

OTDK THESIS

**On the helicity flux
transport of MHD waves**

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Astronomy MSc., I. year



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A naptevékenység érdekes, látványos jelenségekkel szolgál az emberiség számára, azonban a lenyűgöző szépsége mellett kockázatokat is rejt magában. A kockázatok és károk nem csak anyagi, technológiai és gazdasági szférákban merülnek ki, hanem akár súlyos egészségügyi következményei is lehetnek. Egy-egy napkitörés folyamán jelentős energia szabadul fel, amely plazmafelhőként terjedve potenciálisan a Földet is eltalálhatja. Ezáltal geomágneses viharok keletkezhetnek, amelyek kockázati és károkozó hatásait megfelelő intézkedésekkel minimalizálni lehet. A napkitörések előrejelzéséhez szükséges egy elméleti modellt alkotni, amely megfelelő analitikai alapot nyújt a megfigyelésekhez. [Korsós et al. \(2020b\)](#) azt találta, hogy a mágneses helicitás-fluxus (MHF) periodikussága alapján meg lehet különböztetni különböző aktív régiók flerprodukáló képességét, ezáltal elősegítve az előrejelzések módszerét és pontosságát. Az említett módszerhez viszont jelenlegi tudásunk szerint nincs kidolgozott elméleti háttér, amellyel különböző magnetohidrodinamikai hullámok MHF szállítását lehet modellezni. Munkám célja, hogy ezen analitikai háttérrel kidolgozzam. Ennek érdekében tömören bemutatom a napkitörésekre jellemző különböző jelenségeket, illetve röviden beszámolok az elsőként rögzített nagy napkitörésről és annak földi hatásairól, amit [Carrington \(1859\)](#) figyelt meg és írt le. Néhány képpel szemléltetem, hogy milyen - a napkitöréshez köthető - jelenségek zajlanak a Napon és ennek milyen látványos hatásai vannak. Felvázolom a naptevékenységet leíró magnetohidrodinamika (MHD) elméletét, említésszerűen prezentálom a dolgozatban fontos MHD hullámokat, azok tulajdonságait, majd tárgyalom az úgynevezett mágneses helicitás-fluxus számolását és annak alkalmazását.

A helicitás számoló integrálok meghatároznak egy számot, ami jellemzi az adott erővonalak topológiai tulajdonságait. Ezáltal a mágneses helicitás-fluxus (MHF) azt méri, hogy mennyire tekerednek, csavarodnak egymás körül a mágneses erővonalak egy egységnyi felületre nézve. Ahogy korábban említettem, egyre fontosabb szerepet játszik ennek számítása és alkalmazása. Mivel MHD hullámok energiát szállíthatnak a koronába, így érdemes azt is megvizsgálni, hogy mennyi MHF-et és milyen módon szállítanak. [Woltjer \(1958\)](#) tétele alapján zárt területekre a mágneses helicitás megmaradó mennyiség, ha azt egy mágneses felület határolja ($\mathbf{B} \cdot \mathbf{n}|_S = 0$). Ez az eredmény ideális MHD esetén teljesül. Mivel a mágneses tér forrásmentes, azaz $\nabla \cdot \mathbf{B} = 0$, így ha $\mathbf{B} = \nabla \times \mathbf{A}$, tehát ha a mágneses mező egy mágneses vektorpotenciál rotációja, akkor minden esetben teljesül az eredeti feltétel, hiszen a rotáció divergenciája 0. A mágneses helicitás definíció szerint

$$H = \int_V \mathbf{B} \cdot \mathbf{A} \, d^3x,$$

viszont munkám során a [Korsós et al. \(2020b\)](#) cikk által bevezetett MHF számoló egyenletet fogom használni, amit eredetileg [Berger \(1984\)](#) vezetett be:

$$T \equiv \left. \frac{dH}{dt} \right|_S = 2 \int_S (\mathbf{A}_P \cdot \mathbf{B}) \mathbf{v} \cdot \mathbf{n} \, dS - 2 \int_S (\mathbf{A}_P \cdot \mathbf{v}) \mathbf{B} \cdot \mathbf{n} \, dS,$$

ahol T a teljes MHF, $\mathbf{A}_P$ a mágneses vektorpotenciál, $\mathbf{B}$ a mágneses tér, $\mathbf{n}$ a normálvektor, $\mathbf{v}$ a plazma sebessége, dS pedig a felület ami szerint integrálunk. [Korsós et al. \(2020b\)](#) azt találta, hogy korreláció van az MHF periodicitása és az aktív régiók (AR) flerprodukálási képessége között. Összehasonlították a flerprodukáló (legalább egy X-osztályú) és flert nem produkáló (M5,0-nál kisebb intenzitású) AR-ek MHF fejlődését. Ehhez SDO/HMI vektor magnetogramokat használtak, amikből minden fontos adatot kinyertek az MHF integrálhoz. A teljes MHF-et (T) felbontották egy nyíró (SH) és egy felbukkanó (EM) komponensre, majd wavelet analízist alkalmaztak. Az időintervallumok hozzávetőleg a fler előtt 100 órától a fler után 100 óráig terjedtek. [Soós et al. \(2022\)](#) bebizonyította sejtésüket egy nagyobb adathalmaz segítségével, hogy a különböző komponensek valóban adott módon oszcillálnak egy flerkitörés előtt.

A bevezetés után bemutatom az alapvető MHD egyenleteket, majd felírom az ideális MHD egyenleteket is, ami alapján a számításaim végzem. Röviden értekezem a perturbációszámítás elméletéről és gyakorlatáról, valamint hogy ezt miként használok fel analitikai levezetéseim során. Ezáltal linearizálom a plazma viselkedését leíró differenciálegyenleteket. [Roberts \(1981\)](#)

cikkéhez hasonlóan részletesen levezetem a fontosabb diszperziós relációkat, amely némi tapasztalatot és betekintést nyújt az MHD hullámok viselkedésébe és az őket leíró egyenletekbe. A cikkben található egyenletekhez hasonlóan járok el a diszperziós relációk levezetése során, amely magába foglalja a végtelen, homogén mágneses tér, valamint a mágneses interfész esetét is. A mágneses interfész egy olyan rétegződött mágneses tér, amely az interfészen ugrást szenved, tehát az egyik tartományt más mágneses mezőerősség jellemez, mint ami a másik tartományt. A sebesség- és nyomásperturbációkat Fourier alakban tárgyalom, amellyel különböző összefüggéseket vezetek le a dolgozat során.

$$\begin{aligned}v_x &= \hat{v}_x(x)e^{i(\omega t + ly + kz)}, \\v_y &= \hat{v}_y(x)e^{i(\omega t + ly + kz)}, \\v_z &= \hat{v}_z(x)e^{i(\omega t + ly + kz)}.\end{aligned}$$

Itt a kalappal ellátott mennyiségek x -től függő amplitudók. Ezután összehasonlítom és diszkutálom a különböző MHD hullámokra vonatkozó diszperziós relációkat, amelyeket a kezdeti 6 egyenlet alapján mátrix formalizmus segítségével vezetek le, valamint y irányba homogenitást feltételezek ($l = 0$). Feltételezem továbbá azt is, hogy a hullámok pusztán $\hat{\mathbf{z}}$ irányban terjednek, ami lényegesen leegyszerűsíti a későbbi számolásokat. A diszperziós relációk levezetésével szerzett tapasztalatokat utána felhasználom az MHF kiszámolásához is, amelyhez [Berger \(1984\)](#) teljes MHF számoló képletét használom fel. Ezen integrálokat elvégzem homogén mágneses mező esetén általános hullámra határozatlan és határozott integrálként, majd hasonló módon kiszámolom a teljes MHF-et mágneses interfész esetén is úgy szintén határozatlan és határozott integrálként. A tömörség kedvéért a nagyobb helyet igénylő eredmények esetében bevezetek egy jelölésrendszert. Ezután alkalmazom az adott MHD hullámra vonatkozó diszperziós relációt az MHF számolása során, valamint a végső képletek tömörségeért itt is bevezetek egy rövidítő jelölésrendszert.

Elsősorban a homogén, végtelen plazma esetére vonatkozó eredményeket ismertetném. Minden alábbi képlet az xy -sikon történő határozott integrálás utáni eredmény, ahol a különböző hullámok $\hat{\mathbf{z}}$, azaz a mágneses térrel megegyező irányba terjednek. Homogén eset Alfvén hullámra:

$$T|_{x_1, x_2, y_1, y_2}(t, z) = -B_0 e^{i(kv_A t + kz)} [\beta \hat{v}_x(y_2^2 - y_1^2)(x_2 - x_1) + \alpha \hat{v}_y(x_2^2 - x_1^2)(y_2 - y_1)],$$

ahol x_1, x_2, y_1, y_2 a tetszőleges határok ($y_2 > y_1, x_2 > x_1$), B_0 a mágneses mező erőssége, k a hullámszámvektor $\hat{\mathbf{z}}$ irányú komponense, v_A az Alfvén sebesség, a kalapos mennyiségek konstans sebesség amplitudók, β és α pedig a mágneses vektorpotenciálra jellemző állandók úgy, hogy $\alpha - \beta = B_0$. A következő eredmény a magnetoakusztikus hullámok esete, viszont magnetoakusztikus hullámból kétféle létezik: gyors és lassú. Azonban, ahogy azt a 3. fejezet végén levezetem, az egyik eset ekvivalens az Alfvén hullámra kapott megoldással, a másik megoldásnál pedig a hangsebesség jelenik meg. Ezáltal a következő eset a homogén, Alfvéntől különböző magnetoakusztikus hullám:

$$T|_{x_1, x_2, y_1, y_2}(t, z) = -B_0 e^{i(kc_s t + kz)} [\beta \hat{v}_x(y_2^2 - y_1^2)(x_2 - x_1) + \alpha \hat{v}_y(x_2^2 - x_1^2)(y_2 - y_1)],$$

ahol c_s a hangsebesség, a többi paramétert pedig fentebb definiáltam. A következő eredmény a mágneses interfész esete Alfvén hullámra:

$$T|_{x_1, x_2, y_1, y_2}(t, z) = a_0(y_2^2 - y_1^2)(-1)^{r+1}h_{r1},$$

ahol a_0 egy konstans amplitudó,

$$h_{rq}(t, z) = \frac{B_0(x_r)}{n(x_r)} e^{n(x_r)x_r} e^{i(kv_q(x_r)t + kz)}$$

egy olyan rövidítés, ahol a mágneses térerősség

$$B_0(x) = \begin{cases} B_e, & x > 0, \\ B_0, & x < 0, \end{cases}$$

az r és q indexek

$$r \in \{1, 2\}, \quad q \in \{1, 2\},$$

az adott hullám sebessége

$$v_q = \begin{cases} v_A, & q = 1, \\ c_s, & q = 2. \end{cases}$$

$$n(x) = \begin{cases} -m_e, & x > 0, \\ m_0, & x < 0, \end{cases}$$

egy írásmódot elősegítő rövidítés, ahol

$$m_0^2 = \frac{(k^2 c_s^2 - \omega^2)(k^2 v_A^2 - \omega^2)}{(c_s^2 + v_A^2)(k^2 c_T^2 - \omega^2)},$$

$$c_T^2 = \frac{c_s^2 v_A^2}{c_s^2 + v_A^2}.$$

Valamint m_e definíciója m_0 -hoz hasonló annyi különbséggel, hogy ott mind a hangsebesség, mind az Alfvén sebesség az arra a közegre vonatkozó sebesség. Tehát $x > 0$ tartományra $v_{A,e}$, illetve $c_{s,e}$ sebességek vonatkoznak. Ezáltal a végső eredmény a fentebb említett módon az az esete a magnetoakusztikus hullámoknak, amely az Alfvéntől különböző megoldást ad, tehát ahol a hangsebesség jelenik meg.

$$T|_{x_1, x_2, y_1, y_2}(t, z) = a_0(y_2^2 - y_1^2)(-1)^{r+1} h_{r2},$$

ahol minden paraméter a korábbiakban meg lett határozva.

Ezen analitikai számolások új perspektívából világítják meg az MHF számolásának témakörét, amely számoló képleteket sikeresen levezettem dolgozatom során. A mágneses jelenségek közvetlenül befolyásolhatják a napkitöréseket, így kulcsfontosságú, hogy jobban megértjük ezen jelenségeket és analitikusan modellezzük. Ahogy azt [Korsós et al. \(2020b\)](#) is tárgyalta, az MHF periodikussága összefüggésben állhat az aktív régiók flerprodukáló képességével, így ezen a területen is segítséget nyújthatnak az általam levezetett képletek. Természetesen az egyik legfontosabb feladat a jövőre tekintve az, hogy egyeztessük a képlet által kapott mágneses helicitás-fluxus értékeket a mérési adatokkal validáció gyanánt. Azonban meg kell jegyeznem, hogy a levezetés során számos közelítést és feltételezést alkalmaztam, amelynek köszönhetően bizonyos fizikai és matematikai problémák is felléphetnek, így a jövőben ezek további vizsgálatára van szükség. Továbbá, a jövőben nem csak ezen fizikai, esetleg matematikai problémák megoldására van szükség, hanem más, komplexebb modellekre is célszerű levezetni a teljes MHF analitikus alakját. Az egyik ilyen probléma, hogy a mágneses vektorpotenciál a végtelenhez közelítve nem 0. Egy másik probléma, hogy mi történik az MHF-fel az interfészen, ahol diszkontinuitás található.

Abstract

The solar activity provides interesting and spectacular phenomena for humanity, however, besides its associated beauty, it also bears considerable risks. First, in my thesis, I briefly discuss these risks and the important phenomena as motivation for the modelling calculations that I derived as part of the present work. The risks and damages are not only material, technological, and economic, but there may also be severe health consequences. I briefly discuss the first well-documented great solar eruption and its impact on Earth, which was observed and described by [Carrington \(1859\)](#). I demonstrate with a few examples what kind of phenomena - linked to solar eruptions - can be seen on the Sun, and what kind of spectacular impacts they may have. Next, I briefly explain the theory of magnetohydrodynamics (MHD), which is a practical approximation to investigate and model solar activity, and use it for the calculation for the application of analysing the so-called magnetic helicity flux (MHF).

After the introduction, I embark on the basic MHD equations, then I move on to the ideal MHD equations that are the base of my calculations. I briefly discuss the theory and application of perturbation theory, as well as how I will use it for my analytic derivations. I concisely present the important MHD waves and their properties that are relevant for this work, then I derive the more important dispersion relations following the technique of [Roberts \(1981\)](#), which yield some insight into the wave solutions of the governing equations. I focus on the dispersion relations that address the case of infinite, homogeneous magnetic field, as well as the case of a single magnetic interface. A magnetic interface is a layered magnetic field that experiences a discontinuity at the interface, meaning that one region is characterised by a different magnetic field strength than the other region. I discuss the velocity and pressure perturbations in Fourier decomposed form that I use throughout the work. I use the experience obtained by deriving the dispersion relations to calculate the total MHF defined by [Berger \(1984\)](#). I evaluate the obtained indefinite and definite integrals in the case of homogeneous magnetic field for a general wave, then I calculate the total MHF in the case of a magnetic interface in a similar manner as well.

These analytical calculations illuminate the topic of MHF computation from a new perspective, for which I have successfully derived computational formulas in my work. Dynamic magnetic phenomena like MHD waves may directly influence solar eruptions, so it is crucial to better understand these phenomena and model them analytically. The periodicity of MHF, as also investigated by [Korsós et al. \(2020b\)](#), may be related to the flare-producing capability of active regions. Therefore, the modelling that I have developed may also provide assistance in this area. However, it must be noted that I used a number of approximations and assumptions during the derivations, which may lead to certain physical and mathematical issues, thus caution must be exercised and further investigation is needed in the future. Furthermore, future efforts should focus not only on resolving these physical and potential mathematical issues, but also on deriving the analytical form of total MHF for more complex models. One of the problems may be that the magnetic vector potential is not 0 as we approach infinity. Another problem may be the question: what happens at the place of the magnetic interface, where the discontinuity exists. Naturally, one of the most important tasks for the future is to compare the MHF values calculated from my formulae with the measured data as validation.

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1 Introduction

1.1 Motivation

Space weather and its forecasting (Korsós et al., 2020a) are becoming increasingly important issues, as modern society would be unimaginable without today’s technology, which largely relies on electric power. In addition to the harmful radiation produced during solar flares, coronal mass ejections (CMEs) can release several million tons of ionised material, potentially propagating directly towards Earth. When this charged plasma enters the magnetosphere, it can trigger geomagnetic storms that may cause substantial damage to both satellite systems orbiting Earth and e.g. terrestrial power grids. From everyday logistics to water supply, security systems, street lighting, and even hospitals, all require electrical power to function properly, and a lack of power would have far-reaching, unforeseen consequences. In 2003, the so-called Halloween Solar Storm was one of the most powerful one to reach Earth. A spectacular example of a CME can be seen in Figure 1, which left behind the Sun’s corona on October 28, 2003. Disturbances in the ionosphere can also be significant, leading to issues with radio wave propagation and potentially causing disruptions or even outages in telecommunications. Spacecraft orbiting our planet, including GPS satellites, experience the effects of CMEs first hand. Their instruments may become inaccurate, making precise location determination impossible. In extreme cases, these spacecraft can even suffer permanent damage. Additionally, increasing numbers of crewed space missions are planned to distant celestial bodies, where astronauts will no longer be protected by Earth’s magnetosphere. As a result, they would be exposed to extreme radiation levels if their path intersected with the material ejected by a CME. Similarly, plans for space stations being developed by both government and private entities are increasingly appearing, such as those around the Moon (e.g., the Gateway space station¹). The permanent crew of these stations will be gradually exposed to the effects of space weather, and the risks associated with it must be minimised, as its consequences could be potentially fatal for the human body.

