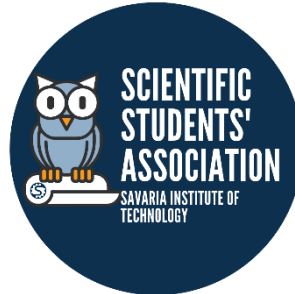




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SCIENTIFIC STUDENTS'S ASSOCIATION ESSAY

***Optimizing Satellite Autonomy for
Collision Prevention***

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NOTATIONS

Latin letters

Notation	Name of the notation	Unit
a	measurement of a sensor	1
A	system matrix	rad^2/s^2
B	control matrix	$\text{Kg}^{-1}.\text{m}^{-2}$
C	covariance matrix	$\text{Kg}.\text{m}^2.\text{rad}^2/\text{s}^2$
d	input disturbance	N.m
D	disturbance matrix	1
E	expected value or statical expectation	1
F	force	N
F	final configuration function	1
g	gravitational acceleration (9.8)	m/s^2
H	defined matrix	$\text{Kg}.\text{m}^2.\text{rad}/\text{s}$
I	inertia matrix	$\text{Kg}.\text{m}^2$
J	identity matrix	1
K	transformation matrix	1
m	mass of the satellite	Kg
N	number of sensors	1
P	error covariance matrix	1
Q	quaternion vector	1
Q	weight matrices of states	1
R	resistance	Ω
R	weight matrices of inputs	1
s	Laplace variable	1
t	time	s
T	torque	N.m
u	input control	N.m
U	input control matrix	1
v	velocity	m/s
x	composite measurement of sensors	1
x	state variable vector	1
X	state variable matrix	1

y	admittance	Ω^{-1}
Y	admittance matrix	Ω^{-1}
z	state variable vector	rad/s ²
Z	state variable matrix	1

Greek letters

Notation	Name of the notation	Unit
α	angular rotation vector	rad
δ	indicator function	1
θ	angle	rad
λ	longitude	rad
$\mathfrak{Z}$	defined matrix	Kg.m ² .rad ² /s ²
φ	angle	rad
ψ	angle	rad
Ψ	Initial configuration function	1
ω	angular velocity	rad/s

Indexes, exponents

Notation	Name, explanation
cov	covariance
dist	disturbance
g	gravitation
i	general index (integer)
j	general index (integer)
k	general index (integer)
m	general index (integer)
n	general index (integer)
p	general index (integer)
s	Laplace variable
thru	thrust
var	variance

1. INTRODUCTION

1.1. Problem

The reasons leading to this research work are the increasing number of space debris and satellites in Low Earth Orbit (LEO) between 160 and 1,600 km above sea level [1], which heightens the risk of collisions and threatens satellite security [2]. Our space is clustered with over 500,000 space debris between 1 and 10 cm orbiting the Earth [3]. With the rising number of LEO satellites, the importance of addressing security-related threats of satellite collisions becomes highly prioritized [4]. The collisions between active satellites and space debris with a speed that can reach 14 km/s can cause a tremendous impact on the satellite's fragile systems, and possibly more space debris due to fragmentation [5]. The current systems that detect and predict satellite collisions are based on Earth, which was proven to be insufficient and lack autonomy where human control is mandatory for the satellite maneuver [6].

1.2. Aim

The research seeks to answer questions such as how to accurately assess the risk of satellite collisions in LEO, how to develop autonomous, efficient maneuvering techniques with no human intervention to mitigate these risks, and how to design control models to implement these maneuvers effectively.

The relevance of this work lies in ensuring the safety and efficiency of satellite operations among the growing challenges posed by space debris in LEO and expanding satellite constellations to reduce the impact of human error. This will keep operational expenses to a minimum and highlight the satellite autonomy advantage in threat detection and decision-making.

The study was structured around three primary goals aimed at achieving effective collision avoidance. Initially, it required constructing a threat analyzer capable of evaluating collision probabilities by considering satellite dynamics and space threat parameters. Subsequently, attention was directed towards designing a dynamic satellite model that can be implemented in the control objective. Finally, the focus shifted to crafting a control objective to execute satellite collision avoidance maneuvers autonomously, emphasizing energy conservation and station stability.

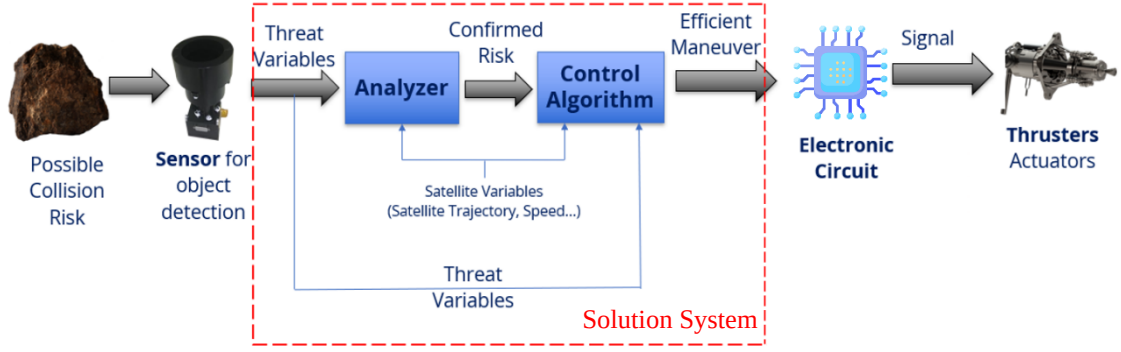


Figure 1. The solution system proposed for autonomous satellite collision avoidance [7], [8], [9], [10].

1.3. Methods

This research work is structured around three main objectives as explained previously in the Aim section, of which two methods are distinguished for approaching the desired goals. The first method includes the analytical analysis of the problem where the probability collision analyzer of space threats and the dynamic model of the satellite collision maneuver techniques are developed. The second method is based on simulation analysis where the collision maneuver techniques are simulated with MATLAB 2023b software to better confirm the analytical analysis.

1.3.1. Analytical Analysis Method

The analytical analysis method of the Threat Analyzer focuses on integrating data from multiple independent sensors with varying error characteristics to reduce overall estimation errors and improve collision probability evaluation. By modeling error covariances as analogous to electrical resistance in parallel circuits, it constructs a combined error covariance matrix C_N to track threats effectively during maneuvers. This matrix includes allo-combined covariances for tracking partially surpassed threats and mutual covariances for refining initial detections, enabling precise, efficient threat analysis and autonomous satellite operations.

Furthermore, an analytical analysis method was used to create the dynamic model of the satellite where the control variables are derived in differential equations describing the satellite motion. Energy saving and station keeping were studied during the satellite maneuver to ensure efficiency.

1.3.2. Simulation Analysis Method

The simulation analysis method utilizes MATLAB 2023b, with the help of YALMIP and SeDuMi, to solve the Linear Matrix Inequality (LMI) associated with the Kalman-Popov-Yakubovich (KYP) lemma. This approach, combined with the Linear

Quadratic Regulator (LQR) method, determines the disturbance attenuation level, computes the feedback gain matrix K , and calculates the closed-loop system eigenvalues to achieve the optimal multi-objective controller for maneuver execution.

1.3.3. Assumptions

- **Sensor Independence:**

The sensors used for threat detection operate independently, with their error covariances being uncorrelated.

- **Perfect Measurement Availability:**

All the sensors' data is assumed to be available and reliable within the predefined error margins.

- **Error Covariance Validity:**

The combined error covariance matrix C_N accurately represents the system's overall uncertainty, adhering to the mathematical analogy with electrical systems.

- **Linear System Dynamics:**

The satellite's motion and control systems are modeled as linear systems to simplify analysis and controller design.

- **Simplified Satellite Geometry:**

The satellite's geometry is approximated for calculating inertia and collision probabilities, neglecting minor structural irregularities.

- **Isolated Satellite System:**

The satellite is treated as a single isolated system without interactions or communications with other satellites during maneuvers.

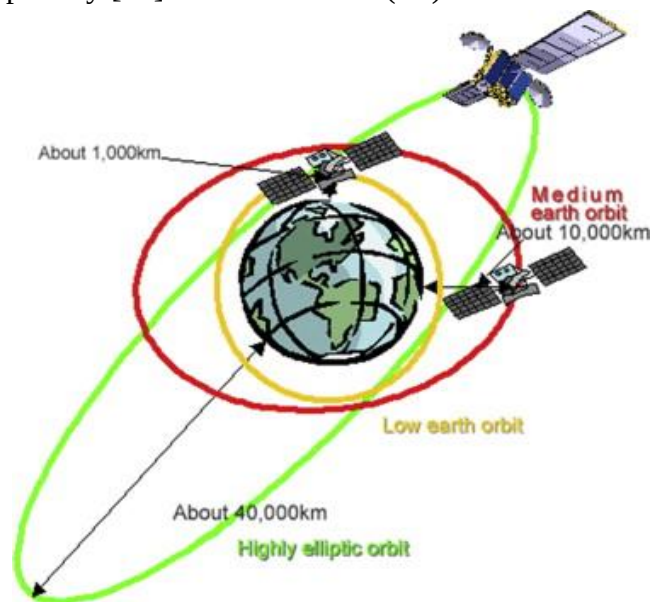
- **Neglect of Small Debris:**

The analysis focuses on debris of trackable size, excluding objects below detection thresholds.

2. LITERATURE REVIEW

2.1. Overview

Satellites in LEO orbit the Earth in a range of 160 to 1,600 km above sea level [1], they can have elliptical or circular trajectories where their orbital periods vary between 90 to 120 min [11]. Compared to other orbitals like the “Highly Elliptical Orbit (HEO)” [11] or the “Geostationary Earth Orbit (GEO)” [11], the altitude of LEO is low enough to provide the satellites with higher velocities of more than 25,000 km/h and they make between 12 to 16 Earth turns per day [11]. As the velocities of these satellites are very high, the time they can last above Earth’s visible horizon for an observer from Earth is as low as 20 minutes per day [11]. Small satellites (SS) are defined as EPSA class (180kg)



or less.

Figure 2. GEO, LEO, MEO, and HEO orbit positions around the Earth [11].

The short period that an LEO satellite lasts above Earth’s visible horizon may cause communication problems and delays with ground stations, especially for urgent communications. The constellation of satellites was introduced to compensate for the lack of direct communications with the satellite. A group of satellites is distributed in different trajectories enough away from each other to create a network of satellites that has constant signal communication with one another where instead of direct communication with the ground station, the information is transferred from a satellite that is not in its period of transit above the ground station to the nearest satellite that is located above the desired ground station. The information may travel through more than one satellite before getting sent to its last destination. In contrast, these constellations introduce risks associated with

the deposit of more space debris [12] and raise the amount of aluminum residue in the Earth's upper atmosphere more than meteoroids do [13].

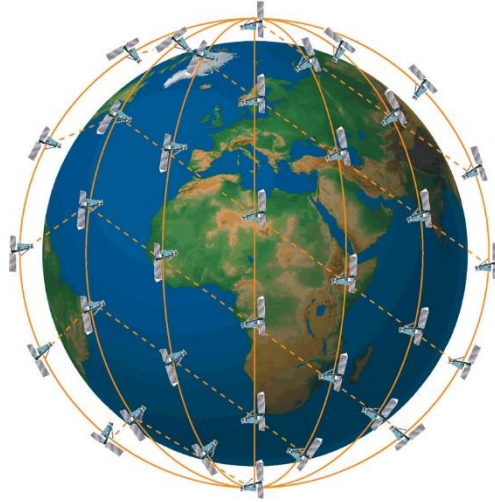


Figure 3. Low Earth Orbit (LEO) satellite constellation [14].

Due to the complexity of space traffic management, human intervention is becoming weak and inefficient in the increasing number of satellites in LEO due to fuel costs, operational costs, and frequent human errors [15]. Methods have emerged in recent years with the development of artificial intelligence to provide the satellite with enough autonomy over collision avoidance and minimize human interventions to leave a feedback-controlled system that is partially or entirely commanded by machine learning [16].

The lack of direct communication with the satellite, the constant human intervention costs and errors, and the overly crowded LEO with satellite constellations that adduce more risks, lead to the search for autonomous collision avoidance maneuvers that takes into consideration energy efficiency, operational costs, and accuracy.

2.2. Threat Analyzer for Collision Probability Evaluation

2.2.1. Conjunction Event Prediction

One important method for evaluating collision risk and assisting satellites in planning maneuvers to reduce that risk, if required, is NASA's "Conjunction Assessment Risk Analysis (CARA)" [17]. All active robotic satellites under NASA's control receive this service from the CARA team at NASA-GSFC [18], which also acts as a provider for a few outside agencies and organizations.

The first step in the CARA collision risk assessment and avoidance is the Conjunction Event Prediction. In this stage, all other known space objects (the "secondary" objects) are studied in conjunction with NASA-protected assets (the "primary" objects) to predict their future positions [19]. The US Space Force's 18th Space Defense Squadron

conducts screenings thrice daily. They analyze volumetric regions around the primary object to determine if any cataloged objects penetrate this volume, which is then assessed as conjunction threats [20]. Finding out which space objects will be in close vicinity to a protected space asset, like one of NASA's scientific missions, approximately one week from now is the first stage in collision avoidance orbital safety [21]. The expected future positions of the protected asset (referred to as the "primary" object) and every other known object in space must be known to detect these close approaches [21]. The protected asset is typically an operational payload, and these satellites generate predicted ephemerides that show the payload's "state" (position and velocity) and "covariance" (expected uncertainty about that state) at regular intervals several days into the future (typically one-minute time steps in Low Earth Orbit, or LEO) [21]. For the remaining known objects, positional statements are either ephemerides created from a space catalog updated with tracking information gathered by space surveillance sensors, or ephemerides that the objects themselves produce and communicate if they are operational satellites [21]. NASA uses data from the US Department of Defense space catalog; other countries and certain commercial companies also have space catalogs [21]. Building a volumetric zone around the position of the primary item, shifting that volume along the anticipated ephemeris, and assessing whether any of the other listed objects pierce this volume are the usual steps in Conjunction Assessment (CA) screens [21], [22].

