



PÁZMÁNY

Pázmány Péter Catholic University

Faculty of Information Technology and Bionics

Scientific Students' Associations Conference

Analysis and observer design for networked compartmental flow models

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2024

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Abstract

This research provides a systematic approach to modeling and estimation of nonlinear compartmental systems, contributing valuable tools for managing complex networked flow processes with finite capacities.

The behavior of dynamical models is often governed by nonlinearities, which limit the direct applicability of linear systems and control theory. A recently introduced class of compartmental systems with nonlinear dynamics and finite capacities offers a promising framework for modeling networked flow processes. Motivated by this, my research focuses on understanding and analyzing a class of nonlinear compartmental models where each compartment has finite capacity, and on developing effective estimation methods for these systems.

My analysis focused on the nonlinear observability of the Traffic Reaction Model (TRM), both with and without external inputs. The study demonstrated that the proposed state estimators, a continuous-time nonlinear state estimator and an Extended Kalman Filter were generally capable of tracking the system's states, although the EKF was not always successful. These results support the applicability of the developed methods for estimation in nonlinear compartmental systems with finite capacities, providing a robust framework for managing networked flow processes.

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1. Introduction

The main objective of my research is to study the nonlinear compartmental traffic reaction model and to investigate its basic system theoretical properties. Due to the nonlinearities (coming from physical, chemical, biological laws) determining the behavior of dynamical models, the applicability of linear systems and control theory is often limited.

Compartmental dynamical models are used in many disciplines. The two main features of these models from a systems theory point of view are that, on the one hand, the solutions can only be non-negative, and on the other hand, all units of the described individuals, substances, particles, vehicles, etc. can be present in exactly one compartment at any given time. For more than 50 years, linear compartmental models have been intensively studied, and much is known about the relation between compartmental structure and qualitative dynamics [1], [2]. However, much less is known about nonlinear compartmental systems due to the generality of this system class [3]. The motivation for my research is to explore a class of nonlinear compartmental models where each compartment has a finite capacity.

The nonlinear compartmental traffic reaction model (TRM) has been recently introduced to describe mathematical models of flow (e.g., road) networks with different topologies [4]. The compartments of such a system are obtained by decomposing the modeled network into several road segments of equal length. The mathematical models use a system of differential equations to describe how the number of modeled objects (e.g., vehicles or ribosomes) and the available space slots change over time, based on the number of compartments. The sum of the number of vehicles and available free space units in a compartment is always equal to the maximum vehicle capacity. The constant maximum vehicle density value guarantees that the solutions of the differential equation system equations are bounded. The states of the traffic reaction model are the continuous density of vehicles in the compartments and the density of available free space slots. Several interesting properties of TRMs have been shown such as the nonnegativity, persistence and contractivity of the solutions [5]. Moreover, we can compute logarithmic Lyapunov functions for TRMs even if the transition rates are quite general and possibly time-varying nonlinear functions [6]. However, other important control related properties of TRMs like observability or controllability have not been studied yet. Therefore, the main goal of this research is the design of controllers and observers for TRMs, along with observability analysis. As a sub-problem, it is important to determine which compartments are to be observed (measured) to be able to fully reconstruct the state of the system.

For nonlinear systems, assessing observability becomes more complex due to the nonlinearity. Generally, observability becomes a local property in the state space, where it may change. Moreover, the unobservability manifold is no longer a subspace, but generally a more complicated set in the state space [7]. Nonlinear observability co-distributions provide a framework for understanding the observability of nonlinear systems by analyzing the distributions of observable quantities [8]. However, determining and evaluating the co-distributions require symbolic computations which is a significant difficulty compared to the linear case.

From a systems theory point of view, Matlab and Simulink provide a suitable environment for simulating the models, estimating the states. In these environments, by choosing the appropriate model parameters and initial values, the investigated models can be well simulated and examined in terms of the desired result. Moreover, the symbolic toolbox of Matlab is powerful enough to perform the required calculations.

1.1. Outline

The summary of the chapters of the thesis work:

Chapter 2 This chapter explains the systems, theories, models, methods, and processes that were used to solve each task. These form the theoretical background that supports and guides the work presented in the thesis.

Chapter 3 An overview of the article [5] that forms the foundation of the model used in this thesis.

Chapter 4 This chapter presents the achieved results, including symbolic computational findings and simulation outcomes. It also details the results of observability analysis and observer design.

Chapter 5 This chapter provides an overview of the results achieved and outlines future plans, exploring potential improvements, extensions, and applications of the work.

2. Systems theory background

In this section, the most important tools and techniques will be summarized, which later will be used for the analysis of the studied models.

2.1. Linear systems

A differential equation is linear if the coefficients are constants or functions only of the independent variable. Dynamic systems that are composed of linear time invariant lumped-parameter components may be described by linear time-invariant (LTI) differential equations that is, constant-coefficient differential equations [9].

The state-space approach, also known as the modern or time-domain approach, is a method for modeling, analyzing and designing a wide range of linear and non-linear systems. State space models use state variables to describe a system using a set of first order differential equations [10].

A linear time invariant system is represented in state space by the following equations

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t),\end{aligned}\tag{1}$$

where:

- $x(t) \in \mathbb{R}^n$ is the state vector,
- $u(t) \in \mathbb{R}^r$ is the input vector,
- $y(t) \in \mathbb{R}^p$ is the output vector,
- $A \in \mathbb{R}^{n \times n}$ is the system matrix,
- $B \in \mathbb{R}^{n \times r}$ is the input matrix,
- $C \in \mathbb{R}^{p \times n}$ is the output matrix,
- $D \in \mathbb{R}^{p \times r}$ is the feed-through matrix.

The block diagram representation of the system described in (1) is shown in Figure 1.

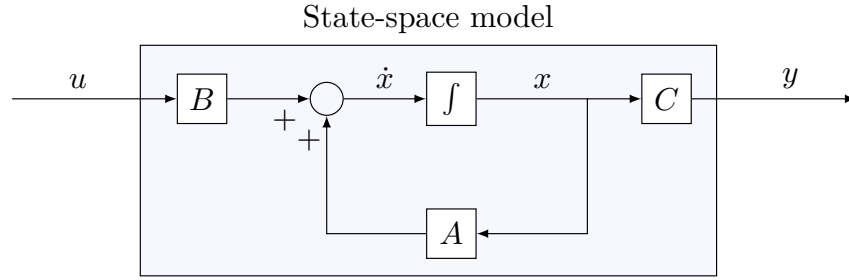


Figure 1.: Block diagram representation of the linear state-space model.

2.1.1. Complete observability

The system described by (1) is considered completely observable if every state $x(t)$ can be determined from the observation of the output $y(t)$ over a finite time interval. The system is, therefore, completely observable if every transition of the state eventually affects every element of the output vector. The concept of observability is useful in solving the problem of reconstructing unmeasurable state variables from measurable variables. For complete observability, the observability matrix must have full rank [9].

Complete observability of a system is achieved when the observability matrix

$$\mathcal{O}_n = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has full rank, meaning its rank is equal to n

$$\text{rank}(\mathcal{O}_n) = n \text{ [9].}$$

2.2. Nonlinear systems

Nonlinear control systems can be described by either ordinary differential equations or difference equations. The systems under consideration are of the following type

$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t)), \\ y(t) &= h(x(t), u(t)),\end{aligned}$$

or

$$\begin{aligned}x(k+1) &= f(x(k), u(k)), \\ y(k) &= h(x(k), u(k)),\end{aligned}$$

where $x \in \mathbb{R}^n$ denotes the state of the system, $u \in \mathbb{R}$ is the control input and $y \in \mathbb{R}$ is the output of the system [7].

Multivariable nonlinear control systems with r inputs $u_1, \dots, u_r$ and p outputs $y_1, \dots, y_p$ can be described in state space form by equations of the following type

$$\begin{aligned}\dot{x} &= f(x) + \sum_{i=1}^r g_i(x)u_i \\ y_1 &= h_1(x) \\ &\vdots \\ y_p &= h_p(x)\end{aligned}\tag{2}$$

in which

- $x(t) \in \mathbb{R}^n \quad \forall t \geq 0,$
- $u(t) \in \mathbb{R}^r \quad \forall t \geq 0,$
- $y(t) \in \mathbb{R}^p \quad \forall t \geq 0,$
- $f \in \mathbb{R}^n \rightarrow \mathbb{R}^n,$
- $g_i \in \mathbb{R}^n \rightarrow \mathbb{R}^n,$
- $h \in \mathbb{R}^n \rightarrow \mathbb{R}^p.$

These equations will be rewritten in a more condensed form

$$\begin{aligned}\dot{x} &= f(x) + g(x)u, \\ y &= h(x),\end{aligned}$$

where

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix},$$

and

$$g(x) = \begin{bmatrix} g_1(x) & \dots & g_r(x) \end{bmatrix}, \quad h(x) = \begin{bmatrix} h_1(x) \\ \vdots \\ h_p(x) \end{bmatrix}$$

are respectively an $n \times r$ matrix and an $p \times 1$ matrix [8].

2.2.1. Nonlinear observability co-distributions

For linear systems, observability can be assessed using well-known tools and conditions like observability matrices and observability rank. However, for nonlinear systems, assessing observability is more complex due to the nonlinearity. Nonlinear observability co-distributions provide a framework for understanding the observability of nonlinear systems by analyzing the distributions of observable quantities in the state space [8].

The co-distributions Ω are defined by smooth co-vector fields $\omega_1, \dots, \omega_n$, where

$$\Omega(x) = \text{span} \{ \omega_1(x), \dots, \omega_n(x) \}$$

denotes the span of these fields at the point x of the manifold. Considered point-wise, the co-distributions are subspaces of $(\mathbb{R}^n)^*$, and can therefore be expressed as

$$\Omega = \text{span} \{ \omega_1, \dots, \omega_n \} \quad [8].$$

Observability of linear systems expressed with co-distributions

Consider a system with the following dynamics

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned}$$

where x is the state vector, u is the input vector, y is the output vector, A is the system matrix, B is the input matrix and C is the output matrix. Define a co-distribution as

$$\Omega_0 = \text{span} \{ c_1, \dots, c_p \}$$

where $c_1, \dots, c_p$ denote the rows of C .

To extend this co-distribution, consider

$$\Omega_1 = \Omega_0 + L_{Ax}\Omega_0 = \text{span} \{ c_1, \dots, c_p, L_{Ax}c_1, \dots, L_{Ax}c_p \}$$

where L_{Ax} denotes the Lie derivative along Ax .

The Lie derivative of c_i along Ax is given by

$$L_{Ax}c_i = c_i \frac{\partial(Ax)}{\partial x} = c_i A.$$

The co-distribution Ω_1 becomes

$$\Omega_1(x) = \text{span}\{c_1, \dots, c_p, c_1 A, \dots, c_p A\}.$$

In general, the co-distribution $\Omega_k(x)$ for any $k \geq 1$ is

$$\Omega_k(x) = \text{span}\{c_1, \dots, c_p, c_1 A, \dots, c_p A, \dots, c_1 A^k, \dots, c_p A^k\},$$

which is the well-known classical observability matrix [8].

Algorithm for constructing co-distributions

Consider the nonlinear input-affine system described by (2). Similarly to observability matrices in the linear case, observability co-distributions can be algorithmically generated as follows

1. The starting co-distribution is

$$\Omega_0 = \text{span}\{dh_1, \dots, dh_p\}$$

where $dh_i = \left(\frac{\partial h_i}{\partial x_1} \quad \frac{\partial h_i}{\partial x_2} \quad \dots \quad \frac{\partial h_i}{\partial x_n} \right)$ is the gradient of the real-valued function h_i .

2. Developing the observability co-distribution

$$\Omega_k = \Omega_{k-1} + \sum_{i=0}^m L_{g_i} \Omega_{k-1}$$

where $g_0 = f$ and $g_1, \dots, g_m$ is a set of vector fields.

3. Stopping criterion

If there exists an integer k^* such that $\Omega_{k^*} = \Omega_{k^*+1}$, then

$$\Omega_n = \Omega_{k^*}.$$

Otherwise, go to step 2 [8].

2.3. Directed graph

A directed graph (or digraph) D consists of a non-empty finite set $V(D)$ of elements called vertices and a finite set $A(D)$ of ordered pairs of distinct vertices called arcs. Let $V(D)$ denote the vertex set and $A(D)$ the arc set of D . The notation $D = (V, A)$ is often used to indicate that V and A represent the vertex set and arc set of D , respectively. The order (size) of D is the number of vertices (arcs) in D [11]. For example, the digraph D in Figure 2. is of order and size 5, $V(D) = \{1, 2, 3, 4, 5\}$, $A(D) = \{(1, 3), (2, 3), (3, 2), (3, 4), (4, 5), (5, 3)\}$.

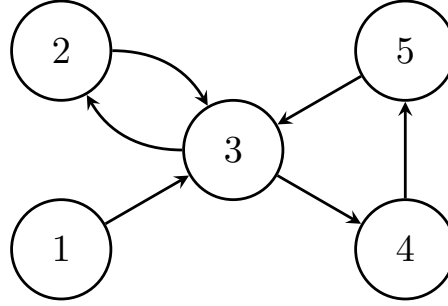


Figure 2.: A digraph D .

A walk in D is an alternating sequence $W = x_1 a_1 x_2 a_2 x_3 \dots x_{k-1} a_{k-1} x_k$ of vertices x_i and arcs a_j from D such that the tail of a_i is x_i and the head of a_i is x_{i+1} for every $i \in [k-1]$. A walk W is closed if $x_1 = x_k$, and open otherwise. The set of vertices $\{x_i : i \in [k]\}$ is denoted by $V(W)$, the set of arcs $a_j : j \in [k-1]$ is denoted by $A(W)$. We say that W is a walk from x_1 to x_k or an (x_1, x_k) -walk [11].

The transpose (or reverse) of a directed graph D is a directed graph D^T with the same set of vertices, where every edge in D is reversed in D^T , meaning the direction of each edge is opposite to its orientation in D [12]. For example, the digraph D^T in Figure 3. is the transpose of the digraph D shown in Figure 2. The order and size of D^T is 5, $V(D^T) = \{1, 2, 3, 4, 5\}$ and $A(D^T) = \{(3, 1), (3, 2), (2, 3), (4, 3), (5, 4), (3, 5)\}$.

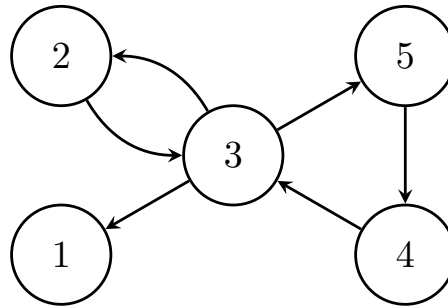


Figure 3.: The transpose graph D^T of digraph D .

2.3.1. Strongly connected directed graph

In a digraph D a vertex y is reachable from a vertex x if D has an (x, y) -walk. In particular, a vertex is reachable from itself. The y is reachable from x if and only if D contains an (x, y) -path. A digraph D is strongly connected (or just strong) if, for every pair x, y of distinct vertices in D , there exists an (x, y) -walk and a (y, x) -walk. In other words, D is strongly connected if every vertex of D is reachable from every other vertex of D [11]. For example, the digraph D in Figure 4. is a strongly connected digraph.

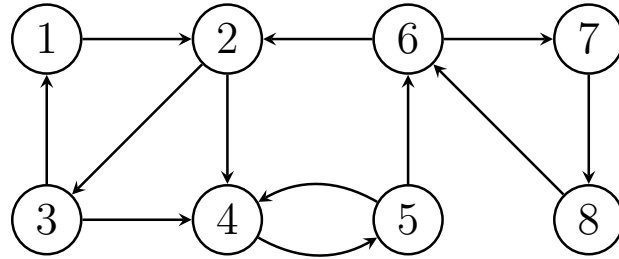


Figure 4.: A strongly connected directed graph D .

2.3.2. Strongly connected component

A strong component of a digraph D is a maximal induced subdigraph of D which is strong. If $D_1, \dots, D_t$ are the strong components of D , then clearly $V(D_1) \cup \dots \cup V(D_t) = V(D)$ (recall that a digraph with only one vertex is strong). Moreover, it is required that $V(D_i) \cap V(D_j) = \emptyset$ for every $i \neq j$ as otherwise all the vertices $V(D_i) \cup V(D_j)$ are reachable from each other, implying that the vertices of $V(D_i) \cup V(D_j)$ belong to the same strong component of D . The union $V(D_1) \cup \dots \cup V(D_t)$ is referred to as the strong decomposition of D . The strong component digraph $SC(D)$ of D is obtained by contracting strong components of D and deleting any parallel arcs obtained in this process. In other words, if $D_1, \dots, D_t$ are the strong components of D , then $V(SC(D)) = \{v_i : i \in [t]\}$ and $A(SC(D)) = \{v_i v_j : (V(D_i), V(D_j))_D = \emptyset\}$. The subdigraph of D induced by the vertices of a cycle in D is strong, i.e., is contained in a strong component of D . Thus, $SC(D)$ is acyclic. The vertices of $SC(D)$ have an acyclic ordering. This implies that the strong components of D can be labelled $D_1, \dots, D_t$ such that there is no arc from D_j to D_i unless $j < i$. The strong components of D that correspond to vertices in $SC(D)$ with in-degree zero, or to elements of $SC(D)$ with no couplings directed toward any of their vertices, are called the source or initial strong components of D . The strong components of D that correspond to vertices in $SC(D)$ with out-degree zero, or to elements of $SC(D)$ with no couplings directed away from any of their vertices, are referred to as sink, trap, or terminal strong components. The remaining strong components of D , which include elements of $SC(D)$ with at least one coupling directed away from and at least one coupling directed toward one of their vertices, are called transit or intermediate strong components of D [11],[13].