To minimise or possibly avoid the harmful effects of flares and CMEs, it is crucial to understand these phenomena, know their precursors, and improve the accuracy of their forecasts.

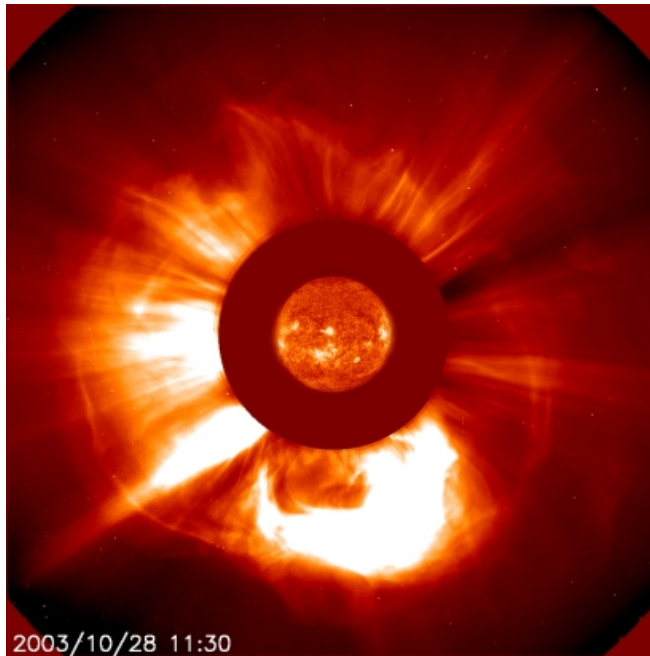


Figure 1: A CME that erupted on October 28, 2003, where material ejected from the Sun is observed moving away into interplanetary space. This composite image was taken using the LASCO C2 coronagraph and the EIT instrument from the NASA/ESA Solar and Heliospheric Observatory (SOHO). (Source: ESA)

¹<https://www.nasa.gov/reference/gateway-about/>

The so-called Carrington Event is widely known, which occurred in the 19th century. This is the first known significant eruption, which [Carrington \(1859\)](#) observed around 11:20 AM (GMT) on September 1, 1859. 18 hours later, a geomagnetic storm of such magnitude was observed that had not been recorded before. The aurora was even observable at latitudes closer to the Earth's equator (such as Hawaii and Colombia), and in the northern parts of the USA, the light was bright enough to read a newspaper by. The telegraph systems used at the time experienced induced currents in their wires, which caused fires and continued to function even when disconnected from the grid; in many cases, the operators also suffered electric shocks. In 2012, an eruption comparable to the 1859 event narrowly missed Earth by only a mere 30 degrees. ([Baker et al., 2013](#)). According to a 2013 report² published by Lloyd's, the North American electrical grid could have suffered up to \$2.6 trillion (equivalent to \$3.35 trillion at today's exchange rate) in damage if the eruption had hit Earth. Moreover, the resulting power shortage would have severely impacted essential, life-supporting systems, potentially leading to a major social catastrophe in today's world, and could even delay humanity's progress by years or even decades.

1.2 Solar eruptions

We can observe various ongoing phenomena of solar activity on the Sun, but in this case, different types of eruptions are the most important. No matter how spectacular they may be, they can pose a significant danger to us. The three space phenomena that generally cause the most problems for the Earth's technosphere are: (i) solar flares, (ii) CMEs, and (iii) eruptive prominences. These can occur independently from each other, but during a solar eruption, usually all three phenomena are observed.

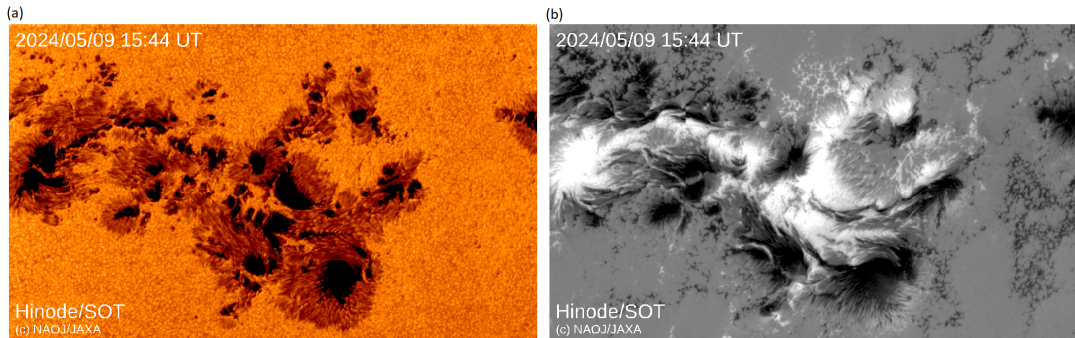


Figure 2: This figure shows the active region AR13664, which produced one of the most powerful solar flares of 2024 with the intensity of X5.8 on May 11, 2024. Panel (a) shows the Sun's surface in continuum, along with the sunspot group responsible for the flare. In contrast, panel (b) displays a magnetogram of the same region taken at the same time, revealing the magnetic field distribution across the entire active region. White locations represent a positive magnetic field value (north polarity), while black represents a negative magnetic field value (south polarity). Grey represents zero magnetic field. (Source: Hinode)

- Flares are sudden (within a few minutes) brightenings of a specific region on the solar disk. These are eruptions resulting from magnetic reconnections occurring in the Sun's atmosphere ([Temmer, 2021](#)). During a flare, the amount of energy released on average is as much as the Sun radiates in 1 second. Their temperature is $< 3 \cdot 10^7$ K.

Flares are classified into different categories based on the maximum X-ray flux (W/m^2), measured in the 1 to 8 Å wavelength range using the Geostationary Operational Environmental Satellites (GOES). These can be classes A, B, C, M, and X, with the exact value indicated by the number following the letter.

²<https://assets.lloyds.com/assets/pdf-solar-storm-risk-to-the-north-american-electric-grid/1/pdf-Solar-Storm-Risk-to-the-North-American-Electric-Grid.pdf>

- We refer to a coronal mass ejection (CME) when the magnetic field lines detach from the Sun's atmosphere, causing a substantial amount of plasma and magnetic flux to be ejected into interplanetary space (Priest, 2014). When the plasma cloud, made of a vast amount of charged particles, leaves the Sun, it becomes an expanding interplanetary magnetic cloud. The amount of departing material ranges between 10^{11} and 10^{14} kg, with speeds varying from a few hundred km/s up to 3000 km/s (Gopalswamy and Thompson, 2000), although the average observed speed is 500 km/s (Yurchyshyn et al., 2005).
- Prominences are constant (quiescent) or variable (active) shaped clouds of hot gas that are less dense than their surroundings and observed at the edge of the solar disk. In the $H\alpha$ range, they appear as bright, curtain-like formations. When prominences are seen on the solar disk by the observer, they appear as dark, thread-like structures, which are then referred to as filaments. Figure 4 clearly shows the difference between a filament and a prominence.

Quiescent prominences can persist even for months with little to no change of their shape. However, they can become active due to some form of instability. In such cases, the quiescent filament can suddenly rise at high speed and may even leave the Sun, in which case it is called an eruptive prominence.

High-energy flares and CMEs can occur independently of each other, but they often happen together, usually erupting from magnetically complex sunspots in active regions. Sunspots are areas with lower temperatures than the surrounding quiet solar surface and possess strong magnetic fields. Sunspots are the footpoint of magnetic field line loops, which enter and exit the photosphere, as shown in Figure 2. This is the active region AR13664 with a magnetically complex sunspot group, which on May 11, 2024, produced an X5.8 flare (causing auroras visible in Hungary), and then on May 14, it produced the strongest flare of the current solar cycle, with an intensity of X8.79.



Figure 3: Aurora in Hungary above Fonyód. (Source: Márton Kárpáti/Telex)

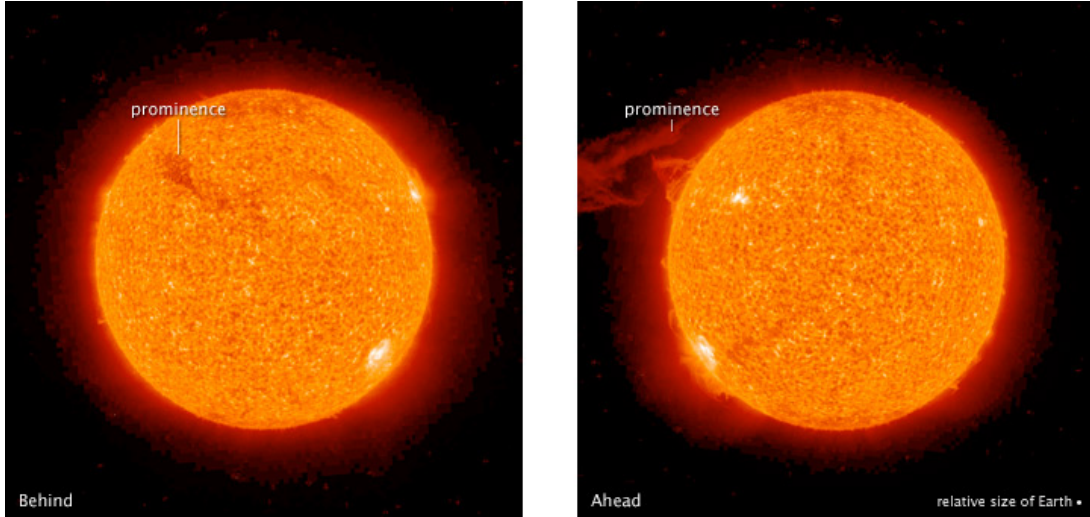


Figure 4: These two images were taken on September 26, 2009, by the STEREO A/B spacecraft. The image on the left shows a filament, while the image on the right shows a prominence, but both actually represent the same phenomenon. (Source: NASA)

1.3 The theory of magnetohydrodynamics

The theory of magnetohydrodynamics (MHD) was developed by [Alfvén \(1942\)](#), for which he received the Nobel Prize in Physics in 1970. He suggested that if an electrically conductive fluid is placed in a constant magnetic field, the motion of the fluid induces an electromagnetic force that generates a current, which alters the behaviour of the conductive fluid, thereby creating MHD waves. The theory combines Maxwell's equations, Ohm's law, the ideal gas law, the mass continuity equation, as well as the equations of motion and energy. It achieved significant success in describing the behaviour of plasmas, whether in terms of the Sun or in the context of experimental fusion reactors on Earth. The complexity arising from both hydrodynamics and electromagnetism makes an analytical approach difficult, so the equations are often simplified with certain assumptions. An example of this is the 'single-fluid' model, which treats the entire plasma as a single fluid regardless of the distribution of electrons and ions within it. It is excellently applicable to most phenomena, however, certain parts of the Sun's surface and corona are better described using a more complex 'three-fluid' model.

Depending on the effects and forces considered, various types of MHD waves can be generated on the Sun ([Roberts, 2019](#)). These waves may partly explain the so-called coronal heating problem, which has long been a concern for solar physicists. Essentially, the temperature of the Sun's surface, the photosphere, is 4,000-5,000 K, while the overlying chromosphere is hotter, at about 25,000 K. In the corona, the temperature then abruptly jumps to the million Kelvin range. However, three types of MHD waves will be of particular importance to us now: the Alfvén wave, and the fast and slow magnetoacoustic waves.

- An Alfvén wave is a type of wave where energy propagates in the direction of the magnetic field at the Alfvén speed, while it cannot propagate perpendicular to it; the propagation between these two directions scales with the cosine of the angle with respect to the magnetic field. These are transverse waves that carry energy in the direction of the magnetic field at the Alfvén speed (which is the group velocity). This is an incompressible wave, it is commonly referred to as shear or torsional Alfvén waves. More details about these waves can be found in the article of [Mathioudakis et al. \(2013\)](#).
- Magnetoacoustic waves are acoustic origin waves influenced by the magnetic field. Depending on their frequency, there are two cases in an infinite homogeneous plasma: the higher-frequency waves are called fast magnetoacoustic waves, while the lower-frequency ones are called slow magnetoacoustic waves. The frequency of Alfvén waves falls between these two.

1.4 Magnetic helicity flux

1.4.1 Theoretical background

Helicity integrals define a number that describes the topological properties of field lines. Thus, magnetic helicity flux (MHF) measures how much the magnetic field lines twist and coil around each other in a unit surface. The topological aspects and how to calculate the different quantities can be found in [Berger \(1999\)](#)'s article. Its role and calculation are increasingly becoming crucial today. Since MHD waves may transport energy into the corona, it is worth investigating how much and in what manner they carry MHF. This allows us to better understand what happens behind a solar eruption and maybe develop forecasting models and probabilities based on this understanding.

We define the magnetic field as the curl of a magnetic vector potential, thus meeting the expectation $\nabla \cdot \mathbf{B} = 0$, since the divergence of the curl is 0. According to Woltjer's Theorem ([Woltjer, 1958](#)), for a closed volume, magnetic helicity is conserved if it is bounded by a magnetic surface in the ideal MHD case ($\mathbf{B} \cdot \mathbf{n}|_S = 0$).

Thus, let

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (1)$$

the magnetic field be the curl of a magnetic vector potential ($\mathbf{A}$).

Magnetic helicity, by definition, is given by the following integral:

$$H = \int_V \mathbf{B} \cdot \mathbf{A} d^3x. \quad (2)$$

I will use a slightly different calculation method, which I will define before the calculation.

1.4.2 Practical application

[Korsós et al. \(2020b\)](#) found that there is a correlation between the periodicity of MHF and the production of flares in active regions. In the paper the authors compared the evolution of magnetic helicity flux of flaring (at least one X-class) and non-flaring (less than M5.0) active regions to examine their evolution. To do this, they used data from SDO/HMI vector magnetograms, from which they gathered every important component needed in [Berger \(1984\)](#)'s helicity calculation formula, which I will describe later in this work. They split the total helicity flux (T) into two components: shearing (SH) and emerging (EM), then applied wavelet analysis for each one of them on an interval of ca. 100 hours before and after a flare. This helped finding oscillatory patterns in the different components of MHF before a flare in flaring active regions. Their hypothesis was that the different components of MHF oscillate in a given way before flares. Following this, [Soós et al. \(2022\)](#) used a larger sample, performed the same method, and managed to prove the hypothesis. These findings contribute to the refinement and enhancement of existing tools for forecasting flares and CMEs.

I aim to develop a theoretical framework that offers a comprehensive foundation for the observed phenomena. In fact, [Yoshida \(1991\)](#) has already discussed the transport of magnetic helicity by MHD waves within a plasma; however, many of the notations have become outdated. Therefore, it is advisable to further discuss, modernise, and illuminate the topic from a different perspective. In my work, I will outline the MHD and ideal MHD equations based on the book of [Priest \(2014\)](#), and derive the key dispersion relations and connections analogue of [Roberts \(1981\)](#). Then, I will calculate the MHF for several different waves as definite and indefinite integrals in the Cartesian coordinate system, for both the case of homogeneous magnetic field and magnetic interface (where there is a jump in the magnetic field strength at $x = 0$).

2 MHD equations

First and foremost, I will recall Maxwell's equations:

$$\nabla \times \mathbf{B} = \mu \mathbf{j} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}, \quad (3)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (4)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (5)$$

$$\nabla \cdot \mathbf{E} = \frac{\rho^*}{\epsilon}, \quad (6)$$

where $\mathbf{B}$, $\mathbf{E}$ are the magnetic and electric fields, c is the speed of light, μ is the permeability, ϵ is the permittivity, ρ^* is the charge density, and $\mathbf{j}$ is the current density, respectively. Then, I will approximate the values everywhere using the vacuum permeability and permittivity, denoted by μ_0 , ϵ_0 , respectively, but for simplicity, I will omit the '0' index. I should also note that formally $\mathbf{B}$ denotes the magnetic flux density, but in astrophysics, it is commonly referred to as the magnetic field.

The next necessary equation is Ohm's law:

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \quad (7)$$

where σ is the specific conductivity, while $\mathbf{v}$ is the plasma velocity.

It is advisable to eliminate $\mathbf{E}$ and $\mathbf{j}$, so we combine the appropriate Maxwell's equations and Ohm's law into a single equation. This yields the induction equation:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times (\eta \nabla \times \mathbf{B}), \quad (8)$$

where $\eta = \frac{1}{\mu\sigma}$ is the magnetic diffusivity. In Equation (8), the second term can be simplified using relation (4) and the curl-curl vector identity; however, I will omit this term in my calculations.

Also important is the use of the ideal gas law in the following form:

$$p = \frac{k_B}{m} \rho T, \quad (9)$$

where p is the pressure, k_B is Boltzmann's constant, m is the mean particle mass, ρ is the density, T is the temperature.

The continuity equation derived from the conservation of mass:

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0, \quad (10)$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$ is the substantial derivative operator.

The plasma's equation of motion:

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \mathbf{j} \times \mathbf{B} + \mathbf{F}, \quad (11)$$

where we assume p is a scalar, $\mathbf{j} \times \mathbf{B}$ is the Lorentz force per unit volume, whereas $\mathbf{F}$ is all other forces combined.