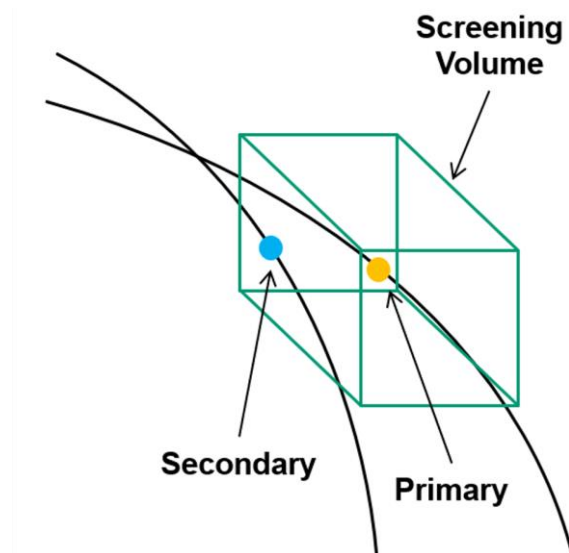


Figure 4. Conjunction Assessment by volume screening [21].

Conjunctors are things that have been detected as potentially dangerous conjunctions; they are also referred to as "secondary" objects in conjunctions, as opposed to the primary object, which is the asset that is protected [21], [22]. Every screening volume penetration for the CA screenings conducted by the US Space Force's 18th Space Defense

Squadron, which currently serves as the primary source of screening data for the space community, produces a Conjunction Data Message (CDM) [21], [22], [23]. The states and covariances at the two objects' "Time of Closest Approach (TCA)", conjunction summary parameters, and additional details of the orbit determination utilized to get these states and covariances are all included in this communication [21], [23]. It's critical to understand that CDMs are not collision warnings because no real collision risk assessment was done to create them [21], [23]. Instead, they serve as proximity alarms, highlighting circumstances that may lead to a dangerously near encounter [21], [23]. Similar to the backup radar used in high-end vehicles, when the radar begins to beep, it does not indicate imminent collision; rather, it alerts the driver to a potential crash situation and suggests that they take extra precautions [21]. For this reason, any CDMs that are received must go through a collision risk assessment process at the following stage in the CA process [21], [23].

2.2.2. Close Approach Risk Assessment

The second step in CARA conjunction assessment risk analysis is the Close Approach Risk Assessment [24]. Conjunction Assessment (CA) screening results only reveal that two items will be close to one another; they do not accurately estimate the collision risk [21]. Risk is determined by combining the probability of an unpleasant event with its predicted effect; hence, it is necessary to assess both aspects of a satellite conjunction to calculate the risk [24]. A vital variable in spacecraft collision avoidance problems is the "miss distance", which is the distance that separates the primary object (the satellite) and the secondary object which is the threat that could be a non-active satellite or space debris.

The miss distance is calculated from two characteristic points of the first and second objects. Due to the complex geometrical shapes of both objects, the need for a minimized shape that can describe and simulate the satellite and the space threat is a must. Therefore, the Hard Body Radius (HBR) notion is introduced as the smallest radius of a sphere that could encompass the entire object. There are two HBRs in a collision assessment, one for the satellite and the other for the space threat, and the distance between each sphere center (defined based on the HBR definition) is the miss distance.



Figure 5. Miss distance and HBRs of primary and secondary objects [25].

Based on the provided definitions of HBR and miss distance, to have a collision between the two objects, the below equation (1) needs to be fulfilled at TCA:

$$Miss\ Distance = r_{miss} < HBR = HBR_1 + HBR_2 \quad (1)$$

There are other methods to construct the HBR for asymmetric-shaped primary satellites. For example, a 3D CAD model can be designed for the satellite then a simplified version of the model is constructed. The projected cross-sectional area, which can be more than double the defined circumscribing circle, is calculated using a mix of orthonormal functions over a unit sphere and spherical harmonics in conjunction with the "Ray Tracing" approach [26].

When two objects in space come close to each other, CDMs are generated. These messages include information about the encounter, such as the time, predicted positions, miss distance, and covariance matrices. The covariance matrix represents the uncertainty associated with the position determination of an object at the time of the encounter [27]. When using a 3D method, the covariance matrix is shown as an error ellipsoid, and it is assumed that the object's HBR could be anywhere inside the ellipsoid [28]. Therefore, there are two error ellipsoids, one for the primary object and the other for the secondary object. For calculation simplification, a 2D representation of the covariance matrix as an ellipse is commonly used [19]. In general, the covariance matrix on (x,y,z) coordinate system is defined as:

$$C = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_y^2 & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_z^2 \end{bmatrix} \quad (2)$$

Where $var(i) = \sigma_i^2$ and $cov(i, j) = \sigma_{ij}$.

To better estimate the results of uncertainty associated with the position determination of the primary or secondary object, a 1-sigma ($1-\sigma$ or 1σ) confidence interval is used, which represents a range around the mean (average) value in a normal distribution where 68% of the data points lie within this range [29]. Therefore, the representation of the covariance matrix is a 1-sigma error ellipsoid.

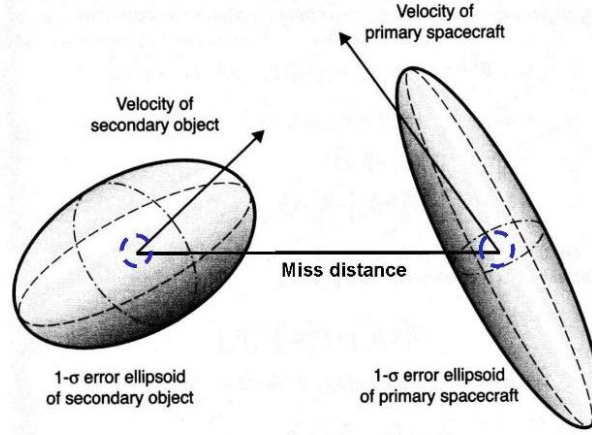


Figure 6. $1-\sigma$ error ellipsoids of both primary and secondary objects with their velocity vectors (blue dashed circles are the HBRs of the objects) [28].

The two objects' coordinates are subtracted to provide the relative position vector, whose magnitude is equal to the miss distance between them (at TCA) [28]. The covariances of the two objects can also be combined to create a joint (relative) covariance, as shown in Equation (3), since the relative position uncertainty is more important than the absolute position uncertainty. This covariance can then be assigned, if desired, to one "end" of the relative position vector (usually the end for the secondary object), and Chan's book "Spacecraft Collision Probability" (2008) provides proof of this [28].

$$C_r = C_1 + C_2 \quad (3)$$

Assuming independence and combining the covariances in this manner is often acceptable [28].

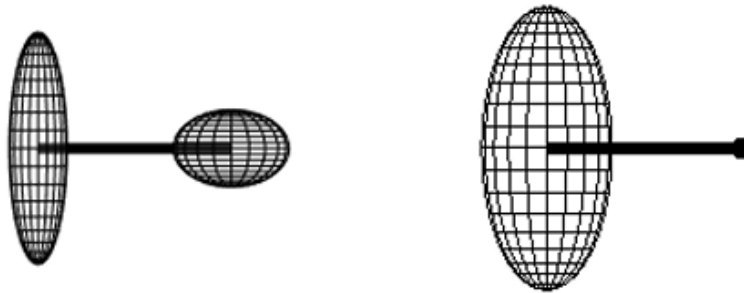


Figure 7. Relative positions and uncertainties of the two objects [19].

After having the relative position uncertainties of the two objects, the combined size of the two needs to be modeled. As described in the notion of the HBR, placing a circumscribing sphere around the primary object is the initial stage [19]. The procedure is then repeated for the secondary, which is usually a debris object for which the exact size is unknown and frequently entails an estimation technique [19]. A possible collision has been detected if at any time throughout the encounter, these two spheres approach and start to overlap. The size spheres of the two objects can be joined to form a "collision sphere" and positioned at one end of the relative miss vector, which is typically the principal object end [19]. If the miss vector dropped to a size smaller than the collision sphere's radius, the two original spheres would intersect at TCA [19].

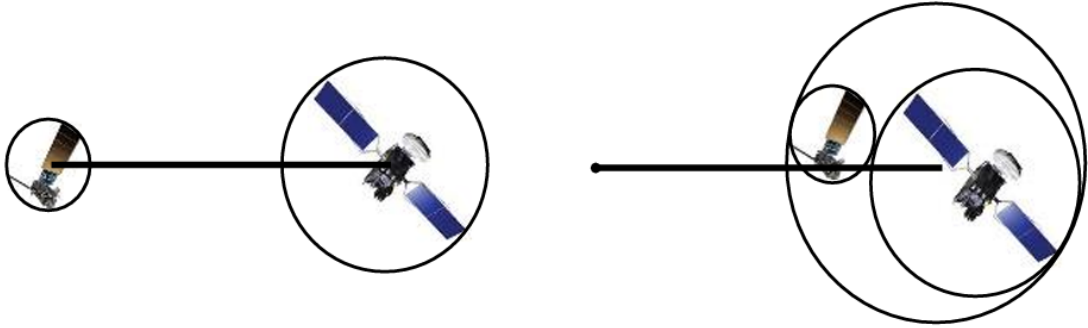


Figure 8. Combining the Sizes of the Primary and Secondary Objects [19].

Therefore, equation (1) can be rewritten to equation (4) which describes the same collision notion but considering the collision sphere technique.

$$r_{miss} < r_{collision} \quad (4)$$

Where $r_{collision}$ and r_{miss} are the radius of the collision sphere and the miss distance, respectively.

The procedure can be resumed as shown below by having the trajectory of the primary object and the collision sphere (representing both objects' combined size) form a "soda straw". The secondary object is potentially anywhere in the relative conjunction frame ellipsoid away from the "soda straw" by the miss distance. The probability of collision is then defined by the chance that the secondary object falls along the cylindrical path of the "soda straw".

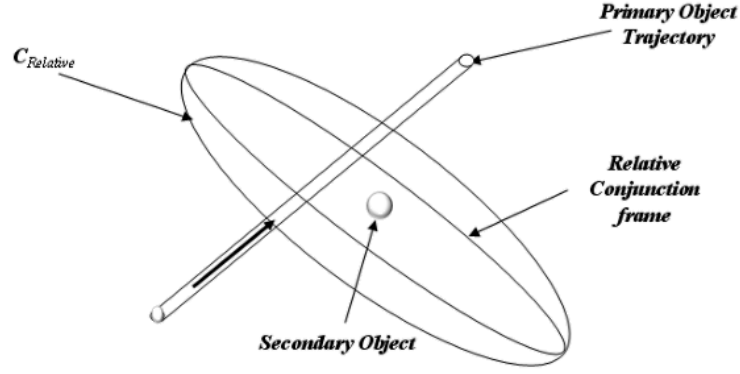


Figure 9. 3D geometry of collision at TCA [19].

In an (x,y,z) coordinate system, the 3D collision geometry at TCA can be simplified by considering the direction of the “soda straw” as one of the axes of the coordinate system (e.g. z -axis) [19]. Thus, the problem can be reduced to a 2D problem where the secondary object is placed at the origin of the coordinate system and the cylindrical path of the primary object is shown as a circle with a radius equal to the collision radius (a cross-sectional profile of the “soda straw”) [19]. On the left side of Figure 10, the 2D model is shown where the primary object is placed at the point $(r_{\text{miss}}, 0)$. On the right side, the coordinate system comprises the principal axes (x', y') that render the combined covariance ellipse with no rotation angle because the covariance matrix's cross terms are all equal to 0. The position of the primary object is defined by the miss distance vector $(x_{\text{miss}}, y_{\text{miss}})$ [19].

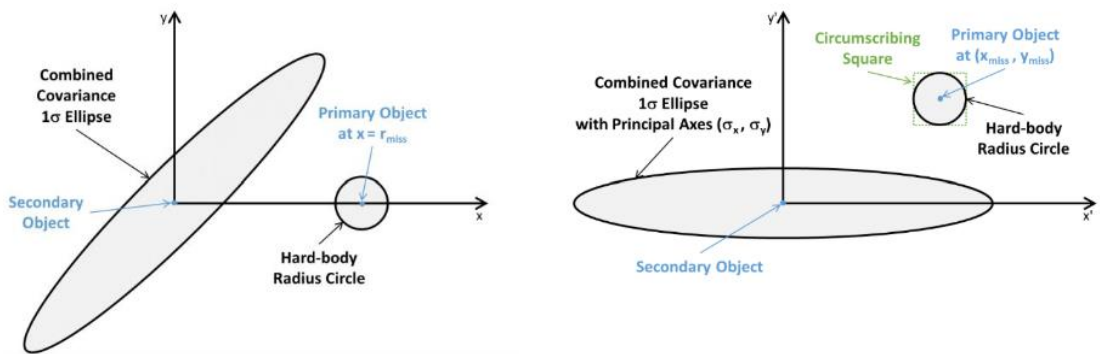


Figure 10. Two Equivalent Representations of the Conjunction Plane [19].