Each node of $SC(D)$ falls into one of the following categories:

- Source: a node in $SC(D)$ that has no incoming couplings
- Transit: a node in $SC(D)$ that has both incoming and outgoing couplings
- Sink/Trap: a node in $SC(D)$ that has no outgoing couplings

A compartmental model's digraph can always be decomposed into collections of sources, transits, traps, and couplings [13].

For example, the directed graph in Figure 5b. is the strong component digraph of the directed graph in Figure 5a.

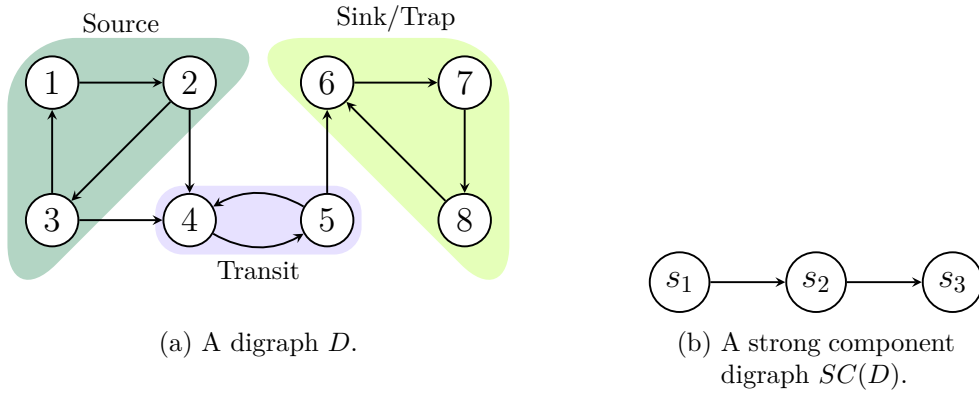


Figure 5.: A digraph D and its strong component digraph $SC(D)$.

The vertices s_1, s_2, s_3 are obtained by contracting the sets $\{1, 2, 3\}$, $\{4, 5\}$ and $\{6, 7, 8\}$ which correspond to the strong components of D . The digraph D has a source component, D_1 with vertices $V(D_1) = \{1, 2, 3\}$, a transit component D_2 with vertices $V(D_2) = \{4, 5\}$ and a trap component D_3 with vertices $V(D_3) = \{6, 7, 8\}$.

2.4. Compartmental models

Compartmental differential equations were first defined in physiology to describe systems in which there are flows of material between storage units called compartments. Compartmental systems form a subclass of non negative dynamical systems. The conservation law that drives the dynamics of such systems is the law of conservation of mass. A compartment stores an amount of some material that is homogeneous, which means that the material of a compartment is assumed to be homogeneous, any material entering it is instantaneously mixed with the material of the compartment [1], [3].

Non-negative and compartmental dynamical system models are usually derived from mass and energy balance considerations that involve dynamic states whose values are non-negative. These models are widespread in biology, chemistry, ecology, economics, genetics, medicine, sociology, and engineering, and play a key role in the understanding of these disciplines. Specifically, since such disciplines give rise to systems that have numerous input, state, and output properties related to conservation, dissipation, and transport of mass and energy, non-negative and compartmental models are conceptually simple yet remarkably effective in describing the essential phenomenological features of these dynamical systems [1].

Figure 6. represents the i -th compartment and its neighbors of an n compartment system. Let q_i represent the mass stored in compartment i . The arrows in Figure 6. indicate flows into and out of the compartment. The flows into the compartment from outside the system, or inflows, are represented by $I_i \geq 0$. The outflows to the environment and, therefore, out of the system represented by $F_{0,i}$. The transfers from compartment i to compartment $i + 1$ represented by $F_{i+1,i}$ and the transfers from $i + 1$ to i represented by $F_{i,i+1}$. All flows are non-negative [3].

The general equations for such a system are obtained by writing the instantaneous mass balance equations for each compartment

$$\dot{q}_i = \sum_{\substack{i \neq i+1 \\ i \neq i-1}} (-F_{i,i+1} + F_{i+1,i} - F_{i-1,i} + F_{i,i-1}) + I_i - F_{0,i}. \quad (3)$$

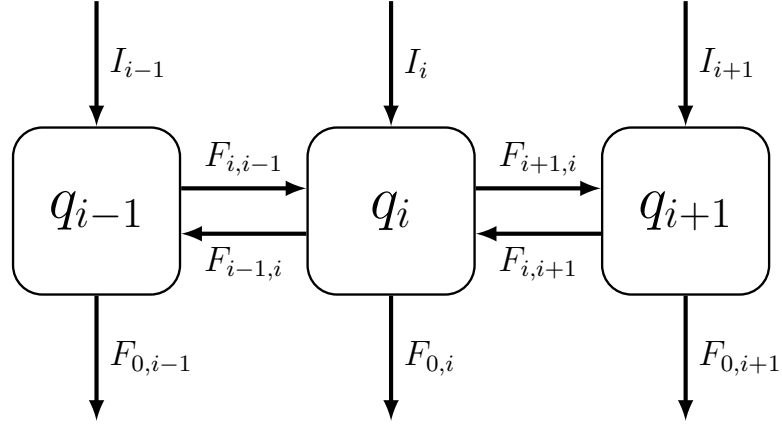


Figure 6.: A compartmental system with the possible flows [3].

2.4.1. Directed graph representation of compartmental models

In the context of compartmental models, nodes in a directed graph represent different compartments, while directed edges indicate the transitions or flows between these compartments or with the environment. Each edge is typically labeled with a rate or probability, which quantifies the speed at which individuals move from one state to another. The directed edge from compartment i to compartment j on Figure 7., labeled with k_{ij} , denotes the rate at which individuals transition from state i to state j . The directed edges to and from the environment represent the transition rates between a compartment and the environment, acting as inputs in the model. The flow from the environment into compartment i on Figure 7. is described by the on-ramp transition rate coefficient, k_{on_i} , where i denotes the compartment receiving the inflow. Conversely, the flow from compartment j to the environment on Figure 7. is represented by the off-ramp transition rate coefficient, k_{off_j} , where j indicates the compartment from which the outflow occurs.

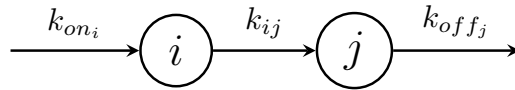


Figure 7.: Directed graph representation of a compartmental model.

The necessary and sufficient condition for the observability of linear compartmental models is well-known from the literature:

Theorem 1 *A compartmental model is completely observable if and only if a node in every trap is observed [13].*

2.5. State estimation

In control theory, a state estimator is a system that uses a model of a real system to provide an estimate of the internal state of the real system based on its input and output measurements. In practice, not all state variables can be measured directly. In this case, an estimate is needed to solve several tasks like process monitoring, fault detection, or feedback control. The estimation of state variables that cannot be measured is called observation. A state observer estimates state variables based on output measurements and input variables. If the state observer can unambiguously determine all state variables of the system, regardless of whether some state variables are available for direct measurement, it is called a full state observer [9].

2.5.1. Linear state estimation

Suppose a linear, time invariant system described by (1). To estimate the states of this system, an equivalent system known as an observer is constructed. The purpose of the observer is to provide an accurate estimation of all the states of the original system. The construction of an observer begins with the same equations previously introduced in (1), namely

$$\begin{aligned}\dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t), \\ \hat{y}(t) &= C\hat{x}(t).\end{aligned}$$

In order to set the error dynamics as required, the observer is additionally fed with the error signal between the outputs, i.e. $y(t) - \hat{y}(t)$, multiplied by a gain matrix L . The new system takes the form

$$\begin{aligned}\dot{\hat{x}}(t) &= (A - LC)\hat{x}(t) + Bu(t) + Ly(t), \\ \hat{y}(t) &= C\hat{x}(t).\end{aligned}\tag{4}$$

The use of the output feedback can be understood considering the approximation error. Let $e(t)$ denote the error of the approximation, i.e. $e(t) = x(t) - \hat{x}(t)$. By substituting the expressions for the actual state and the estimated state, the resulting error dynamics are given by

$$\dot{e}(t) = (A - LC)e(t).$$

If the feedback matrix L is chosen to have a stable system matrix $(A - LC)$, the error $e(t) \rightarrow 0$ and the approximation is accurate. In addition, depending on the value of L , we can set the convergence rate of the estimate [10].

The block diagram representation of the system described in (4) is shown in Figure 8.

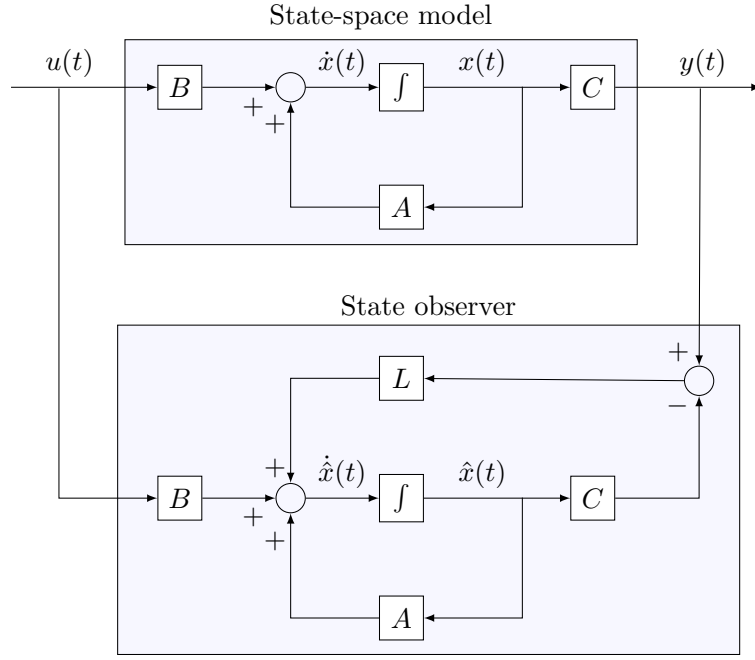


Figure 8.: Block diagram representation of linear state observer.

2.5.2. Nonlinear state estimation with Extended Kalman Filter

For nonlinear systems, a state estimator based on a linearized state-space model often provides poor-quality estimates or may even fail to converge. The Extended Kalman Filter (EKF) improves this by linearizing the system around the current state estimate and its uncertainty [14],[15].

The system and measurement equations in discrete time can be written as follows

$$\begin{aligned} x_k &= f(x_{k-1}, u_{k-1}, w_{k-1}), \\ z_k &= h(x_k, v_k), \end{aligned}$$

where w_k and v_k are uncorrelated zero mean white noise processes having known covariance matrices Q_k and R_k respectively [14].

In the studied model, the Q_k and R_k matrices will be used just for tuning purposes and not for noise filtering.

EKF algorithm

The process begins by specifying the initial values for the estimated states, $x_{0|0}$, and the initial covariance matrix, $P_{0|0}$.

The state transition matrix and observation matrix are defined as the following Jacobians

$$F_k = \left. \frac{\partial f}{\partial x} \right|_{\hat{x}_{k+1|k}}, \quad H_k = \left. \frac{\partial h}{\partial x} \right|_{\hat{x}_{k+1|k}}.$$

The prediction steps are expressed as

$$\begin{aligned} \hat{x}_{k+1|k} &= f(\hat{x}_{k|k}, u_k, 0), \\ P_{k+1|k} &= F_k P_{k|k} F_k^T + Q_k. \end{aligned}$$

The prediction equations project the state and covariance estimates from the previous time step to the current time step [14].

The update steps are expressed as

$$\begin{aligned} K_{k+1} &= P_{k+1|k} H_k^T \left(H_k P_{k+1|k} H_k^T + R_k \right)^{-1}, \\ \hat{x}_{k+1|k+1} &= \hat{x}_{k+1|k} + K_{k+1} \left(z_k - h(\hat{x}_{k+1|k}, 0) \right), \\ P_{k+1|k+1} &= (I - K_{k+1} H_k) P_{k+1|k}. \end{aligned}$$

The update equations correct the state and covariance estimates with the measurement z_k [14].

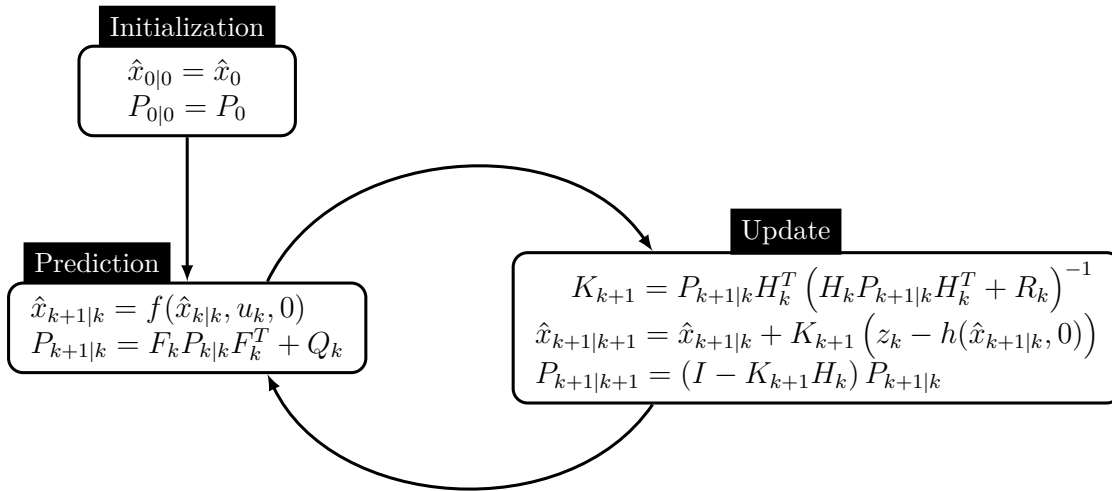


Figure 9.: Computation scheme of the Extended Kalman Filter.

2.5.3. Continuous time nonlinear state observer

Consider a nonlinear system described by the state-space representation

$$\begin{aligned}\dot{x} &= f(x, u), \\ y &= h(x).\end{aligned}$$

For a nonlinear system, the Luenberger observer dynamics can be written as

$$\dot{\hat{x}} = f(\hat{x}, u) + L(y - h(\hat{x}))$$

where L is the observer gain. The error dynamics, which represent the difference between the actual state x and the estimated state $\hat{x}$, are given by

$$\dot{e} = \dot{x} - \dot{\hat{x}} = f(x, u) - (f(\hat{x}, u) + L(y - h(\hat{x}))).$$

These error dynamics describe how the estimation error evolves over time, with the correction term $L(y - h(\hat{x}))$ driving the estimate closer to the true state.

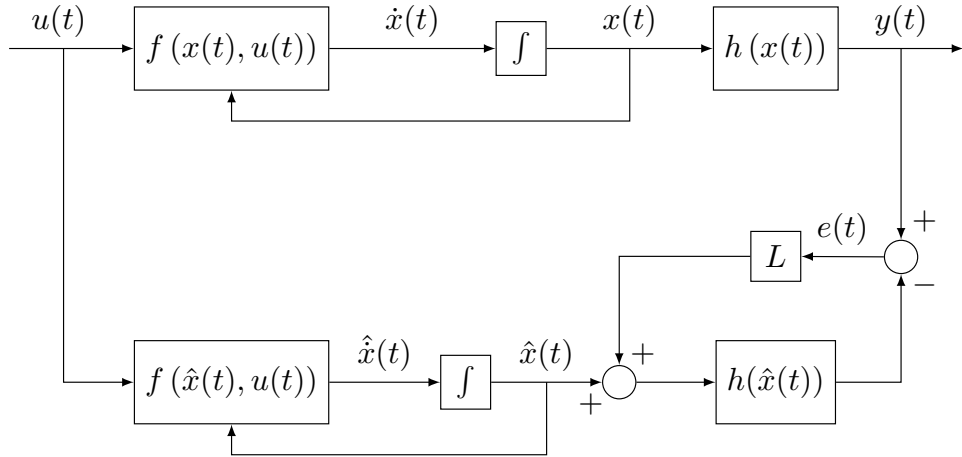


Figure 10.: Graphical representation of the continuous time nonlinear state observer.