Finally, the energy equation will also be needed. Since many different effects can be considered, but I will use the ideal MHD equations, writing down the formal energy equation goes beyond the core discussion of this work. Therefore, I will directly present the adiabatic case here:

$$\frac{d}{dt} \left(\frac{p}{\rho^\gamma} \right) = 0, \quad (12)$$

where γ is the adiabatic exponent, i.e. the ratio of the specific heat capacities.

2.1 Ideal MHD equations

Starting with the ideal MHD equations significantly simplifies the calculations. This requires that the magnetic Reynolds number be much greater than one. This can be interpreted as the plasma's specific conductivity being infinite or at least very large; however, in reality, the assumption that magnetic diffusivity is negligible is also generally applicable. This is because, in astrophysics, we usually examine large spatial scales, which makes this term negligibly small. In addition, I consider the forces $\mathbf{F}$ in the equation of motion (11) to be zero, meaning that we neglect the effects from rotation and gravity. Since the denominator of the second term in Equation (3) contains the square of the speed of light, we can neglect this term. Thus, the equations of interest to us are:

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0, \quad (13)$$

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \frac{1}{\mu} (\nabla \times \mathbf{B}) \times \mathbf{B}, \quad (14)$$

$$\frac{d}{dt} \left(\frac{p}{\rho^\gamma} \right) = 0, \quad (15)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}), \quad (16)$$

$$\nabla \cdot \mathbf{B} = 0. \quad (17)$$

Furthermore, it is true that

$$\mathbf{j} = \frac{1}{\mu} \nabla \times \mathbf{B}$$

and

$$T = \frac{mp}{k_B \rho}.$$

2.1.1 Perturbation theory

Perturbation theory is essential for the study of waves. Consider a linear perturbation around an equilibrium state:

$$\rho = \rho_0 + \rho_1, \quad p = p_0 + p_1, \quad \mathbf{v} = \mathbf{0} + \mathbf{v}_1, \quad \mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1,$$

then substituting these into the ideal MHD Equations (13)–(17), we obtain the perturbed equations. Since the plasma velocity is small, for simplicity, I will denote it as $\mathbf{v}$ from now on. We neglect the quadratic terms of the small perturbations and use the fact that the derivatives of the equilibrium variables are zero, as they are constants.

The continuity equation:

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{v} = 0,$$

where substituting the perturbation into ρ

$$\frac{\partial(\rho_0 + \rho_1)}{\partial t} + \mathbf{v} \cdot \nabla(\rho_0 + \rho_1) + (\rho_0 + \rho_1) \nabla \cdot \mathbf{v} = 0,$$

finally, neglecting the quadratic terms and considering the derivatives of the equilibrium variables to be 0:

$$\frac{\partial \rho_1}{\partial t} + (\mathbf{v} \cdot \nabla) \rho_0 + \rho_0 (\nabla \cdot \mathbf{v}). \quad (18)$$

Before examining the perturbed form of the equation of motion, let us first reformulate it by deriving a vector identity. For this, we need ε , the Levi-Civita symbol, and δ , the Kronecker delta. In the case of ε , if the indices are cyclically permuted (e.g.: 123, 312), then its value is 1, if two indices coincide, then its value is 0, whereas if the indices are anticyclically permuted (e.g.: 321, 132), then its value is -1 . In the case of δ , if the two indices coincide, then its value is 1, otherwise 0. For two coinciding indices, we automatically sum, but using Einstein's summation convention, I will not explicitly write out the Σ symbols. The symbol of differentiation will

be ∂ . The index notation form of the vector cross product, known from vector calculus, is $(\mathbf{a} \times \mathbf{b})_i = \varepsilon_{ijk} a_j b_k$. Thus, the second term of Equation (14) (currently disregarding μ):

$$[(\nabla \times \mathbf{B}) \times \mathbf{B}]_i = \varepsilon_{ijk} (\nabla \times \mathbf{B})_j B_k = \varepsilon_{ijk} \varepsilon_{jmp} \partial_m B_p B_k.$$

It is known that $\varepsilon_{ijk} \varepsilon_{imp} = \delta_{jm} \delta_{kp} - \delta_{jp} \delta_{km}$, thus, after a cyclic permutation

$$\varepsilon_{ijk} \varepsilon_{jmp} \partial_m B_p B_k = (\delta_{km} \delta_{ip} - \delta_{kp} \delta_{im}) \partial_m B_p B_k.$$

Since the indices of δ need to coincide in order to evaluate to 1, thus, for example $\delta_{km} \partial_m = \partial_k$. Expanding the parentheses and performing certain summations, we find the form

$$\delta_{ip} \partial_k B_p B_k - \delta_{im} \partial_m B_p B_p.$$

Since $\partial_m (B_p B_p) = \partial_m B_p B_p + B_p \partial_m B_p = 2 \partial_m B_p B_p$, as well as $\partial_k B_p B_k = (B_k \partial_k) B_p$, thus, performing the final summation, we obtain the result that

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = (\mathbf{B} \cdot \nabla) \mathbf{B} - \nabla \left(\frac{B^2}{2} \right).$$

We can now write down the perturbed form of the equation of motion:

$$(\rho_0 + \rho_1) \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = -\nabla(p_0 + p_1) + \frac{1}{\mu} [(\mathbf{B}_0 + \mathbf{B}_1) \cdot \nabla] (\mathbf{B}_0 + \mathbf{B}_1) - \nabla \left(\frac{(\mathbf{B}_0 + \mathbf{B}_1)^2}{2\mu} \right).$$

Neglecting again the quadratic terms and considering the derivatives of the equilibrium variables to be zero

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla p_1 + \frac{1}{\mu} (\mathbf{B}_0 \cdot \nabla) \mathbf{B}_1 - \frac{1}{\mu} \nabla (\mathbf{B}_0 \cdot \mathbf{B}_1) \quad (19)$$

will be the perturbed equation of motion.

The derivation of the perturbed form of the adiabatic energy equation is somewhat longer than the derivations of the other perturbed equations. The following steps are required in the derivation:

$$\begin{aligned} \frac{d}{dt} \left(\frac{p}{\rho^\gamma} \right) &= \frac{\frac{dp}{dt} \rho^\gamma - p \frac{d\rho^\gamma}{dt}}{\rho^{2\gamma}} = 0, \\ \frac{dp}{dt} \rho^\gamma - p \frac{d\rho^\gamma}{dt} &= 0, \\ \left(\frac{\partial p}{\partial t} + (\mathbf{v} \cdot \nabla) p \right) \rho^\gamma - p \left(\frac{\partial \rho^\gamma}{\partial t} + (\mathbf{v} \cdot \nabla) \rho^\gamma \right) &= 0, \\ \left(\frac{\partial p}{\partial t} + (\mathbf{v} \cdot \nabla) p \right) \rho^\gamma &= p \left(\gamma \rho^{\gamma-1} \frac{\partial \rho}{\partial t} + \gamma \rho^{\gamma-1} (\mathbf{v} \cdot \nabla) \rho \right), \\ \left(\frac{\partial p}{\partial t} + (\mathbf{v} \cdot \nabla) p \right) \rho^\gamma &= p \gamma \rho^{\gamma-1} \left(\frac{\partial \rho}{\partial t} + (\mathbf{v} \cdot \nabla) \rho \right). \end{aligned}$$

Then, we substitute the perturbation:

$$\begin{aligned} \left(\frac{\partial(p_0 + p_1)}{\partial t} + (\mathbf{v} \cdot \nabla)(p_0 + p_1) \right) (\rho_0 + \rho_1)^\gamma &= (p_0 + p_1) \gamma (\rho_0 + \rho_1)^{\gamma-1} \left(\frac{\partial(\rho_0 + \rho_1)}{\partial t} + (\mathbf{v} \cdot \nabla)(\rho_0 + \rho_1) \right), \\ \left(\frac{\partial p_1}{\partial t} + (\mathbf{v} \cdot \nabla) p_0 \right) (\rho_0 + \rho_1)^\gamma &= (p_0 + p_1) \gamma (\rho_0 + \rho_1)^{\gamma-1} \left(\frac{\partial \rho_1}{\partial t} + (\mathbf{v} \cdot \nabla) \rho_0 \right). \end{aligned}$$

Writing out the power expansion up to the linear term:

$$\left(\frac{\partial p_1}{\partial t} + (\mathbf{v} \cdot \nabla)p_0\right)(\rho_0^\gamma + \gamma p_1 \rho_0^{\gamma-1}) = (p_0 + p_1)\gamma(\rho_0^{\gamma-1} + (\gamma-1)p_1 \rho_0^{\gamma-2})\left(\frac{\partial \rho_1}{\partial t} + (\mathbf{v} \cdot \nabla)\rho_0\right),$$

then neglecting the quadratic terms:

$$\left(\frac{\partial p_1}{\partial t} + (\mathbf{v} \cdot \nabla)p_0\right)\rho_0^\gamma = \frac{p_0 \gamma \rho_0^\gamma}{\rho_0} \left(\frac{\partial \rho_1}{\partial t} + (\mathbf{v} \cdot \nabla)\rho_0\right).$$

Thus, the perturbed form of the adiabatic energy equation:

$$\frac{\partial p_1}{\partial t} + (\mathbf{v} \cdot \nabla)p_0 = c_s^2 \left(\frac{\partial \rho_1}{\partial t} + (\mathbf{v} \cdot \nabla)\rho_0\right), \quad (20)$$

where

$$c_s^2 = \frac{\gamma p_0}{\rho_0}$$

is the square of the sound speed.

Finally, the perturbed form of the induction equation follows:

$$\frac{\partial(\mathbf{B}_0 + \mathbf{B}_1)}{\partial t} = \nabla \times (\mathbf{v} \times (\mathbf{B}_0 + \mathbf{B}_1)).$$

Neglecting the quadratic terms and considering the derivatives of the equilibrium variables to be zero, the perturbed induction equation is

$$\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}_0). \quad (21)$$

Note: since the magnetic field is always source-free (17), this condition has to hold after the perturbation, that is, $\nabla \cdot \mathbf{B}_1 = 0$.

3 Dispersion relations

In this work, I will now focus on a magnetic field pointing in the $\hat{\mathbf{z}}$ -direction, similar to the paper of Roberts (1981). Therefore, deriving the dispersion relations may be instructive, as it provides a better insight into the behaviour and mathematical form of the waves.

3.1 Infinite homogeneous plasma

First and foremost, we take advantage of the fact that

$$\mathbf{B}_0 = \begin{pmatrix} 0 \\ 0 \\ B_0 \end{pmatrix}.$$

Therefore, Equation (19) becomes

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla p_1 + \frac{1}{\mu} \left(B_0 \frac{\partial}{\partial z}\right) \mathbf{B}_1 - \frac{1}{\mu} \nabla (B_0 B_{1,z}).$$

Differentiating this with respect to time gives:

$$\rho_0 \frac{\partial^2 \mathbf{v}}{\partial t^2} = -\nabla \frac{\partial p_1}{\partial t} + \frac{1}{\mu} \left(B_0 \frac{\partial}{\partial z}\right) \frac{\partial \mathbf{B}_1}{\partial t} - \frac{1}{\mu} \nabla \left(B_0 \frac{\partial B_{1,z}}{\partial t}\right).$$

Before simply substituting Equations (20) and (21) into the appropriate places, let us examine what vector the perturbed induction equation corresponds to:

$$\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}_0) = \nabla \times \begin{pmatrix} v_y B_{0,z} - v_z B_{0,y} \\ v_z B_{0,x} - v_x B_{0,z} \\ v_x B_{0,y} - v_y B_{0,x} \end{pmatrix},$$

where only $B_{0,z}$ is non-zero, thus

$$\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}_0) = \nabla \times \begin{pmatrix} v_y B_0 \\ -v_x B_0 \\ 0 \end{pmatrix},$$

and since B_0 is a constant in this case, there is no need to use the product rule during differentiation. Thus, the time derivative of the magnetic field perturbation:

$$\frac{\partial \mathbf{B}_1}{\partial t} = \begin{pmatrix} B_0 \frac{\partial v_x}{\partial z} \\ B_0 \frac{\partial v_y}{\partial z} \\ -B_0 \frac{\partial v_x}{\partial x} - B_0 \frac{\partial v_y}{\partial y} \end{pmatrix}.$$

To obtain the governing equations, we need to take the divergence of the time derivative of the perturbed equation of motion. Based on Equation (17), the divergence of $\mathbf{B}_1$ is 0, thus this term vanishes, furthermore, using the $\nabla \cdot \nabla = \Delta$ vector identity, the Laplace operator appears:

$$\rho_0 \frac{\partial^2 \nabla \cdot \mathbf{v}}{\partial t^2} = -\Delta \frac{\partial p_1}{\partial t} - \frac{1}{\mu} \Delta \left(B_0 \frac{\partial B_{1,z}}{\partial t} \right).$$

Note: I kept the second term on the left side of Equation (20) because it is not zero in the case of a magnetic interface. However, for a homogeneous plasma, where p_0 is constant, this term vanishes. Therefore,

$$\frac{\partial p_1}{\partial t} = -\rho_0 c_s^2 \nabla \cdot \mathbf{v},$$

after using Equation (18). Substituting everything into the modified perturbed equation of motion, we obtain that

$$\rho_0 \frac{\partial^2 \nabla \cdot \mathbf{v}}{\partial t^2} = \Delta (\rho_0 c_s^2 \nabla \cdot \mathbf{v}) + \frac{1}{\mu} \Delta \left(B_0 \frac{\partial v_x}{\partial x} + B_0 \frac{\partial v_y}{\partial y} \right).$$

Since B_0 , ρ_0 , and c_s^2 are constants, they can be factored out in front of the Laplace operator, and if we use the fact that

$$v_A^2 = \frac{B_0^2}{\mu \rho_0}$$

is the square of the Alfvén velocity, then we obtain the first governing equation:

$$\frac{\partial^2 \nabla \cdot \mathbf{v}}{\partial t^2} = (c_s^2 + v_A^2) \Delta \nabla \cdot \mathbf{v} - v_A^2 \Delta \frac{\partial v_z}{\partial z}. \quad (22)$$

If we want to obtain the second equation, we now need to examine the z -component of the time derivative of the perturbed equation of motion. The terms containing the magnetic field cancel each other out, thus, after substituting in $\frac{\partial p_1}{\partial t}$, we obtain the following equation:

$$\frac{\partial^2 v_z}{\partial t^2} = c_s^2 \frac{\partial \nabla \cdot \mathbf{v}}{\partial z}. \quad (23)$$

Let us differentiate Equation (22) with respect to time twice! Therefore, we obtain that

$$\frac{\partial^4 \nabla \cdot \mathbf{v}}{\partial t^4} = (c_s^2 + v_A^2) \frac{\partial^2}{\partial t^2} \Delta \nabla \cdot \mathbf{v} - v_A^2 \frac{\partial^2}{\partial t^2} \Delta \frac{\partial v_z}{\partial z},$$

then using Equation (23) and rearranging to 0, we obtain the third equation from the paper:

$$\frac{\partial^4 \nabla \cdot \mathbf{v}}{\partial t^4} - (c_s^2 + v_A^2) \frac{\partial^2}{\partial t^2} \Delta \nabla \cdot \mathbf{v} + c_s^2 v_A^2 \frac{\partial^2}{\partial z^2} \Delta \nabla \cdot \mathbf{v} = 0. \quad (24)$$

In order to obtain the dispersion relation, the physical quantity appearing in the differential equation needs to be expressed in Fourier form, which in this case is $\nabla \cdot \mathbf{v}$. Therefore,

$$\nabla \cdot \mathbf{v} = \widehat{(\nabla \cdot \mathbf{v})}(x) e^{i(\omega t + l y + k z)},$$

where $\widehat{(\nabla \cdot \mathbf{v})}(x)$ is a constant multiplier depending from the x -direction. Since it would be tedious to write this out constantly, I will replace it with $\alpha(x)$. After the substitution

$$\begin{aligned} \frac{\partial^4}{\partial t^4} \alpha(x) e^{i(\omega t + l y + k z)} - (c_s^2 + v_A^2) \frac{\partial^2}{\partial t^2} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \alpha(x) e^{i(\omega t + l y + k z)} + \\ + c_s^2 v_A^2 \frac{\partial^2}{\partial z^2} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \alpha(x) e^{i(\omega t + l y + k z)} = 0, \end{aligned}$$

$$\begin{aligned} \omega^4 \alpha(x) e^{i(\omega t + l y + k z)} + (c_s^2 + v_A^2) \omega^2 (-l^2 - k^2) \alpha(x) e^{i(\omega t + l y + k z)} + (c_s^2 + v_A^2) \omega^2 \frac{\partial^2 \alpha(x)}{\partial x^2} e^{i(\omega t + l y + k z)} - \\ - k^2 c_s^2 v_A^2 (-l^2 - k^2) \alpha(x) e^{i(\omega t + l y + k z)} - k^2 c_s^2 v_A^2 \frac{\partial^2 \alpha(x)}{\partial x^2} e^{i(\omega t + l y + k z)} = 0. \end{aligned}$$

Since the exponential part is common to all terms, we can simplify by it as it is not zero. Additionally, since $\alpha(x)$ depends only on x , we can also omit the partial derivative notation. Therefore,

$$\begin{aligned} \omega^4 \alpha(x) - (c_s^2 + v_A^2) \left(-\omega^2 \frac{d^2 \alpha(x)}{dx^2} + l^2 \omega^2 \alpha(x) + k^2 \omega^2 \alpha(x) \right) + \\ + c_s^2 v_A^2 \left(-k^2 \frac{d^2 \alpha(x)}{dx^2} + k^2 l^2 \alpha(x) + k^4 \alpha(x) \right) = 0. \end{aligned}$$

If we rearrange the equation by dividing by the factor of $\frac{d^2 \alpha(x)}{dx^2}$ and factoring out $\alpha(x)$ from each term, then we obtain that

$$\frac{d^2 \alpha(x)}{dx^2} - \frac{\omega^4 - (c_s^2 + v_A^2)(l^2 \omega^2 + k^2 \omega^2) + k^2 l^2 c_s^2 v_A^2 + k^4 c_s^2 v_A^2}{(c_s^2 + v_A^2) \left(k^2 \frac{c_s^2 v_A^2}{c_s^2 + v_A^2} - \omega^2 \right)} \alpha(x) = 0.$$

If we examine the numerator more closely, we see that the coefficient of l^2 is exactly the same as the denominator, therefore,

$$\frac{d^2 \alpha(x)}{dx^2} - \left(l^2 + \frac{\omega^4 - k^2 c_s^2 \omega^2 - k^2 v_A^2 \omega^2 + k^4 c_s^2 v_A^2}{(c_s^2 + v_A^2) \left(k^2 \frac{c_s^2 v_A^2}{c_s^2 + v_A^2} - \omega^2 \right)} \right) \alpha(x) = 0.$$

It can be shown through a few simple steps that the numerator of the fraction is $(k^2 c_s^2 - \omega^2)(k^2 v_A^2 - \omega^2)$, thus we obtain:

$$\frac{d^2 \alpha(x)}{dx^2} - (l^2 + m_0^2) \alpha(x) = 0, \quad (25)$$

where

$$m_0^2 = \frac{(k^2 c_s^2 - \omega^2)(k^2 v_A^2 - \omega^2)}{(c_s^2 + v_A^2)(k^2 c_T^2 - \omega^2)},$$

$$c_T^2 = \frac{c_s^2 v_A^2}{c_s^2 + v_A^2}.$$

Here, c_T is the so-called slow speed (or cusp speed), which is the group velocity of the slow magnetoacoustic wave. If $\alpha(x) \sim e^{inx}$, then

$$\omega^4 - \kappa^2 (c_s^2 + v_A^2) \omega^2 + k^2 \kappa^2 c_s^2 v_A^2 = 0,$$

where κ^2 is the square of the wave vector ($\kappa = (n, l, k)$). This is the dispersion relation for the slow and fast magnetoacoustic waves.