The amount of joint covariance probability density falling within the hard-body-radius circle is comparable to the collision probability P_c , which is now the frequency with which the miss distance vector will fall within the hard-body-radius circle [30]. The

following two-dimensional integral provides the amount of the projected joint covariance's probability density that falls inside the hard-body-radius circle since it reflects a bivariate Gaussian distribution of Foster and Estes, 1992 [30].

$$P_c = \frac{1}{\sqrt{\det(2\pi\mathbf{C})}} \iint_A e^{\left(-\frac{\mathbf{r}^T \mathbf{C}^{-1} \mathbf{r}}{2}\right)} dx dy \quad (5)$$

Where $\mathbf{r}$ is the 2×1 miss vector, $\mathbf{C}$ is the 2×2 covariance matrix, and A is the hard-body-radius circular area [30].

The likelihood of a collision The most common technique for estimating the probability of a satellite collision is P_c , which calculates the frequency at which the miss distance vector will fall inside the hard-body-radius circle [19], [24]. Because the ephemerides used to detect the conjunction represent forecasts with uncertainty, the likelihood calculation takes on a probabilistic shape [19], [24]. By considering these two paths and their uncertainties, one can compute the probability of collision [19], [24]. The conjunction plane method of P_c calculation was created for the Space Shuttle Program and was initially published in the literature in 1992 (Foster and Estes) [30]. It is by far the most popular method in the conjunction evaluation industry. Several significant treatments have been conducted since then, such as those by “Akella and Alfriend (2000)” [31], “Patera (2001)” [32], “Alfano (2005)” [33], “Chan (2008)” [28], “Garcia-Pelayo” (2016) [34], and Elrod (2019) [35]. However, all these studies have a common methodology, which is the use of plausible assumptions to facilitate analytical approximations [19]. All these methods—while differing in specifics—combine the idea of estimating the P_c by integrating over a two-dimensional area on a conjunction encounter plane [19]. Acknowledging that P_c computations rely on the relative positions and uncertainty of the two objects is the first step, which is why adopting a paradigm that approaches the issue in this manner is beneficial [19].

2.2.3. Mitigation Planning of Close Approaches

The third step in CARA collision assessment is planning for mitigation of close approaches that show a high degree of collision risk and hence should be anticipated to be started around three days before the time of closest approach (TCA) [36]. At this stage, planning entails starting to talk about strategies for mitigation. Since most protected assets have active propulsion systems, they can reduce the risk of high-risk situations by changing their trajectory and avoiding the secondary item with a propulsive maneuver [36]. A smaller group of missions modify their trajectory by reorienting their satellite to

alter the frontal area in the velocity direction, which in turn alters the atmospheric drag and, consequently, the satellite trajectory. This technique is known as differential drag [37]. Reorienting their satellite to face the direction of the approaching secondary object will lessen the chance of a collision even for a much smaller fraction incapable of modifying its trajectory [37].

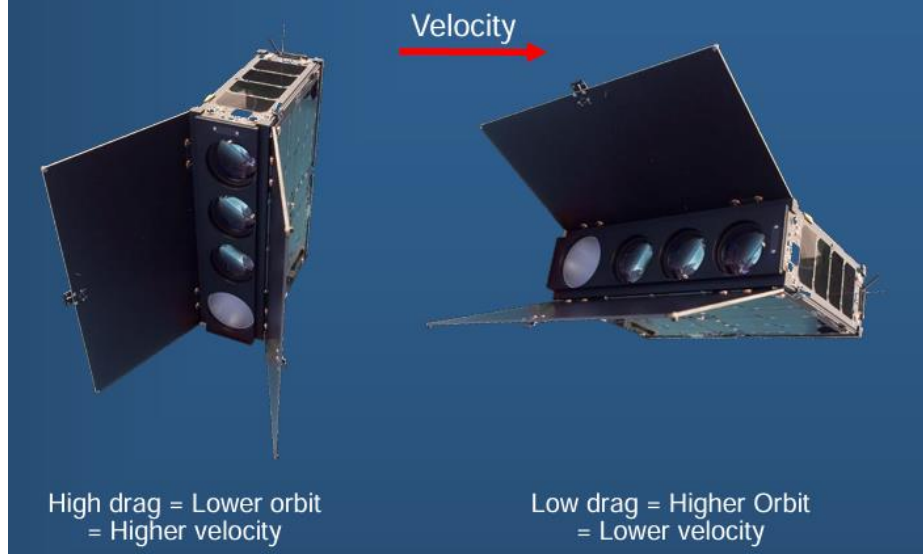


Figure 11. Differential Drag for Small Satellites [37].

2.3. System Dynamics for Satellite Collision Avoidance

2.3.1. Motion of LEO Satellites

The dynamic model of a standard orbiting satellite in LEO needs to be set first to develop the collision avoidance maneuver by introducing the change in its control variables. A satellite with mass “m” is considered orbiting the Earth at a distance “r” from the center of the Earth [38]. The satellite has a centripetal acceleration directed to the center of the Earth. Earth’s gravity is one of the forces acting on the satellite [38]. From Newton’s second law of motion considering the gravitational force only the force applied on the LEO satellite:

$$\vec{F}_g = m \cdot \frac{d^2 \vec{r}}{dt^2} = - \frac{G \cdot M \cdot m}{r^3} \vec{r} \quad (6)$$

Where $\vec{r}$ is the position vector of the satellite relative to the center of the Earth, G is the gravitational constant, and M is the mass of the Earth. Since the satellite trajectory in LEO could be elliptical or nearly circular [1], the equation defining the length of the orbit varies depending on the geometrical shape of the trajectory.

From Equation (6), a differential equation of $\vec{r}$ can be formed which describes the motion of the LEO satellite in an inertial reference frame yet neglecting the disturbance forces.

$$\frac{d^2\vec{r}}{dt^2} = -\frac{GM}{r^3}\vec{r} \quad (7)$$

Another force that needs to be considered for LEO satellites is the aerodynamic drag force, which is expressed as follows [39]:

$$\vec{F}_{drag} = -\frac{1}{2}C_d \cdot \rho \cdot v^2 \cdot A \cdot \hat{v} \quad (8)$$

Where C_d is the drag coefficient, ρ is the atmospheric density, v is the satellite's velocity relative to the atmosphere, A is the cross-sectional area of the satellite, and $\hat{v}$ is the unit vector in the direction of the satellite's velocity [39]. Like other possible disturbances, the aerodynamic force (drag) is considered a disturbance force on the orbit.

All the disturbances on the orbit are non-conservative forces and could be added to the Equation (7) as additional force terms. Therefore, the equation of motion of a LEO satellite considering all disturbances is as follows:

$$\frac{d^2\vec{r}}{dt^2} = -\frac{GM}{r^3}\vec{r} + \vec{F}_{dist} \quad (9)$$

Where $\vec{F}_{dist}$ is the sum of all disturbances modeled as accelerations (m/s²):

$$\vec{F}_{dist} = \vec{F}_{drag} + \vec{F}_{Moon} + \vec{F}_{Sun} + \vec{F}_{solar} \quad (10)$$

Where $\vec{F}_{drag}$, $\vec{F}_{Moon}$, $\vec{F}_{Sun}$, $\vec{F}_{solar}$ are the disturbances due to the aerodynamic drag, gravitational forces of the Moon, the Sun, and the solar radiation force.

2.3.2. Gravitational Potential Energy

On the other hand, the same Equation (9) can be formulated in terms of energy, including the gravitational potential energy (U), where the gravitational force is often represented in terms of the gradient of “ U ” with respect to the position. The gravitational potential energy at a distance “ r ” from the center of the Earth is given by:

$$U(r) = -\frac{G \cdot M \cdot m}{r} \quad (11)$$

The initial and final positions of the satellites' path determine the change in potential energy related to the gravitational force. Thus, the gravitational force is equal to the negative gradient of the gravitational potential energy:

$$\vec{F}_g = -\nabla U(r) \quad (12)$$

Therefore, the equation of motion (9) can be reformulated as follows:

$$m \frac{d^2\vec{r}}{dt^2} = -\nabla U(r) + \vec{F}_{dist} \quad (13)$$

Where Earth is not a perfect sphere; it's slightly flattened at the poles due to its rotation. This deviation from a perfect sphere is known as oblateness. The geopotential model helps us understand Earth's gravitational field and involves analyzing the effects of Earth's gravitational pull.

The integral of the gravitational forces across the whole field—which are the product of the centrifugal force resulting from the rotation of the globe and its atmosphere and the gravitational forces due to Newtonian attraction—is known as the geopotential, or potential function of the earth “ u ”. As a result, the geopotential is equal to the product of the centrifugal force and gravitational field potentials [40].

$$u = -\frac{\mu}{r}(1 + \psi) + \frac{1}{2}\omega^2 r^2 \cos^2 \varphi \quad (14)$$

Where $\mu = GM$, φ geocentric latitude of the satellite, ω is the rotation of the Earth's coordinate system, and the modifier ψ explains why the earth's internal mass distribution is not completely homogeneous and its shape is oblate rather than spherical [40]., which defies fundamental principles of the Newtonian potential [40]. Equation (14), which applies to exo-atmospheric (free-flight) flight phases, represents the gravitational potential appropriate to inertial coordinates if the rotational term ($\frac{1}{2}\omega^2 r^2 \cos^2 \varphi$) is removed [40]. The term is kept for geodesy and all other applications that need a rotating earth coordinate system [40].

The gravitational potential is a function that can be written in the form of $F = R(r) \cdot \Phi(\varphi) \cdot \Lambda(\lambda)$, where (r, φ, λ) are the spherical coordinates, which satisfies the partial differential equation of Laplace ($\nabla^2 u = 0$) [40]. Thus, the harmonic function can be expressed analytically as the sum of a collection of two-dimensional (spherical) harmonics, much like the one-dimensional Fourier series, which depicts a nonanalytical function on a circle [40], and since the spherical harmonics that describe the gravitational potential of the Earth are a function of two variables; the longitude of the satellite is λ , and its geocentric latitude φ , then it involves the so-called “Associated Legendre Polynomials” [41]. The geopotential can be written in the form of spherical harmonic functions [41]:

$$\begin{aligned} f_1(x, y, z) &= \frac{1}{r^{n+1}} P_n^m \cdot \sin\varphi \cdot \cos(m\lambda), \quad 0 \leq m \leq n, \quad n = 0, 1, 2 \dots \\ f_2(x, y, z) &= \frac{1}{r^{n+1}} P_n^m \cdot \sin\varphi \cdot \sin(m\lambda), \quad 1 \leq m \leq n, \quad n = 1, 2 \dots \end{aligned} \quad (15)$$

Where (r, θ, λ) are the spherical coordinates, P_n^0 are the Legendre polynomials [41], and P_n^m for $1 \leq m \leq n$ are the associated Legendre functions [41].

Equation (14) describes the expansion of the “perturbative parameter” ψ of the gravitational potential in an infinite series of spherical harmonics [41].

$$\psi = \sum_{n=1}^{\infty} \sum_{m=0}^n (\bar{R}_e/r)^n \cdot P_n^m(\sin\varphi)(C_n^m \cos(m\lambda) + S_n^m \sin(m\lambda)) \quad (16)$$

Where $\bar{R}_e$ is the mean equatorial radius of the Earth, $P_n^m(\sin\varphi)$ are Associated Legendre Polynomials of degree n and order m , and S_n^m and C_n^m are coefficients that are constants found by satellite data analysis [42]. Legendre polynomials are used to represent the latitude terms, Fourier series are used to model the longitudinal terms, and corresponding Legendre functions are used to combine the terms [42]. Based on Equation (14) and (16), the potential function is a function of combinations of harmonic functions since the earth is not symmetric.

$$u = \frac{\mu}{r} \left[1 + \sum_{n=1}^{\infty} \sum_{m=0}^n \left(\frac{\bar{R}_e}{r} \right)^n P_n^m(\sin\varphi)(C_n^m \cos(m\lambda) + S_n^m \sin(m\lambda)) \right] + \frac{1}{2} \omega^2 r^2 \cos^2 \varphi \quad (17)$$

As is evident, the perturbative potential is only accurately described by the infinite series of harmonics that make up the second aspherical half of Equation (17) when an infinite number of factors are considered [41]. Furthermore, it can be observed that for the first family that exists, when $m = 0$, $1 < n < m$, the terms are taken by families based on the indices, therefore, for the purely gravitational potential, the Equation (17) can be reduced to:

$$u_{m=0} = \frac{\mu}{r} \left[1 + \sum_{n=0}^{\infty} \left(\frac{\bar{R}_e}{r} \right)^n P_n^0(\sin\varphi) \right] \quad (18)$$

Where $P_n^0(\sin\varphi)$ are the (simple) “Legendre Polynomials” of argument $\sin\varphi$, usually referred to as “Zonal Harmonics” [41]. The potential $u_{m=0}$ is independent of the longitude λ , thus, continuous along the parallels of latitude. Inferred is that the sphere's harmonic undulations alternately become positive and negative between these n parallels at a distance π and are equal to zero along parallels [40].

It is recommended by IAU to substitute the term C_n^0 by $(-J_n)$ [43].

$$J_n = -C_n^0 \quad (19)$$

Setting $m = 0$ in Equation (17) assumes that the earth exhibits rotational symmetry, which is not exactly the case. The rotational axis system and the genuine earth's major axes of inertia do not line up exactly [40]. As a result, the earth's angular momentum and angular

velocity do not point precisely in the same direction. As such, greater harmonics indicate the existence of inertia products. If terms that include the longitude λ as a second variable are added to the second portion of Equation (18) by allowing m to assume integral values from 1 to n . As a result, coefficients J_n^m as well as the Associated Legendre Polynomials $P_n^m(\sin\varphi)$ are needed. Except for the two terms that arise from letting $m = n$ ($C_n^n \cos(m\lambda)$ and $S_n^n \sin(n\lambda)$) which are called Sectorial Harmonics, all the other terms are known as Tesseral Harmonics.