The convergence of the EKF and nonlinear Luenberger observer is not guaranteed in theory due to several challenges. The EKF linearizes the system around the state estimate, which can fail in the case of highly nonlinear systems or poor initial estimates. Both methods rely on accurate system models and are sensitive to the characteristics of the noise. Additionally, convergence is typically local, meaning that specific conditions or assumptions are needed to ensure stability.

3. Traffic Reaction Model

The Traffic flow model is formally kinetic and it can be interpreted in a compartmental context. As a consequence, traffic networks can be formally represented as reaction graphs. The model can be equipped with on- and off- ramps in a physically meaningful way [4].

3.1. Chemical reaction networks

The basic building blocks of CRN models are so-called elementary reaction steps of the form $C \rightarrow C'$, which denotes chemical complex C transforming into complex C' [4]. The triplet $(\mathcal{S}, \mathcal{C}, \mathcal{R})$ is called a chemical reaction network, where the components are defined as

- A set of species: $\mathcal{S} = \{X_1, X_2, \dots, X_n\}$
- A set of complexes: $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$
The complexes are formal linear combinations of the species with nonnegative integer coefficients $\alpha_{ji} \in \overline{\mathbb{N}}_+^n$ called the stoichiometric coefficients, i.e. $C_j = \sum_{i=1}^n \alpha_{ji} X_i$.
- A set of reactions: $\mathcal{R} \subseteq \{(C_i, C_j) \mid C_i, C_j \in \mathcal{C}\}$
The ordered pair (C_i, C_j) corresponds to the reaction $C_i \rightarrow C_j$, where C_i is called the source (or reactant) complex and C_j is the product complex.

The complex composition matrix $Y \in \mathbb{R}^{n \times m}$ contains the stoichiometric coefficients as follows: $Y_{ij} = \alpha_{ji}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. The dynamics of a CRN can be described using the following set of ODEs

$$\dot{x} = \sum_{C_i, C_j \in \mathcal{R}} (Y_{.,j} - Y_{.,i}) \cdot R_{ij}(x),$$

where $x_i(t) \in \overline{\mathbb{R}}_+$ denotes the concentration of species X_i in time t , and $R_{ij} : \overline{\mathbb{R}}_+^n \rightarrow \overline{\mathbb{R}}_+$ is the reaction rate function corresponding to the reaction $C_i \rightarrow C_j$ [4].

3.2. Compartmental representation

The transition of vehicles between consecutive road segments can be formally described using reactions involving the number of vehicles N_i and available space slots S_i in each segment.



Equation (5) illustrates that when a vehicle transits from segment $i-1$ to segment i , the number of vehicles decreases by one in segment $i-1$ and increases by one in segment i . Simultaneously, the available space slots decrease by one unit in segment i and increase by one in segment $i-1$. It is important to note that this transition can occur only if there is at least one vehicle in segment $i-1$ and enough space for at least one vehicle in segment i . Equation (6) describes the transition from segment i to segment $i+1$ [4].

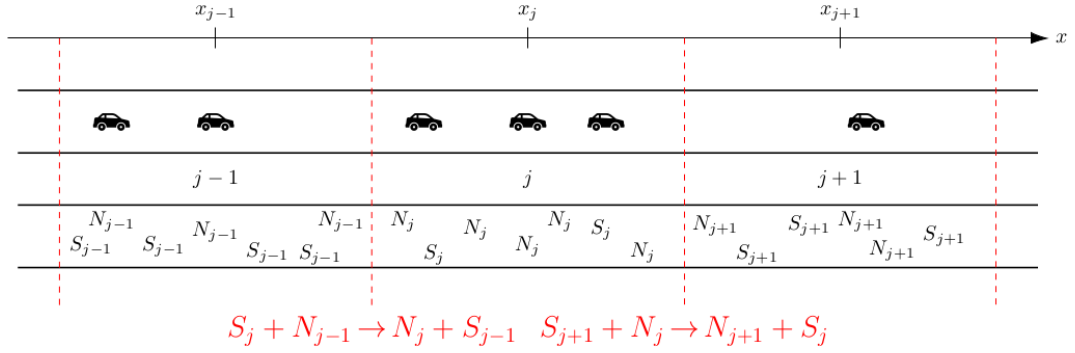


Figure 16.: Representation of the Traffic Reaction
Model interpretation of traffic flow [4].

The variables n_i and s_i denote the continuous number of vehicles and free spaces per unit length of the roadway, respectively. These quantities are analogous to molar concentration in chemistry, which is proportional to the number of molecules per unit volume [4].

The chemical reaction network that corresponds to equations (5) and (6) can be expressed as follows

$$\begin{aligned} \mathcal{S} &= \{N_{i-1}, N_i, N_{i+1}, S_{i-1}, S_i, S_{i+1}\}, \\ \mathcal{C} &= \{N_{i-1} + S_i, N_i + S_{i-1}, N_i + S_{i+1}, N_{i+1} + S_i\}, \\ \mathcal{R} &= \{(C_1, C_2), (C_3, C_4)\}. \end{aligned}$$

The mass-action type kinetic differential equations describing the change of n_i and s_i are the following

$$\dot{n}_i = k_{i-1,i}n_{i-1}s_i - k_{i,i+1}n_is_{i+1}, \quad (7)$$

$$\dot{s}_i = -k_{i-1,i}n_{i-1}s_i + k_{i,i+1}n_is_{i+1}, \quad (8)$$

with $n_i(0) \geq 0$ and $s_i(0) \geq 0$. It is easy to see that $\dot{n} + \dot{s} = 0$. Hence, the quantity $c_i = n_i + s_i$ serves as the first integral, or a conserved quantity, of the dynamics. This represents the maximum density of vehicles that can be accommodated in segment i .

Furthermore, it is evident that the conservation of c_i ensures that the solutions of equations (7) and (8) remain bounded. Substituting $s_i = c_i - n_i$ into equation (7) yields the following expression

$$\dot{n}_i = k_{i-1,i}n_{i-1}c_i - k_{i-1,i}n_{i-1}n_i - k_{i,i+1}n_ic_{i+1} + k_{i,i+1}n_in_{i+1} \quad (9)$$

where the parameters are listed in Table 1.

This alignment perfectly matches the physical basis of the model, as the reaction rate coefficients determining the transition rates between compartments are directly proportional to V_{max} and inversely proportional to ρ_{max} and Δx . It is important to emphasize that we only consider the model (9) with initial conditions that are non-negative, satisfying the condition $0 \leq n_i(0) \leq c_i$.

Parameters	Interpretation
V_{max}	Maximum velocity $\left[\frac{m}{s} \right]$
$n_i = \rho_i$	Vehicle density $\left[\frac{\text{vehicle}}{m} \right]$
$c_i = \rho_{max}$	Maximum vehicle capacity $\left[\frac{\text{vehicle}}{m} \right]$
$\omega = \frac{V_{max}}{c}$	Transition rate between compartments $\left[\frac{m^2}{s \cdot \text{vehicle}} \right]$
Δx	Compartment length $[m]$
$k_{i-1,i} = k_{i,i+1} = \frac{\omega}{\Delta x}$	Transition rate coefficient $\left[\frac{m}{s \cdot \text{vehicle}} \right]$

Table 1.: Physical parameters of Traffic Reaction Model.

For the modeling of on-ramps and off-ramps, respectively, it is straightforward to extend (5)-(6) with the following reactions



where $k_{on,i}$ and $k_{off,i}$ are the corresponding non-negative rate coefficients. The transformation described in equation (10) indicates that during an elementary step of the on-ramp process, an available space slot is utilized (filled in), resulting in an increase in the number of vehicles in segment i . Conversely, equation (11) characterizes the opposite process for the off-ramp, where a vehicle exits the segment, leading to the release of an available space slot [4].

The kinetic differential equations corresponding to (5)-(6) and (10)-(11) are written as

$$\dot{n}_i = k_{i-1,i}n_{i-1}s_i - k_{i,i+1}n_is_{i+1} + k_{on,i}s_i - k_{off,i}n_i, \quad (12)$$

$$\dot{s}_i = -k_{i-1,i}n_{i-1}s_i + k_{i,i+1}n_is_{i+1} - k_{on,i}s_i + k_{off,i}n_i, \quad (13)$$

with $n_i(0) \geq 0$ and $s_i(0) \geq 0$. It is visible from (12)-(13) that $s_i = c_i - n_i$ with c_i being constant in this configuration, too. Substituting this into (12) gives

$$\dot{n}_i = k_{i-1,i}c_in_{i-1} - k_{i,i+1}c_{i+1}n_i + k_{i,i+1}n_in_{i+1} - k_{i-1,i}n_{i-1}n_i + k_{on,i}(c_i - n_i) - k_{off,i}n_i$$

which matches with $n_i = \rho_i$, $c_i = \rho_{max}$, $k_{i-1,i} = k_{i,i+1} = \frac{\omega}{\Delta x}$, $k_{on,i} = 1_{on,i}u_{on}$, and $k_{off,i} = 1_{off,i}u_{off}$ [4].

4. Results

4.1. General model representation of TRM

Solving differential equations can be simplified if one can represent them in a uniform way. This is particularly important in systems with numerous compartments and multiple transitions between them, as the resulting equations can become complex and prone to errors.

It is assumed that the transition rates between compartments may be different. Denote the transition rate from the i -th to the j -th cell by k_{ij} . Equation (14) denotes an $N \times N$ matrix containing all the transition rates, where N is the number of compartments.

$$\underline{\underline{K}} = \begin{bmatrix} 0 & k_{12} & k_{13} & \dots & k_{1N} \\ k_{21} & 0 & k_{23} & \dots & k_{2N} \\ k_{31} & k_{32} & 0 & \dots & k_{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_{N1} & k_{N2} & k_{N3} & \dots & 0 \end{bmatrix}_{[N \times N]} \quad (14)$$

The states, on-ramp rate coefficients, and off-ramp rate coefficients can be represented as column vectors, as shown in equation (15). Here, n_i represents the number of vehicles in the i -th compartment, and $s_i = c - n_i$ denotes the number of free spaces in the i -th compartment, where c is the maximum vehicle capacity. The on-ramp rate coefficient for the i -th compartment is k_{on_i} , while k_{off_i} represents the off-ramp rate coefficient for the i -th compartment.

$$\underline{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ \vdots \\ n_N \end{bmatrix}_{[N \times 1]}, \quad \underline{s} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ \vdots \\ s_N \end{bmatrix}_{[N \times 1]}, \quad \underline{k}_{on} = \begin{bmatrix} k_{on_1} \\ k_{on_2} \\ k_{on_3} \\ \vdots \\ k_{on_N} \end{bmatrix}_{[N \times 1]}, \quad \underline{k}_{off} = \begin{bmatrix} k_{off_1} \\ k_{off_2} \\ k_{off_3} \\ \vdots \\ k_{off_N} \end{bmatrix}_{[N \times 1]}. \quad (15)$$

With the matrix in equation (14) and the vectors in equation (15), the general model can be represented as follows

$$\begin{bmatrix} \dot{n}_1 \\ \dot{n}_2 \\ \dot{n}_3 \\ \vdots \\ \dot{n}_N \end{bmatrix}_{[N \times 1]} = \left[\text{diag}(\underline{n})(-\underline{\underline{K}})\underline{s} + \text{diag}(\underline{s})\underline{\underline{K}}^T \underline{n} + \text{diag}(\underline{s}) \underline{k}_{on} - \text{diag}(\underline{n}) \underline{k}_{off} \right]. \quad (16)$$

In equation (16), the matrix on the right side includes outflow transitions to other compartments, inflow transitions from other compartments, inflow transitions from the environment, and outflow transitions to the environment, in that order.

Example

Consider the graph structure shown in Figure 17., which consists of three compartments and includes two inputs, an on-ramp in the first compartment and an off-ramp in the third compartment.

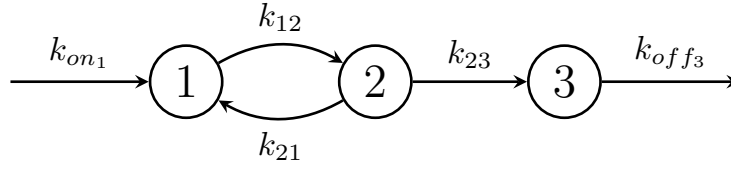


Figure 17.: Graph representation of a compartmental model with three compartments and two inputs.

The differential equations of the model are shown in Figure 17. can be written as follows

$$\begin{aligned}\dot{n}_1 &= -k_{12}n_1s_2 + k_{21}n_2s_1 + k_{on1}s_1, \\ \dot{n}_2 &= k_{12}n_1s_2 - k_{21}n_2s_1 - k_{23}n_2s_3, \\ \dot{n}_3 &= k_{23}n_2s_3 - k_{off3}n_3.\end{aligned}\tag{17}$$

Using the general model representation, the equations can be written as follows

$$\begin{aligned}\begin{bmatrix} \dot{n}_1 \\ \dot{n}_2 \\ \dot{n}_3 \end{bmatrix} &= \left(\begin{bmatrix} n_1 & 0 & 0 \\ 0 & n_2 & 0 \\ 0 & 0 & n_3 \end{bmatrix} \begin{bmatrix} 0 & -k_{12} & 0 \\ -k_{21} & 0 & -k_{23} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} + \right. \\ &\quad \left. + \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix} \begin{bmatrix} 0 & k_{21} & 0 \\ k_{12} & 0 & 0 \\ 0 & k_{23} & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} + \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix} \begin{bmatrix} k_{on1} \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} n_1 & 0 & 0 \\ 0 & n_2 & 0 \\ 0 & 0 & n_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ k_{off3} \end{bmatrix} \right).\end{aligned}\tag{18}$$

Performing the multiplications in (18) gives

$$\begin{bmatrix} \dot{n}_1 \\ \dot{n}_2 \\ \dot{n}_3 \end{bmatrix} = \left(\begin{bmatrix} -k_{12}n_1s_2 \\ -k_{21}n_2s_1 - k_{23}n_2s_3 \\ 0 \end{bmatrix} + \begin{bmatrix} k_{21}n_2s_1 \\ k_{12}n_1s_2 \\ k_{23}n_2s_3 \end{bmatrix} + \begin{bmatrix} k_{on1}s_1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ k_{off3}n_3 \end{bmatrix} \right).\tag{19}$$

It can be seen that when the matrices on the right side of (19) are added together, the resulting equations match those in (17).

4.2. Nonlinear observability analysis of TRMs

4.2.1. Observability without inputs

To examine the complete observability of a nonlinear TRM system without input, let's consider its basic graph structure shown in Figure 18. This graph includes a source, a transit, and a trap component, each containing a single compartment.

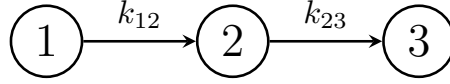


Figure 18.: Graph representation of a TRM with three compartments.

Following [13], let us first consider the linear compartmental differential equations corresponding to Figure 18., along with its Jacobian matrix and complete observability.

4.2.1.1. Complete observability in the linear case

The linear compartmental model corresponding to the graph in Figure 18. is given by

$$\begin{aligned}\dot{n}_1 &= -k_{12}n_1, \\ \dot{n}_2 &= k_{12}n_1 - k_{23}n_2, \\ \dot{n}_3 &= k_{23}n_2.\end{aligned}\tag{20}$$

The Jacobian matrix for the differential equation system (20) is given by

$$A = \begin{bmatrix} -k_{12} & 0 & 0 \\ k_{12} & -k_{23} & 0 \\ 0 & k_{23} & 0 \end{bmatrix}.$$

According to Theorem 1., the system will be observable only if the third compartment (i.e., one of the compartments in the trap) is measured. Thus, the output matrix should be $C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$. The corresponding observability matrix is as follows

$$\mathcal{O}_3 = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & k_{23} & 0 \\ k_{12}k_{23} & -k_{23}^2 & 0 \end{bmatrix}.$$

The rank of the observability matrix $\mathcal{O}_3$ is full, indicating that the model is completely observable when the compartment in the trap component is measured. However, if one measures the first compartment ($C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$) in the source component or the second compartment ($C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$) in the transit component of Figure 18., the rank of the observability matrix will not be complete, and the model will not be completely observable.

4.2.1.2. Complete observability and observability co-distribution of nonlinear TRM

The nonlinear differential equation system corresponding to the graph in Figure 18. is given by

$$\begin{aligned}\dot{n}_1 &= -k_{12}n_1s_2, \\ \dot{n}_2 &= k_{12}n_1s_2 - k_{23}n_2s_3, \\ \dot{n}_3 &= k_{23}n_2s_3.\end{aligned}\tag{21}$$

The Jacobian matrix for the differential equation system (21) is given by

$$D_{f(n)} = \begin{bmatrix} -k_{12}s_2 & k_{12}n_1 & 0 \\ k_{12}s_2 & -k_{12}n_1 - k_{23}s_3 & k_{23}n_2 \\ 0 & k_{23}s_3 & -k_{23}n_2 \end{bmatrix}.$$

To check the observability of the nonlinear model, co-distributions can be used.