3.2 Infinite inhomogeneous plasma

With the following derivation, I aim to obtain a specific dispersion relation for a magnetic interface. Since we now wish to address the phenomenon of the magnetic interface, the layering of the magnetic field affects the pressure, density, and temperature. The total pressure is constant, so the plasma's equilibrium state is

$$p_0 = p_0(x), \quad \rho_0 = \rho_0(x), \quad \frac{d}{dx} \left(p_0 + \frac{B_0^2}{2\mu} \right) = 0,$$

$$\mathbf{B}_0(x) = \begin{pmatrix} 0 \\ 0 \\ B_0(x) \end{pmatrix}.$$

This modifies Equation (19), such that the term $+\frac{1}{\mu}(\mathbf{B}_1 \cdot \nabla)\mathbf{B}_0$ appears on the right side of the equation:

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla \left(p_1 + \frac{\mathbf{B}_0 \cdot \mathbf{B}_1}{\mu} \right) + \frac{1}{\mu}(\mathbf{B}_0 \cdot \nabla)\mathbf{B}_1 + \frac{1}{\mu}(\mathbf{B}_1 \cdot \nabla)\mathbf{B}_0. \quad (26)$$

Let the total perturbed pressure be

$$p_{1,T} = p_1 + \frac{B_0 B_{1,z}}{\mu},$$

which is inside the gradient on the right side of Equation (26).

Using the fact that $p_0(x)$ depends on x , Equation (20) becomes the following:

$$\frac{\partial p_1}{\partial t} = -\rho_0 c_s^2 \nabla \cdot \mathbf{v} - v_x \frac{dp_0}{dx}.$$

Since $B_0(x)$ depends on x ,

$$\frac{\partial \mathbf{B}_1}{\partial t} = \begin{pmatrix} B_0 \frac{\partial v_x}{\partial z} \\ B_0 \frac{\partial v_y}{\partial z} \\ -B_0 \frac{\partial v_x}{\partial x} - v_x \frac{dB_0}{dx} - B_0 \frac{\partial v_y}{\partial y} \end{pmatrix}.$$

Therefore,

$$\frac{\partial p_{1,T}}{\partial t} = \frac{\partial p_1}{\partial t} + \frac{B_0}{\mu} \frac{\partial B_{1,z}}{\partial t},$$

substituting the above relationships into this, we obtain that

$$\frac{\partial p_{1,T}}{\partial t} = \rho_0 v_A^2 \frac{\partial v_z}{\partial z} - \rho_0 (c_s^2 + v_A^2) \nabla \cdot \mathbf{v} - v_x \frac{dp_0}{dx} - v_x \frac{B_0}{\mu} \frac{dB_0}{dx},$$

$$-v_x \left(\frac{dp_0}{dx} + \frac{B_0}{\mu} \frac{dB_0}{dx} \right) = -v_x \frac{d}{dx} \left(p_0 + \frac{B_0^2}{2\mu} \right) = 0,$$

therefore, the total perturbed pressure satisfies that

$$\frac{\partial p_{1,T}}{\partial t} = \rho_0 v_A^2 \frac{\partial v_z}{\partial z} - \rho_0 (c_s^2 + v_A^2) \nabla \cdot \mathbf{v}.$$

Decomposing Equation (26) into separate components,

$$\rho_0 \frac{\partial v_x}{\partial t} = -\frac{\partial}{\partial x} \left(p_1 + \frac{B_0 B_{1,z}}{\mu} \right) + \frac{1}{\mu} B_0 \frac{\partial B_{1,x}}{\partial z},$$

$$\rho_0 \frac{\partial v_y}{\partial t} = -\frac{\partial}{\partial y} \left(p_1 + \frac{B_0 B_{1,z}}{\mu} \right) + \frac{1}{\mu} B_0 \frac{\partial B_{1,y}}{\partial z},$$

$$\rho_0 \frac{\partial v_z}{\partial t} = -\frac{\partial p_1}{\partial z} + \frac{1}{\mu} B_{1,x} \frac{dB_0}{dx}.$$

In order to obtain the required dispersion relation, the above three equations must be differentiated with respect to time individually. I will start with the v_x component, which then becomes

$$\rho_0 \frac{\partial^2 v_x}{\partial t^2} = -\frac{\partial}{\partial x} \frac{\partial}{\partial t} \left(p_1 + \frac{B_0 B_{1,z}}{\mu} \right) + \frac{1}{\mu} B_0 \frac{\partial^2 B_{1,x}}{\partial z \partial t}.$$

Substituting the appropriate relations:

$$\rho_0 \frac{\partial^2 v_x}{\partial t^2} = -\frac{\partial}{\partial x} \left(\rho_0 v_A^2 \frac{\partial v_z}{\partial z} - \rho_0 (c_s^2 + v_A^2) \nabla \cdot \mathbf{v} \right) + \frac{B_0}{\mu} \frac{\partial}{\partial z} B_0 \frac{\partial v_x}{\partial z},$$

thus after some rearranging, we find that

$$\rho_0 \left(\frac{\partial^2}{\partial t^2} - v_A^2 \frac{\partial^2}{\partial z^2} \right) v_x = \frac{\partial}{\partial x} \left(\rho_0 (c_s^2 + v_A^2) \nabla \cdot \mathbf{v} - \rho_0 v_A^2 \frac{\partial v_z}{\partial z} \right). \quad (27)$$

One must proceed similarly to Equation (27) regarding the equation for the v_y component:

$$\rho_0 \frac{\partial^2 v_y}{\partial t^2} = -\frac{\partial}{\partial y} \frac{\partial}{\partial t} \left(p_1 + \frac{B_0 B_{1,z}}{\mu} \right) + \frac{1}{\mu} B_0 \frac{\partial^2 B_{1,y}}{\partial z \partial t}.$$

Similarly, by substituting the relations and rearranging the equation, we obtain that

$$\rho_0 \left(\frac{\partial^2}{\partial t^2} - v_A^2 \frac{\partial^2}{\partial z^2} \right) v_y = \frac{\partial}{\partial y} \left(\rho_0 (c_s^2 + v_A^2) \nabla \cdot \mathbf{v} - \rho_0 v_A^2 \frac{\partial v_z}{\partial z} \right). \quad (28)$$

And finally, we also take the time derivative of the equation for the v_z component:

$$\rho_0 \frac{\partial^2 v_z}{\partial t^2} = -\frac{\partial^2 p_1}{\partial z \partial t} + \frac{1}{\mu} \frac{dB_0}{dx} \frac{\partial B_{1,x}}{\partial t}.$$

Substituting the appropriate relations:

$$\rho_0 \frac{\partial^2 v_z}{\partial t^2} = -\frac{\partial}{\partial z} \left(-\rho_0 c_s^2 \nabla \cdot \mathbf{v} - v_x \frac{dp_0}{dx} \right) + \frac{1}{\mu} \frac{dB_0}{dx} B_0 \frac{\partial v_x}{\partial z},$$

which, after some rearranging, becomes

$$\rho_0 \frac{\partial^2 v_z}{\partial t^2} = \rho_0 c_s^2 \frac{\partial \nabla \cdot \mathbf{v}}{\partial z} + \frac{\partial v_x}{\partial z} \left(\frac{dp_0}{dx} + \frac{B_0}{\mu} \frac{dB_0}{dx} \right).$$

As derived above, the term in the last parentheses is zero, so we obtain that

$$\frac{\partial^2 v_z}{\partial t^2} = c_s^2 \frac{\partial \nabla \cdot \mathbf{v}}{\partial z}. \quad (29)$$

Using Equations (27), (28), and (29), one can derive the dispersion relation; however, this requires more extensive calculations and a lot of algebra. I will use the appropriate Fourier forms of the components, then eliminate the variables as a system of three equations, resulting in a differential equation for $\hat{v}_x$ that yields the dispersion relation. Here $\hat{v}_x$ is an x -dependent amplitude of the perturbation. The appropriate Fourier forms:

$$v_x = \hat{v}_x(x) e^{i(\omega t + ly + kz)},$$

$$v_y = \hat{v}_y(x) e^{i(\omega t + ly + kz)},$$

$$v_z = \hat{v}_z(x) e^{i(\omega t + ly + kz)}.$$

First, substitute the Fourier forms into the equations, and then only algebraic operations remain. Thus, Equation (27) becomes

$$\rho_0 \left(\frac{\partial^2}{\partial t^2} - v_A^2 \frac{\partial^2}{\partial z^2} \right) \hat{v}_x e^{i(\omega t + l y + k z)} = \frac{\partial}{\partial x} \left(\rho_0 (c_s^2 + v_A^2) \left(\frac{\partial}{\partial x} \hat{v}_x e^{i(\omega t + l y + k z)} + \frac{\partial}{\partial y} \hat{v}_y e^{i(\omega t + l y + k z)} + \frac{\partial}{\partial z} \hat{v}_z e^{i(\omega t + l y + k z)} \right) - \rho_0 v_A^2 \frac{\partial}{\partial z} \hat{v}_z e^{i(\omega t + l y + k z)} \right),$$

where, after performing the appropriate derivatives and factoring out the common exponential term, we arrive at

$$\rho_0 (-\omega^2 + k^2 v_A^2) \hat{v}_x e^{i(\omega t + l y + k z)} = e^{i(\omega t + l y + k z)} \frac{\partial}{\partial x} \left(\rho_0 (c_s^2 + v_A^2) \left(\frac{d\hat{v}_x}{dx} + il\hat{v}_y + ik\hat{v}_z \right) - \rho_0 v_A^2 ik\hat{v}_z \right).$$

After some rearrangement of the bracketed terms and simplification with the exponential term, Equation (27) takes the following form:

$$\rho_0 (k^2 v_A^2 - \omega^2) \hat{v}_x = \frac{d}{dx} \left(\rho_0 c_s^2 \left(\frac{d\hat{v}_x}{dx} + il\hat{v}_y + ik\hat{v}_z \right) + \rho_0 v_A^2 \left(\frac{d\hat{v}_x}{dx} + il\hat{v}_y \right) \right), \quad (30)$$

where we can omit the partial derivative sign, as within the parentheses, every quantity depends only on x .

The following is the equation for the v_y component, where the only change during the derivation is that before factoring out the exponential term on the right-hand side, it must first be differentiated with respect to y . Thus, Equation (28) becomes

$$\rho_0 \left(\frac{\partial^2}{\partial t^2} - v_A^2 \frac{\partial^2}{\partial z^2} \right) \hat{v}_y e^{i(\omega t + l y + k z)} = \frac{\partial}{\partial y} \left(\rho_0 (c_s^2 + v_A^2) \left(\frac{\partial}{\partial x} \hat{v}_x e^{i(\omega t + l y + k z)} + \frac{\partial}{\partial y} \hat{v}_y e^{i(\omega t + l y + k z)} + \frac{\partial}{\partial z} \hat{v}_z e^{i(\omega t + l y + k z)} \right) - \rho_0 v_A^2 \frac{\partial}{\partial z} \hat{v}_z e^{i(\omega t + l y + k z)} \right),$$

where, if we look closely, we essentially get the same result inside the parentheses as in Equation (30), except that after differentiating with respect to y on the right-hand side, the entire parentheses is multiplied by an additional il . Multiplying every term inside the parentheses by this factor, we obtain that

$$\rho_0 (k^2 v_A^2 - \omega^2) \hat{v}_y = \rho_0 c_s^2 \left(il \frac{d\hat{v}_x}{dx} - l^2 \hat{v}_y - lk\hat{v}_z \right) + \rho_0 v_A^2 \left(il \frac{d\hat{v}_x}{dx} - l^2 \hat{v}_y \right), \quad (31)$$

where again every quantity depends only on x .

Finally, a somewhat shorter derivation for the v_z component equation, where attention must be paid to the differentiation with respect to z as well. Equation (29) then becomes

$$\frac{\partial^2}{\partial t^2} \hat{v}_z e^{i(\omega t + l y + k z)} = c_s^2 \frac{\partial}{\partial z} \left(\frac{\partial}{\partial x} \hat{v}_x e^{i(\omega t + l y + k z)} + \frac{\partial}{\partial y} \hat{v}_y e^{i(\omega t + l y + k z)} + \frac{\partial}{\partial z} \hat{v}_z e^{i(\omega t + l y + k z)} \right),$$

where, after performing the derivations,

$$-\omega^2 \hat{v}_z e^{i(\omega t + l y + k z)} = c_s^2 ik e^{i(\omega t + l y + k z)} \left(\frac{d\hat{v}_x}{dx} + il\hat{v}_y + ik\hat{v}_z \right).$$

After simplifying with the exponential term, performing the multiplications, and some rearrangement, Equation (29) takes the following form:

$$(k^2 c_s^2 - \omega^2) \hat{v}_z = c_s^2 \left(ik \frac{d\hat{v}_x}{dx} - lk\hat{v}_y \right), \quad (32)$$

where again every quantity depends only on x .