Table 1 mentions the ratio to 2-body dynamics contribution of each gravitational and non-gravitational terms that is applied on the satellite.

Term	Contribution
2-body (Earth)	100%
J2 term (Earth)	0.14%
J3 term (Earth)	0.00081%
J4 term (Earth)	0.00101%
Sun	0.000006%
Moon	0.000015%
Aerodynamics	0.00659%
Solar Radiation	0.00068%

Table 1. Ratio to 2-body dynamics contribution of each gravitational and non-gravitational term.

2.4. Multi-Objective Optimal Control for Maneuver Execution

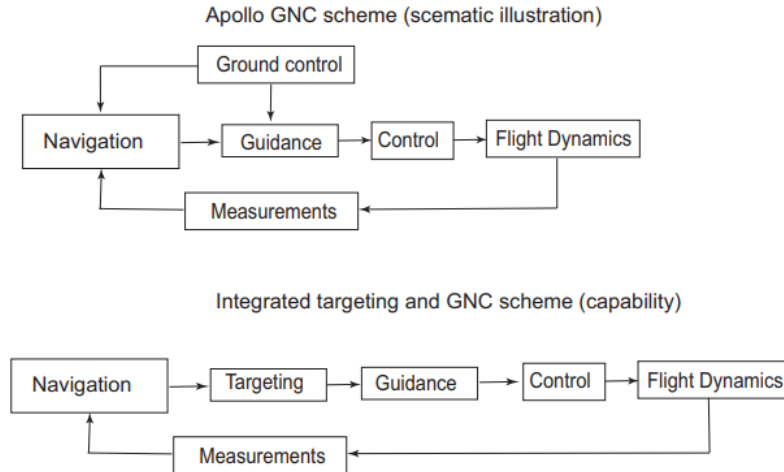
2.4.1. Overview of Guidance, Navigation, and Control (GNC) Systems

In recent years, significant advancements have occurred in developing lightweight micro-thrusters. With an increasing number of satellites orbiting Earth, there is a critical demand for rapid and responsive maneuvers to adapt to changing conditions. Consequently, satellites require thrusters for swift attitude adjustments, whether aligning solar panels or avoiding collisions [44]. This review aims to spotlight research in attitude Guidance, “Navigation, and Control (GNC)” for orbital satellites. Although some aspects of the current state-of-the-art Guidance, Navigation, and Control (GNC) systems for small satellites have been demonstrated in missions like PRISMA, FASTRAC, and OCULUS, these missions do not utilize real-time GNC software for attitude maneuvers [45]. Innovative ideas and self-governing GNC system topologies can improve attitude maneuver precision, effectiveness, and adaptability to overcome these obstacles [45]. Processing

lidar, inertial measurement unit (IMU), and GPS data yields real-time navigation solutions [44]. By integrating real-time GNC functionality, these techniques enhance the current state-of-the-art on-board Guidance, Navigation, and Control (GNC) technologies [46]. Developing onboard software to control and guide spacecraft using navigation solutions is essential.

Figure 12. Functional diagrams of conventional GNC and GNC [44].

2.4.2. Control Schemes and Actuators



Satellite control can be performed in different approaches with the same results. These control strategies will mostly concentrate on navigation filters or control mechanisms. They frequently either assume that the sensors are faultless or estimate the position and orientation of the satellite using an instrument like an Extended Kalman Filter [47]. These tests used a variety of torque sources, including response wheels, magnetic torque rods, and even a mini-thruster. When used with a Kalman Filter, it was discovered that the Riccati controller (State Dependent Riccati Equation) outperformed the Linear Quadratic Regulator [48]. If the desired state feedback control system is included to gain satellite control autonomy, the controller must be calibrated to avoid overshooting and settling in the cycle. Possible satellite controllers are summarized in the table below:

Type of Controller	Description
PID controller	These controllers have developed into incredibly practical and efficient devices that are utilized in a variety of controllable systems, including submarines, unmanned aircraft vehicles, spaceships, and mussels. Their effectiveness depends on our ability

	to choose the coefficients freely, which enables us to meet a range of criteria and terminal conditions.
Classical Linear Controller	Represented as $u = -K \cdot x$, where x is the state vector and K is the controller gain. It has been demonstrated that these controllers are effective and valuable for real-time applications.
H-infinity Controller	Although the effectiveness of these controllers has been demonstrated in certain systems, their usefulness for onboard systems is severely hampered by the need for extensive analysis.
Other nonlinear controllers	The relevance of the canonical equations to current nonlinear control problems, which are usually shown to be integrable by numerical methods [49], [50] .

Table 2. Different types of satellite system controllers and their descriptions.

As can be observed in numerous works [51], [52], the Euler angles serve as the foundation for the kinematic models. But, if necessary, this can subsequently be changed into a quaternion, transformation matrices, or Euler-Rodrigues parameters. Simultaneously, the satellite's body can be represented as either a flexible body if it has solar panel booms or a stiff body, which is more typical for smaller spacecraft.

There are numerous techniques for controlling satellites. Thrusters are typically turned on or off using a control. The control mechanism is the primary distinction. “Pulse-Width Pulse-Frequency Modulation” [53] can be implemented to determine when the thruster is operational and can be adjusted by modulation. Due to the excessive thruster action, control techniques such as bang-bang will not be included in this research [54].

In recent years, research has been conducted on satellite thrusters, where new technologies emerged based on the size, mass, and satellite mission. In the table below, a summary of the most common thruster types and their functionalities.

Thruster type	Description
Cold Gas Thrusters	• These thrusters use pressurized gas (such as nitrogen or xenon) expelled through a nozzle to generate thrust. • They are simple, reliable, and suitable for small satellites. • It is ideal for attitude control and orbit adjustments. • Commonly used in CubeSats and nanosatellites [55].
Bi-Propellant Thrusters	• Combine two chemical propellants (oxidizer and fuel) for higher efficiency. • Used for precise orbital maneuvers and station-keeping. • Commonly found on larger satellites.
Resistojet Thrusters	• Heat a resistive element to vaporize a propellant (usually water). • Generate low thrust but are efficient for long-duration missions. • Suitable for small satellites.
Pulsed Plasma Thrusters (PPTs)	• Use ionized gas (plasma) accelerated by an electric field. • Provide moderate thrust and specific impulse. • Suitable for CubeSats and microsatellites [56].
Hall Effect Thrusters	• Use closed electron drift and feature high thrust density and specific impulse.

	• Efficient for deep-space missions. • Commonly used on larger satellites [57].
Ion Thrusters	• Accelerate ions using high electric potential applied to a metallic grid. • High specific impulse but low thrust density. • Used for long-duration missions and station-keeping.

Table 3. Different satellite thrusters and their description.

2.4.3. Navigation and Sensors

The satellite's status must be understood by the GNC system. To bring the satellite to its desired condition, the guidance system must know, or at least estimate, its present angular location, velocity, and acceleration.

As mentioned previously in the control schemes and actuators section, Euler angles are the easiest way to represent and model satellite kinematic attitude, yet a Gimbal lock can occur when two of the three rotational axes become aligned, resulting in a loss of one degree of freedom [58]. However, because quaternions are free of this issue, they are frequently utilized in a variety of missions, such as the “Student Space Exploration and Technology Initiative (SSETI)” of the “European Space Agency (ESA)” [59].

Multiple sensors can be used to determine the exact orientation of a satellite such as sun sensors that work based on the magnitude of sunlight hitting the satellite [60], or star sensors can also be used based on the detection of the location of a specific star due to its brightness [60].

3. THREAT ANALYZER FOR COLLISION PROBABILITY EVALUATION

3.1. Threat Analyzer Setup

The threat analyzer should be autonomous, operating independently in the satellite without human intervention. The objective is to design a system that analyzes error measurement data from nearby objects, providing an optimal estimate of the aggregated measurements from all sensors along with the associated error covariance. Additionally, the system should define a relationship between error covariances to better enhance the threat tracking and mitigation.

In many systems, different sensors operate on different principles (e.g., radar vs. lidar, infrared vs. visual camera), and each has its own unique sources of error: different fields of view, resolution, or sensing capabilities. By combining the measurements and error covariances of multiple sensor modalities, it is possible to reduce the overall estimation error. When the system is tracking an object over time, combining the sensor error covariances leads to more accurate tracking by reducing the uncertainty in the object's trajectory. This is particularly useful for tracking moving objects, where errors in position and velocity estimates can compound over time. If the errors of the sensors are independent, combining measurements from multiple sensors can lead to a smaller overall error.

In this setup, a and x represent the measurement as observed from a sensor A and the combined estimate of all sensors, respectively. For a perfect measurement, the observed measurement should equal the combined estimate of all sensors $a = x$. However, due to measurement errors, the error in the estimated measurements $\hat{a}$ and $\hat{x}$ are denoted by $\tilde{a}$ and $\tilde{x}$ respectively. The individual measurement of the sensor A is combined such as $a = \hat{a} + \tilde{a}$, and the combined measurement of all sensors is $x = \hat{x} + \tilde{x}$.

Given estimate measurement from the sensor A , denoted as $\hat{a}$, with error covariance $P_a = E(\tilde{a}\tilde{a}^T)$ (where $E(.)$ denotes the expected value or statistical expectation, and $\tilde{a}\tilde{a}^T$ is the outer products of the error vector), the aim is to find the best estimate $\hat{x}$ of the measurement by combining the information from all sensors, as well as the associated error covariance $P_x = E(\tilde{x}\tilde{x}^T)$.

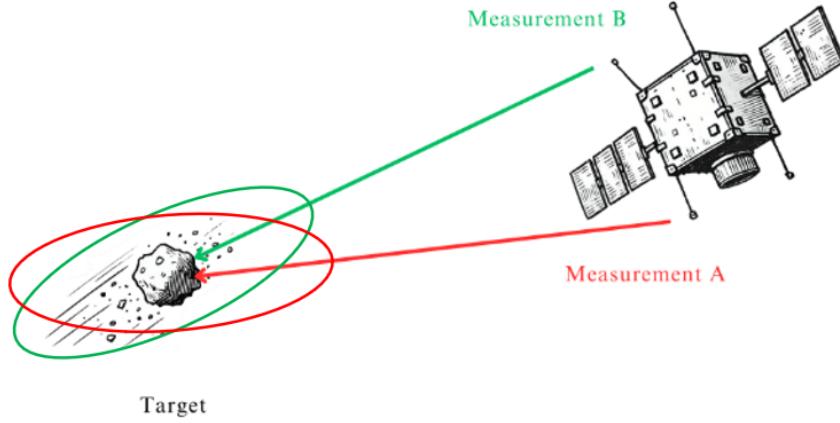


Figure 13. The setup of combining simultaneously measurements and their covariances with state estimates.

3.2. Combined Covariance Matrix and Error Measurement

All mathematical steps taken in this subsection have been verified and proven in multiple research work [61], [62], [63].

Let the estimated measurement and its error covariance from the m^{th} sensor from a total of N sensors be denoted as $\hat{\underline{a}}_m$ and P_m respectively, with components $\hat{a}_m^k$ and $P_m^{ij} = E(\tilde{a}_m^i \tilde{a}_m^j)$, ($1 \leq i, j, k \leq N_e$ and $1 \leq m \leq N$), where N_e is the number of measurement vector elements for each sensor. Also, the components of $\hat{\underline{x}}$ and P_x are denoted as $\hat{x}^r$ and $P_x^{pq} = E(\tilde{x}^p \tilde{x}^q)$, ($1 \leq p, q, r \leq N_e$).

The combined estimated measurement $\hat{\underline{x}}$ is calculated by the linear combination of all $\hat{\underline{a}}_m$ with unknown transformation matrices K_m :

$$\hat{\underline{x}} = \sum_{m=1}^N K_m \hat{\underline{a}}_m \quad (20)$$

In an exact or perfect measurement ($\underline{a} = \underline{x}$):

$$\begin{aligned} \underline{x} &= \hat{\underline{x}} + \tilde{\underline{x}} \\ \underline{x} &= \hat{\underline{a}}_m + \tilde{\underline{a}}_m \end{aligned} \quad (21)$$

From Equation (20) and (21):

$$\begin{aligned} \underline{x} - \tilde{\underline{x}} &= \sum_{m=1}^N K_m (\underline{x} - \hat{\underline{a}}_m) = \sum_{m=1}^N K_m \underline{x} - \sum_{m=1}^N K_m \hat{\underline{a}}_m \\ \tilde{\underline{x}} &= \sum_{m=1}^N K_m \hat{\underline{a}}_m + \left(J - \sum_{m=1}^N K_m \right) \underline{x} \end{aligned} \quad (22)$$

In a no-bias measurement from sensors, $E(\tilde{\underline{a}}_m) = 0$, the estimate of the combination must be unbiased as well $E(\hat{\underline{x}}) = 0$, therefore, the sum of all K_m in an exact or perfect measurement must equal to 1:

$$\sum_{m=1}^N K_m = 1 \quad (23)$$

The above rule must be followed to make sure the error in the combined measurement estimate does not depend on the actual true value. This assumes that each measurement estimate also has errors that do not rely on the true value. By true measurement, we mean the value that would be recorded if the radars were perfectly accurate.