Observing in trap

To build the observability co-distribution, one can start by measuring $y = n_3$, which represents the compartment in the trap component. The initial co-distribution is given by

$$\Omega_0 = \text{span} \left\{ \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \right\}.$$

Next, the co-distribution is further developed as follows

$$\Omega_0 D_{f(n)} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} D_{f(n)} = \begin{bmatrix} 0 & k_{23}s_3 & -k_{23}n_2 \end{bmatrix}.$$

After the development the updated co-distribution matrix is

$$\Omega_1 = \text{span} \left\{ \begin{bmatrix} 0 & 0 & 1 \\ 0 & k_{23}s_3 & -k_{23}n_2 \end{bmatrix} \right\}.$$

Since the rank has increased by one, the process can continue as the stopping criterion has not yet been reached. The next step involves

$$\begin{aligned}\Omega_1 D_{f(n)} &= \begin{bmatrix} 0 & k_{23}s_3 & -k_{23}n_2 \end{bmatrix} \cdot D_{f(n)} = \\ &= \begin{bmatrix} k_{12}k_{23}s_2s_3 & -k_{23}s_3(k_{23}n_2 + k_{12}n_1 + k_{23}s_3) & k_{23}^2n_2(n_2 + s_3) \end{bmatrix}.\end{aligned}$$

After this second development step, the co-distribution matrix is given by

$$\begin{aligned}\Omega_2 = \text{span} \Big\{ & \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \\ & \begin{bmatrix} 0 & k_{23}s_3 & -k_{23}n_2 \end{bmatrix}, \\ & \begin{bmatrix} k_{12}k_{23}s_2s_3 & -k_{23}s_3(k_{23}n_2 + k_{12}n_1 + k_{23}s_3) & k_{23}^2n_2(n_2 + s_3) \end{bmatrix} \Big\}.\end{aligned}\tag{22}$$

Since the rank of Ω_2 is full, the system can be completely observable by measuring the compartment in the trap component, only if the sum of the initial values is at least zero, but not more than the number of compartments in the trap multiplied by the maximum vehicle capacity. In mathematical form, this condition can be expressed as

$$c \geq \sum_i n_i(0) \geq 0.$$

If the sum of the initial values is too large, the trap compartment is filled ($n_3 = c$), causing the first and second columns of the (22) co-distribution matrix to become zero, which results in a loss of rank. In this case, the co-distribution matrix is

$$\begin{aligned}\Omega_2 = \text{span} \Big\{ & \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \\ & \begin{bmatrix} 0 & 0 & -k_{23}n_2 \end{bmatrix}, \\ & \begin{bmatrix} 0 & 0 & k_{23}^2n_2^2 \end{bmatrix} \Big\}.\end{aligned}$$

Observing in transit

If the co-distribution matrix is constructed by measuring $y = n_2$, which corresponds to the compartment in the transit component, it is given by

$$\begin{aligned}\Omega_2 = \text{span} \Big\{ & \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \\ & \begin{bmatrix} k_{12}s_2 & -k_{12}n_1 - k_{23}s_3 & k_{23}n_2 \end{bmatrix}, \\ & \begin{bmatrix} -k_{12}s_2(k_{12}s_2 + \sigma_1) & \sigma_1^2 + k_{12}^2n_1s_2 + k_{23}^2n_2s_3 & -k_{23}n_2(k_{23}n_2 + \sigma_1) \end{bmatrix} \Big\}\end{aligned}\tag{23}$$

where

$$\sigma_1 = k_{12}n_1 + k_{23}s_3.$$

Since the rank of Ω_2 is full, the system can be completely observable by measuring the compartment in the transit component, only if the sum is at least the number of compartments reachable from the source multiplied by the maximum vehicle capacity, but not more than the number of compartments that can reach the transit multiplied by the maximum vehicle capacity.

In mathematical form, this condition can be expressed as

$$2 \cdot c \geq \sum_i n_i(0) \geq c.$$

If the sum of the initial values is too low, the transit compartment is emptied ($n_2 = 0$), causing the third column of the (23) co-distribution matrix to become zero, which results in a loss of rank. In this case, the co-distribution matrix is

$$\Omega_2 = \text{span} \left\{ \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \right. \\ \left. \begin{bmatrix} k_{12}s_2 & -k_{12}n_1 - k_{23}s_3 & 0 \end{bmatrix}, \right. \\ \left. \begin{bmatrix} -k_{12}^2s_2^2 - k_{12}s_2\sigma_1 & \sigma_1^2 + k_{12}^2n_1s_2 + k_{23}^2n_2s_3 & 0 \end{bmatrix} \right\}$$

where

$$\sigma_1 = k_{12}n_1 + k_{23}s_3.$$

If the sum of the initial values is too large, the transit compartment is filled ($n_2 = c$), causing the first column of the (23) co-distribution matrix to become zero, which results in a loss of rank. In this case, the co-distribution matrix is

$$\Omega_2 = \text{span} \left\{ \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \right. \\ \left. \begin{bmatrix} 0 & -k_{12}n_1 - k_{23}s_3 & k_{23}n_2 \end{bmatrix}, \right. \\ \left. \begin{bmatrix} 0 & (k_{12}n_1 + k_{23}s_3)^2 + k_{23}^2n_2s_3 & -k_{23}^2n_2^2 - k_{23}n_2(k_{12}n_1 + k_{23}s_3) \end{bmatrix} \right\}.$$

Observing in source

If the co-distribution matrix is constructed by measuring $y = n_1$, which corresponds to the compartment in the source component, it is given by

$$\Omega_2 = \text{span} \left\{ \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \right. \\ \left. \begin{bmatrix} -k_{12}s_2 & k_{12}n_1 & 0 \end{bmatrix}, \right. \\ \left. \begin{bmatrix} k_{12}^2s_2(s_2 + n_1) & -k_{12}n_1(k_{12}s_2 + k_{12}n_1 + k_{23}s_3) & k_{12}k_{23}n_1n_2 \end{bmatrix} \right\}. \quad (24)$$

Since Ω_2 has a full rank, the system is completely observable by measuring the compartment in the source component, only if the sum of the initial values is greater or equal than the number of compartments reachable from the source multiplied by the maximum vehicle capacity, but less than or equal than the number of all compartments multiplied by the maximum vehicle capacity.

In mathematical form, this condition can be expressed as

$$3 \cdot c \geq \sum_i n_i(0) \geq 2 \cdot c.$$

If the sum of the initial values is too low, the source compartment is emptied ($n_1 = 0$), causing the second and third columns of the (24) co-distribution matrix to become zero, which results in a loss of rank. In this case, the co-distribution matrix is

$$\Omega_2 = \text{span} \left\{ \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -k_{12}s_2 & 0 & 0 \end{bmatrix}, \begin{bmatrix} k_{12}^2 s_2^2 & 0 & 0 \end{bmatrix} \right\}.$$

Analyzing the TRM graph structure in Figure 18. for both linear and nonlinear cases shows that the linear model is fully observable only if the compartment in the trap component is measured. In contrast, the nonlinear model becomes fully observable if any compartment is measured, provided the initial values meet the required constraints.

4.2.1.3. Multiple components from the same category

Multiple trap components

Consider the simple graph structure shown in Figure 19. This graph includes a source and two trap components, each containing a single compartment.

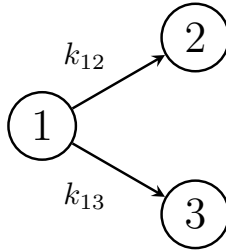


Figure 19.: Graph with two trap components.

The nonlinear differential equation system of the graph in Figure 19. is as follows

$$\begin{aligned} \dot{n}_1 &= -k_{12}n_1s_2 - k_{23}n_2s_3, \\ \dot{n}_2 &= k_{12}n_1s_2, \\ \dot{n}_3 &= k_{23}n_2s_3. \end{aligned} \tag{25}$$

If the co-distribution matrix is constructed by measuring $y = n_2$, it is given by

$$\Omega_2 = \text{span} \left\{ \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \right. \\ \begin{bmatrix} k_{12}s_2 & -k_{12}n_1 & 0 \end{bmatrix}, \\ \left. \begin{bmatrix} -k_{12}^2 n_1 s_2 - k_{12} (k_{12} s_2 + k_{13} s_3) s_2 & k_{12}^2 n_1^2 + s_2 k_{12}^2 n_1 & k_{12} k_{13} n_1 s_2 \end{bmatrix} \right\}.$$

When measuring compartment 2, the initial values shall be chosen so that their sum is less than or equal than the maximum vehicle density.

- If the sum of the initial values is less than or equal than the maximum vehicle density, then the third column of the co-distribution matrix is zeroed out because the source is empty ($n_1 = 0$).
- If the sum is greater, the third column is also zeroed out because the measured compartment is full ($n_2 = c$).

Therefore, if compartment 2 is measured, it is not possible to choose initial values that allow the estimation of all states.

If the co-distribution matrix is constructed by measuring $y = n_3$, it is given by

$$\Omega_2 = \text{span} \left\{ \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \right. \\ \begin{bmatrix} k_{13}s_3 & 0 & -k_{13}n_1 \end{bmatrix}, \\ \left. \begin{bmatrix} -k_{13}^2 n_1 s_3 - k_{13} (k_{12} s_2 + k_{13} s_3) s_3 & k_{12} k_{13} n_1 s_3 & k_{13}^2 n_1^2 + s_3 k_{13}^2 n_1 \end{bmatrix} \right\}.$$

When measuring compartment 3, the initial values shall be chosen so that their sum is less than or equal than the maximum vehicle density.

- If the sum of the initial values is less than or equal than the maximum vehicle density, then the third column of the co-distribution matrix is zeroed out because the source is empty ($n_1 = 0$).
- If the sum is greater, the third column is also zeroed out because the measured compartment is full ($n_3 = c$).

Therefore, if compartment 3 is measured, it is not possible to choose initial values that allow estimation of all states.

If the co-distribution matrix is constructed by measuring $y = [n_2, n_3]$, it is given by

$$\Omega_1 = \text{span} \left\{ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ k_{12}s_2 & -k_{12}n_1 & 0 \\ k_{13}s_3 & 0 & -k_{13}n_1 \end{bmatrix} \right\}.$$

Since Ω_1 has a full rank, the system is completely observable by measuring compartments in the trap components, only if the sum of the initial values is greater than or equal to zero but less than or equal than the number of compartments in the measured trap components multiplied by the maximum vehicle capacity.

If measurements are to be taken within a trap component in a model containing multiple trap components, at least one compartment must be measured in each trap component.

Multiple transit components

Consider another simple graph structure shown in Figure 20. This graph includes of a source, two transit, and a trap component, each containing a single compartment. The two transit components are not on the same path.

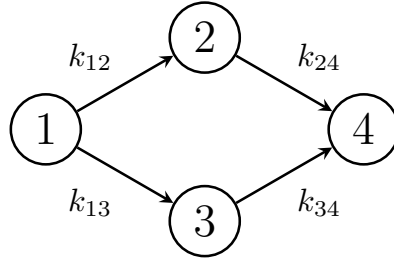


Figure 20.: Graph with two transit components.

The nonlinear differential equation system of the graph in Figure 20. is as follows

$$\begin{aligned} \dot{n}_1 &= -k_{12}n_1s_2 - k_{13}n_1s_3, \\ \dot{n}_2 &= k_{12}n_1s_2 - k_{24}n_2s_4, \\ \dot{n}_3 &= k_{13}n_1s_3 - k_{34}n_3s_4, \\ \dot{n}_4 &= k_{24}n_2s_4 + k_{34}n_3s_4. \end{aligned} \tag{26}$$

When measuring compartment 2, the initial values should be chosen so that their sum is not larger than twice the maximum vehicle capacity but not less than the maximum vehicle capacity. If these conditions are fulfilled, then the third column of the co-distribution matrix is zeroed out because the source is empty ($n_1 = 0$) and the trap is full ($n_4 = c$).

When measuring compartment 3, the second column of the co-distribution matrix is ze-

roed out because the source is empty ($n_1 = 0$) and the trap is full ($n_4 = c$).

If the co-distribution matrix is constructed by measuring $y = [n_2, n_3]$, it is given by

$$\Omega_1 = \text{span} \left\{ \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} k_{12}s_2 & -k_{12}n_1 - k_{24}s_4 & 0 & k_{24}n_2 \end{bmatrix}, \begin{bmatrix} k_{13}s_3 & 0 & -k_{13}n_1 - k_{34}s_4 & k_{34}n_3 \end{bmatrix} \right\}.$$

Since Ω_1 has a full rank, the system is completely observable by measuring the compartments in the transit components, only if the sum is not larger than three times the maximum vehicle capacity but not less than the maximum vehicle capacity.

If measurements are to be taken within a transit component in a model containing multiple transit components that are not on the same path, at least one compartment must be measured in each transit component.

Multiple source components

Consider another simple graph structure shown in Figure 21. This graph includes two source components and a trap component, each containing a single compartment.

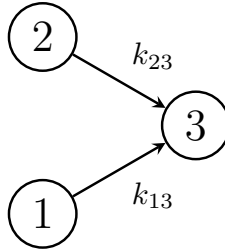


Figure 21.: Graph with two source components.

The nonlinear differential equation system of the graph in Figure 21. is as follows

$$\begin{aligned} \dot{n}_1 &= -k_{13}n_1s_3, \\ \dot{n}_2 &= -k_{23}n_2s_3, \\ \dot{n}_3 &= k_{13}n_1s_3 + k_{23}n_2s_3. \end{aligned} \tag{27}$$

If the co-distribution matrix is constructed by measuring $y = n_1$, it is given by

$$\Omega_2 = \text{span} \left\{ \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -k_{13}s_3 & 0 & k_{13}n_1 \end{bmatrix}, \begin{bmatrix} k_{13}^2s_3^2 + k_{13}^2n_1s_3 & k_{13}k_{23}n_1s_3 & -k_{13}n_1(k_{13}n_1 + k_{23}n_2) - k_{13}^2n_1s_3 \end{bmatrix} \right\}.$$

When measuring compartment 1, the initial values should be chosen so that their sum is not larger than twice the maximum vehicle capacity but not less than the maximum vehicle capacity. If these conditions are fulfilled, then the second column of the co-distribution matrix is zeroed out because the trap is full ($n_3 = c$).

If the co-distribution matrix is constructed by measuring $y = n_2$, it is given by

$$\Omega_2 = \text{span} \left\{ \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \right. \\ \left. \begin{bmatrix} 0 & -k_{23} s_3 & k_{23} n_2 \end{bmatrix}, \right. \\ \left. \begin{bmatrix} k_{13} k_{23} n_2 s_3 & k_{23}^2 s_3^2 + k_{23}^2 n_2 s_3 & -k_{23} n_2 (k_{13} n_1 + k_{23} n_2) - k_{23}^2 n_2 s_3 \end{bmatrix} \right\}.$$

When measuring compartment 2, the first column of the co-distribution matrix is also zeroed out because the trap is full ($n_3 = c$).

If the co-distribution matrix is constructed by measuring $y = [n_1, n_2]$, it is given by

$$\Omega_1 = \text{span} \left\{ \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \right. \\ \left. \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \right. \\ \left. \begin{bmatrix} -k_{13} s_3 & 0 & k_{13} n_1 \end{bmatrix}, \right. \\ \left. \begin{bmatrix} 0 & -k_{23} s_3 & k_{23} n_2 \end{bmatrix} \right\}.$$

Since Ω_1 has a full rank, the system is completely observable by measuring the compartments in the source components, only if the sum is not larger than three times the maximum vehicle capacity but not less than the maximum vehicle capacity.

If measurements are to be taken within a source component in a model containing multiple source components, at least one compartment must be measured in each source component.

4.2.1.4. Finding the minimum covering set

In the graph representation of the model, strong components can be compressed into a single node, forming a directed acyclic graph (DAG) [11]. The goal is to identify the smallest set of nodes that collectively cover all others, representing the minimal measurement points needed. These nodes correspond to the strongly connected components of the original graph. Measurement of a compartment within each component, together with appropriate initial values, allows the system to be fully observable.

In a DAG, coverage relationships between nodes can be determined using depth-first search (DFS). Performing a DFS from each node identifies its reachable nodes, and reversing the edges allows another DFS to find the nodes that can reach it. Combining these results for each node gives the set of nodes it can reach or be reached by. The objective is to find the smallest subset of nodes that collectively cover all others, equivalent to solving an NP-hard minimum set cover problem [16]. Exact solutions can be determined using integer linear programming (ILP), however, this approach becomes computationally intensive when applied to large graphs. Alternatively, a greedy algorithm provides a faster approximation, often yielding near-optimal solutions efficiently [12].