The Equations (30), (31), and (32) form a system of three equations, which can be combined through some algebra into a single differential equation. To do this, I first isolate $\hat{v}_z$, and then substitute it into Equation (31). After that, I also substitute $\hat{v}_z$ into Equation (30), and then $\hat{v}_y$ as well, and then, after some rearranging, we obtain the desired dispersion relation. Isolating $\hat{v}_z$:

$$\hat{v}_z = \frac{c_s^2}{k^2 c_s^2 - \omega^2} \left(ik \frac{d\hat{v}_x}{dx} - lk \hat{v}_y \right),$$

thus,

$$(k^2 v_A^2 - \omega^2) \hat{v}_y = c_s^2 \left(il \frac{d\hat{v}_x}{dx} - l^2 \hat{v}_y - lk \frac{c_s^2}{k^2 c_s^2 - \omega^2} \left(ik \frac{d\hat{v}_x}{dx} - lk \hat{v}_y \right) \right) + v_A^2 \left(il \frac{d\hat{v}_x}{dx} - l^2 \hat{v}_y \right).$$

After some rearranging:

$$\left(k^2 v_A^2 - \omega^2 + l^2 c_s^2 + l^2 v_A^2 - l^2 k^2 \frac{c_s^4}{k^2 c_s^2 - \omega^2} \right) \hat{v}_y = c_s^2 \left(il \frac{d\hat{v}_x}{dx} - il k^2 \frac{c_s^2}{k^2 c_s^2 - \omega^2} \frac{d\hat{v}_x}{dx} \right) + il v_A^2 \frac{d\hat{v}_x}{dx},$$

thus,

$$\hat{v}_y = \frac{c_s^2 \left(il \frac{d\hat{v}_x}{dx} - il k^2 \frac{c_s^2}{k^2 c_s^2 - \omega^2} \frac{d\hat{v}_x}{dx} \right) + il v_A^2 \frac{d\hat{v}_x}{dx}}{k^2 v_A^2 - \omega^2 + l^2 c_s^2 + l^2 v_A^2 - l^2 k^2 \frac{c_s^4}{k^2 c_s^2 - \omega^2}}.$$

First, substitute $\hat{v}_z$ into Equation (30):

$$\rho_0 (k^2 v_A^2 - \omega^2) \hat{v}_x = \frac{d}{dx} \left(\rho_0 c_s^2 \left(\frac{d\hat{v}_x}{dx} + il \hat{v}_y + ik \frac{c_s^2}{k^2 c_s^2 - \omega^2} \left(ik \frac{d\hat{v}_x}{dx} - lk \hat{v}_y \right) \right) + \right. \\ \left. + \rho_0 v_A^2 \left(\frac{d\hat{v}_x}{dx} + il \hat{v}_y \right) \right).$$

Now substitute $\hat{v}_y$ into the previously obtained relation:

$$\rho_0 (k^2 v_A^2 - \omega^2) \hat{v}_x = \frac{d}{dx} \left(\rho_0 c_s^2 \left(\frac{d\hat{v}_x}{dx} + il \frac{c_s^2 \left(il \frac{d\hat{v}_x}{dx} - il k^2 \frac{c_s^2}{k^2 c_s^2 - \omega^2} \frac{d\hat{v}_x}{dx} \right) + il v_A^2 \frac{d\hat{v}_x}{dx}}{k^2 v_A^2 - \omega^2 + l^2 c_s^2 + l^2 v_A^2 - l^2 k^2 \frac{c_s^4}{k^2 c_s^2 - \omega^2}} + \right. \right. \\ \left. + ik \frac{c_s^2}{k^2 c_s^2 - \omega^2} \left(ik \frac{d\hat{v}_x}{dx} - lk \frac{c_s^2 \left(il \frac{d\hat{v}_x}{dx} - il k^2 \frac{c_s^2}{k^2 c_s^2 - \omega^2} \frac{d\hat{v}_x}{dx} \right) + il v_A^2 \frac{d\hat{v}_x}{dx}}{k^2 v_A^2 - \omega^2 + l^2 c_s^2 + l^2 v_A^2 - l^2 k^2 \frac{c_s^4}{k^2 c_s^2 - \omega^2}} \right) \right) + \\ \left. + \rho_0 v_A^2 \left(\frac{d\hat{v}_x}{dx} + il \frac{c_s^2 \left(il \frac{d\hat{v}_x}{dx} - il k^2 \frac{c_s^2}{k^2 c_s^2 - \omega^2} \frac{d\hat{v}_x}{dx} \right) + il v_A^2 \frac{d\hat{v}_x}{dx}}{k^2 v_A^2 - \omega^2 + l^2 c_s^2 + l^2 v_A^2 - l^2 k^2 \frac{c_s^4}{k^2 c_s^2 - \omega^2}} \right) \right).$$

With particular attention given to simplifying the terms in the large parentheses, if we perform the appropriate multiplications and factor out $\frac{d\hat{v}_x}{dx}$ from every term, then the expression in the parentheses becomes

$$\frac{d\hat{v}_x}{dx} \left(\rho_0 c_s^2 \left(1 + \frac{c_s^2 \left(-l^2 + l^2 k^2 \frac{c_s^2}{k^2 c_s^2 - \omega^2} \right) - l^2 v_A^2}{k^2 v_A^2 - \omega^2 + l^2 c_s^2 + l^2 v_A^2 - l^2 k^2 \frac{c_s^4}{k^2 c_s^2 - \omega^2}} + \right. \right. \\ \left. + \frac{c_s^2}{k^2 c_s^2 - \omega^2} \left(-k^2 - lk \frac{c_s^2 \left(-lk + lk^3 \frac{c_s^2}{k^2 c_s^2 - \omega^2} \right) - lk v_A^2}{k^2 v_A^2 - \omega^2 + l^2 c_s^2 + l^2 v_A^2 - l^2 k^2 \frac{c_s^4}{k^2 c_s^2 - \omega^2}} \right) \right) + \\ \left. + \rho_0 v_A^2 \left(1 + \frac{c_s^2 \left(-l^2 + l^2 k^2 \frac{c_s^2}{k^2 c_s^2 - \omega^2} \right) - l^2 v_A^2}{k^2 v_A^2 - \omega^2 + l^2 c_s^2 + l^2 v_A^2 - l^2 k^2 \frac{c_s^4}{k^2 c_s^2 - \omega^2}} \right) \right).$$

Factoring out ρ_0 , and after bringing to a common denominator, this expression simplifies considerably:

$$\begin{aligned} \frac{d\hat{v}_x}{dx} \rho_0 \left(c_s^2 \left(\frac{k^2 v_A^2 - \omega^2}{k^2 v_A^2 - \omega^2 + l^2 c_s^2 + l^2 v_A^2 - l^2 k^2 \frac{c_s^4}{k^2 c_s^2 - \omega^2}} - \right. \right. \\ \left. \left. - \frac{c_s^2}{k^2 c_s^2 - \omega^2} \frac{k^4 v_A^2 - k^2 \omega^2}{k^2 v_A^2 - \omega^2 + l^2 c_s^2 + l^2 v_A^2 - l^2 k^2 \frac{c_s^4}{k^2 c_s^2 - \omega^2}} \right) + \right. \\ \left. + v_A^2 \frac{k^2 v_A^2 - \omega^2}{k^2 v_A^2 - \omega^2 + l^2 c_s^2 + l^2 v_A^2 - l^2 k^2 \frac{c_s^4}{k^2 c_s^2 - \omega^2}} \right). \end{aligned}$$

After bringing to a common denominator, we obtain that

$$\frac{d\hat{v}_x}{dx} \rho_0 \frac{(c_s^2 + v_A^2)(k^2 c_s^2 - \omega^2)(k^2 v_A^2 - \omega^2) - k^2 c_s^4 (k^2 v_A^2 - \omega^2)}{(k^2 v_A^2 - \omega^2)(k^2 c_s^2 - \omega^2) + l^2 (c_s^2 + v_A^2) \left(k^2 c_s^2 - \omega^2 - \frac{k^2 c_s^4}{c_s^2 + v_A^2} \right)},$$

as well as, after some rearranging,

$$\frac{d\hat{v}_x}{dx} \rho_0 \frac{(c_s^2 + v_A^2)(k^2 v_A^2 - \omega^2) \left(k^2 \frac{c_s^2 v_A^2}{c_s^2 + v_A^2} - \omega^2 \right)}{(k^2 v_A^2 - \omega^2)(k^2 c_s^2 - \omega^2) + l^2 (c_s^2 + v_A^2) \left(k^2 \frac{c_s^2 v_A^2}{c_s^2 + v_A^2} - \omega^2 \right)},$$

where if we use the expressions m_0^2 and c_T^2 , which were also used in Equation (25), then this is nothing more than

$$\frac{d\hat{v}_x}{dx} \rho_0 \frac{k^2 v_A^2 - \omega^2}{m_0^2 + l^2}.$$

Therefore, we obtain the dispersion equation:

$$\frac{d}{dx} \left[\frac{\rho_0(x)(k^2 v_A^2(x) - \omega^2)}{m_0^2(x) + l^2} \frac{d\hat{v}_x}{dx} \right] - \rho_0(x)(k^2 v_A^2(x) - \omega^2) \hat{v}_x = 0. \quad (33)$$

This is an ordinary differential equation for $\hat{v}_x$, where both the density and the different velocities depend on the x coordinate. However, if the stratification is weak, then these can be approximated as constants, in which case Equation (33) becomes

$$(k^2 v_A^2 - \omega^2) \left(\frac{d^2 \hat{v}_x}{dx^2} - (m_0^2 + l^2) \hat{v}_x \right) = 0, \quad (34)$$

where the left-side parentheses describe the Alfvén waves, while the differential equation within the parentheses describes the magnetoacoustic waves (Roberts, 1981).

3.3 Magnetic interface

The following derivations will be the most important for us, as we will now introduce the magnetic interface and the corresponding dispersion relation. The magnetic field:

$$B_0(x) = \begin{cases} B_e, & x > 0, \\ B_0, & x < 0, \end{cases} \quad (35)$$

where B_e and B_0 are constants. This defines a discontinuity in the magnetic field at position $x = 0$; however, the interface is homogeneous on both sides, so the velocities do not depend on x and can be treated as constants. The total pressure is continuous, so

$$p_e + \frac{B_e^2}{2\mu} = p_0 + \frac{B_0^2}{2\mu}. \quad (36)$$

Based on Equation (34), if we isolate the part related to the Alfvén wave, we obtain a differential equation that can be easily solved, and its solutions are

$$\hat{v}_x(x) = \begin{cases} a_e e^{-(m_e^2 + l^2)^{\frac{1}{2}} x}, & x > 0, \\ a_0 e^{(m_0^2 + l^2)^{\frac{1}{2}} x}, & x < 0, \end{cases} \quad (37)$$

where both $(m_e^2 + l^2)^{\frac{1}{2}}$, and $(m_0^2 + l^2)^{\frac{1}{2}}$ are positive, whereas m_e^2 is defined in a similar manner to m_0^2 ; however, it consists of velocities determined in the region with a magnetic field of B_e .

To derive the dispersion relation, the pressure perturbation must be written in the following form:

$$p_1 = \hat{p}_1(x) e^{i(\omega t + l y + k z)},$$

however, it would be cumbersome to explicitly write the index '1' every time to indicate that this is the perturbed pressure, so I will omit it. $\hat{p}_1(x)$ is the x -dependent amplitude of the perturbed pressure.

Using the fact that the total perturbed pressure is

$$\frac{\partial p_{1,T}}{\partial t} = \rho_0 v_A^2 \frac{\partial v_z}{\partial z} - \rho_0 (c_s^2 + v_A^2) \nabla \cdot \mathbf{v},$$

therefore, from this point on, omitting the index '1', the Fourier form looks as follows after substitution:

$$\begin{aligned} \frac{\partial}{\partial t} \hat{p}_T e^{i(\omega t + l y + k z)} = \rho_0 v_A^2 \frac{\partial}{\partial z} \hat{v}_z e^{i(\omega t + l y + k z)} - \rho_0 (c_s^2 + v_A^2) \left(\frac{\partial}{\partial x} \hat{v}_x e^{i(\omega t + l y + k z)} + \frac{\partial}{\partial y} \hat{v}_y e^{i(\omega t + l y + k z)} + \right. \\ \left. + \frac{\partial}{\partial z} \hat{v}_z e^{i(\omega t + l y + k z)} \right). \end{aligned}$$

Performing the derivations and factoring out the common exponential term,

$$i\omega \hat{p}_T e^{i(\omega t + l y + k z)} = e^{i(\omega t + l y + k z)} \rho_0 \left(-c_s^2 \left(\frac{\partial \hat{v}_x}{\partial x} + i l \hat{v}_y + i k \hat{v}_z \right) - v_A^2 \left(\frac{\partial \hat{v}_x}{\partial x} + i l \hat{v}_y \right) \right),$$

which, after simplifying with the exponential term, becomes

$$i\omega \hat{p}_T = -\rho_0 \left(c_s^2 \left(\frac{d\hat{v}_x}{dx} + i l \hat{v}_y + i k \hat{v}_z \right) + v_A^2 \left(\frac{d\hat{v}_x}{dx} + i l \hat{v}_y \right) \right).$$

Substituting in the previously obtained forms of $\hat{v}_z$, and then $\hat{v}_y$,

$$i\omega \hat{p}_T = -\rho_0 \left(c_s^2 \left(\frac{d\hat{v}_x}{dx} + i l \hat{v}_y + i k \frac{c_s^2}{k^2 c_s^2 - \omega^2} \left(i k \frac{d\hat{v}_x}{dx} - l k \hat{v}_y \right) \right) + v_A^2 \left(\frac{d\hat{v}_x}{dx} + i l \hat{v}_y \right) \right),$$

then, after substituting $\hat{v}_y$,

$$\begin{aligned} i\omega \hat{p}_T = -\rho_0 \left(c_s^2 \left(\frac{d\hat{v}_x}{dx} + i l \frac{c_s^2 \left(i l \frac{d\hat{v}_x}{dx} - i l k^2 \frac{c_s^2}{k^2 c_s^2 - \omega^2} \frac{d\hat{v}_x}{dx} \right) + i l v_A^2 \frac{d\hat{v}_x}{dx}}{k^2 v_A^2 - \omega^2 + l^2 c_s^2 + l^2 v_A^2 - l^2 k^2 \frac{c_s^4}{k^2 c_s^2 - \omega^2}} + \right. \right. \\ \left. + i k \frac{c_s^2}{k^2 c_s^2 - \omega^2} \left(i k \frac{d\hat{v}_x}{dx} - l k \frac{c_s^2 \left(i l \frac{d\hat{v}_x}{dx} - i l k^2 \frac{c_s^2}{k^2 c_s^2 - \omega^2} \frac{d\hat{v}_x}{dx} \right) + i l v_A^2 \frac{d\hat{v}_x}{dx}}{k^2 v_A^2 - \omega^2 + l^2 c_s^2 + l^2 v_A^2 - l^2 k^2 \frac{c_s^4}{k^2 c_s^2 - \omega^2}} \right) \right) + \\ \left. + v_A^2 \left(\frac{d\hat{v}_x}{dx} + i l \frac{c_s^2 \left(i l \frac{d\hat{v}_x}{dx} - i l k^2 \frac{c_s^2}{k^2 c_s^2 - \omega^2} \frac{d\hat{v}_x}{dx} \right) + i l v_A^2 \frac{d\hat{v}_x}{dx}}{k^2 v_A^2 - \omega^2 + l^2 c_s^2 + l^2 v_A^2 - l^2 k^2 \frac{c_s^4}{k^2 c_s^2 - \omega^2}} \right) \right). \end{aligned}$$

Factoring out $\frac{d\hat{v}_x}{dx}$ and performing the multiplications,

$$i\omega\hat{p}_T = -\frac{d\hat{v}_x}{dx}\rho_0\left(c_s^2\left(1 + \frac{c_s^2\left(-l^2 + l^2k^2\frac{c_s^2}{k^2c_s^2-\omega^2}\right) - l^2v_A^2}{k^2v_A^2 - \omega^2 + l^2c_s^2 + l^2v_A^2 - l^2k^2\frac{c_s^4}{k^2c_s^2-\omega^2}} + \frac{c_s^2}{k^2c_s^2 - \omega^2}\left(-k^2 - lk\frac{c_s^2\left(-lk + lk^3\frac{c_s^2}{k^2c_s^2-\omega^2}\right) - lk v_A^2}{k^2v_A^2 - \omega^2 + l^2c_s^2 + l^2v_A^2 - l^2k^2\frac{c_s^4}{k^2c_s^2-\omega^2}}\right)\right) + v_A^2\left(1 + \frac{c_s^2\left(-l^2 + l^2k^2\frac{c_s^2}{k^2c_s^2-\omega^2}\right) - l^2v_A^2}{k^2v_A^2 - \omega^2 + l^2c_s^2 + l^2v_A^2 - l^2k^2\frac{c_s^4}{k^2c_s^2-\omega^2}}\right)\right).$$

If we look more closely, this expression appeared exactly during the derivation of Equation (33), which can be significantly simplified, yielding the result that

$$i\omega\hat{p}_T = -\frac{d\hat{v}_x}{dx}\rho_0\frac{k^2v_A^2 - \omega^2}{m_0^2 + l^2}.$$

Substituting in Equation (37), we obtain that

$$i\omega\hat{p}_T = \begin{cases} \rho_e a_e (m_e^2 + l^2)^{\frac{1}{2}} e^{-(m_e^2 + l^2)^{\frac{1}{2}} x} \frac{k^2 v_{A,e}^2 - \omega^2}{m_e^2 + l^2}, & x > 0, \\ -\rho_0 a_0 (m_0^2 + l^2)^{\frac{1}{2}} e^{(m_0^2 + l^2)^{\frac{1}{2}} x} \frac{k^2 v_A^2 - \omega^2}{m_0^2 + l^2}, & x < 0, \end{cases} \quad (38)$$

where I used the fact that m_e^2 applies to the region $x > 0$, and m_0^2 applies to the region $x < 0$.

Since $\hat{v}_x$ and $\hat{p}_T$ have to be continuous at position $x = 0$, $a_0 = a_e$, as well as, we obtain the following dispersion relation based on Equation (38) if we substitute in $x = 0$ and we set the two sides equal:

$$\rho_0(k^2 v_A^2 - \omega^2)(m_e^2 + l^2)^{\frac{1}{2}} + \rho_e(k^2 v_{A,e}^2 - \omega^2)(m_0^2 + l^2)^{\frac{1}{2}} = 0. \quad (39)$$

Thus, we have successfully achieved the goal of this chapter. I have derived the dispersion relations and other relations analogue to Roberts' paper that are now important to us. Before applying the knowledge gained here to calculate the MHF, it is useful to show that in the weakly stratified inhomogeneous case, the dispersion relation for the magnetoacoustic waves will be the same as in the homogeneous case.