The composite estimate measurement is:

$$\begin{aligned} \hat{\underline{x}} &= \sum_{m=1}^N K_m \hat{\underline{a}}_m = \sum_{m=1}^{N-1} K_m \hat{\underline{a}}_m + K_N \hat{\underline{a}}_N \\ &= \sum_{m=1}^{N-1} K_m \hat{\underline{a}}_m + \left(J - \sum_{m=1}^{N-1} K_m \right) \hat{\underline{a}}_N \\ &= \sum_{m=1}^{N-1} K_m (\hat{\underline{a}}_m - \hat{\underline{a}}_N) + \hat{\underline{a}}_N \end{aligned} \quad (24)$$

The composite error measurement is:

$$\tilde{\underline{x}} = \sum_{m=1}^N K_m \tilde{\underline{a}}_m = \sum_{m=1}^{N-1} K_m \tilde{\underline{a}}_m + K_N \tilde{\underline{a}}_N \quad (25)$$

The expected summed squared error in the composite measurement is expressed using indicial notation, considering that the sensors are independent. The error term e is derived below [61]:

$$\begin{aligned} e &= E(\tilde{\underline{x}}^T \tilde{\underline{x}}) = E(\tilde{x}^r \tilde{x}^r) = \sum_{m=1}^N K_m^{rk_1} \sum_{n=1}^N K_n^{rk_2} E(\tilde{a}_m^{k_1} \tilde{a}_n^{k_2}) \\ &= \sum_{m=1}^N K_m^{rk_1} \sum_{n=1}^N K_n^{rk_2} \delta_{mn} E(\tilde{a}_m^{k_1} \tilde{a}_m^{k_2}) \\ &= \sum_{m=1}^N K_m^{rk_1} K_m^{rk_2} E(\tilde{a}_m^{k_1} \tilde{a}_m^{k_2}) \\ &= \sum_{m=1}^{N-1} K_m^{rk_1} K_m^{rk_2} E(\tilde{a}_m^{k_1} \tilde{a}_m^{k_2}) + K_N^{rk_1} K_N^{rk_2} E(\tilde{a}_N^{k_1} \tilde{a}_N^{k_2}) \end{aligned} \quad (26)$$

$$= \sum_{m=1}^{N-1} K_m^{rk_1} K_m^{rk_2} P_m^{k_1 k_2} + K_N^{rk_1} K_N^{rk_2} P_N^{k_1 k_2}$$

Since the last expression does not depend on K_l^{uv} , where $1 \leq l \leq N-1$ and $1 \leq u, v \leq N_e$, the derivative of the error e with respect to K_l^{uv} is equal to 0. After derivation, the expression results in [61]:

$$K_l^{uk_2} P_l^{vk_2} + K_l^{uk_1} P_l^{k_1 v} - K_N^{uk_2} P_N^{vk_2} - K_N^{uk_1} P_N^{k_1 v} = 0 \quad (27)$$

Some terms will cancel out since the covariance matrices are symmetric:

$$\begin{aligned} K_r P_r &= K_N P_N \\ K_r &= K_N P_N P_r^{-1} \end{aligned} \quad (28)$$

From Equation (23) and (28), K_N is expressed as:

$$\begin{aligned} K_N &= J - \sum_{m=1}^{N-1} K_m = J - K_N P_N \sum_{m=1}^{N-1} P_m^{-1} \\ K_N &= \left(J + P_N \sum_{m=1}^{N-1} P_m^{-1} \right)^{-1} \end{aligned} \quad (29)$$

Using the mean vector and the covariance matrix formulas of the product of a Gaussian probability density functions [61], $\forall l$, K_l is expressed as follows:

$$K_l = \left(J + P_N \sum_{m=1}^{N-1} P_m^{-1} \right)^{-1} P_N P_l^{-1}, \quad 1 \leq l \leq N-1 \quad (30)$$

Multiplying the equations (29) by P_N^{-1} :

$$K_N = \left(P_N^{-1} + \sum_{m=1}^{N-1} P_m^{-1} \right)^{-1} P_N^{-1} = \left(\sum_{m=1}^N P_m^{-1} \right)^{-1} P_N^{-1} \quad (31)$$

Therefore, the general formula for the covariance matrices is:

$$K_l = \left(\sum_{m=1}^N P_m^{-1} \right)^{-1} P_l^{-1}, \quad 1 \leq l \leq N-1 \quad (32)$$

After solving for the formula of the covariance matrices, the aim is to find the error covariance of the combined measurements from all sensors P_x . From Equation (25), P_x can be expressed as:

$$P_x = E(\tilde{\mathbf{x}}\tilde{\mathbf{x}}^T) = E\left(\sum_{m=1}^N K_m \tilde{\mathbf{a}}_m \sum_{m=1}^N \tilde{\mathbf{a}}_m^T K_m^T\right) \quad (33)$$

$$= \sum_{m=1}^N K_m E(\tilde{\underline{a}}_m \tilde{\underline{a}}_m^T) K_m^T = \sum_{m=1}^N K_m P_m K_m^T$$

From the Equation (32) and (33):

$$\begin{aligned} P_x &= \sum_{m=1}^N K_m P_m K_m^T = \left(\sum_{n=1}^N P_n^{-1} \right)^{-1} \sum_{m=1}^N (P_m^{-1})^T \left(\left(\sum_{m=1}^N P_m^{-1} \right)^{-1} \right)^T \\ &= \left(\sum_{n=1}^N P_n^{-1} \right)^{-1} \end{aligned} \quad (34)$$

Finally, the error covariance of the combined sensors' measurement is [61], [62], [63]:

$$P_x = \left(\sum_{n=1}^N P_n^{-1} \right)^{-1} \quad (35)$$

3.3. Input Matrix of The Threat Analyzer

If the error covariance P_n of the n^{th} sensor is represented as a resistance, then the Equation (35) is similar to the total resistance R_N of N resistors in parallel, $R_N = (\sum_{n=1}^N R_n^{-1})^{-1}$, which can be useful to simplify the association of sensors in terms of covariance to have one input data to the threat analyzer rather than manipulating all inputs of sensors separately.

Since the resistance is the reverse of admittance Y , $Y = \frac{1}{R}$, an analogy between the combined error covariance P_x and the total resistance R_N of N resistors in parallel can be modeled by the admittance matrix Y_N of a linear power system of N bus network of resistors, without reference or ground resistors connected between the nodes and the ground cables, where its elements are defined as follows:

$$y_{ij} = \begin{cases} \sum_{k=1, k \neq i}^N y_{ik}, & \text{if } i = j \\ -y_{ij}, & \text{if } i \neq j \end{cases} \quad (36)$$

$$Y_N = [y_{ij}]$$

Where y_{ij} is the admittance between nodes i and j , $y_{ij} = \frac{1}{R_{ij}}$, and R_{ij} is the resistance between nodes i and j . It is worth mentioning that the admittance matrix is symmetric, thus, $y_{ij} = y_{ji}$ if $i \neq j$.

An example of an admittance diagram of a three-bus network ($N = 3$), without reference or ground resistors connected between the nodes and the ground cables, is shown below:

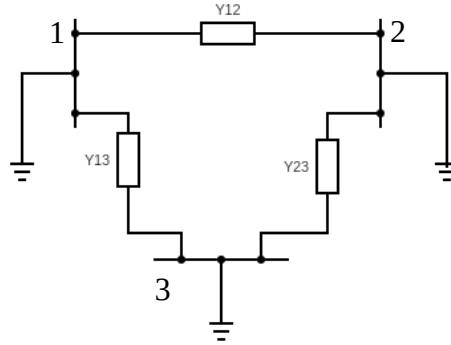


Figure 14. Admittance circuit of a three-bus network (nodes 1, 2, and 3) without ground resistors [64].

The admittance matrix of the Figure 14 is giving as:

$$Y_3 = \begin{bmatrix} y_{12} + y_{13} & -y_{12} & -y_{13} \\ -y_{12} & y_{12} + y_{23} & -y_{23} \\ -y_{13} & -y_{23} & y_{13} + y_{23} \end{bmatrix} \quad (37)$$

Since the combined error covariance of sensors is analogue to an admittance matrix of N bus network without ground resistors, then, the input matrix C_N of the threat analyzer of N sensors can be expressed as:

$$c_{ij} = \begin{cases} \sum_{k=1, k \neq i}^N P_k^{-1}, & \text{if } i = j \\ -P_s^{-1} = -(P_i^{-1} + P_j^{-1}), & \text{if } i \neq j \end{cases} \quad (38)$$

$$C_N = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \dots & c_{1N} \\ \bullet & c_{22} & c_{23} & \dots & c_{2N} \\ \bullet & \bullet & c_{33} & \dots & c_{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bullet & \bullet & \bullet & \dots & c_{NN} \end{bmatrix}, \text{ where } c_{ij} = c_{ji} (i \neq j).$$

Where P_s , based on the Equation (35), is the combined error covariance matrix of a system of two independent sensors ($N = 2$), the i^{th} and j^{th} sensor. Similarly to the admittance matrix, the input matrix of the threat analyzer C_N is symmetric.

The sensors will be distributed in different places on the satellite to cover a 360° view. The input matrix C_N will simplify the threat analysis by decomposing the matrix into submatrices or elements depending on the orientation and position of the target (threat). Depending on the position of the threat with respect to the satellite, some sensors facing the target will need to have their data processed more than others. Therefore, the threat analyzer will read only the necessary column vector or row vector from C_N that has elements corresponding to the needed sensors.

The ideal case is to have a minimum of six sensors ($N = 6$) on the satellite in the shape of an octahedron, while the target is facing mostly certain sensors. Thus, to reduce the calculation error and avoid unnecessary compilations, the threat analyzer will only take the row vector from C_6 that has only the elements of those needed sensors.

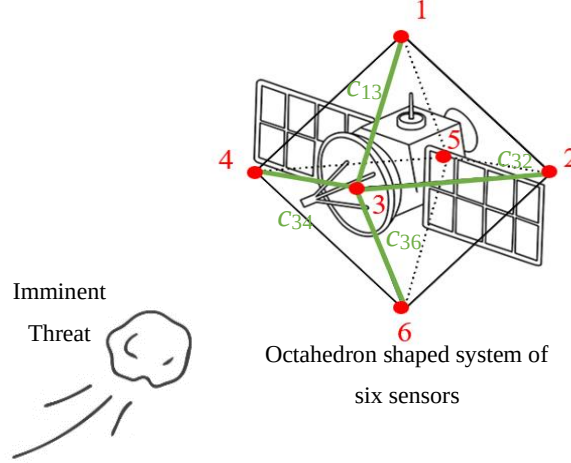


Figure 15. Graphic representation of the mutual combined error covariance matrices elements

$(c_{3j})_{1 \leq j \leq 6, j \neq 3}$ in green, in an octahedron-shaped system of six sensors (drawn in red dots) [65], [66], [67].

According to Figure 15, if the target is facing sensor number 3, the data of the threat needed to be analyzed is coming primarily from sensor 3 and secondly from its adjacent sensors (sensors 1, 2, 4, and 6), therefore, the row vector to be inputted into the threat analyzer, that has the most information about the target, is c_{3j} where $1 \leq j \leq 6$ (also equal to $c_{i3}^T, 1 \leq i \leq 6$).

The symmetric matrix C_6 of six sensors and the primary row vector $(c_{3j})_{1 \leq j \leq 6}$ to be inputted into the threat analyzer, are expressed as follows:

$$C_6 = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ \bullet & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ \bullet & \bullet & c_{34} & c_{44} & c_{45} & c_{46} \\ \vdots & \vdots & c_{35} & \vdots & c_{55} & c_{56} \\ \bullet & \bullet & c_{36} & \bullet & \bullet & c_{66} \end{bmatrix} \begin{matrix} c_{1j} \\ c_{2j} \\ c_{3j} \\ c_{4j} \\ c_{5j} \\ c_{6j} \end{matrix} \quad (39)$$

$$c_{3j} = [-(P_1^{-1} + P_3^{-1}) \quad -(P_2^{-1} + P_3^{-1}) \quad P_1^{-1} + P_2^{-1} + P_4^{-1} + P_5^{-1} + P_6^{-1} \quad -(P_3^{-1} + P_4^{-1}) \quad -(P_3^{-1} + P_5^{-1}) \quad -(P_3^{-1} + P_6^{-1})]$$

It is worth mentioning that in a general case of a threat analyzer's input matrix C_N of N sensors, the primary row vector $(c_{pj})_{1 \leq j \leq N}$ (or column vector $(c_{ip})_{1 \leq i \leq N}$) to be inputted into the threat analyzer, where p is the reference number of the primary sensor (facing the target), has two types of elements based on Equation (38):

- **Combined Allo-Error Covariance Matrix:**

If $j = p$, $c_{pj} = \sum_{k=1, k \neq p}^N P_k^{-1}$ is the combined allo-error covariance matrix as represented in

Figure 16, which is the inverse of the combined error covariance matrix of a system of $N - 1$ independent arbitrarily placed sensors excluding the primary sensor p , $(1, 2, \dots, p - 1, p + 1, \dots, N)$. This element can be useful to keep track of the target while the satellite is performing the collision avoidance maneuver or gain slight supplementary data on the threat depending on the satellite orientation, giving the chance to the feedback controller to adjust the trajectory based on the relative position of the target during the collision avoidance maneuver (when the target is partially surpassed by the satellite and the data of the primary sensor p is partially or no longer needed).

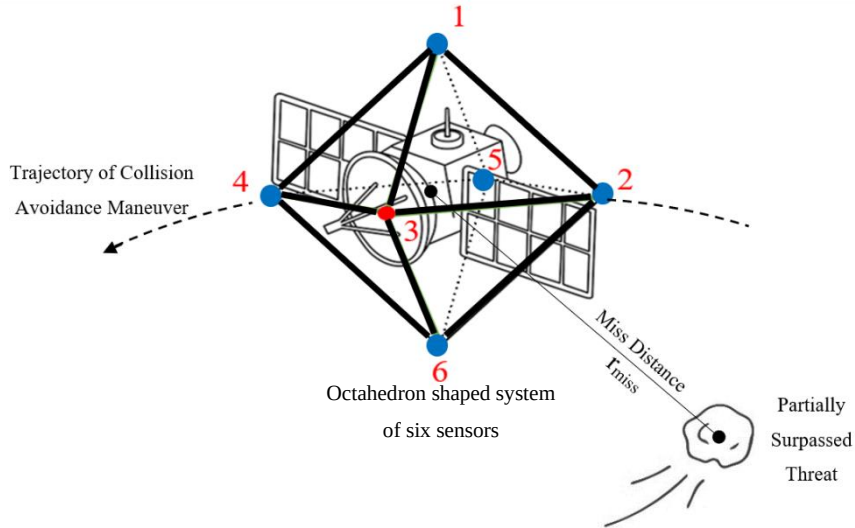


Figure 16. Graphic representation of the sensors (drawn in blue dots) concerned by the combined allo-error covariance matrix element c_{33} while tracking the target to correct the miss distance during collision avoidance [65], [66], [67].