For example, consider the directed graph D in Figure 22. and its strong component directed graph $SC(D)$ in Figure 23a. The vertices s_1, s_2, s_3, s_4, s_5 are obtained by the sets $\{1, 2, 3\}, \{4, 5\}, \{6, 7\}, \{8, 9, 10\}$ and $\{11, 12\}$ which corresponds to the strong components of D . The digraph D has two source components, D_1 with vertices $V(D_1) = \{1, 2, 3\}$ and D_2 with vertices $V(D_2) = \{4, 5\}$, a transit component, D_3 with vertices $V(D_3) = \{6, 7\}$ and two trap components, D_4 with vertices $V(D_4) = \{8, 9, 10\}$ and D_5 with vertices $V(D_5) = \{11, 12\}$.

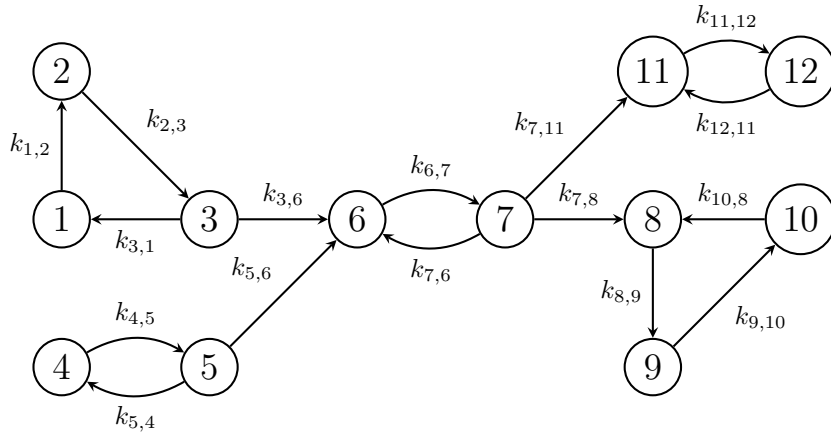


Figure 22.: A digraph D .

The transpose strong component digraph $SC(D)^T$ of $SC(D)$ can be seen in Figure 23b.

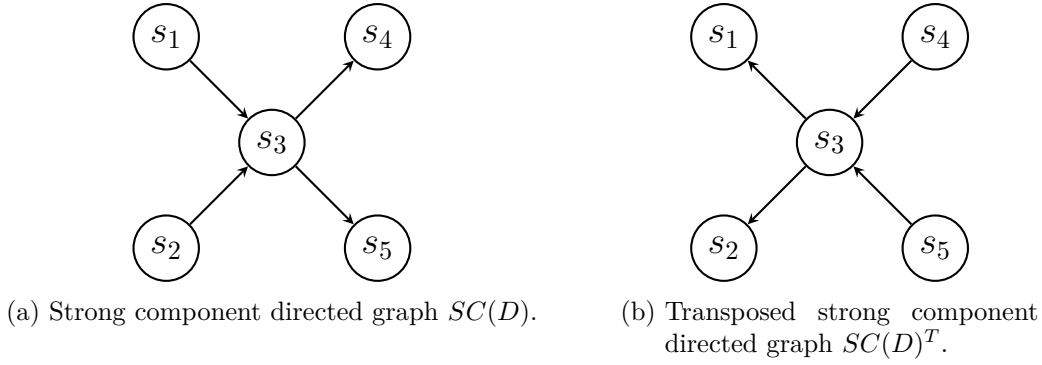


Figure 23.: The strong component digraph $SC(D)$ and the transpose strong component digraph $SC(D)^T$ of digraph D .

Node (n)	$\text{DFS}(SC(D), n)$	$\text{DFS}(SC(D)^T, n)$	$\text{DFS}(SC(D), n) \cup \text{DFS}(SC(D)^T, n)$
s_1	s_1, s_3, s_4, s_5	s_1	s_1, s_3, s_4, s_5
s_2	s_2, s_3, s_4, s_5	s_2	s_2, s_3, s_4, s_5
s_3	s_3, s_4, s_5	s_1, s_2, s_3	s_1, s_2, s_3, s_4, s_5
s_4	s_4	s_1, s_2, s_3, s_4	s_1, s_2, s_3, s_4
s_5	s_5	s_1, s_2, s_3, s_5	s_1, s_2, s_3, s_5

Table 2.: The nodes covered by each node.

In Table 2., the first column lists the examined vertices, while the second and third columns show the nodes that can reach or be reached by each vertex in the strong and transposed strong component digraphs. The last column combines these sets, revealing that node s_3 covers all other nodes. Thus, measuring either compartment 6 or 7 ensures complete observability which can be seen in Figure 24.

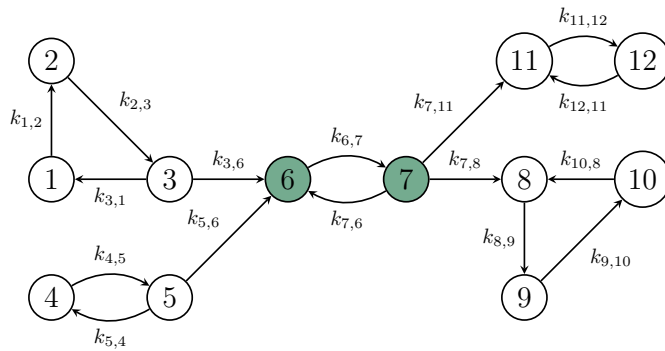


Figure 24.: A digraph D with potential measurement points (green nodes) ensuring full observability.

The sum of the initial values must satisfy the condition

$$7 \cdot c \geq \sum_i n_i(0) \geq 5 \cdot c.$$

For another example, consider the directed graph D on Figure 25. and its strong component directed graph $SC(D)$ on Figure 26a. The vertices $s_1, s_2, s_3, s_4, s_5, s_6, s_7$ are obtained by the sets $\{1, 2\}, \{3, 4\}, \{10, 11, 12\}, \{5, 6\}, \{7, 8, 9\}, \{13, 14, 15\}$ and $\{16, 17\}$ which corresponds to the strong components of D . The digraph D has three source components, D_1 with vertices $V(D_1) = \{1, 2\}$, D_2 with vertices $V(D_2) = \{3, 4\}$ and D_3 with vertices $V(D_3) = \{10, 11, 12\}$, a transit component, D_4 with vertices $V(D_4) = \{5, 6\}$ and three trap components, D_5 with vertices $V(D_5) = \{7, 8, 9\}$, D_6 with vertices $V(D_6) = \{13, 14, 15\}$ and D_7 with vertices $V(D_7) = \{16, 17\}$.

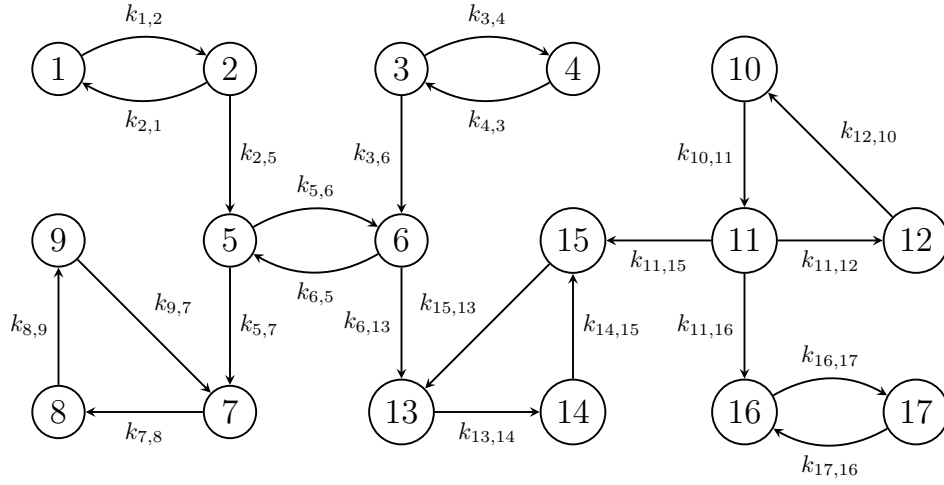


Figure 25.: A digraph D .

The transpose strong component graph $SC(D)^T$ of $SC(D)$ can be seen in Figure 26b.

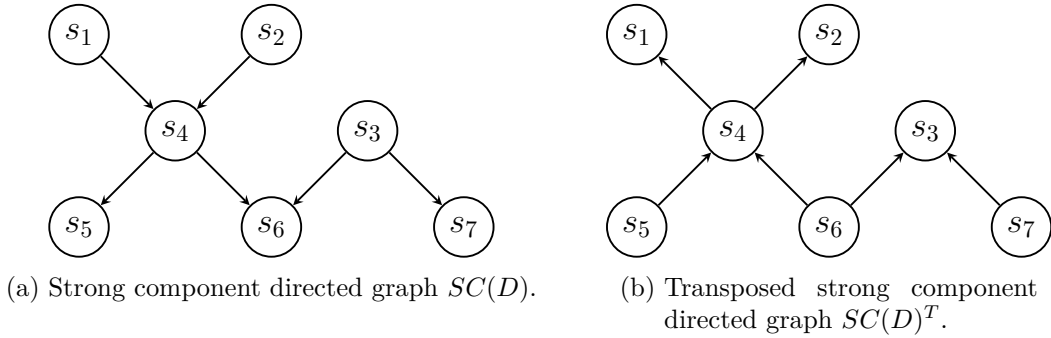


Figure 26.: The strong component digraph $SC(D)$ and the transpose strong component digraph $SC(D)^T$ of digraph D .

Node (n)	$\text{DFS}(SC(D), n)$	$\text{DFS}(SC(D)^T, n)$	$\text{DFS}(SC(D), n) \cup \text{DFS}(SC(D)^T, n)$
s_1	s_1, s_4, s_5, s_6	s_1	s_1, s_4, s_5, s_6
s_2	s_2, s_4, s_5, s_6	s_2	s_2, s_4, s_5, s_6
s_3	s_3, s_6, s_7	s_3	s_3, s_6, s_7
s_4	s_4, s_5, s_6	s_1, s_2, s_4	s_1, s_2, s_4, s_5, s_6
s_5	s_5	s_1, s_2, s_4, s_5	s_1, s_2, s_4, s_5
s_6	s_6	s_1, s_2, s_3, s_4, s_6	s_1, s_2, s_3, s_4, s_6
s_7	s_7	s_3, s_7	s_3, s_7

Table 3.: The nodes covered by each node.

In Table 3. the first column lists the examined vertices, while the second and third columns show the nodes that can reach or be reached by each vertex in the strong and transposed strong component digraphs. The last column combines these sets, revealing that nodes s_3 and s_4 or s_4 and s_7 cover all other nodes. Consequently, the system will be fully observable by measuring from component s_3 , selecting compartment 10, 11, or 12 as shown in Figure 27b., or from component s_4 , selecting compartment 5 or 6 as shown in Figure 27a.

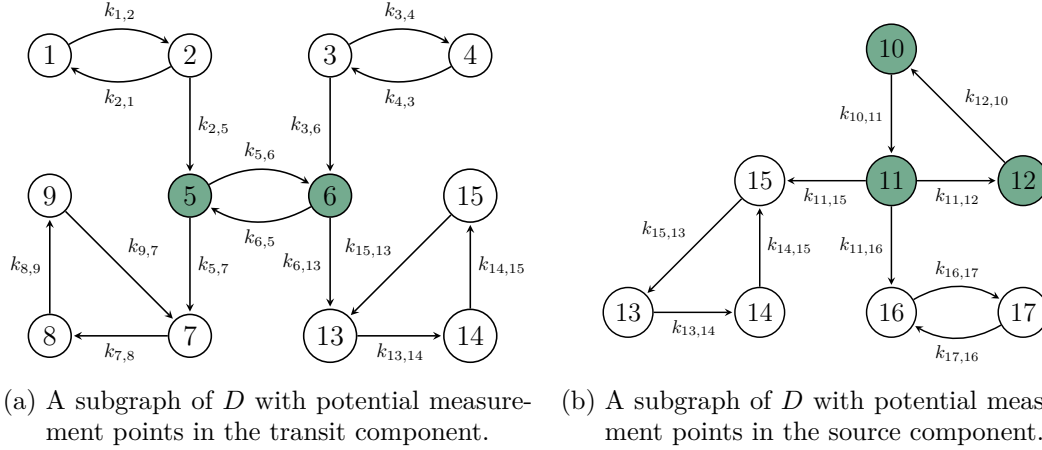


Figure 27.: Subgraphs of D with potential measurement points (green nodes) ensuring full observability.

The total initial values of the graph of Figure 27a. must satisfy the condition

$$8 \cdot c \geq \sum_i n_i(0) \geq 6 \cdot c,$$

and the total initial values of the graph of Figure 27b. must satisfy the condition

$$8 \cdot c \geq \sum_i n_i(0) \geq 5 \cdot c.$$

4.2.1.5. Conditions that must be met to estimate all states

For the input-free nonlinear compartmental model, the following conditions must be met to estimate all states by measuring in the fewest strong components:

- The observability co-distribution matrix must be full rank across the entire state space, given the specific measurement configuration and initial values under consideration.
- The smallest set of nodes that collectively cover all nodes in the strongly connected component directed acyclic graph (DAG) of the TRM's graph representation must be identified.
- The initial values should be chosen such that their sum in the subgraph covered by the measured compartment exceeds the number of reachable compartments from the measured component multiplied by the maximum vehicle capacity, but is less than the product of the sum of reachable compartments and the compartments in the measured component with the maximum vehicle capacity.

Assume that measuring a single compartment in the examined subgraph is sufficient. Let S_i denote the subgraph covered by the measured compartment i , N_m the number of compartments in the measured component, and N_r the number of compartments reachable from the measured component. The initial values $n_j(0)$ for the nodes j in these subgraphs should be chosen such that

$$c \cdot (N_r + N_m) \geq \sum_{j \in S_i} n_j(0) \geq c \cdot N_r$$

where c is the maximum vehicle capacity.

4.2.2. Observability with inputs

To examine the complete observability of a nonlinear TRM system with inputs, let's consider its basic graph structure shown in Figure 28. This graph includes two inputs, an on-ramp in the first compartment and an off-ramp in the third compartment.

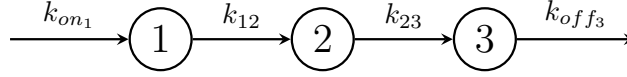


Figure 28.: Graph representation of a TRM with three compartments and inputs.

Following [13], let us first consider the linear compartmental differential equations corresponding to Figure 28., along with its Jacobian matrix and complete observability.

4.2.2.1. Complete observability in the linear case

The linear compartmental model corresponding to the graph in Figure 28. is given by

$$\begin{aligned}\dot{n}_1 &= -k_{12}n_1 + k_{on1}, \\ \dot{n}_2 &= k_{12}n_1 - k_{23}n_2, \\ \dot{n}_3 &= k_{23}n_2 - k_{off3}n_3.\end{aligned}\tag{28}$$

The Jacobian matrix for the differential equation system (28) is given by

$$A = \begin{bmatrix} -k_{12} & 0 & 0 \\ k_{12} & -k_{23} & 0 \\ 0 & k_{23} & -k_{off3} \end{bmatrix}.$$

According to Theorem 1., the system will be observable only if the third compartment (i.e., one of the compartments in the trap) is measured. Thus, the output matrix should be $C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$. The observability matrix is as follows

$$\mathcal{O}_3 = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & k_{23} & -k_{off3} \\ k_{12}k_{23} & -k_{23}(k_{23} + k_{off3}) & -k_{off3}^2 \end{bmatrix}.$$

The rank of the observability matrix $\mathcal{O}_3$ is full, indicating that the model is completely observable if the compartment in the trap component is measured. However, if one measures the first compartment ($C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$) in the source component or the second compartment ($C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$) in the transit component of Figure 28., the rank of the observability matrix will not be complete, and the model will not be completely observable.