3.4 Comparison of dispersion relations

Using Equations (26), and (21), the following 6 equations govern the components of the velocity and magnetic field perturbations.

$$\begin{aligned} \rho_0 \frac{\partial v_x}{\partial t} &= -\frac{\partial}{\partial x} \left(p_1 + \frac{B_0 B_{1,z}}{\mu} \right) + \frac{1}{\mu} B_0 \frac{\partial B_{1,x}}{\partial z}, \\ \rho_0 \frac{\partial v_y}{\partial t} &= -\frac{\partial}{\partial y} \left(p_1 + \frac{B_0 B_{1,z}}{\mu} \right) + \frac{1}{\mu} B_0 \frac{\partial B_{1,y}}{\partial z}, \\ \rho_0 \frac{\partial v_z}{\partial t} &= -\frac{\partial p_1}{\partial z} + \frac{1}{\mu} B_{1,x} \frac{dB_0}{dx}, \\ \frac{\partial B_{1,x}}{\partial t} &= B_0 \frac{\partial v_x}{\partial z}, \\ \frac{\partial B_{1,y}}{\partial t} &= B_0 \frac{\partial v_y}{\partial z}, \\ \frac{\partial B_{1,z}}{\partial t} &= -B_0 \frac{\partial v_x}{\partial x} - v_x \frac{dB_0}{dx} - B_0 \frac{\partial v_y}{\partial y}. \end{aligned}$$

After careful observation, and using uniformity in the y direction (or simply $l = 0$), we may notice that the equations for v_y and $B_{1,y}$ are decoupled from the rest. Therefore, writing these equations in matrix form, taking the determinant of the matrix and equating it to 0 will result in the dispersion relation for v_y and $B_{1,y}$. For the sake of simplicity, I will notate differentiation with respect to a given variable with $\partial_{\text{variable}}$. Taking the derivative with respect to a variable will result in multiplication, whereas integration by a variable will result in division by $i \cdot \text{variable}$, respectively.

$$\begin{pmatrix} \rho_0 \partial_t & -\frac{B_0}{\mu} \partial_z \\ -B_0 \partial_z & \partial_t \end{pmatrix} \begin{pmatrix} v_y \\ B_{1,y} \end{pmatrix} = \mathbf{0} \quad (40)$$

Taking the determinant of the matrix results in

$$\partial_t^2 - v_A^2 \partial_z^2 = 0.$$

Using the aforementioned notation, we obtain

$$\omega^2 - k^2 v_A^2 = 0, \quad (41)$$

which is the dispersion relation for Alfvén waves. The 4 equations governing components v_x , v_z , $B_{1,x}$, $B_{1,z}$ are coupled. If we use a modified equation for v_z , then the formula for the time derivative of the total perturbed pressure can be used.

$$\rho_0 \frac{\partial v_z}{\partial t} = -\frac{\partial}{\partial z} \left(p_1 + \frac{B_0 B_{1,z}}{\mu} \right) + \frac{1}{\mu} B_0 \frac{\partial B_{1,z}}{\partial z} + \frac{1}{\mu} B_{1,x} \frac{dB_0}{dx},$$

$$\frac{\partial p_{1,T}}{\partial t} = \rho_0 v_A^2 \frac{\partial v_z}{\partial z} - \rho_0 (c_s^2 + v_A^2) \nabla \cdot \mathbf{v}.$$

Therefore, using the fact that $\partial_y = 0$ and $\partial_t \int \partial_t$ is analogue to multiplying by 1, the constructed matrix takes the following form.

$$\begin{pmatrix} \rho_0 \partial_t + \partial_x \int \partial_t (-\rho_0 c_s^2 \partial_x - \rho_0 v_A^2 \partial_x) & \partial_x \int \partial_t (-\rho_0 c_s^2 \partial_z) & -\frac{1}{\mu} B_0 \partial_z & 0 \\ \partial_z \int \partial_t (-\rho_0 c_s^2 \partial_x - \rho_0 v_A^2 \partial_x) & \rho_0 \partial_t + \partial_z \int \partial_t (-\rho_0 c_s^2 \partial_z) & -\frac{1}{\mu} \frac{dB_0}{dx} & -\frac{1}{\mu} B_0 \partial_z \\ -B_0 \partial_z & 0 & \partial_t & 0 \\ B_0 \partial_x + \frac{dB_0}{dx} & 0 & 0 & \partial_t \end{pmatrix} \quad (42)$$

If we multiply

$$\begin{pmatrix} v_x \\ v_z \\ B_{1,x} \\ B_{1,z} \end{pmatrix}$$

by Equation (42), then the result will be $\mathbf{0}$. Taking the determinant of a 4x4 matrix requires the method of cofactor expansion, which breaks down the determinant of the 4x4 matrix into 3x3 determinants. Using the checkerboard signs and the 3x3 minor matrices, the determinant of the 4x4 matrix using the third row can be calculated as follows.

$$\begin{aligned} \det\{A\} = & -B_0 \partial_z \begin{vmatrix} \partial_x \int \partial_t (-\rho_0 c_s^2 \partial_z) & -\frac{1}{\mu} B_0 \partial_z & 0 \\ \rho_0 \partial_t + \partial_z \int \partial_t (-\rho_0 c_s^2 \partial_z) & -\frac{1}{\mu} \frac{dB_0}{dx} & -\frac{1}{\mu} B_0 \partial_z \\ 0 & 0 & \partial_t \end{vmatrix} + \\ & + \partial_t \begin{vmatrix} \rho_0 \partial_t + \partial_x \int \partial_t (-\rho_0 c_s^2 \partial_x - \rho_0 v_A^2 \partial_x) & \partial_x \int \partial_t (-\rho_0 c_s^2 \partial_z) & 0 \\ \partial_z \int \partial_t (-\rho_0 c_s^2 \partial_x - \rho_0 v_A^2 \partial_x) & \rho_0 \partial_t + \partial_z \int \partial_t (-\rho_0 c_s^2 \partial_z) & -\frac{1}{\mu} B_0 \partial_z \\ B_0 \partial_x + \frac{dB_0}{dx} & 0 & \partial_t \end{vmatrix}, \end{aligned}$$

where if we consider the case of an interface (see Equation (35)), then the derivative of B_0 is zero, as B_0 is constant. Therefore, by expressing the determinants using Sarrus' 3x3 rule and

taking $\frac{dB_0}{dx} = 0$, we obtain

$$\begin{aligned} & -B_0 \partial_z \frac{1}{\mu} B_0 \partial_z \left(\rho_0 \partial_t + \partial_z \int \partial_t (-\rho_0 c_s^2 \partial_z) \right) \partial_t + \\ & + \partial_t \left[\left(\rho_0 \partial_t + \partial_x \int \partial_t (-\rho_0 c_s^2 \partial_x - \rho_0 v_A^2 \partial_x) \right) \left(\rho_0 \partial_t + \partial_z \int \partial_t (-\rho_0 c_s^2 \partial_z) \right) \partial_t + \right. \\ & \left. + \partial_x \int \partial_t (-\rho_0 c_s^2 \partial_z) \left(-\frac{1}{\mu} B_0 \partial_z \right) B_0 \partial_x - \partial_x \int \partial_t (-\rho_0 c_s^2 \partial_z) \partial_z \int \partial_t (-\rho_0 c_s^2 \partial_x - \rho_0 v_A^2 \partial_x) \partial_t \right]. \end{aligned}$$

Expanding the brackets, performing the multiplications and cancelling out the time integrals with the time derivatives, we obtain

$$\begin{aligned} & -\frac{B_0^2}{\mu} \rho_0 \partial_z^2 \partial_t^2 + \frac{B_0^2}{\mu} \rho_0 c_s^2 \partial_z^4 + \rho_0^2 \partial_t^4 - \rho_0^2 c_s^2 \partial_z^2 \partial_t^2 - \rho_0^2 c_s^2 \partial_x^2 \partial_t^2 - \rho_0^2 v_A^2 \partial_x^2 \partial_t^2 + \rho_0^2 c_s^2 v_A^2 \partial_x^2 \partial_z^2 + \\ & + \rho_0^2 c_s^4 \partial_x^2 \partial_z^2 + \frac{B_0^2}{\mu} \rho_0 c_s^2 \partial_x^2 \partial_z^2 - \rho_0^2 c_s^4 \partial_x^2 \partial_z^2 - \rho_0^2 c_s^2 v_A^2 \partial_x^2 \partial_z^2. \end{aligned}$$

Collecting the terms and dividing by ρ_0^2 results in

$$\partial_t^4 - \partial^2 (c_s^2 + v_A^2) (\partial_x^2 + \partial_z^2) + c_s^2 v_A^2 \partial_z^2 (\partial_x^2 + \partial_z^2) = 0.$$

Using the notation mentioned above, we obtain

$$\omega^4 - \kappa^2 (c_s^2 + v_A^2) \omega^2 + k^2 \kappa^2 c_s^2 v_A^2 = 0, \quad (43)$$

$$\kappa^2 = n^2 + k^2,$$

$$\kappa = \begin{pmatrix} n \\ 0 \\ k \end{pmatrix}.$$

3.4.1 Solutions for magnetoacoustic waves

I will now use a simpler case for the wave propagation vector, namely

$$\kappa = \begin{pmatrix} 0 \\ 0 \\ k \end{pmatrix},$$

thus, Equation (43) modifies to

$$\omega^4 - k^2 (c_s^2 + v_A^2) \omega^2 + k^4 c_s^2 v_A^2 = 0,$$

then dividing by k^4 :

$$\left(\frac{\omega^2}{k^2} \right)^2 - (c_s^2 + v_A^2) \frac{\omega^2}{k^2} + c_s^2 v_A^2 = 0. \quad (44)$$

The solution to Equation (44) takes the form

$$\frac{\omega^2}{k^2} = \frac{c_s^2 + v_A^2 \pm |c_s^2 - v_A^2|}{2},$$

from which the 4 solutions are

$$\begin{aligned} \omega_{1,2} &= \pm k c_s, \\ \omega_{3,4} &= \pm k v_A, \end{aligned} \quad (45)$$

where the fast magnetoacoustic wave propagates with $\max(c_s, v_A)$, whereas the slow magnetoacoustic wave propagates with $\min(c_s, v_A)$ in the $\hat{\mathbf{z}}$ direction.

4 Magnetic helicity flux of MHD waves

Based on the previous chapter, I will now investigate two cases: i) the case of a homogeneous magnetic field and ii) the case of a magnetic interface placed at $x = 0$. I will also derive the MHF for both cases as both indefinite and definite integrals. As mentioned earlier, this chapter will be the main part of my work.

Note: In my calculations, I will use the most general wave form, although outside of this work, I also examined the case of $l = 0$, what significantly simplifies the computations and provides a relatively compact expression for the MHF.

I assume an infinite plane geometry in the $[xy]$ plane, where the magnetic field is in the $\hat{\mathbf{z}}$ direction. A sketch of this can be seen in Figure 5.

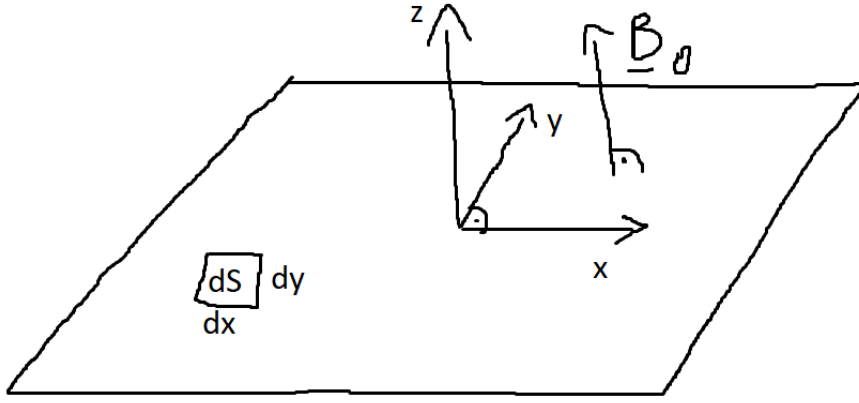


Figure 5: A sketch of the magnetic field's direction and the geometry of the plane.

During the calculations, I will use the MHF calculation formula introduced in the paper of [Korsós et al. \(2022\)](#), which was originally derived by [Berger \(1984\)](#):

$$T \equiv \left. \frac{dH}{dt} \right|_S = 2 \int_S (\mathbf{A}_p \cdot \mathbf{B}) \mathbf{v} \cdot \mathbf{n} dS - 2 \int_S (\mathbf{A}_p \cdot \mathbf{v}) \mathbf{B} \cdot \mathbf{n} dS, \quad (46)$$

where

$$\mathbf{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

is the normal vector of the plane, $\mathbf{A}_p$ is the magnetic vector potential,

$$\mathbf{B}_0 = \begin{pmatrix} 0 \\ 0 \\ B_0 \end{pmatrix}$$

is the magnetic field, which satisfies Equation (1), $\mathbf{v}$ is the plasma velocity, whereas $dS = dx dy$ is a small surface element over which we integrate.

Note: $\mathbf{A}_p$ satisfies the Coulomb gauge, that is, $\nabla \cdot \mathbf{A}_p = 0$, additionally, as [Berger \(1984\)](#) found, $\mathbf{A}_p \cdot \mathbf{n} = 0$, therefore, $\mathbf{A}_p$ has no component in the $\hat{\mathbf{z}}$ direction. Consequently, $\mathbf{A}_p \cdot \mathbf{B} = 0$, thus, the first integral of Equation (46) vanishes.

4.1 Deriving the magnetic vector potential

Before we can calculate the integral for MHF (46), it is necessary to derive $\mathbf{A}_p$ to understand how each component depends on other factors, and this must be reconciled with the magnetic field $\mathbf{B}_0$. Thus,

$$\begin{pmatrix} 0 \\ 0 \\ B_0 \end{pmatrix} = \nabla \times \begin{pmatrix} A_x \\ A_y \\ 0 \end{pmatrix}. \quad (47)$$

Expanding Equation (47), we obtain that

$$\begin{pmatrix} 0 \\ 0 \\ B_0 \end{pmatrix} = \begin{pmatrix} -\frac{\partial A_y}{\partial z} \\ \frac{\partial A_x}{\partial z} \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \end{pmatrix},$$

from which it follows that $\mathbf{A}_p(x, y)$ does not depend on z , as well as,

$$\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = B_0.$$

Since in the case of the interface the magnetic field does not depend on x in such a way that it cannot be considered constant, $B_0 = \text{const.}$, and thus the derivatives of the magnetic vector potential must also be constant. Let

$$\alpha - \beta = B_0,$$

thus, the simplest magnetic vector potential is

$$\mathbf{A}_p = \begin{pmatrix} \beta y \\ \alpha x \\ 0 \end{pmatrix}, \quad (48)$$

where α and β are constants.

Note: The non-zero components of the magnetic vector potential can be shifted by a constant, but I will not address this case in my calculations. Equation (48), and based on the above notes, Equation (46) modify into

$$T = -2 \int_S (\beta y v_x + \alpha x v_y) B_0 dS. \quad (49)$$

4.2 Magnetic helicity flux of a homogeneous magnetic field

Since the magnetic field does not depend on x , there is no stratification that would require more attention to the variables during integration. Therefore, it is possible to freely express all spatial variables in Fourier form for v_x and v_y quantities, so I will examine the following general wave form:

$$\begin{aligned} v_x &= \hat{v}_x e^{i(\omega t + nx + ly + kz)}, \\ v_y &= \hat{v}_y e^{i(\omega t + nx + ly + kz)}, \end{aligned}$$

where $\hat{v}_x$ and $\hat{v}_y$ are constant amplitudes of the wave, and we perform the integration over the xy plane.

4.2.1 Indefinite integral

Substituting the appropriate expressions into Equation (49), we obtain that

$$T = -2 \int \int \left(\beta y \hat{v}_x e^{i(\omega t + nx + ly + kz)} + \alpha x \hat{v}_y e^{i(\omega t + nx + ly + kz)} \right) B_0 dx dy.$$

B_0 can be factored out of this, as it is a constant. First, I will carry out the integration with respect to x , for which there is need for the method of integration by parts (IBP). It would be time-consuming to write out the process of IBP every single time, so I will only describe its

starting point and result. Nevertheless, I will try to describe the applied operations in words for easier followability. Thus, the integral with respect to x :

$$\int \left(\beta y \hat{v}_x e^{i(\omega t + nx + ly + kz)} + \alpha x \hat{v}_y e^{i(\omega t + nx + ly + kz)} \right) dx.$$

The second term in the parentheses needs the use of IBP, which has the following result:

$$\left(\frac{\alpha \hat{v}_y x}{in} + \frac{\alpha \hat{v}_y}{n^2} \right) e^{i(\omega t + nx + ly + kz)}.$$

Additionally, during the integration of the first term inside the integral's parentheses, the exponent of the exponential function moves to the denominator, so the result of the integration with respect to x is

$$\left(\frac{\beta \hat{v}_x y}{in} + \frac{\alpha \hat{v}_y x}{in} + \frac{\alpha \hat{v}_y}{n^2} \right) e^{i(\omega t + nx + ly + kz)}.$$

Now this result has to be integrated with respect to y , thus,

$$\int \left(\frac{\beta \hat{v}_x y}{in} + \frac{\alpha \hat{v}_y x}{in} + \frac{\alpha \hat{v}_y}{n^2} \right) e^{i(\omega t + nx + ly + kz)} dy.$$

The first term in the parentheses needs the use of IBP, which results in

$$\left(-\frac{\beta \hat{v}_x y}{nl} + \frac{\beta \hat{v}_x}{inl^2} \right) e^{i(\omega t + nx + ly + kz)},$$

for the others, one shall proceed in a manner similar to the previous ones, so the result of the integration is

$$\left(-\frac{\beta \hat{v}_x y}{nl} + \frac{\beta \hat{v}_x}{inl^2} - \frac{\alpha \hat{v}_y x}{nl} + \frac{\alpha \hat{v}_y}{in^2 l} \right) e^{i(\omega t + nx + ly + kz)}.$$

Bringing it to the common denominator, we find that

$$\frac{-nl\beta \hat{v}_x y - in\beta \hat{v}_x - nl\alpha \hat{v}_y x - il\alpha \hat{v}_y}{n^2 l^2} e^{i(\omega t + nx + ly + kz)},$$

based on which the total MHF is

$$T = 2B_0 e^{i(\omega t + nx + ly + kz)} \frac{nl(\beta \hat{v}_x y + \alpha \hat{v}_y x) + i(n\beta \hat{v}_x + l\alpha \hat{v}_y)}{n^2 l^2}. \quad (50)$$

