa, akkor ezt a shift operációt *maximális shift operációnak* nevezzük. Azt mondjuk, hogy A *átalakítható (maximális) shift operációk sorozatával az A^* halmazzá*, ha létezik egy olyan halmazsorozat, amelynek első eleme A , minden másik elem megkapható az előzőből egy (maximális) shift operációval, és ebben a sorozatban megjelenik A^* elemként.

A [2] cikket követve definiálunk egy speciális halmazt, amely a hipertéglatestgráfok teszőleges paraméterezése mellett minimális elemszámmal perkolál. Első nyitott kérdésünk az, hogy mely d, t, r paraméterek esetén alakítható át shift operációk sorozatával bármely perkoláló halmaz a feljebb definiált speciális halmazt tartalmazó halmazzá. Ezt a kérdést szigorítva ugyanezt megkérdezzük maximális shift operációk sorozatáról is. Ha a d, t, r paraméterek meghatározott értékére a válasz igenleges, az egy bizonyítást ad az éles alsó korlátra ezen paraméterértékek esetén, hiszen a speciális halmaz elemszáma éppen a bizonyítandó alsó korlát.

Ezt követően megfogalmazzuk a kérdés egy gyengített változatát is. Tegyük fel, hogy $d = r$. Ekkor d és t mely értékeire alakítható át bármely perkoláló halmaz shift operációk sorozatával egy olyan halmazzá, amelyik egy lépésben perkolál? Ha $d = 2$ és t valamely értékére a válasz igenleges, akkor ez az általunk igazolt két dimenzióban egy fázisban perkoláló halmazokról szóló tétellel együtt egy bizonyítást

ad az éles alsó korlátra általános esetben. Ez motiválta azt, hogy az egy fázisban perkoláló halmazokról szóló tételt igazoljuk. Azt sejtjük, hogy erre a kérdésre a válasz igenleges t tetszőleges értékére. Különösen érdekelne minket ezen sejtés kombinatorikus igazolása akár speciális t értékekre is, hiszen ez eredményünkkel együtt egy kombinatorikus bizonyítást adna a kétdimenziós eset további aleteire.

A dolgozatot az imént feltett kérdések igenleges megválaszolásával zárjuk a $d = t = r = 2$ esetben. Egyúttal karakterizáljuk a lehetséges A halmazok teljes alakjait.

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