4.2.2.2. Complete observability and observability co-distribution of nonlinear TRM with input

The nonlinear differential equation system corresponding to the graph in Figure 28. is given by

$$\begin{aligned}\dot{n}_1 &= -k_{12}n_1s_2 + k_{on_1}s_1, \\ \dot{n}_2 &= k_{12}n_1s_2 - k_{23}n_2s_3, \\ \dot{n}_3 &= k_{23}n_2s_3 - k_{off_3}n_3.\end{aligned}\tag{29}$$

The Jacobian matrices for the differential equation system (29) are as follows

$$\begin{aligned}D_{f(n)} &= \begin{bmatrix} -k_{12}s_2 & k_{12}n_1 & 0 \\ k_{12}s_2 & -k_{12}n_1 - k_{23}s_3 & k_{23}n_2 \\ 0 & k_{23}s_3 & -k_{23}n_2 \end{bmatrix}, \\ D_{g_{on_1}(n)} &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ D_{g_{off_3}(n)} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.\end{aligned}$$

Observing in trap

To build the observability co-distribution with inputs in the model, one can start by measuring $y = n_3$, which represents the compartment in the trap component. The initial co-distribution is given by

$$\Omega_0 = \text{span} \left\{ \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \right\}.$$

Next, the co-distribution is further developed as follows

$$\begin{aligned}\Omega_0 D_{f(n)} &= \begin{bmatrix} 0 & k_{23}s_3 & -k_{23}n_2 \end{bmatrix}, \\ \Omega_0 D_{g_{on_1}(n)} &= \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \\ \Omega_0 D_{g_{off_3}(n)} &= \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}.\end{aligned}$$

After the development the updated co-distribution matrix is

$$\begin{aligned}\Omega_1 = \text{span} \left\{ \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \right. \\ \begin{bmatrix} 0 & k_{23}s_3 & -k_{23}n_2 \end{bmatrix}, \\ \left. \begin{bmatrix} 0 & 0 & -1 \end{bmatrix} \right\}.\end{aligned}$$

Since the rank has increased by one, the process can continue as the stopping criterion has not yet been reached. The next step with the second vector of Ω_1 is

$$\begin{aligned}\Omega_1 Df(n) &= \begin{bmatrix} 0 & k_{23}s_3 & -k_{23}n_2 \end{bmatrix} \cdot Df(n) = \\ &= \begin{bmatrix} k_{12}k_{23}s_2s_3 & -k_{23}s_3(k_{23}n_2 + k_{12}n_1 + k_{23}s_3) & k_{23}^2n_2(n_2 + s_3) \end{bmatrix}, \\ \Omega_1 D_{g_{on_1}}(n) &= \begin{bmatrix} 0 & k_{23}s_3 & -k_{23}n_2 \end{bmatrix} \cdot D_{g_{on_1}}(n) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \\ \Omega_1 D_{g_{off_3}}(n) &= \begin{bmatrix} 0 & k_{23}s_3 & -k_{23}n_2 \end{bmatrix} \cdot D_{g_{off_3}}(n) = \begin{bmatrix} 0 & 0 & k_{23}n_2 \end{bmatrix}.\end{aligned}$$

The next step with the third vector of Ω_1 is

$$\begin{aligned}\Omega_1 Df(n) &= \begin{bmatrix} 0 & 0 & -k_{off_3} \end{bmatrix} \cdot Df(n) = \begin{bmatrix} 0 & -k_{23}s_3 & k_{23}n_2 \end{bmatrix}, \\ \Omega_1 D_{g_{on_1}}(n) &= \begin{bmatrix} 0 & 0 & -k_{off_3} \end{bmatrix} \cdot D_{g_{on_1}}(n) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \\ \Omega_1 D_{g_{off_3}}(n) &= \begin{bmatrix} 0 & 0 & -k_{off_3} \end{bmatrix} \cdot D_{g_{off_3}}(n) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.\end{aligned}$$

After the second development step, the co-distribution matrix is given by

$$\begin{aligned}\Omega_2 = \text{span} \Big\{ & \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \\ & \begin{bmatrix} 0 & k_{23}s_3 & -k_{23}n_2 \end{bmatrix}, \\ & \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}, \\ & \begin{bmatrix} k_{12}k_{23}s_2s_3 & -k_{23}s_3(k_{23}n_2 + k_{12}n_1 + k_{23}s_3) & k_{23}^2n_2(n_2 + s_3) \end{bmatrix}, \\ & \begin{bmatrix} 0 & 0 & k_{23}n_2 \end{bmatrix}, \\ & \begin{bmatrix} 0 & -k_{23}s_3 & k_{23}n_2 \end{bmatrix}, \\ & \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \Big\}.\end{aligned}$$

Since Ω_2 has full rank, the system is completely observable by measuring compartment 3 in Figure 28.

Observing in transit

If the co-distribution matrix is constructed by measuring $y = n_2$, which corresponds to the compartment in the transit component, is given by

$$\begin{aligned}\Omega_2 = \text{span} \Big\{ & \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \\ & \begin{bmatrix} k_{12}s_2 & -k_{12}n_1 - k_{23}s_3 & k_{23}n_2 \end{bmatrix}, \\ & \begin{bmatrix} -k_{12}^2s_2^2 - k_{12}s_2\sigma_1 & \sigma_1^2 + k_{12}^2n_1s_2 + k_{23}^2n_2s_3 & -k_{23}^2n_2^2 - k_{23}n_2\sigma_1 \end{bmatrix}, \\ & \begin{bmatrix} -k_{12}s_2 & 0 & 0 \end{bmatrix}, \\ & \begin{bmatrix} 0 & 0 & -k_{23}n_2 \end{bmatrix} \Big\}.\end{aligned}$$

where

$$\sigma_1 = k_{12} n_1 + k_{23} s_3.$$

Since Ω_2 has full rank, the system is completely observable by measuring compartment 2 in Figure 28.

Observing in source

If the co-distribution matrix is constructed by measuring $y = n_1$, which corresponds to the compartment in the source component, is given by

$$\begin{aligned} \Omega_2 = \text{span} \Big\{ & \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \\ & \begin{bmatrix} -k_{12}s_2 & k_{12}n_1 & 0 \end{bmatrix}, \\ & \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}, \\ & \begin{bmatrix} k_{12}^2 s_2 (s_2 + n_1) & -k_{12}^2 n_1 s_2 - k_{12} n_1 (k_{12} n_1 + k_{23} s_3) & k_{12} k_{23} n_1 n_2 \end{bmatrix}, \\ & \begin{bmatrix} k_{12}s_2 & 0 & 0 \end{bmatrix}, \\ & \begin{bmatrix} k_{12}s_2 & -k_{12}n_1 & 0 \end{bmatrix}, \\ & \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \Big\}. \end{aligned}$$

Since Ω_2 has full rank, the system is completely observable by measuring compartment 1 in Figure 28.

Including inputs in the model can improve observability because the function $g(n)$ can increase the rank of the co-distribution. When inputs are well-placed - meaning each source component has an on-ramp and each trap component has an off-ramp - the model avoids the limitations of an input-free model due to the continuous vehicle flow. Observability no longer depends on the initial values when inputs are present, and when measurements are needed for a component type with multiple instances, it is not necessary to measure each one.

For example, in Figure 29., the model includes 3 source components and 3 trap components, resulting in 3 on-ramps and 3 off-ramps.

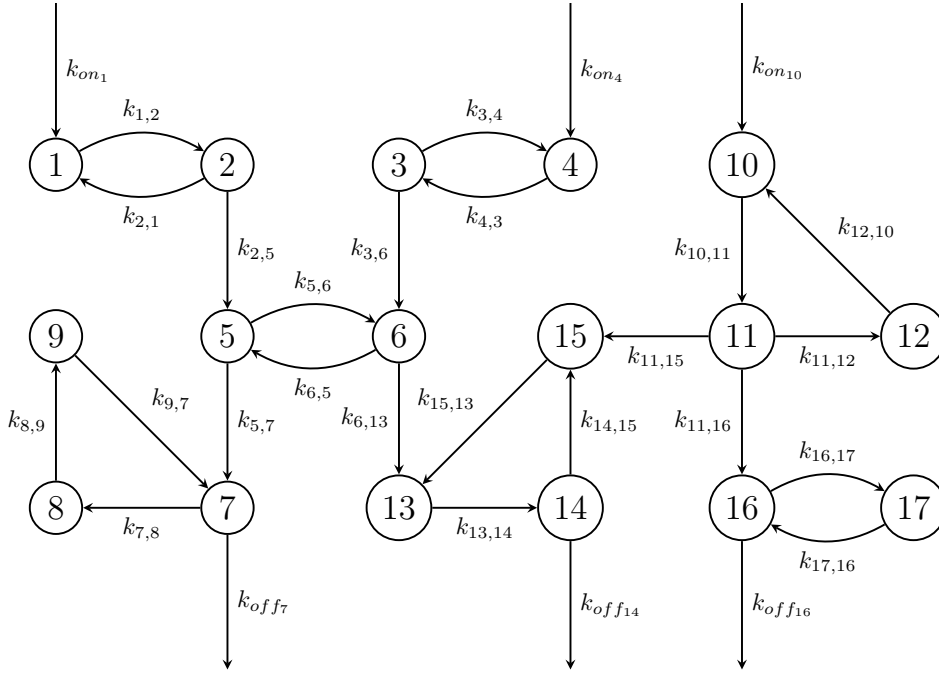


Figure 29.: A digraph D with on-ramps in the source components and with off-ramps in the trap components.

4.3. Observer design simulations

The physical parameters used for the simulation are listed in Table 4.

Parameters	Interpretation	Value
V_{max}	Maximum velocity $\left[\frac{m}{s}\right]$	25
c	Maximum vehicle capacity $\left[\frac{1m}{\text{average vehicle length}(\approx 4.5m)}\right]$	0.23
$\omega = \frac{V_{max}}{c}$	Transition rate between compartments $\left[\frac{m^2}{s \cdot \text{vehicle}}\right]$	108.7
Δx	Compartment length $[m]$	200
$k = \frac{\omega}{\Delta x}$	Transition rate coefficient $\left[\frac{m}{s \cdot \text{vehicle}}\right]$	0.5435

Table 4.: Physical parameters of the Traffic Reaction Model.

4.3.1. Continuous time nonlinear state observer without input

4.3.1.1. Simple model

The simulations were performed using the simple model illustrated in Figure 18.

Figure 30a. shows the model dynamics alongside the estimated dynamics when measurements are taken from the compartment in the trap component. The initial values for both the true states and the estimated states are the following

$$n(0) = \begin{bmatrix} n_1(0) \\ n_2(0) \\ n_3(0) \end{bmatrix} = \begin{bmatrix} 0.03 \\ 0.15 \\ 0.04 \end{bmatrix}, \quad \hat{n}(0) = \begin{bmatrix} \hat{n}_1(0) \\ \hat{n}_2(0) \\ \hat{n}_3(0) \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.05 \\ 0.04 \end{bmatrix}. \quad (30)$$

The sum of the initial values is given by

$$0.23 \geq \sum_{i=1}^3 n_i(0) = 0.22 \geq 0$$

which satisfies the constraints on the initial values. Each estimated state trajectory converges to its corresponding actual trajectory, so every state can be estimated.

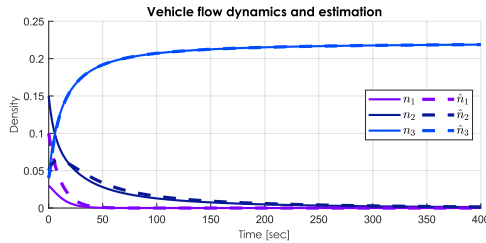
Figure 30b. shows the model dynamics alongside the estimated dynamics when measurements are taken from the compartment in the trap component. The initial values for both the true states and the estimated states are the following

$$n(0) = \begin{bmatrix} n_1(0) \\ n_2(0) \\ n_3(0) \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.1 \\ 0.22 \end{bmatrix}, \quad \hat{n}(0) = \begin{bmatrix} \hat{n}_1(0) \\ \hat{n}_2(0) \\ \hat{n}_3(0) \end{bmatrix} = \begin{bmatrix} 0.12 \\ 0.05 \\ 0.22 \end{bmatrix}. \quad (31)$$

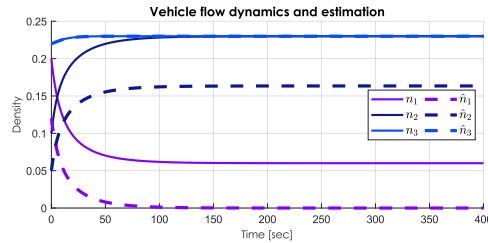
The sum of the initial values is given by

$$0.23 \not\geq \sum_{i=1}^3 n_i(0) = 0.52 \geq 0$$

indicating that the sum falls outside the desired range. As a result, not all estimated state trajectories converge to the actual trajectory because the measured trap component is filled.



(a) Measurement in the trap component with well-chosen initial values.



(b) Measurement in the trap component with poorly chosen initial values.

Figure 30.: Simulation of state estimation when measuring in trap component.

Figure 31a. shows the model dynamics alongside the estimated dynamics when measurements are taken from the compartment in the transit component. The initial values for both the true states and the estimated states are the following

$$n(0) = \begin{bmatrix} n_1(0) \\ n_2(0) \\ n_3(0) \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.06 \\ 0.13 \end{bmatrix}, \quad \hat{n}(0) = \begin{bmatrix} \hat{n}_1(0) \\ \hat{n}_2(0) \\ \hat{n}_3(0) \end{bmatrix} = \begin{bmatrix} 0.21 \\ 0.06 \\ 0 \end{bmatrix}. \quad (32)$$

The sum of the initial values is given by

$$0.46 \geq \sum_{i=1}^3 n_i(0) = 0.29 \geq 0.23$$

which satisfies the constraints on the initial values. Each estimated state trajectory converges to its corresponding actual trajectory, so every state can be estimated.

Figure 31b. shows the model dynamics alongside the estimated dynamics when measurements are taken from the compartment in the transit component. The initial values for

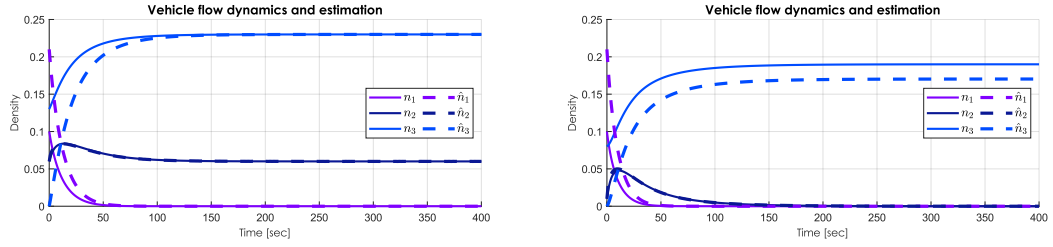
both the true states and the estimated states are the following

$$n(0) = \begin{bmatrix} n_1(0) \\ n_2(0) \\ n_3(0) \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.01 \\ 0.08 \end{bmatrix}, \quad \hat{n}(0) = \begin{bmatrix} \hat{n}_1(0) \\ \hat{n}_2(0) \\ \hat{n}_3(0) \end{bmatrix} = \begin{bmatrix} 0.21 \\ 0.01 \\ 0 \end{bmatrix}. \quad (33)$$

The sum of the initial values is given by

$$0.46 \geq \sum_{i=1}^3 n_i(0) = 0.19 \not\geq 0.23$$

indicating that the sum falls outside the desired range. As a result, not all estimated state trajectories converge to the actual trajectory because the measured transit component is emptied.



(a) Measurement in the transit component with well-chosen initial values.

(b) Measurement in the transit component with poorly chosen initial values.

Figure 31.: Simulation of state estimation when measuring in transit component.

Figure 32a. shows the model dynamics alongside the estimated dynamics when measurements are taken from the compartment in the source component. The initial values for both the true states and the estimated states are the following

$$n(0) = \begin{bmatrix} n_1(0) \\ n_2(0) \\ n_3(0) \end{bmatrix} = \begin{bmatrix} 0.21 \\ 0.17 \\ 0.12 \end{bmatrix}, \quad \hat{n}(0) = \begin{bmatrix} \hat{n}_1(0) \\ \hat{n}_2(0) \\ \hat{n}_3(0) \end{bmatrix} = \begin{bmatrix} 0.21 \\ 0.15 \\ 0.0 \end{bmatrix}. \quad (34)$$

The sum of the initial values is given by

$$0.69 \geq \sum_{i=1}^3 n_i(0) = 0.5 \geq 0.46$$

which satisfies the constraints on the initial values. Each estimated state trajectory converges to its corresponding actual trajectory, so every state can be estimated.

Figure 32b. shows the model dynamics alongside the estimated dynamics when measurements are taken from the compartment in the source component. The initial values

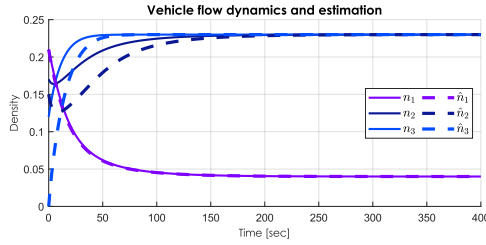
for both the true states and the estimated states are the following

$$n(0) = \begin{bmatrix} n_1(0) \\ n_2(0) \\ n_3(0) \end{bmatrix} = \begin{bmatrix} 0.05 \\ 0.2 \\ 0.18 \end{bmatrix}, \quad \hat{n}(0) = \begin{bmatrix} \hat{n}_1(0) \\ \hat{n}_2(0) \\ \hat{n}_3(0) \end{bmatrix} = \begin{bmatrix} 0.05 \\ 0.02 \\ 0.08 \end{bmatrix}. \quad (35)$$

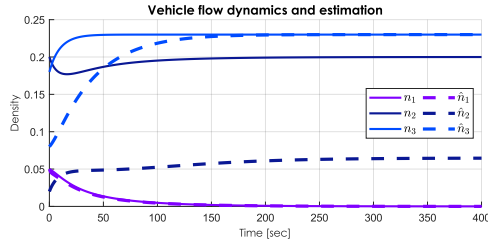
The sum of the initial values is given by

$$0.69 \geq \sum_{i=1}^3 n_i(0) = 0.43 \not\geq 0.46$$

indicating that the sum falls outside the desired range. As a result, not all estimated state trajectories converge to the actual trajectory because the measured source component is emptied.