4.2.2 Definite integral

I will now derive the previously derived MHF as a definite integral with arbitrary limits x_1 , x_2 , and y_1 , y_2 . Thus,

$$T = -2 \int_{y_1}^{y_2} \int_{x_1}^{x_2} \left(\beta y \hat{v}_x e^{i(\omega t + nx + ly + kz)} + \alpha x \hat{v}_y e^{i(\omega t + nx + ly + kz)} \right) B_0 dx dy.$$

Using the result of the integration with respect to x from the previous chapter, let us now take the limits into account as well! Thus,

$$\left[\left(\frac{\beta \hat{v}_x y}{in} + \frac{\alpha \hat{v}_y x}{in} + \frac{\alpha \hat{v}_y}{n^2} \right) e^{i(\omega t + nx + ly + kz)} \right]_{x_1}^{x_2},$$

based on which the definite integral with respect to x results in

$$\left(\frac{\beta \hat{v}_x y}{in} + \frac{\alpha \hat{v}_y x_2}{in} + \frac{\alpha \hat{v}_y}{n^2} \right) e^{i(\omega t + nx_2 + ly + kz)} - \left(\frac{\beta \hat{v}_x y}{in} + \frac{\alpha \hat{v}_y x_1}{in} + \frac{\alpha \hat{v}_y}{n^2} \right) e^{i(\omega t + nx_1 + ly + kz)}.$$

Now this expression needs to be integrated with respect to y , where, as in the previous cases, some parts of the expression need IBP. Thus, after IBP, we obtain that

$$\left(-\frac{\beta \hat{v}_x y}{ln} + \frac{\beta \hat{v}_x}{inl^2} \right) \left(e^{i(\omega t + nx_2 + ly + kz)} - e^{i(\omega t + nx_1 + ly + kz)} \right),$$

then carrying out the remaining integration,

$$\left[\left(-\frac{\beta \hat{v}_x y}{ln} + \frac{\beta \hat{v}_x}{inl^2} + \frac{\alpha \hat{v}_y}{iln^2} \right) \left(e^{i(\omega t + nx_2 + ly + kz)} - e^{i(\omega t + nx_1 + ly + kz)} \right) - \frac{\alpha \hat{v}_y x_2 e^{i(\omega t + nx_2 + ly + kz)}}{ln} + \frac{\alpha \hat{v}_y x_1 e^{i(\omega t + nx_1 + ly + kz)}}{ln} \right]_{y_1}^{y_2}.$$

Well, many different terms will appear here, so it is advisable to group them according to some criterion. I will group the expressions here based on the principle that for a given pair x_i, y_j , if $i = j$, I will consider the term as positive, while if $i \neq j$, I will consider it as negative. This will result in four different groups, with 2 having a positive sign and 2 having a negative sign. It will also be useful to introduce a simple notation that describes each group accordingly. However, it is useful to first bring everything to a common denominator, which will be $n^2 l^2$ as in the previous calculation. Bringing to a common denominator:

$$\left[\frac{(-nl\beta \hat{v}_x y - in\beta \hat{v}_x - il\alpha \hat{v}_y) (e^{i(\omega t + nx_2 + ly + kz)} - e^{i(\omega t + nx_1 + ly + kz)})}{n^2 l^2} + \frac{nl\alpha \hat{v}_y x_1 e^{i(\omega t + nx_1 + ly + kz)} - nl\alpha \hat{v}_y x_2 e^{i(\omega t + nx_2 + ly + kz)}}{n^2 l^2} \right]_{y_1}^{y_2}.$$

Now substitute the limits and immediately perform the grouping! Thus, we obtain that

$$\begin{aligned} \frac{1}{n^2 l^2} & \left[e^{i(\omega t + nx_2 + ly_2 + kz)} (-nl\beta \hat{v}_x y_2 - in\beta \hat{v}_x - il\alpha \hat{v}_y - nl\alpha \hat{v}_y x_2) - \right. \\ & - e^{i(\omega t + nx_1 + ly_2 + kz)} (-nl\beta \hat{v}_x y_2 - in\beta \hat{v}_x - il\alpha \hat{v}_y - nl\alpha \hat{v}_y x_1) - \\ & - e^{i(\omega t + nx_2 + ly_1 + kz)} (-nl\beta \hat{v}_x y_1 - in\beta \hat{v}_x - il\alpha \hat{v}_y - nl\alpha \hat{v}_y x_2) + \\ & \left. + e^{i(\omega t + nx_1 + ly_1 + kz)} (-nl\beta \hat{v}_x y_1 - in\beta \hat{v}_x - il\alpha \hat{v}_y - nl\alpha \hat{v}_y x_1) \right]. \end{aligned}$$

I now introduce the following notation:

$$f_{rq}(t, z) = e^{i(\omega t + nx_r + ly_q + kz)} (nl\beta\hat{v}_x y_q + in\beta\hat{v}_x + il\alpha\hat{v}_y + nl\alpha\hat{v}_y x_r), \quad (51)$$

$$r \in \{1, 2\}, \quad q \in \{1, 2\},$$

thus, the group can be simply written with the help of Equation (51) as

$$\frac{1}{n^2 l^2} (-1)^{r+q-1} f_{rq}.$$

Multiplying this expression by $-2B_0$, we find the MHF:

$$T|_{x_1, x_2, y_1, y_2}(t, z) = \frac{2B_0}{n^2 l^2} (-1)^{(r+q)} f_{rq} \quad (52)$$

Note: If T is integrated with respect to time over one period, the result is 0. This will also hold true for the MHF formulas derived in the thesis.

4.3 Magnetic helicity flux of a magnetic interface

In this section, I will derive the MHF for a magnetic interface at $x = 0$. For this, we need Equations (35) and (37), as well as the relation for $\hat{v}_y(x)$. Note: due to the weak stratification, the different velocities can be considered constant. Since

$$\hat{v}_y(x) = \frac{c_s^2 \left(il \frac{d\hat{v}_x}{dx} - il k^2 \frac{c_s^2}{k^2 c_s^2 - \omega^2} \frac{d\hat{v}_x}{dx} \right) + il v_A^2 \frac{d\hat{v}_x}{dx}}{k^2 v_A^2 - \omega^2 + l^2 c_s^2 + l^2 v_A^2 - l^2 k^2 \frac{c_s^4}{k^2 c_s^2 - \omega^2}},$$

and if we multiply based on previous experience by $\frac{k^2 c_s^2 - \omega^2}{k^2 c_s^2 - \omega^2}$, then the fraction will result in the form $\frac{1}{m_0^2 + l^2}$. Thus,

$$\hat{v}_y(x) = il \frac{d\hat{v}_x}{dx} \frac{c_s^2 ((k^2 c_s^2 - \omega^2) - k^2 c_s^2) + (k^2 c_s^2 - \omega^2) v_A^2}{(k^2 v_A^2 - \omega^2)(k^2 c_s^2 - \omega^2) + l^2 ((c_s^2 + v_A^2)(k^2 c_s^2 - \omega^2) - k^2 c_s^4)},$$

then expanding the numerator, we find that

$$\hat{v}_y(x) = il \frac{d\hat{v}_x}{dx} \frac{(c_s^2 + v_A^2) (k^2 c_s^2 - \omega^2) - k^2 c_s^4}{(k^2 v_A^2 - \omega^2)(k^2 c_s^2 - \omega^2) + l^2 ((c_s^2 + v_A^2)(k^2 c_s^2 - \omega^2) - k^2 c_s^4)}.$$

If we expand the parentheses in the numerator, it can be seen with a bit of algebra that

$$(c_s^2 + v_A^2) \left(k^2 \frac{c_s^2 v_A^2}{c_s^2 + v_A^2} - \omega^2 \right)$$

appears there, so

$$\hat{v}_y(x) = \frac{il}{m_0^2 + l^2} \frac{d\hat{v}_x}{dx}. \quad (53)$$

For convenience, let us introduce the following notation:

$$n(x) = \begin{cases} -(m_e^2 + l^2)^{\frac{1}{2}}, & x > 0, \\ (m_0^2 + l^2)^{\frac{1}{2}}, & x < 0, \end{cases} \quad (54)$$

where $n(x)$ should not be confused with the n introduced for x in the Fourier form. However, for ease of notation, it is useful to denote the term in the exponent of the exponential function with some symbol, so I have chosen this one. It can be somewhat distinguished that when treating the product of exponential terms as an exponential here, unlike the other terms, there will be a $-i$ factor in front of $n(x)$ in the parentheses multiplied by i . I will treat the magnetic field in the same way, because it actually depends on x only in the sense that, due to the interface, it takes on different constant values in the two separate regions. Additionally, as discussed earlier, $a_e = a_0$, so it essentially does not matter which one I use.

4.3.1 Indefinite integral

Substituting the appropriate expressions into Equation (49), we obtain that

$$T = -2 \int \int \left(\beta y a_0 e^{n(x)x} e^{i(\omega t + ly + kz)} + \alpha x a_0 \frac{il}{n(x)} e^{n(x)x} e^{i(\omega t + ly + kz)} \right) B_0(x) dx dy.$$

Since $B_0(x)$ actually depends on x only in the sense that it takes on the constant values B_0 in one region and B_e in the other, it can be factored out of the integral just like a_0 , as we are not yet considering the limits. First, perform the integration with respect to x ! Thus,

$$\int \left(\beta y e^{i(\omega t - in(x)x + ly + kz)} + \alpha x \frac{il}{n(x)} e^{i(\omega t - in(x)x + ly + kz)} \right) dx.$$

The second term in the parentheses needs IBP, while the first term can be integrated traditionally. The second term after IBP is:

$$\left(\frac{il\alpha x}{n^2(x)} - \frac{il\alpha}{n^3(x)} \right) e^{i(\omega t - in(x)x + ly + kz)},$$

and then performing the integration on the first term as well, the result of the integration with respect to x is

$$\left(\frac{\beta y}{n(x)} + \frac{il\alpha x}{n^2(x)} - \frac{il\alpha}{n^3(x)} \right) e^{i(\omega t - in(x)x + ly + kz)}.$$

Now the resulting expression needs to be integrated with respect to y . So

$$\int \left(\frac{\beta y}{n(x)} + \frac{il\alpha x}{n^2(x)} - \frac{il\alpha}{n^3(x)} \right) e^{i(\omega t - in(x)x + ly + kz)} dy.$$

The first term needs IBP, which will result in

$$\left(\frac{\beta y}{iln(x)} + \frac{\beta}{l^2 n(x)} \right) e^{i(\omega t - in(x)x + ly + kz)},$$

then performing the remaining integration with respect to y , the result of the integration is

$$\left(\frac{\beta y}{iln(x)} + \frac{\beta}{l^2 n(x)} + \frac{\alpha x}{n^2(x)} - \frac{\alpha}{n^3(x)} \right) e^{i(\omega t - in(x)x + ly + kz)}.$$

Bringing this to a common denominator, we obtain that

$$\frac{-iln^2(x)\beta y + n^2(x)\beta + l^2 n(x)\alpha x - l^2 \alpha}{l^2 n^3(x)} e^{i(\omega t - in(x)x + ly + kz)},$$

based on which the total MHF is

$$T = 2B_0(x)a_0 e^{n(x)x} e^{i(\omega t + ly + kz)} \frac{l^2 \alpha - n^2(x)\beta - l^2 n(x)\alpha x + iln^2(x)\beta y}{l^2 n^3(x)}, \quad (55)$$

where $n(x)$ and $B_0(x)$ are defined according to Equations (54) and (35), respectively. Note: here the term $a_0 e^{n(x)x}$ could be rewritten as $\hat{v}_x(x)$, but this way it is clearer what the total MHF depends on, as well as how, in the case of an indefinite integral for a magnetic interface. Problem: What happens to the magnetic helicity flux at $x = 0$? I address this issue in my thesis by considering $x \neq 0$, as a more detailed discussion of this is beyond the scope of my thesis. However, the equation is meaningful for all other values of x .

4.3.2 Definite integral

I will now derive the previously obtained MHF as a definite integral with arbitrary limits x_1 , x_2 , and y_1 , y_2 . So

$$T = -2 \int_{y_1}^{y_2} \int_{x_1}^{x_2} \left(\beta y a_0 e^{n(x)x} e^{i(\omega t + ly + kz)} + \alpha x a_0 \frac{il}{n(x)} e^{n(x)x} e^{i(\omega t + ly + kz)} \right) B_0(x) dx dy.$$

Since it matters whether $B_0(x_1)$ or $B_0(x_2)$ appears here, it cannot be factored out in front of the integral with certainty. In exchange, a_0 can still be factored out.

Using the result of the integration with respect to x from the previous chapter, let us now take the limits into account as well! Thus,

$$\left[B_0(x) \left(\frac{\beta y}{n(x)} + \frac{il\alpha x}{n^2(x)} - \frac{il\alpha}{n^3(x)} \right) e^{i(\omega t - in(x)x + ly + kz)} \right]_{x_1}^{x_2}.$$

The result of the definite integral with respect to x is thus

$$B_0(x_2) \left(\frac{\beta y}{n(x_2)} + \frac{il\alpha x_2}{n^2(x_2)} - \frac{il\alpha}{n^3(x_2)} \right) e^{i(\omega t - in(x_2)x_2 + ly + kz)} - \\ - B_0(x_1) \left(\frac{\beta y}{n(x_1)} + \frac{il\alpha x_1}{n^2(x_1)} - \frac{il\alpha}{n^3(x_1)} \right) e^{i(\omega t - in(x_1)x_1 + ly + kz)}.$$

The resulting expression now needs to be integrated with respect to y , where, as in the previous cases, some terms of the expression require IBP. After IBP, we obtain that

$$\left(\frac{\beta y}{iln(x_2)} + \frac{\beta}{l^2 n(x_2)} \right) B_0(x_2) e^{i(\omega t - in(x_2)x_2 + ly + kz)} - \\ - \left(\frac{\beta y}{iln(x_1)} + \frac{\beta}{l^2 n(x_1)} \right) B_0(x_1) e^{i(\omega t - in(x_1)x_1 + ly + kz)},$$

then performing the remaining integration:

$$\left[\left(\frac{\beta y}{iln(x_2)} + \frac{\beta}{l^2 n(x_2)} \right) B_0(x_2) e^{i(\omega t - in(x_2)x_2 + ly + kz)} - \right. \\ \left. - \left(\frac{\beta y}{iln(x_1)} + \frac{\beta}{l^2 n(x_1)} \right) B_0(x_1) e^{i(\omega t - in(x_1)x_1 + ly + kz)} + \right. \\ \left. + \left(\frac{\alpha x_2}{n^2(x_2)} - \frac{\alpha}{n^3(x_2)} \right) B_0(x_2) e^{i(\omega t - in(x_2)x_2 + ly + kz)} - \right. \\ \left. - \left(\frac{\alpha x_1}{n^2(x_1)} - \frac{\alpha}{n^3(x_1)} \right) B_0(x_1) e^{i(\omega t - in(x_1)x_1 + ly + kz)} \right]_{y_1}^{y_2}.$$

Substitute the limits and perform the grouping applied earlier! This gives us the following expression:

$$B_0(x_2) e^{i(\omega t - in(x_2)x_2 + ly_2 + kz)} \left(\frac{\beta y_2}{iln(x_2)} + \frac{\beta}{l^2 n(x_2)} + \frac{\alpha x_2}{n^2(x_2)} - \frac{\alpha}{n^3(x_2)} \right) - \\ - B_0(x_1) e^{i(\omega t - in(x_1)x_1 + ly_2 + kz)} \left(\frac{\beta y_2}{iln(x_1)} + \frac{\beta}{l^2 n(x_1)} + \frac{\alpha x_1}{n^2(x_1)} - \frac{\alpha}{n^3(x_1)} \right) - \\ - B_0(x_2) e^{i(\omega t - in(x_2)x_2 + ly_1 + kz)} \left(\frac{\beta y_1}{iln(x_2)} + \frac{\beta}{l^2 n(x_2)} + \frac{\alpha x_2}{n^2(x_2)} - \frac{\alpha}{n^3(x_2)} \right) + \\ + B_0(x_1) e^{i(\omega t - in(x_1)x_1 + ly_1 + kz)} \left(\frac{\beta y_1}{iln(x_1)} + \frac{\beta}{l^2 n(x_1)} + \frac{\alpha x_1}{n^2(x_1)} - \frac{\alpha}{n^3(x_1)} \right).$$

Within the large parentheses, bring everything to a common denominator, which is $l^2 n^3(x_i)$. After bringing to a common denominator:

$$\begin{aligned}
& B_0(x_2)e^{i(\omega t - in(x_2)x_2 + ly_2 + kz)} \frac{-iln^2(x_2)\beta y_2 + n^2(x_2)\beta + l^2n(x_2)\alpha x_2 - l^2\alpha}{l^2n^3(x_2)} - \\
& - B_0(x_1)e^{i(\omega t - in(x_1)x_1 + ly_2 + kz)} \frac{-iln^2(x_1)\beta y_2 + n^2(x_1)\beta + l^2n(x_1)\alpha x_1 - l^2\alpha}{l^2n^3(x_1)} - \\
& - B_0(x_2)e^{i(\omega t - in(x_2)x_2 + ly_1 + kz)} \frac{-iln^2(x_2)\beta y_1 + n^2(x_2)\beta + l^2n(x_2)\alpha x_2 - l^2\alpha}{l^2n^3(x_2)} + \\
& + B_0(x_1)e^{i(\omega t - in(x_1)x_1 + ly_1 + kz)} \frac{-iln^2(x_1)\beta y_1 + n^2(x_1)\beta + l^2n(x_1)\alpha x_1 - l^2\alpha}{l^2n^3(x_1)}.
\end{aligned}$$

It is useful to introduce a new notation to make the formula more concise, therefore, I now introduce the following notation:

$$g_{rq}(t, z) = \frac{B_0(x_r)}{l^2n^3(x_r)} e^{n(x_r)x_r} e^{i(\omega t + ly_q + kz)} (l^2\alpha - n^2(x_r)\beta - l^2n(x_r)\alpha x_r + iln^2(x_r)\beta y_q), \quad (56)$$

$$r \in \{1, 2\}, \quad q \in \{1, 2\},$$

thus, the group can be simply written with the help of Equation (56) as

$$(-1)^{r+q-1} g_{rq}.$$

If we now multiply the resulting expression by $-2a_0$, we obtain the total MHF:

$$T|_{x_1, x_2, y_1, y_2}(t, z) = 2a_0(-1)^{r+q} g_{rq}. \quad (57)$$

Note: When substituting the limits after integrating with respect to x , I checked outside the thesis that if we handle this in the form $\int_{x_1}^0 dx + \int_0^{x_2} dx$, the quantities evaluated at $x = 0$ cancel out. Thus, it is valid to simply substitute x_2 as the upper limit and x_1 as the lower limit.