- **Mutual Combined Error Covariance Matrices:**

If $j \neq p$, $(c_{pj})_{1 \leq j \leq N, j \neq p} = -(P_p^{-1} + P_j^{-1})$ are the mutual combined error covariance matrices

as shown in Figure 15 for the case of an octahedron-shaped system of six sensors, which are the negative inverse of the combined error covariance matrix of a system of two independent sensors ($N = 2$), the p^{th} and j^{th} sensor. These elements are necessary to correct the first detection data of the target, depending on the satellite orientation and the relative position of the threat.

4. SYSTEM DYNAMICS FOR SATELLITE COLLISION AVOIDANCE

4.1. Defining System Dynamics

Motions with respect to the Earth's center are described using the spacecraft's local orbital frame. One axis of this is always pointing toward the center of the Earth and is utilized for entities in orbit. The spacecraft's center of mass serves as the coordinate system's origin. The axis X_{L0} is tangent to the vehicle's trajectory and represents the direction of motion since it aligns with the linear velocity vector. Axis Z_{L0} is always oriented directly toward the Earth's center, while Axis Y_{L0} is always perpendicular to the plane on which the orbit is located.

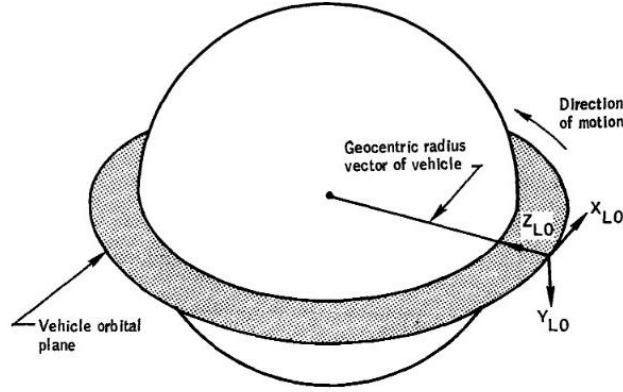


Figure 17. Local Orbit Coordinate System for the Space Shuttle Program [68].

Euler vectors can be used to represent the kinematic attitude of the satellite. Using the three single-axis rotation matrices with angles ψ , θ , and ϕ , the Euler angle can be expressed in terms of angular rotation rate (ω_x , ω_y , ω_z) [44]:

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} -\sin \theta & 0 & 1 \\ \sin \phi \cos \theta & \cos \phi & 0 \\ \cos \phi \cos \theta & -\sin \phi & 0 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} \quad (40)$$

Due to no Gimbal lock issue, the quaternion parameter system can be used too which is composed of three vectors and a rotation about an axis. The quaternion equation is expressed as the following [44]:

$$\mathbf{Q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} a_1 \cos \frac{\phi}{2} \\ a_2 \cos \frac{\phi}{2} \\ a_3 \cos \frac{\phi}{2} \\ \cos \frac{\phi}{2} \end{bmatrix} = \begin{bmatrix} a \sin \frac{\phi}{2} \\ \cos \frac{\phi}{2} \end{bmatrix}, \quad \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1}{2 \sin \phi} \begin{bmatrix} C_{23} - C_{32} \\ C_{31} - C_{13} \\ C_{12} - C_{21} \end{bmatrix} \quad (41)$$

Where C_{ij} are the elements of the Direction Cosine Matrix (DCM) which is the multiplication of the axis rotation matrices.

In the following work, the chosen method to represent the kinematic attitude of the satellite is the three single-axis rotation matrices with angles ψ , θ , and ϕ , which will express the Euler vector in terms of angular rotation rate $(\omega_x, \omega_y, \omega_z)$.

4.2. Control and Objective

The state vector of the satellite can be expressed as $x = [\mathbf{r}, \mathbf{v}, m, \mathbf{q}, \boldsymbol{\omega}]^T$, therefore the equation of motion of the spacecraft can be given as [44]:

$$\dot{x} = f(x, u) = \begin{bmatrix} \mathbf{v} \\ -\frac{\partial U}{\partial \mathbf{r}} + \mathbf{a}_{total} \\ -\frac{\|\mathbf{F}\|}{I_{sp}g_0} \\ \frac{1}{2}\mathbf{B}(\boldsymbol{\omega})\mathbf{q} \\ \mathbf{I}^{-1}[-\boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega}) + \mathbf{T}_{dist} + \mathbf{T}_{thru}] \end{bmatrix}, \quad u = \begin{bmatrix} \mathbf{F} \\ I_{sp} \\ \mathbf{T} \end{bmatrix} \quad (42)$$

Where $\mathbf{a}_{all}$ is the acceleration due to all disturbances and the propulsive thrust, U is the Earth's potential function, $\mathbf{I}$ is the inertia matrix, g_0 is the sea-level gravitational acceleration, $\mathbf{q}$ is the quaternion consisting of quaternion vector (q_1, q_2, q_3) and scalar, q_4 , and u is the control input vector, $\mathbf{F}$ and $\mathbf{T}_{thru}$ are the thrust control and torque control vectors, $\mathbf{T}_{dist}$ is the torque of the perturbation forces [44].

The objective is to bring the system at time t_0 from an initial configuration given by:

$$\Psi_l(x_{01}, x_{02}, \dots, x_{0n}, t_0) = 0, \quad l = 1, \dots, q_1, \quad q_1 \leq n + 1 \quad (43)$$

to the final configuration at time t_1 given by:

$$F_k(x_{11}, x_{12}, \dots, x_{1n}, t_1) = 0, \quad l = 1, \dots, q_2, \quad q_2 \leq n + 1 \quad (44)$$

4.3. Satellite's Dynamic Model

The steps taken in this subsection have been verified and proven in multiple research work [44], [69].

The dynamical attitude of a satellite in its inertial coordinate system is given by $\dot{\mathbf{M}}$ the derivative of the satellite's overall momentum:

$$\dot{\mathbf{M}} = \mathbf{T}_{dist} + \mathbf{T}_g + \mathbf{T}_{thru} \quad (45)$$

Where $\mathbf{T}_{dist}$, $\mathbf{T}_g$, and $\mathbf{T}_{thru}$ are the disturbance torque, gravitational torque, and thrust torque respectively.

The momentum's derivative can be expressed by:

$$\dot{\mathbf{M}} = \mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega}) = \mathbf{T}_{dist} + \mathbf{T}_g + \mathbf{T}_{thru} \quad (46)$$

To simplify the calculations, the inertia matrix $\mathbf{I}$ is as follows:

$$\mathbf{I} = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \quad (47)$$

Equation (46) can be expressed in the following system of equations:

$$\begin{aligned} I_x \dot{\omega}_x + (I_z - I_y) \omega_y \omega_z &= \mathbf{T}_{dist,x} + \mathbf{T}_{gx} + \mathbf{T}_{thru,x} \\ I_y \dot{\omega}_y + (I_x - I_z) \omega_z \omega_x &= \mathbf{T}_{dist,y} + \mathbf{T}_{gy} + \mathbf{T}_{thru,y} \\ I_z \dot{\omega}_z + (I_y - I_x) \omega_x \omega_y &= \mathbf{T}_{dist,z} + \mathbf{T}_{gz} + \mathbf{T}_{thru,z} \end{aligned} \quad (48)$$

The angular velocity of the satellite can be given by small angle approximation of Equation (40):

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \dot{\phi} - \omega_0 \psi \\ \dot{\theta} - \omega_0 \varphi \\ \dot{\psi} + \omega_0 \phi \end{bmatrix} \quad (49)$$

Where ω_0 is the rotation due to the orbital motion of the spacecraft.

Given all the above, the satellite's dynamical attitude can be expressed as:

$$\begin{aligned} I_x \ddot{\phi} + (I_y - I_z) \omega_0^2 \phi + (I_y - I_z - I_x) \omega_0 \dot{\psi} &= \mathbf{T}_{dist,x} + \mathbf{T}_{gx} + \mathbf{T}_{thru,x} \\ I_y \ddot{\theta} &= \mathbf{T}_{dist,y} + \mathbf{T}_{gy} + \mathbf{T}_{thru,y} \\ I_z \ddot{\psi} + (I_x + I_z - I_y) \omega_0 \dot{\phi} + (I_y - I_x) \omega_0^2 \psi &= \mathbf{T}_{dist,z} + \mathbf{T}_{gz} + \mathbf{T}_{thru,z} \end{aligned} \quad (50)$$

The gravitational torques can be expressed as:

$$\begin{aligned} T_{gz} &= 0 \\ T_{gy} &= -3\omega_0^2(I_y - I_z)\phi \\ T_{gx} &= -3\omega_0^2(I_x - I_z)\phi \end{aligned} \quad (51)$$

From (45) and (46), the satellite' dynamic model is finally obtained by:

$$\begin{aligned} I_x \ddot{\phi} + 4(I_y - I_z) \omega_0^2 \phi + (I_y - I_z - I_x) \omega_0 \dot{\psi} &= \mathbf{T}_{dist,x} + \mathbf{T}_{gx} + \mathbf{T}_{thru,x} \\ I_y \ddot{\theta} + 3\omega_0^2(I_x - I_z) \theta &= \mathbf{T}_{dist,y} + \mathbf{T}_{thru,y} \\ I_z \ddot{\psi} + (I_x + I_z - I_y) \omega_0 \dot{\phi} + (I_y - I_x) \omega_0^2 \psi &= \mathbf{T}_{dist,z} + \mathbf{T}_{thru,z} \end{aligned} \quad (52)$$

The above system of equations (52) can be compacted in a matrix form to [69]:

$$\mathbf{I}\ddot{\boldsymbol{\alpha}} + \mathbf{H}\dot{\boldsymbol{\alpha}} + \mathbf{\Im}\boldsymbol{\alpha} = \mathbf{J}\mathbf{u} + \mathbf{J}\mathbf{d} \quad (53)$$

Where

$$\begin{aligned}
\boldsymbol{\alpha} &= \begin{bmatrix} \varphi \\ \theta \\ \psi \end{bmatrix}, \quad \boldsymbol{u} = \begin{bmatrix} \boldsymbol{T}_{thru,x} \\ \boldsymbol{T}_{thru,y} \\ \boldsymbol{T}_{thru,z} \end{bmatrix}, \quad \boldsymbol{d} = \begin{bmatrix} \boldsymbol{T}_{dist,x} \\ \boldsymbol{T}_{dist,y} \\ \boldsymbol{T}_{dist,z} \end{bmatrix} \\
\boldsymbol{J} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{\mathfrak{J}} = \begin{bmatrix} 4(I_y - I_z)\omega_0^2 & 0 & 0 \\ 0 & 3\omega_0^2(I_x - I_z) & 0 \\ 0 & 0 & (I_y - I_x)\omega_0^2 \end{bmatrix} \\
\boldsymbol{H} &= (I_y - I_z - I_x)\omega_0 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}
\end{aligned} \tag{54}$$

5. MULTI-OBJECTIVE OPTIMAL CONTROL FOR MANEUVER EXECUTION

5.1. Defining State Variables

The state variables are defined as such [69]:

$$\mathbf{z} = I\ddot{\alpha} * 10^{-3}, \quad \mathbf{x} = \begin{bmatrix} \alpha \\ \dot{\alpha} \end{bmatrix} \quad (55)$$

The state space form of the system is generally represented as:

$$\dot{\mathbf{x}} = A\mathbf{x} + B_1\mathbf{u} + B_2\mathbf{d} \quad (56)$$

To derive this, the state space form needs to be divided into components according to equation (53):

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{\alpha} \\ \ddot{\alpha} \end{bmatrix} = \begin{bmatrix} I^{-1}(J\mathbf{u} + J\mathbf{d} - H\dot{\alpha} - \mathfrak{I}\alpha) \\ \dot{\alpha} \end{bmatrix} \quad (57)$$

The previous matrix can be written in the form:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & J \\ -I^{-1}\mathfrak{I} & -I^{-1}H \end{bmatrix} \begin{bmatrix} \alpha \\ \dot{\alpha} \end{bmatrix} + \begin{bmatrix} 0 \\ I^{-1} \end{bmatrix} \mathbf{u} + \begin{bmatrix} 0 \\ I^{-1} \end{bmatrix} \mathbf{d} \quad (58)$$

Thus,

$$A = \begin{bmatrix} 0 & J \\ -I^{-1}\mathfrak{I} & -I^{-1}H \end{bmatrix}, \quad B = B_1 = B_2 = \begin{bmatrix} 0 \\ I^{-1} \end{bmatrix} \quad (59)$$

The output state variable $\mathbf{z}$ can be also written as follows:

$$\mathbf{z} = C\mathbf{x} + D_1\mathbf{u} + D_2\mathbf{d} \quad (60)$$

From the initial definition of state variables, the variable $\mathbf{z}$ can be expressed as:

$$\mathbf{z} = I\ddot{\alpha} * 10^{-3} = (J\mathbf{u} + J\mathbf{d} - H\dot{\alpha} - \mathfrak{I}\alpha) * 10^{-3} \quad (61)$$