(a) Measurement in the source component with well-chosen initial values.



(b) Measurement in the source component with poorly chosen initial values.

Figure 32.: Simulation of state estimation when measuring in source component.

4.3.1.2. Multiple components from the same category

The following simulations were performed using the model illustrated in Figure 19.

Figure 33a. and Figure 33b. illustrate the model dynamics and estimated dynamics when measurements are taken from the second and third compartments, respectively. The initial values of the states and estimated states, with measurements taken from the second compartment ($y = n_2$), are as follows

$$n(0) = \begin{bmatrix} n_1(0) \\ n_2(0) \\ n_3(0) \end{bmatrix} = \begin{bmatrix} 0.13 \\ 0.01 \\ 0.02 \end{bmatrix}, \quad \hat{n}(0) = \begin{bmatrix} \hat{n}_1(0) \\ \hat{n}_2(0) \\ \hat{n}_3(0) \end{bmatrix} = \begin{bmatrix} 0.21 \\ 0.01 \\ 0.14 \end{bmatrix}. \quad (36)$$

The initial values of the states and estimated states, with measurements taken from the third compartment ($y = n_3$), are as follows

$$n(0) = \begin{bmatrix} n_1(0) \\ n_2(0) \\ n_3(0) \end{bmatrix} = \begin{bmatrix} 0.15 \\ 0.02 \\ 0.03 \end{bmatrix}, \quad \hat{n}(0) = \begin{bmatrix} \hat{n}_1(0) \\ \hat{n}_2(0) \\ \hat{n}_3(0) \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.06 \\ 0.03 \end{bmatrix}. \quad (37)$$

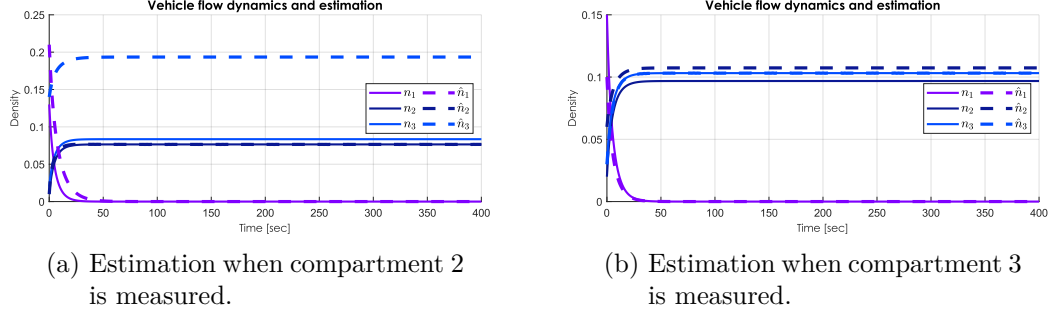


Figure 33.: Estimation when compartments 2 and 3 are measured separately.

Figure 34. shows the model dynamics and estimated dynamics when measurements are taken from both the second and third compartments ($y = [n_2, n_3]$). The initial values of the states and estimated states are as follows

$$n(0) = \begin{bmatrix} n_1(0) \\ n_2(0) \\ n_3(0) \end{bmatrix} = \begin{bmatrix} 0.08 \\ 0.04 \\ 0.01 \end{bmatrix}, \quad \hat{n}(0) = \begin{bmatrix} \hat{n}_1(0) \\ \hat{n}_2(0) \\ \hat{n}_3(0) \end{bmatrix} = \begin{bmatrix} 0.21 \\ 0.04 \\ 0.01 \end{bmatrix}. \quad (38)$$

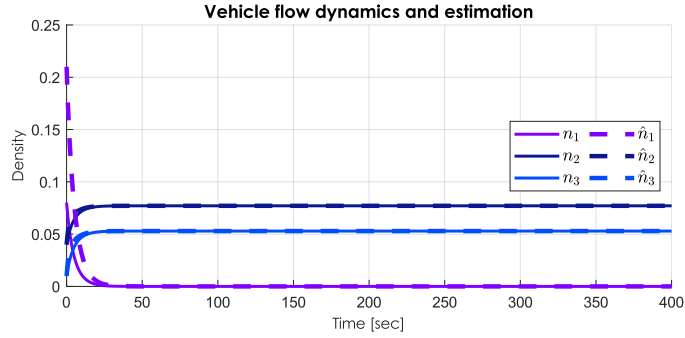


Figure 34.: Estimation when compartments 2 and 3 are measured together.

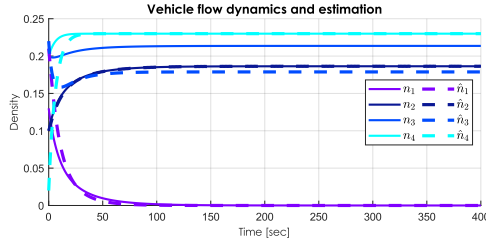
The following simulations were performed using the model illustrated in Figure 20.

Figure 35a. and Figure 35b. illustrate the model dynamics and estimated dynamics when measurements are taken from the second and third compartments, respectively. The initial values of the states and estimated states, with measurements taken from the second compartment ($y = n_2$), are as follows

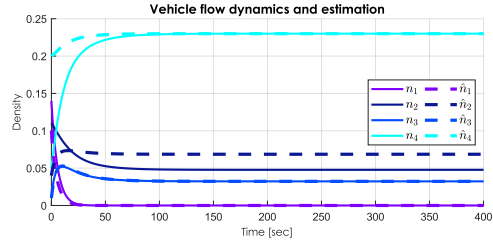
$$n(0) = \begin{bmatrix} n_1(0) \\ n_2(0) \\ n_3(0) \\ n_4(0) \end{bmatrix} = \begin{bmatrix} 0.13 \\ 0.01 \\ 0.2 \\ 0.2 \end{bmatrix}, \quad \hat{n}(0) = \begin{bmatrix} \hat{n}_1(0) \\ \hat{n}_2(0) \\ \hat{n}_3(0) \\ \hat{n}_4(0) \end{bmatrix} = \begin{bmatrix} 0.21 \\ 0.01 \\ 0.22 \\ 0.02 \end{bmatrix}. \quad (39)$$

The initial values of the states and estimated states, with measurements taken from the third compartment ($y = n_3$), are as follows

$$n(0) = \begin{bmatrix} n_1(0) \\ n_2(0) \\ n_3(0) \\ n_4(0) \end{bmatrix} = \begin{bmatrix} 0.14 \\ 0.11 \\ 0.01 \\ 0.05 \end{bmatrix}, \quad \hat{n}(0) = \begin{bmatrix} \hat{n}_1(0) \\ \hat{n}_2(0) \\ \hat{n}_3(0) \\ \hat{n}_4(0) \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.04 \\ 0.01 \\ 0.2 \end{bmatrix}. \quad (40)$$



(a) Estimation when compartment 2 is measured.



(b) Estimation when compartment 3 is measured.

Figure 35.: Estimation when compartments 2 and 3 are measured separately.

Figure 36. shows the model dynamics and estimated dynamics when measurements are taken from both the second and third compartments ($y = [n_2, n_3]$). The initial values of the states and estimated states are as follows

$$n(0) = \begin{bmatrix} n_1(0) \\ n_2(0) \\ n_3(0) \\ n_4(0) \end{bmatrix} = \begin{bmatrix} 0.18 \\ 0.04 \\ 0.09 \\ 0.05 \end{bmatrix}, \quad \hat{n}(0) = \begin{bmatrix} \hat{n}_1(0) \\ \hat{n}_2(0) \\ \hat{n}_3(0) \\ \hat{n}_4(0) \end{bmatrix} = \begin{bmatrix} 0.21 \\ 0.04 \\ 0.09 \\ 0.12 \end{bmatrix}. \quad (41)$$

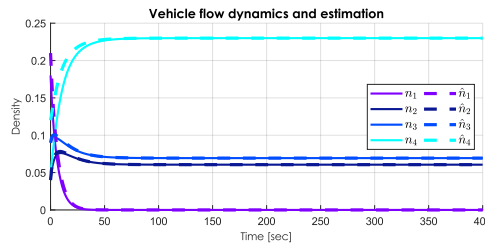


Figure 36.: Estimation when compartments 2 and 3 are measured together.

The following simulations were performed using the model illustrated in Figure 21.

Figure 37a. and Figure 37b. illustrate the model dynamics and estimated dynamics when measurements are taken from the first and second compartments, respectively.

The initial values of the states and estimated states, with measurements taken from the first compartment ($y = n_1$), are as follows

$$n(0) = \begin{bmatrix} n_1(0) \\ n_2(0) \\ n_3(0) \end{bmatrix} = \begin{bmatrix} 0.13 \\ 0.01 \\ 0.02 \end{bmatrix}, \quad \hat{n}(0) = \begin{bmatrix} \hat{n}_1(0) \\ \hat{n}_2(0) \\ \hat{n}_3(0) \end{bmatrix} = \begin{bmatrix} 0.13 \\ 0.14 \\ 0.14 \end{bmatrix}. \quad (42)$$

The initial values of the states and estimated states, with measurements taken from the second compartment ($y = n_2$), are as follows

$$n(0) = \begin{bmatrix} n_1(0) \\ n_2(0) \\ n_3(0) \end{bmatrix} = \begin{bmatrix} 0.15 \\ 0.12 \\ 0.15 \end{bmatrix}, \quad \hat{n}(0) = \begin{bmatrix} \hat{n}_1(0) \\ \hat{n}_2(0) \\ \hat{n}_3(0) \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.12 \\ 0.12 \end{bmatrix}. \quad (43)$$

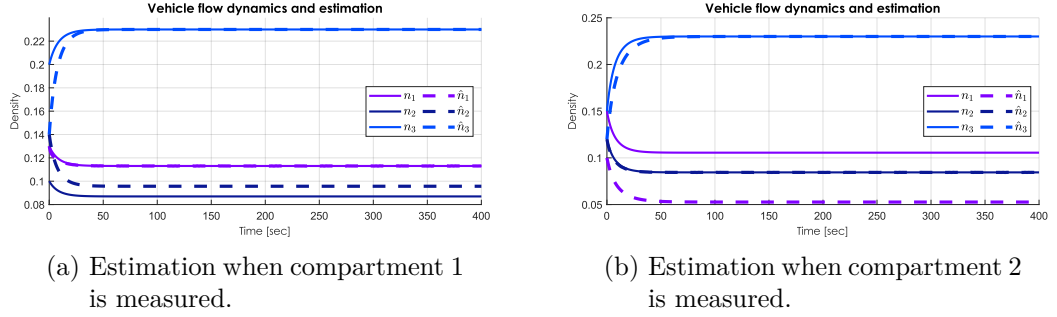


Figure 37.: Estimation when compartments 1 and 2 are measured separately.

Figure 38. shows the model dynamics and estimated dynamics when measurements are taken from both the second and third compartments ($y = [n_1, n_2]$). The initial values of the states and estimated states are as follows

$$n(0) = \begin{bmatrix} n_1(0) \\ n_2(0) \\ n_3(0) \end{bmatrix} = \begin{bmatrix} 0.18 \\ 0.04 \\ 0.11 \end{bmatrix}, \quad \hat{n}(0) = \begin{bmatrix} \hat{n}_1(0) \\ \hat{n}_2(0) \\ \hat{n}_3(0) \end{bmatrix} = \begin{bmatrix} 0.18 \\ 0.04 \\ 0 \end{bmatrix} \quad (44)$$

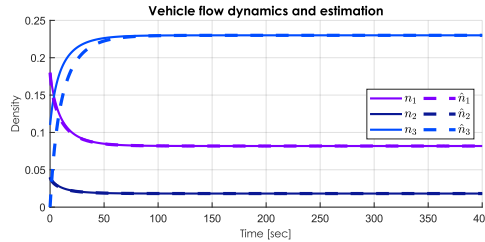


Figure 38.: Estimation when compartments 1 and 2 are measured together.

4.3.1.3. General model

The following simulations were performed using the model illustrated in Figure 22. Figure 39a. and Figure 39b. illustrate the model dynamics and estimated dynamics when measurements are taken from the sixth compartment. These figures demonstrate cases with well-chosen and poorly chosen initial value sums, respectively. The initial values of the states and estimated states with well-chosen initial value sums, with measurements taken from the sixth compartment ($y = n_6$), are as follows

$$\begin{aligned} n(0)^T &= [0.0511 \quad 0.2003 \quad 0.0475 \quad 0.2113 \quad 0.1123 \quad 0.1407 \\ &\quad 0.1762 \quad 0.1192 \quad 0.0683 \quad 0.0432 \quad 0.0186 \quad 0.1698], \\ \hat{n}(0)^T &= [0.1015 \quad 0.0364 \quad 0.2024 \quad 0.0630 \quad 0.0953 \quad 0.0681 \\ &\quad 0.1446 \quad 0.1334 \quad 0.1380 \quad 0.0611 \quad 0.0655 \quad 0.0583]. \end{aligned}$$

The sum of the initial values is given by

$$7 \cdot c = 1.61 \geq \sum_{i=1}^{12} n_i(0) = 1.3584 \geq 5 \cdot c = 1.15.$$

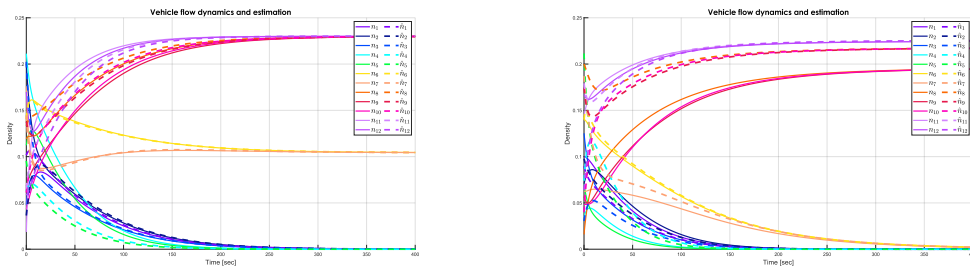
The initial values of the states and estimated states with poorly chosen initial value sums, with measurements taken from the sixth compartment ($y = n_6$), are as follows

$$\begin{aligned} n(0)^T &= [0.0996 \quad 0.0707 \quad 0.1252 \quad 0.0345 \quad 0.0709 \quad 0.1400 \\ &\quad 0.0627 \quad 0.0160 \quad 0.0569 \quad 0.0485 \quad 0.1444 \quad 0.1673], \\ \hat{n}(0)^T &= [0.0897 \quad 0.0977 \quad 0.0306 \quad 0.0740 \quad 0.2114 \quad 0.0039 \\ &\quad 0.1870 \quad 0.1999 \quad 0.1746 \quad 0.0401 \quad 0.1791 \quad 0.0641]. \end{aligned}$$

The sum of the initial values is given by

$$7 \cdot c = 1.61 \geq \sum_{i=1}^{12} n_i(0) = 1.0367 \not\geq 5 \cdot c = 1.15.$$

means that the transit component is emptied and not all the estimated state trajectory converges to its corresponding actual trajectory.



(a) Estimation when compartment 6 is measured with well-chosen initial values. (b) Estimation when compartment 6 is measured with poorly chosen initial values.

Figure 39.: Estimations when compartment 6 is measured with different initial values.

4.3.2. EKF observer without input

The tuning parameters for the EKF simulations are as follows:

- Initial Covariance Estimate: The matrix $P = I_N$, where I_N is the $N \times N$ identity matrix.
- Process Noise and Covariance: The process noise is $w = 10^{-3}$ incorporated exclusively into the covariance matrix Q defined as $Q = w \cdot I_N$.
- Observation Noise and Covariance: The observation noise is $v = 10^{-4}$ incorporated exclusively into the covariance matrix R defined as $R = v \cdot I_N$.

The simulations in Figure 40. based on the graph in Figure 18. , illustrate that not all trajectories converge to their corresponding paths, even with well-chosen initial values and a full-rank co-distribution matrix. Figure 40a. and Figure 40b. show the simulations of the EKF estimations starting from the initial values (30) and (31), Figure 40c. and Figure 40d. show the simulations of the EKF estimations starting from the initial values (32) and (33) and Figure 40e. and Figure 40f. show the simulations of the EKF estimations starting from the initial values (34) and (35).