5 Magnetic helicity flux with dispersion relation

The following examples will greatly rely on the derivations in Section 4. However, there are a number of changes in the derivation process. I have fixed the wave propagation vector to be in the $\hat{\mathbf{z}}$ direction, therefore, the Fourier form will look as follows.

$$\begin{aligned}
v_x &= \hat{v}_x e^{i(\omega(k)t + kz)}, \\
v_y &= \hat{v}_y e^{i(\omega(k)t + kz)},
\end{aligned}$$

where $\hat{v}_x$ and $\hat{v}_y$ are constant amplitudes of the wave. Furthermore, I will take only the definite integrals into consideration, and I will use only the positive branch of the given dispersion relation.

5.1 Magnetic helicity flux of a homogeneous magnetic field

5.1.1 Alfvén mode

$$T = -2 \int_{y_1}^{y_2} \int_{x_1}^{x_2} \left(\beta y \hat{v}_x e^{i(kv_A t + kz)} + \alpha x \hat{v}_y e^{i(kv_A t + kz)} \right) B_0 dx dy.$$

As the wave part does not depend either on x or y , therefore, it can be brought out of the integration along with B_0 .

$$\int_{x_1}^{x_2} (\beta y \hat{v}_x + \alpha x \hat{v}_y) dx = \left[\beta y x \hat{v}_x + \frac{\alpha}{2} x^2 \hat{v}_y \right]_{x_1}^{x_2} = \left[\beta y \hat{v}_x (x_2 - x_1) + \frac{\alpha}{2} \hat{v}_y (x_2^2 - x_1^2) \right]$$

Integrating with respect to y :

$$\int_{y_1}^{y_2} \left[\beta y \hat{v}_x (x_2 - x_1) + \frac{\alpha}{2} \hat{v}_y (x_2^2 - x_1^2) \right] dy = \left[\frac{\beta}{2} y^2 \hat{v}_x (x_2 - x_1) + \frac{\alpha}{2} \hat{v}_y (x_2^2 - x_1^2) y \right]_{y_1}^{y_2}.$$

Evaluating the limits and multiplying the result with the brought out factors, we obtain that

$$T|_{x_1, x_2, y_1, y_2}(t, z) = -B_0 e^{i(kv_A t + kz)} [\beta \hat{v}_x (y_2^2 - y_1^2)(x_2 - x_1) + \alpha \hat{v}_y (x_2^2 - x_1^2)(y_2 - y_1)]. \quad (58)$$

Problem: if the limits are not symmetric with the origin being the center of the rectangle, then going further from the origin, the MHF will grow indefinitely, which shall not be physically possible.

5.1.2 Magnetoacoustic modes

The solutions for the dispersion relation for the magnetoacoustic modes are given by Equation (45). Considering only the positive branch, there are 2 solutions: $\omega = kv_A$ or $\omega = kc_s$. Due to this, the only difference to the Alfvén mode occurs when the wave propagates with the sound speed. Without loss of generality, one can write c_s instead of v_A , therefore, the starting equation will be

$$T = -2 \int_{y_1}^{y_2} \int_{x_1}^{x_2} (\beta y \hat{v}_x e^{i(kc_s t + kz)} + \alpha x \hat{v}_y e^{i(kc_s t + kz)}) B_0 dx dy.$$

As this difference does not make a change in the derivation process, we obtain that

$$T|_{x_1, x_2, y_1, y_2}(t, z) = -B_0 e^{i(kc_s t + kz)} [\beta \hat{v}_x (y_2^2 - y_1^2)(x_2 - x_1) + \alpha \hat{v}_y (x_2^2 - x_1^2)(y_2 - y_1)]. \quad (59)$$

5.2 Magnetic helicity flux of a magnetic interface

As κ does not have a y -component, thus, $l = 0$. This changes Equation (53) to be equal to 0, therefore, $\hat{v}_y = 0 \implies v_y = 0$. Equation (54) changes, as well as Equation (49). The changed MHF integral will take the following form.

$$T = -2 \int \int (\beta y v_x) B_0 dx dy. \quad (60)$$

$$n(x) = \begin{cases} -m_e, & x > 0, \\ m_0, & x < 0, \end{cases} \quad (61)$$

where $n(x)$ is used for ease of notation. Both the Alfvén and sound speed take a similar form to Equation (35) due to the interface.

5.2.1 Alfvén mode

Applying the change and the previous experience, the starting equation will be:

$$T = -2 \int_{y_1}^{y_2} \int_{x_1}^{x_2} (\beta y a_0 e^{n(x)x} e^{i(kv_A(x)t + kz)}) B_0(x) dx dy.$$

$$\int_{x_1}^{x_2} (\beta y a_0 e^{n(x)x} e^{i(kv_A(x)t + kz)}) B_0(x) dx = \left[a_0 \beta \frac{y}{n(x)} e^{n(x)x} e^{i(kv_A(x)t + kz)} B_0(x) \right]_{x_1}^{x_2},$$

then evaluating the limits:

$$a_0 \beta \left[\frac{y}{n(x_2)} e^{n(x_2)x_2} e^{i(kv_A(x_2)t + kz)} B_0(x_2) - \frac{y}{n(x_1)} e^{n(x_1)x_1} e^{i(kv_A(x_1)t + kz)} B_0(x_1) \right].$$

Integrating with respect to y :

$$\int_{y_1}^{y_2} a_0 \beta \left[\frac{y}{n(x_2)} e^{n(x_2)x_2} e^{i(kv_A(x_2)t + kz)} B_0(x_2) - \frac{y}{n(x_1)} e^{n(x_1)x_1} e^{i(kv_A(x_1)t + kz)} B_0(x_1) \right] dy.$$

After performing the integration, we obtain that

$$a_0 \beta \left[\frac{y^2}{2n(x_2)} e^{n(x_2)x_2} e^{i(kv_A(x_2)t + kz)} B_0(x_2) - \frac{y^2}{2n(x_1)} e^{n(x_1)x_1} e^{i(kv_A(x_1)t + kz)} B_0(x_1) \right]_{y_1}^{y_2},$$

then taking the limits results in

$$a_0 \beta \left[\frac{y_2^2 - y_1^2}{2n(x_2)} e^{n(x_2)x_2} e^{i(kv_A(x_2)t+kz)} B_0(x_2) - \frac{y_2^2 - y_1^2}{2n(x_1)} e^{n(x_1)x_1} e^{i(kv_A(x_1)t+kz)} B_0(x_1) \right].$$

To make the formula more concise, I will now introduce the following notation:

$$h_{rq}(t, z) = \frac{B_0(x_r)}{n(x_r)} e^{n(x_r)x_r} e^{i(kv_q(x_r)t+kz)}, \quad (62)$$

$$r \in \{1, 2\}, \quad q \in \{1, 2\},$$

$$v_q = \begin{cases} v_A, & q = 1, \\ c_s, & q = 2, \end{cases}$$

thus, the expression can be simply written with the help of Equation (62) as

$$a_0 \frac{y_2^2 - y_1^2}{2} (-1)^r h_{r1}.$$

Multiplying the expression by -2 , we obtain the total MHF. Thus,

$$T|_{x_1, x_2, y_1, y_2}(t, z) = a_0(y_2^2 - y_1^2)(-1)^{r+1} h_{r1}. \quad (63)$$

5.2.2 Magnetoacoustic modes

The solutions for the dispersion relation for the magnetoacoustic modes are given by Equation (45) in the case of a single magnetic interface as well. Considering only the positive branch, there are 2 solutions: $\omega = kv_A$ or $\omega = kc_s$. Due to this, the only difference to the Alfvén mode occurs when the wave propagates with the sound speed. Without loss of generality, one can write c_s instead of v_A , therefore, the starting equation will be

$$T = -2 \int_{y_1}^{y_2} \int_{x_1}^{x_2} \left(\beta y a_0 e^{n(x)x} e^{i(kc_s(x)t+kz)} \right) B_0(x) dx dy.$$

As this difference does not make a change in the derivation process, using Equation (62), we obtain that

$$T|_{x_1, x_2, y_1, y_2}(t, z) = a_0(y_2^2 - y_1^2)(-1)^{r+1} h_{r2}. \quad (64)$$

Deriving Equations (57), (63), and (64) was the main goal of my work, so these chapters provide a new perspective on the MHF transport in MHD waves.

6 Summary

This work concerns solar eruptions, their history, effect, and the importance of magnetic helicity flux transport by magnetohydrodynamic waves. MHF was found to be a useful quantity in the field of flare and CME forecast, observations have been made, however, to the best of our knowledge, there is no developed theoretical framework that offers a comprehensive foundation for the observed phenomena. In order to obtain a governing equation for the MHF transport of MHD waves, I introduce the equations of MHD and ideal MHD. After that, I apply the ideal MHD equations and perturbation theory to linearise the governing equations and derive the dispersion relations of different MHD waves. I introduce the MHF calculating formula derived by Berger (1984), then consider the case of a homogeneous medium, as well as a magnetic interface. After deriving the general forms, I examine the obtained formulae with regards to the derived dispersion relations. To conclude the work, I discuss the different problems and the potential of future application and improvement.

In Section 2, I briefly describe the theory of magnetohydrodynamics and introduce the governing equations. In that section I use perturbation theory and linearisation to obtain the perturbed, linearised equations of MHD. In Section 3, I consider a constant magnetic field pointing in the $\hat{\mathbf{z}}$ -direction similar to the paper of Roberts (1981). Then I apply the perturbed

ideal MHD equations to obtain the dispersion relation for MHD waves. The first case is an infinite homogeneous plasma, the second case is a single magnetic interface at $x = 0$. After I obtained the dispersion relations for Alfvén and Magnetoacoustic modes, I considered wave propagation along the magnetic field in the $\hat{\mathbf{z}}$ direction, which vastly simplified the given dispersion relations.

In Section 4, I sketch the planar geometry (xy-plane) I assumed, then introduce the aforementioned MHF calculating formula. I derive the magnetic vector potential with the given boundary conditions, which are assuming Coulomb gauge, no component in the $\hat{\mathbf{z}}$ -direction and using the simplest solution to the differential equation. I modify the original MHF integral according to the obtained magnetic vector potential, which simplifies the calculations. I assume a general wave in Fourier form, then perform the integration for infinite homogeneous plasma. I introduce a notation to simplify the result of the definite integral with arbitrary limits x_1, x_2, y_1, y_2 . After that, I use the results obtained in Section 3 to derive the MHF for a single magnetic interface. I introduce a new notation to simplify the writing, then I follow the calculations in Section 5.

In Section 5, I apply the different dispersion relations to obtain the following results for MHF. The first result is the Alfvén mode in a homogeneous magnetic field.

$$T|_{x_1, x_2, y_1, y_2}(t, z) = -B_0 e^{i(kv_A t + kz)} [\beta \hat{v}_x (y_2^2 - y_1^2)(x_2 - x_1) + \alpha \hat{v}_y (x_2^2 - x_1^2)(y_2 - y_1)],$$

where x_i and y_i are the limits, B_0 is the magnetic field strength, k is the wavenumber in the $\hat{\mathbf{z}}$ -direction, v_A is the Alfvén speed, $\hat{v}_x$ and $\hat{v}_y$ are amplitudes of the speeds in the x and y direction, respectively, while β and α are constants such that $\alpha - \beta = B_0$.

The magnetoacoustic modes have two different solutions. One is with sound speed and the other with Alfvén speed. As the result with Alfvén speed is equivalent to the aforementioned result, only the differing result is mentioned here. This result utilizes sound speed.

$$T|_{x_1, x_2, y_1, y_2}(t, z) = -B_0 e^{i(kc_s t + kz)} [\beta \hat{v}_x (y_2^2 - y_1^2)(x_2 - x_1) + \alpha \hat{v}_y (x_2^2 - x_1^2)(y_2 - y_1)],$$

where c_s represents the sound speed, and all other variables have been defined above.

After this, I calculated the MHF for Alfvén and magnetoacoustic modes in the case of a single magnetic interface. The result for the Alfvén mode is:

$$T|_{x_1, x_2, y_1, y_2}(t, z) = a_0 (y_2^2 - y_1^2) (-1)^{r+1} h_{r1}.$$

The definition of the parameters can be found in Sections 5 and 3 (directly under Equation (25))

Here, the same situation applies for the magnetoacoustic modes: one of them has the Alfvén speed and the other has the sound speed. As the result with Alfvén speed is equivalent to the aforementioned result, only the differing result is mentioned here. This result utilizes sound speed.

$$T|_{x_1, x_2, y_1, y_2}(t, z) = a_0 (y_2^2 - y_1^2) (-1)^{r+1} h_{r2}.$$

The parameters can be found in Sections 5 and 3 as mentioned above.

7 Problems and what is next?

7.1 Formula length and the interface problem

Although I have successfully derived how much MHF the different MHD waves may transport, it must unfortunately be mentioned as a problem that at $x = 0$ — that is, at the location of the interface — the formula expressing MHF is not well-defined. Additionally, some of the computational formulae can likely be written in a more compact form using trigonometric functions. In this work, I have currently addressed this issue by considering the condition $x \neq 0$. However, it may be worth investigating in the future how MHF behaves around $x = 0$.

7.2 Homogeneous case for z -directed propagation

Although the calculation process and formula are greatly simplified if we consider a wave propagating only in the $\hat{z}$ direction, the MHF grows indefinitely as we go further from the origin. However, if the medium is uniform, the MHF transport may need to be the same everywhere on the xy -plane. A potential solution to this problem is that there have been a number of approximations and assumptions, as well as, in physical reality, homogeneity appears only on relatively small scales, therefore, going far from the origin may result in breaking the starting assumptions.

7.3 The form of the magnetic vector potential

Another concern may be that the magnetic vector potential obtained does not decay to zero at infinity, so it may not be physically relevant. The potential solution to this lies in the fact that I imposed many strict conditions on the form of the potential and approached the problem with an infinite plane and a constant magnetic field in the $\hat{z}$ direction. Thus, the approximation may have local physical relevance, but this will need to be validated with experimental data. Note: Since α and β determine the strength of the resulting magnetic field, their values actually depend on whether we are considering the region where $x > 0$ or $x < 0$ in the case of an interface, as $B_0 \neq B_e$. Thus, the α - β pair applied in one region does not match the pair applied in the other region.

7.4 Further geometries and applicability

My work addressed the homogeneous magnetic field and magnetic interface, but solar activity often requires geometries resembling a magnetic slab (which can be constructed as two separate magnetic interfaces) or a more realistic magnetic flux tube. The analytical derivation should examine its practical use in determining the periodicity of MHF discussed by [Korsós et al. \(2022\)](#), and whether it can improve forecasting models. Consequently, my work could open new avenues for solving the problems mentioned in the introduction's motivation, which could ultimately refine space weather forecasts and bring us closer to a more comprehensive understanding of solar activity.

Appendix A

Own work in the thesis

I have done the following calculations by hand with pen and paper:

- I have linearised the ideal MHD equations
- I have derived the dispersion relation for MHD waves from the linearised ideal MHD equations in the case of a $\hat{\mathbf{z}}$ -directed magnetic field and homogeneous, infinite plasma
- I have derived the dispersion relation for an inhomogeneous plasma
- I have derived the dispersion relation for a single magnetic interface set at the place $x = 0$
- I have assumed an infinite plane geometry in the xy -plane, then derived the magnetic vector potential
- Using the previous results and Berger (1984)'s magnetic helicity flux (MHF) calculating formula, I have modified the integral
- I have calculated the total MHF for a general wave in a homogeneous medium as an indefinite integral
- I have calculated the definite integral with arbitrary limits for the same wave and introduced a new notation for ease of writing
- I have calculated the total MHF for a wave in the case of a single magnetic interface as an indefinite integral
- I have calculated the definite integral with arbitrary limits as well, and introduced a new notation for ease of writing
- I have derived the total MHF calculating formula for the Alfvén and magnetoacoustic modes as a definite integral with arbitrary limits in the case of a homogeneous, infinite plasma
- I have derived the total MHF calculating formula for the Alfvén and magnetoacoustic modes as a definite integral with arbitrary limits in the case of a single magnetic interface, and introduced a new notation for ease of writing

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