Hence:

$$C = [-\mathfrak{I} \quad -H], \quad D = D_1 = D_2 = 10^{-3}J \quad (62)$$

Therefore, the state space form is derived:

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u} + B\mathbf{d} \\ \mathbf{z} &= C\mathbf{x} + D\mathbf{u} + D\mathbf{d} \end{aligned} \quad (63)$$

Where

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{4(I_y - I_z)\omega_0^2}{I_x} & 0 & 0 & 0 & 0 & -\frac{(I_y - I_x - I_z)\omega_0}{I_x} \\ 0 & -\frac{3(I_x - I_z)\omega_0^2}{I_y} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{(I_y - I_x)\omega_0^2}{I_z} & \frac{(I_y - I_x - I_z)\omega_0}{I_z} & 0 & 0 \end{bmatrix} \quad (64)$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{I_x} & 0 & 0 \\ 0 & \frac{1}{I_y} & 0 \\ 0 & 0 & \frac{1}{I_z} \end{bmatrix}$$

$$C = \begin{bmatrix} -4(I_y - I_z)\omega_0^2 & 0 & 0 & 0 & 0 & -(I_y - I_z - I_x)\omega_0 \\ 0 & -3(I_x - I_z)\omega_0^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(I_y - I_x)\omega_0^2 & (I_y - I_z - I_x)\omega_0 & 0 & 0 \end{bmatrix} \cdot 10^{-3}$$

5.2. Transfer Function of the System Subjected to Disturbances

After having the previous equations that relate the state variables ($\mathbf{x}$ and $\mathbf{z}$), disturbance ($\mathbf{d}$) and the control ($\mathbf{u}$), the Laplace Transform is necessary for the control study and analysis. The previous equations are therefore converted to the Laplace domain with the Laplace variable “ s ” as follows:

$$s\mathbf{X}_{(s)} = A \cdot \mathbf{X}_{(s)} + B(\mathbf{U}_{(s)} + \mathbf{D}_{(s)}) \quad (65)$$

$$\mathbf{Z}_{(s)} = C \cdot \mathbf{X}_{(s)} + D(\mathbf{U}_{(s)} + \mathbf{D}_{(s)})$$

Where, if $\mathcal{L}$ is the Laplace operator, then, $\mathcal{L}(\mathbf{i}) = \mathbf{I}_{(s)}$, $\mathbf{i} \in \{\mathbf{x}, \mathbf{z}, \mathbf{u}, \mathbf{d}\}$ and $\mathcal{L}(\dot{\mathbf{x}}) = s\mathbf{X}_{(s)}$.

Rearranging the above:

$$(sI - A)\mathbf{X}_{(s)} = B(\mathbf{U}_{(s)} + \mathbf{D}_{(s)}) \quad (66)$$

Therefore:

$$\mathbf{X}_{(s)} = (sI - A)^{-1}B(\mathbf{U}_{(s)} + \mathbf{D}_{(s)}) \quad (67)$$

Substituting $\mathbf{X}_{(s)}$ in Equation (67) into $\mathbf{Z}_{(s)}$ in Equation (65):

$$\begin{aligned} \mathbf{Z}_{(s)} &= C((sI - A)^{-1}B(\mathbf{U}_{(s)} + \mathbf{D}_{(s)})) + D(\mathbf{U}_{(s)} + \mathbf{D}_{(s)}) \\ &= (C(sI - A)^{-1}B + D)\mathbf{U}_{(s)} + (C(sI - A)^{-1}B + D)\mathbf{D}_{(s)} \end{aligned} \quad (68)$$

To simplify the above equation, $T(s)$ is defined as:

$$T_{(s)} = C(sI - A)^{-1}B + D \quad (69)$$

As a result, the control system of the satellite without feedback controller can be expressed as the following:

$$\mathbf{Z}_{(s)} = T_{(s)} \cdot \mathbf{U}_{(s)} + T_{(s)}\mathbf{D}_{(s)} \quad (70)$$

The transfer function without feedback controller $G_{zd,K=0}(s)$ of the output $\mathbf{Z}_{(s)}$ and the disturbance input $\mathbf{D}_{(s)}$ is given by:

$$G_{zd,(K=0)}(s) = T_{(s)} \quad (71)$$

Stabilizing the closed loop system with a feedback controller K such as $\mathbf{U}_{(s)} = -K\mathbf{X}_{(s)}$ to control the satellite is the objective. Substituting the $\mathbf{U}_{(s)} = -K\mathbf{X}_{(s)}$ into the state space equations changes the system's dynamics:

$$\begin{aligned} \mathbf{X}_{(s)} &= (sJ - A + B.K)^{-1}B\mathbf{D}_{(s)} \\ \mathbf{Z}_{(s)} &= (C - D.K)\mathbf{X}_{(s)} + D\mathbf{D}_{(s)} \end{aligned} \quad (72)$$

Substituting the new $\mathbf{X}_{(s)}$ equation into $\mathbf{Z}_{(s)}$:

$$\mathbf{Z}_{(s)} = ((C - D.K)(sJ + (A + B.K))^{-1}B + D)\mathbf{D}_{(s)} \quad (73)$$

The modified transfer function in the presence of the feedback controller K becomes:

$$G_{zd}(s) = B(C - DK)(sJ - A + BK)^{-1} + D \quad (74)$$

5.3. Multi-Objective Controller

The control objective is to design K such that the closed loop system is stable and the infinity norm of the transfer function $\mathcal{H}_\infty = \|G_{zd}(s)\|_\infty$ is minimized, therefore bounded, to ensure:

$$\mathcal{H}_\infty \leq \gamma \quad (75)$$

Where γ is a small positive real number.

The $\mathcal{H}_\infty$ performance requirement translates into ensuring that:

$$\|z(t)\| \leq \gamma \|d(t)\| \quad (76)$$

To achieve this objective, the problem is converted into a Linear Matrix Inequality (LMI) form [70], which is easier to solve computationally. The ‘‘Kalman-Popov-Yakubovich (KYP) Lemma’’ also known as ‘‘Bounded Real Lemma (BRL)’’ [71] [72] can be used to prove many LMI results.

For the given system $\hat{G}_{(t)}(A, B, C, D)$ below:

$$\begin{aligned} \dot{x} &= Ax + Bu + Bd \\ z &= Cx + Du + Dd \end{aligned} \quad (77)$$

Where $x(t) \in \mathbb{R}^n, z(t) \in \mathbb{R}^m, u(t) \in \mathbb{R}^q$ and $d(t) \in \mathbb{R}^p$ at any time $t \in \mathbb{R}$.

Using KYP Lemma (Lemma 1) applied on $\hat{G}_{(t)}(A, B, C, D)$ solves the bounded $\mathcal{H}_\infty$ problem:

Lemma 1.

If a symmetric positive-definite matrix X and a matrix W both exist and satisfy the following optimization problem,

minimize γ subject to

$$X > 0, \begin{bmatrix} AX + BW + (AX + BW)^T & B & (CX + DW)^T \\ B^T & -\gamma J & D^T \\ CX + DW & D & 1 \end{bmatrix} < 0 \quad (78)$$

Then $\mathcal{H}_\infty$ is bounded by γ ($\|G_{zd}(s)\|_\infty = \mathcal{H}_\infty \leq \gamma$), where γ is a small real number.

After bounding the $\mathcal{H}_\infty$ with γ , the feedback controller K needs to obey the optimal control law $U(s) = -K X(s)$ to stabilize the system, therefore, the Lemma 2 refers to the Linear Quadratic Regulator (LQR) method which minimizes the cost function of the system to compute the optimal feedback controller K .

Lemma 2.

Given the time-invariant system $\hat{G}_{(t)}(A, B, C, D)$, the LQR aims to find a control law $u(t) = -Kx(t)$ that minimizes the following quadratic cost function [73]:

$$J = \int_0^\infty (x(t)^T Q x(t) + u(t)^T R u(t)) dt \quad (79)$$

Where Q is a symmetric positive semi-definite matrix that penalizes deviations of the state and R is a symmetric positive definite matrix that penalizes the control effort [73].

Thus, the feedback K is computed as:

$$K = R^{-1} B^T S \quad (80)$$

Where S is the solution to the ‘‘Algebraic Riccati Equation (ARE)’’ [73]:

$$A^T S + SA - SBR^{-1}B^T S + Q = 0 \quad (81)$$

Lemmas 1 and 2 must be joined together to achieve the desired multi-objective controller. Theorem 1 below describes the optimal control that combines the two previous lemmas:

Theorem 1.

Given the time-invariant system $\hat{G}_{(t)}(A, B, C, D)$, the optimal solution for the multi-objective controller problem exists if there is a symmetric positive-definite matrix X , where $U = -K X$, and a matrix W both exist and satisfy the following optimization problem,

minimize γ subject to

$$X > 0, \begin{bmatrix} AX + BW + (AX + BW)^T & B & (CX + DW)^T \\ B^T & -\gamma J & D^T \\ CX + DW & D & 1 \end{bmatrix} < 0 \quad (82)$$

Then $\mathcal{H}_\infty$ is bounded by γ ($\|G_{zd}(s)\|_\infty = \mathcal{H}_\infty \leq \gamma$), where γ is a small real number, and the cost function below can be minimized,

$$J = \int_0^{\infty} (x(t)^T Q x(t) + u(t)^T R u(t)) dt \quad (83)$$

with Q and R are two symmetric positive semi-definite matrices which penalizes deviations of the state and the control effort, respectively, to get a feedback controller $K = R^{-1}B^T S$ [73], where S is the solution to the “Algebraic Riccati Equation (ARE)” [73]:

$$A^T S + SA - SBR^{-1}B^T S + Q = 0 \quad (84)$$

Theorem 1 combines the KYP Lemma (Lemma 1) and the LQR method (Lemma 2) without interfering with the results of the two lemmas. In other words, each lemma has a separate task in Theorem 1. The Lemma 1 was needed to bound $\mathcal{H}_{\infty}$ and compute γ , and the Lemma 2 was chosen to solve the feedback gain K . Thus, each result is separate and not based on the other. Therefore, Theorem 1 lays under triviality and proving it is not a matter of concern.

Another approach to the problem is to set LMI regions as further constraints and solve for the static feedback gain K using Lemma 1 by supposing $K = WX^{-1}$ while minimizing γ subject to the Equation (82). In that case, the theorem needs to be proven.

6. SUMMARY

6.1. Results

6.1.1. *Threat Analyzer for Collision Probability Evaluation*

Different sensors operate based on various principles (e.g., radar vs. lidar, or infrared vs. visual cameras), each introducing distinct errors, such as differences in field of view, resolution, and sensing capabilities. By integrating measurements and error covariances from multiple independent sensor types, the overall estimation error can be reduced. Combining the error covariances of the independent different sensors implemented on the satellite demonstrated similarity to the total resistance of N resistors in parallel. Since the resistance is inversely proportional to the admittance, therefore, the combined error covariance of N independent sensors provided an analogy between it and a linear power system of N bus network of resistors. The analogy was implemented to design a matrix C_N to be an input to the threat analyzer, constructed from elements of combined error covariance matrices of the sensors where two types of elements are distinguished: allo-combined error covariance matrix used to keep track of the threat when it is partially surpassed by the satellite (during collision avoidance maneuver) while the data of the primary sensor (the sensor that was in an angle primarily facing the threat in the early stage of detection) is no longer needed, and the mutual-combined covariance matrices which are necessary to correct first detection data of the target, depending on the satellite orientation and the relative position of the threat (the threat can be facing partially two sensors in the early stage of detection).

6.1.2. *System Dynamics for Collision Avoidance*

This section presents a detailed mathematical model for the dynamics of a satellite. The equations presented describe the dynamics of the satellite and the control inputs required to achieve the desired maneuvers. The model represents the satellite's inertia matrix, angular velocity, and various thrust torques acting upon it to perform the collision avoidance maneuver. The equations derived provide the expressions of state variables to be used in the next section of the work for the multi-objective optimal control. The dynamic system's control and objective differ from the optimal control section due to the separate approach to the problem. The dynamic system control deals with the physical and dynamical attitude control of the satellite during collision avoidance, on the other hand, the multi-objective optimal control designs a feedback gain for better robustness against external disturbances and lower energy consumption during the maneuver.

6.1.3. Multi-Objective Optimal Control for Maneuver Execution

The inertia matrix data was calculated based on the shape approximation of the satellite Iridium-33 which collided with the unfunctional satellite Cosmos-2251 at extremely high speed in 2009, leaving a large amount of relatively big trackable debris.

The simplified inertia matrix of the satellite was considered as follows:

$$I = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \quad (85)$$

Where $I_x = 4376.57 \text{ kg/m}^2$, $I_y = 4380.57 \text{ kg/m}^2$, $I_z = 106.21 \text{ kg/m}^2$.

The MATLAB 2023b code was written with the help of YALMIP and SeDuMi, to calculate the bounding real number γ of $\mathcal{H}_\infty$ and the static feedback gain K using KYP lemma and the LQR method, respectively, which led to a fully controllable and observable system of both ranks equal to 6. The system was found to achieve an attenuation of $\gamma = 0.001$ indicating a high level of robustness against external disturbances, therefore, the energy of the disturbance is drastically reduced before affecting the system's output. The feedback controller K is obtained below by penalizing all states and inputs equally with weigh matrices of states and inputs $Q = R = J_3$ (J_3 is the identity matrix 3x3) and after scaling the system matrix A by 10^3 to ensure better performance, adjust response time, and make the system more sensitive to control actions.

$$K = \begin{bmatrix} 0.9132 & 3.2935e-16 & -0.0042 & 2827.2557 & 8.6884e-13 & -0.0117 \\ -5.7070e-17 & 0.9342 & 5.4643e-16 & -1.4864e-12 & 2860.8762 & 1.8506e-13 \\ 0.0042 & -2.0382e-15 & 1 & -0.4840 & -24277e-12 & 460.8844 \end{bmatrix} \quad (86)$$

The closed-loop system eigenvalues are as follows:

$$\lambda = [-0.3230 \pm 0.3537i \quad -0.3265 \pm 0.3495i \quad -2.1697 \pm 2.1697i] \quad (87)$$

The step responses of all inputs to all outputs subjected to the feedback gain K are generated using MATLAB 2023b and represented in Annex 2 (Figures 18 to 26).

6.2. Suggestions / Conclusions

The thesis concludes that autonomous systems for satellites to avoid collisions are both possible and necessary in preventing the increasing issues of space traffic and debris in Low Earth Orbit (LEO). By using analytical methods that combine data from multiple sensors, the error measurement in the detection and tracking of threats is improved. The dynamic model ensures a satellite follows a precise trajectory, and adding a multi-objective controller ensures that the collision avoidance maneuver is energy efficient and can handle disturbances, making the autonomous satellite system robust against external perturbations.

Several areas need to be improved. First, adding artificial intelligence to the threat analyzer could benefit threat error prediction and classification. Second, generalizing the multi-objective controller system to manage satellite networks while using shared positions and trajectories. Additionally, improving sensors' sensibility with higher resolution and wider range could help track smaller pieces of debris more accurately.

Future efforts should highlight implementing these systems into existing satellite networks, making sure they can work together and follow international space traffic management rules. By fulfilling these suggestions, this work sparks the start of the next generation of satellite systems that not only avoid collision risks but also promote the responsible use of space.

Szombathely, 2024, December 1st.

Amine Zribi

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8. SHORT SUMMARY

The results of this thesis highlight advancements in satellite collision avoidance strategies utilizing multi-sensor data, dynamic modeling, and optimal control techniques.

The threat analyzer effectively combines independent sensor measurements, reducing error in the detection of threats by making an analogy with electrical circuits, where error covariance matrices resemble the resistance of parallel resistors. This approach enhances the error calculation during the collision avoidance maneuver, making it applicable to lower human intervention and computational errors.

The system dynamics part developed a detailed mathematical model that considers simplified satellite inertia, angular velocity, and thrust torques as inputs. This model generates the dynamic equation to be used for the multi-objective controller, needed to ensure the desired disturbance attenuation even while maneuvering. Additionally, the system dynamics clearly state the torques and forces acting on the satellite and the input variables to be considered, which are needed to maintain stability during collision avoidance.

The multi-objective optimal controller calculated the feedback gain matrix K using MATLAB 2023b with the help of YALMIP and SeDuMi. The solution obtained a good attenuation ($\gamma = 0.001$) that can lead to better energy efficiency and precise satellite maneuvering. The closed-loop eigenvalues far enough from the origin, confirmed the results obtained, making sure that the satellite maintains its reliability under different conditions.

The gathered results highlight the importance and feasibility of autonomous collision avoidance techniques for satellites, lowering human intervention. This work shows that combining resource optimization and enhancing precision is the key to the next generation of autonomous satellites. This thesis contributes a practical solution to the challenges of space traffic and debris, leading the way to more innovative techniques in space sustainability.

Keywords: *Satellite Collision Avoidance, Threat Analyzer, Dynamic Modeling, Optimal Control*

APPENDIX

1. Annex: MATLAB 2023b (YALMIP, SeDuMi) code for multi-objective optimal control:

```
clear all
clc
% Elements of diagonal inertia matrix (kg.m²)
I_x=4376.57;
I_y=4380.57;
I_z=106.21;

% Simplification formulas
I_xy=I_x-I_y;
I_yx=I_y-I_x;
I_xz=I_x-I_z;
I_yz=I_y-I_z;
I_yzx=I_y-I_z-I_x;
I_yxz=I_y-I_x-I_z;
omega=7.292115*10^-5; % Earth's rotational angular velocity

% Define the system matrices (u: control / x,z: state variables / w:disturbance)
% dx/dt = A*x + B1*u + B2*w
% z = C*x + D1*u + D2*w
addpath(genpath('C:\Program Files\MATLAB\R2023b\YALMIP-master'));
addpath(genpath('C:\Program Files\MATLAB\R2023b\sedumi'));

A = [0 0 0 1 0 0;
     0 0 0 0 1 0;
     0 0 0 0 0 1;
     -(4*I_yz*omega^2)/I_x 0 0 0 0 -(I_yxz*omega)/I_x;
     0 -(3*I_xz*omega^2)/I_y 0 0 0 0;
     0 0 -(I_yx*omega^2)/I_z (I_yxz*omega)/I_z 0 0]*10^3 % System matrix modified
B1 = [0 0 0;
      0 0 0;
```

```

0 0 0;
1/I_x 0 0;
0 1/I_y 0;
0 0 1/I_z]; % Disturbance matrix
B2 = B1; % Control matrix
C = [-4*I_yz*omega^2 0 0 0 0 -I_yzx*omega;
0 -3*I_xz*omega^2 0 0 0 0;
0 0 -I_yx*omega^2 I_yzx*omega 0 0] *10e-3; % Output matrix
D1 = [10^-3 0 0;
0 10^-3 0;
0 0 10^-3]; % Direct transmission matrix for disturbances
D2 = D1; % Direct transmission matrix for control input

% Define the weighting matrices for LQR
Q = eye(size(A)); % State weighting matrix (can be adjusted)
R = eye(size(B1, 2)); % Control weighting matrix

% Compute the LQR gain
[K, S, e] = lqr(A, B1, Q, R);

% Display the LQR gain matrix
disp('LQR gain K is:');
disp(K);

% Display the closed-loop poles (eigenvalues of A-BK)
disp('Closed-loop poles are:');
disp(e);

% Closed-loop system: A_cl = A - B1*K
A_cl = A - B1*K;

% Create the state-space system for closed-loop
sys = ss(A_cl, B1, C, D1);

```

```

figure; % Open a new figure window
[m, p] = size(D1); % Get the number of outputs (m) and inputs (p)

for i = 1:p % Loop over the number of inputs
    for j = 1:m % Loop over the number of outputs
        subplot(m, p, (i-1)*m + j); % Create a subplot for each input-output pair
        step(sys(j, i)); % Plot step response for input i to output j
        title(['Step Response: Input ' num2str(i) ' to Output ' num2str(j)]);
        xlabel('Time');
        ylabel('Amplitude');
    end
end

% Define the LMI variables using YALMIP
W = sdpvar(size(B1,2),size(B1,1)); % Symmetric positive definite matrix W
X = sdpvar(size(A,1)); % Symmetric positive definite matrix X
gamma = sdpvar(1); % Scalar variable gamma

% Define the LMI conditions
LMI1 = [A*X+B1*W+transpose(A*X+B1*W) B2 transpose(C*X+D1*W);
        transpose(B2) -gamma*eye(size(C,1)) D2;
        C*X+D1*W transpose(D2) -gamma*eye(size(C,1))] <= 0; % H-infinity LMI

% Combine all LMI conditions
LMIs = [LMI1, X >= 0];

% Solve the LMI problem
options = sdpsettings('solver', 'sedumi');
sol = optimize(LMIs, gamma, options);

% Check if the solver was successful and solve for gamma
if sol.problem == 0
    disp('The value of gamma is:');

```

```

disp(value(gamma));
else
    disp('LMI problem is infeasible. ');
    sol.info
end

% Check controllability
controllability_matrix = ctrb(A, B1);
rank_controllability = rank(controllability_matrix);
disp(['Rank of controllability matrix: ', num2str(rank_controllability)]);

% Check observability
observability_matrix = obsv(A, C);
rank_observability = rank(observability_matrix);
disp(['Rank of observability matrix: ', num2str(rank_observability)]);

```

2. Annex: Step response of all inputs to all outputs subjected to the feedback gain K

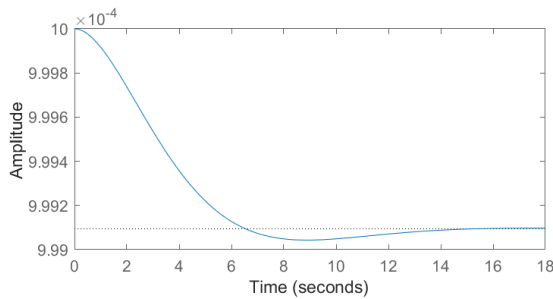


Figure 18. Step response of input 1 to output 1.

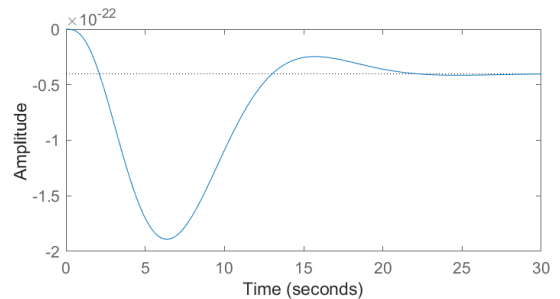


Figure 19. Step response of input 1 to output 2.

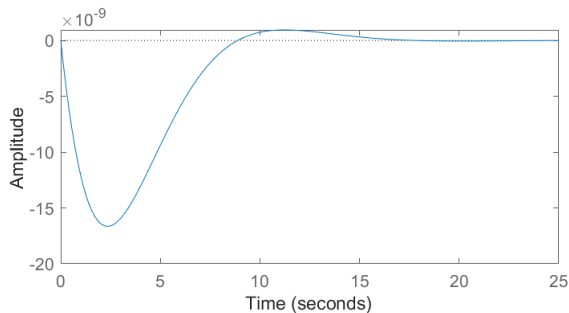


Figure 20. Step response of input 1 to output 3.

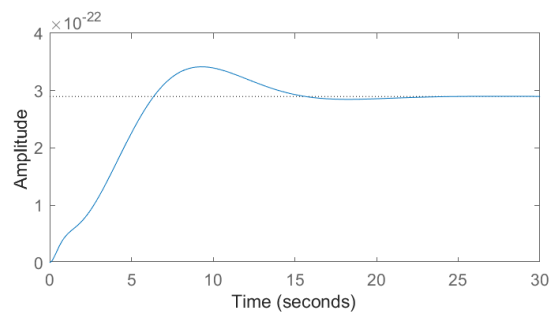


Figure 21. Step response of input 2 to output 1.

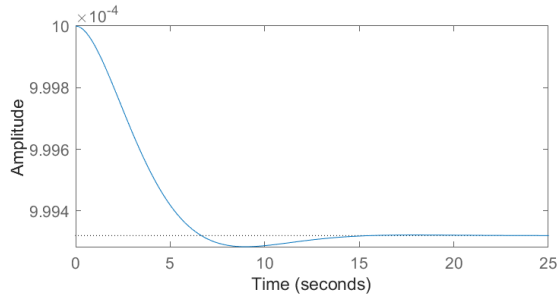


Figure 22. Step response of input 2 to output 2.

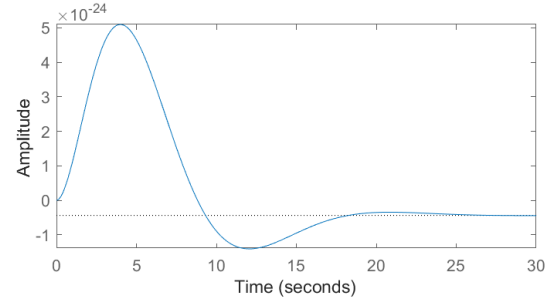


Figure 23. Step response of input 2 to output 3.

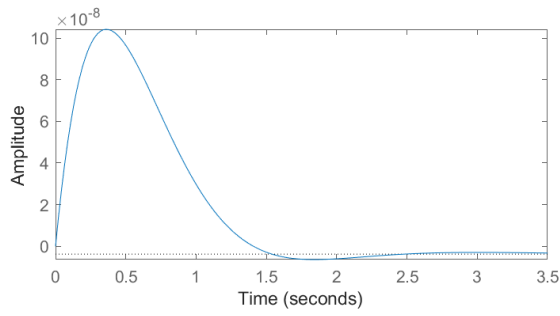


Figure 24. Step response of input 3 to output 1.

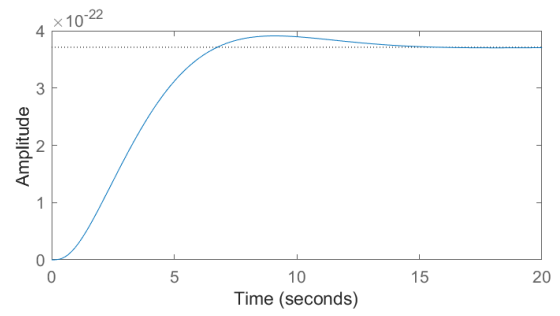


Figure 25. Step response of input 3 to output 2.

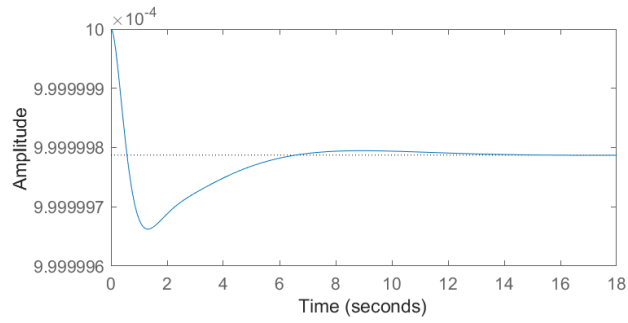


Figure 26. Step response of input 3 to output 3.