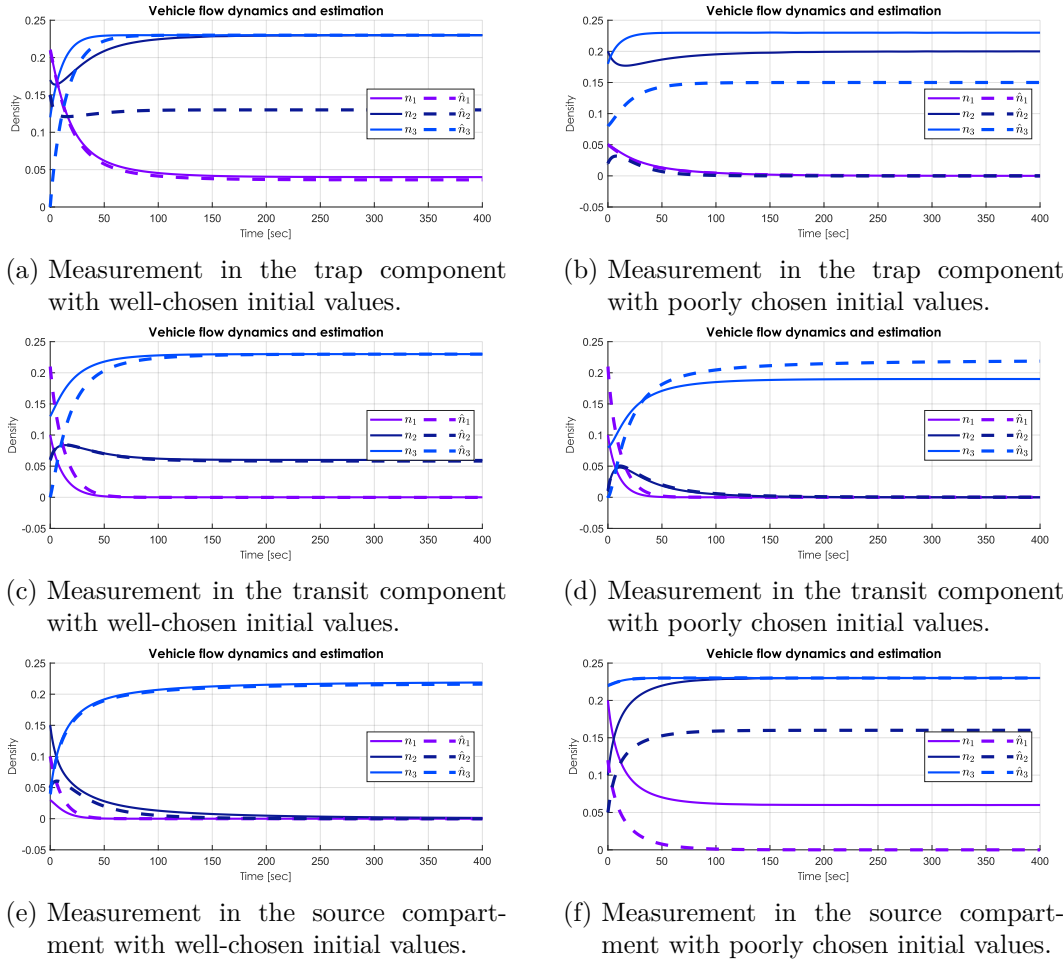
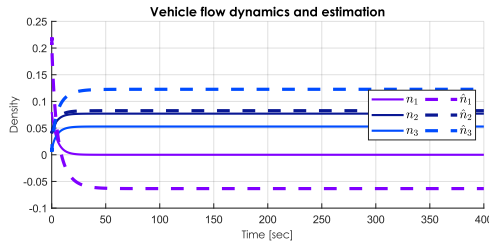


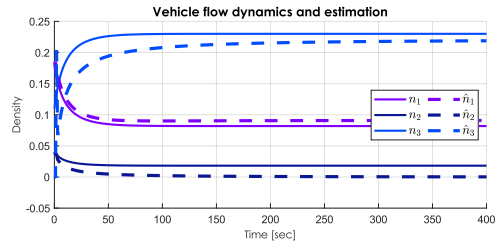
Figure 40.: Simulations of measuring each compartment of the graph in Figure 18. with well-chosen (left column) and poorly chosen (right column) initial values.

The simulations in Figure 41. based on the graph in Figure 19., 20. and 21. illustrate that not all trajectories converge to their corresponding paths, even with well-chosen initial values and a full-rank co-distribution matrix. Figure 41a. show the simulation of the EKF estimation starting from the initial values (36). Figure 41b. show the simulation of the EKF estimation starting from the initial values (37). Figure 41c. show the simulation of the EKF estimation starting from the initial values 38.

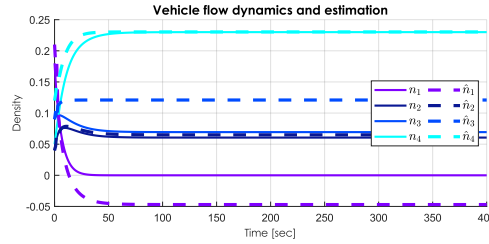
The simulations in Figure 41. show that, despite well-chosen initial values, the estimated state variables do not converge to the correct values.



(a) Estimation when measuring compartments 2 and 3 of the graph in Figure 19. with well-chosen initial values.



(b) Estimation when measuring compartments 2 and 3 of the graph in Figure 20. with well-chosen initial values.



(c) Estimation when measuring compartments 1 and 2 of the graph in Figure 21. with well-chosen initial values.

Figure 41.: Simulations of measuring the multiple components with well-chosen initial values.

When using an EKF to estimate the input-free TRM, even with well-chosen initial values, the EKF can't estimate certain states due to reliance on local decisions. In contrast, a continuous-time state estimator ensures convergence, provided the system is observable and the initial values are properly selected.

4.3.3. Continuous time nonlinear state observer with input

The simulations on Figure 42. were performed using the model illustrated in Figure 28. If an on-ramp is added at the source and an off-ramp at the trap, neither compartment can fully empty or fill up. Consequently, the model's observability is independent of the sum of the initial values. In Figures 42a., 42b. and 42c. the initial values are given by (35), (33) and (31) respectively.

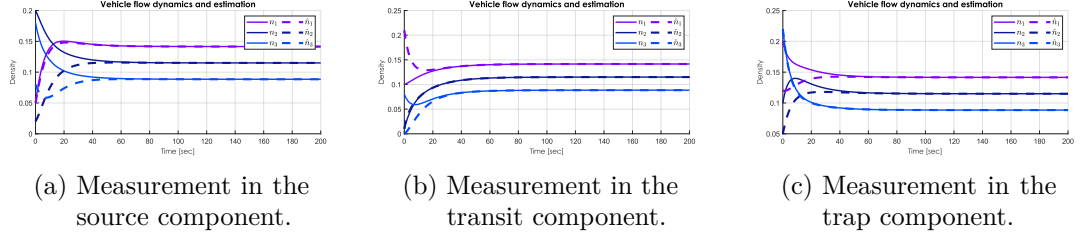


Figure 42.: Simulations of measuring each compartment of the graph in Figure 28.

Figure 43a., 43b. and 43c. illustrates the multiple component models with appropriately placed inputs. Figure 43d., 43e. and 43f. shows the simulations using these multiple component models.

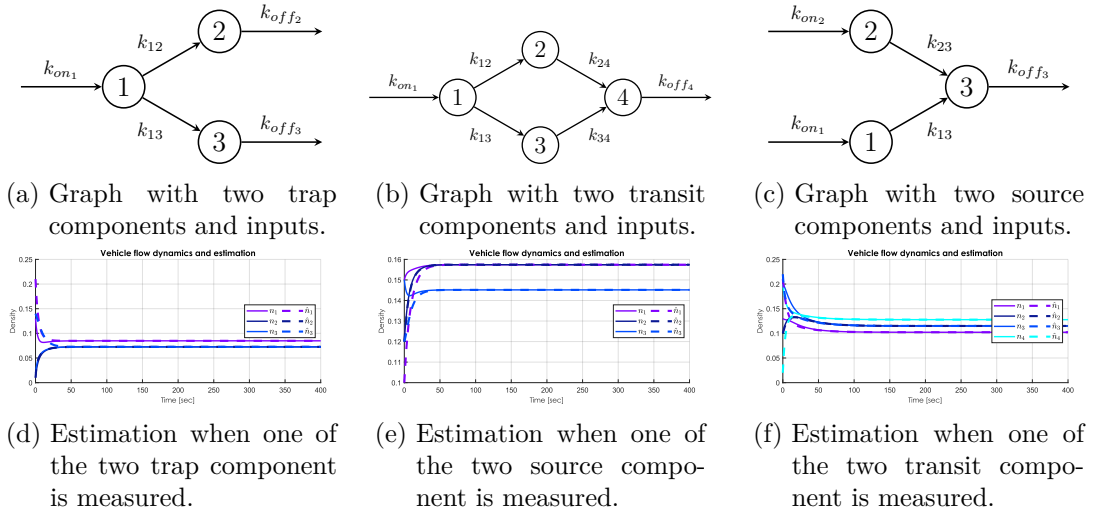


Figure 43.: Graphs and simulations of multiple component graphs with inputs.

By selecting the right input locations, the measurements become independent of the initial values. Additionally, when a category of components has multiple instances in the graph, measuring all of them is unnecessary to estimate all the states.

4.3.3.1. General model

Figure 44. shows a strongly connected graph with 8 compartments with an on-ramp in compartment 3 and an off-ramp in compartment 7.

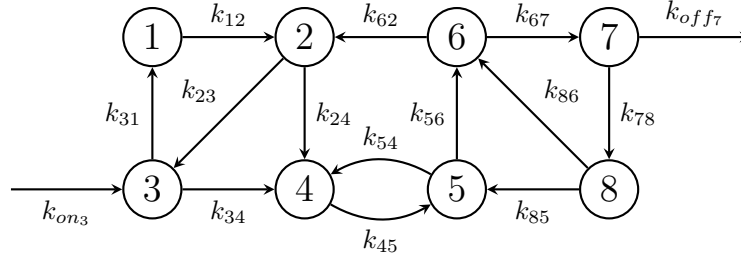
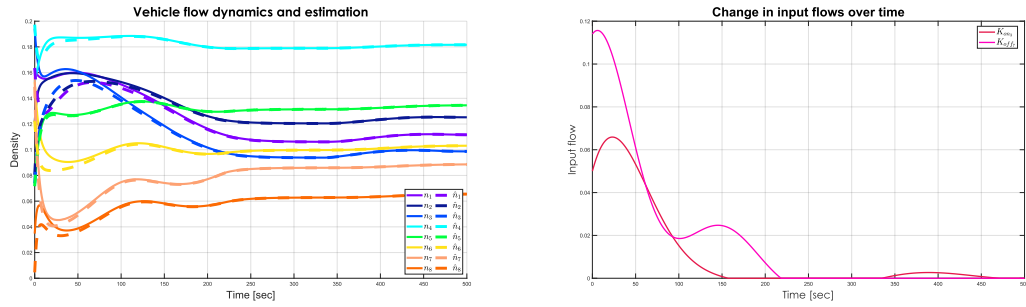


Figure 44.: Strongly connected directed graph with inputs.

Figure 45a. shows the model dynamics and the estimated dynamics when the state of compartment 5 is measured and the inputs vary over time. Figure 45b. shows the change of the input flows in time.



(a) Estimation when compartment 5 is measured.

(b) The time-varying inputs.

Figure 45.: Estimation results when measuring compartment 5 under the influence of time-varying inputs.

Figure 46. shows a graph with 12 compartments with two on-ramps in compartments 1 and 4 and with two off-ramps in compartments 10 and 12.

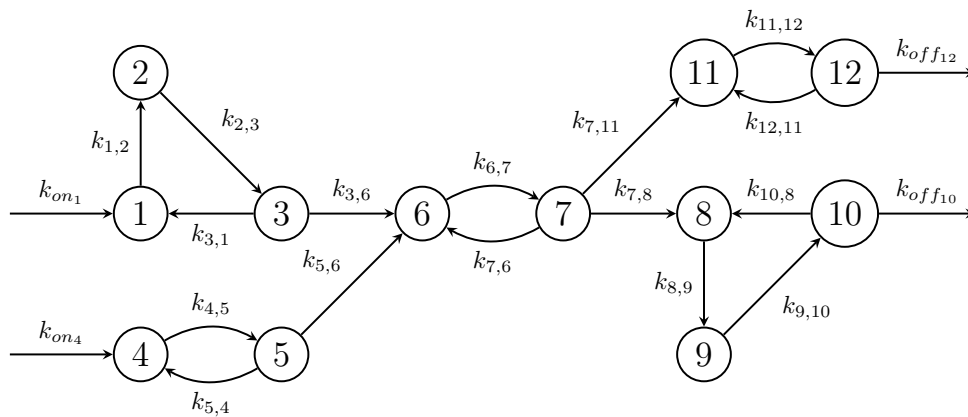


Figure 46.: Graph with strongly connected components and inputs.

Figure 47a., 47b. and 47c. shows the model dynamics and the estimated dynamics when the state of compartments 1, 7 and 8 are measured respectively and the inputs vary over time. Figure 47d. shows the change of the input flows in time.

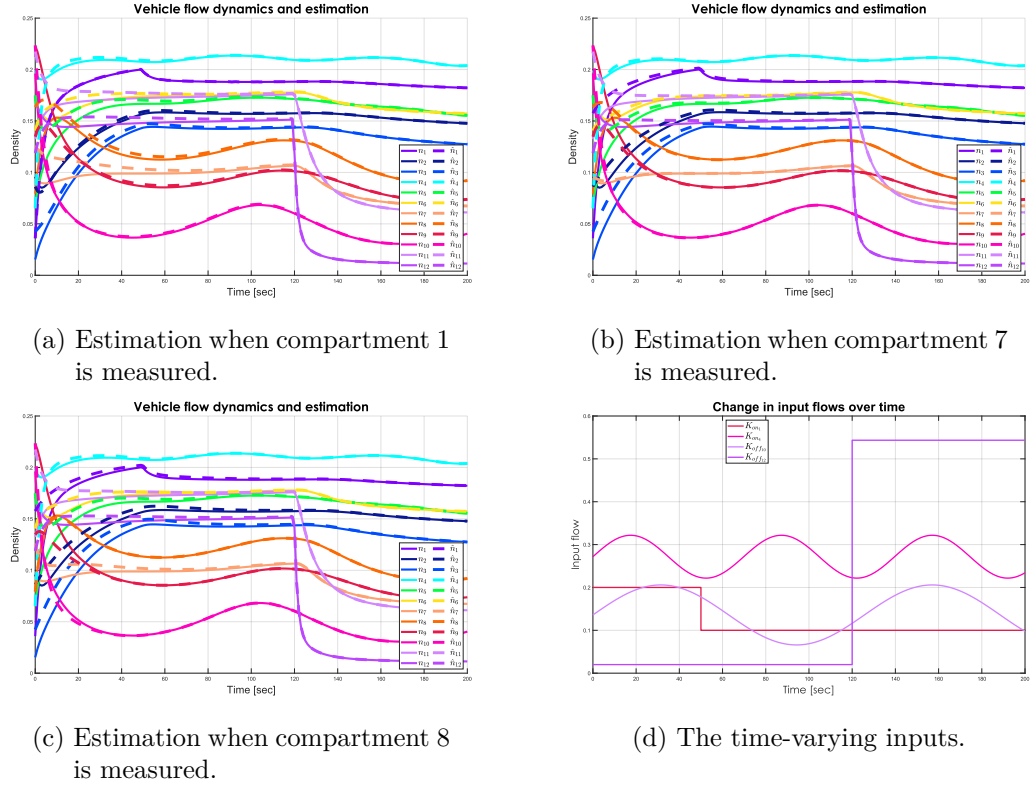
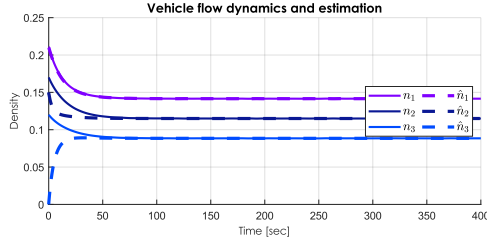


Figure 47.: Estimation results when measuring compartments 1, 7 and 8 under the influence of time-varying inputs.

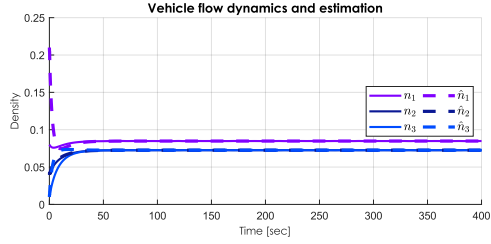
The simulations demonstrate that when the inputs are well-positioned, all states can be effectively estimated with only a few measurement points.

4.3.4. EKF observer with input

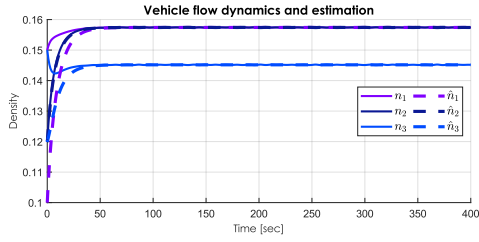
The simulation in Figure 48a. was performed using the model illustrated in Figure 28., where the source compartment is measured. The simulation in Figure 48b. was performed using the model illustrated in Figure 43a., where only compartments 2 and 3 are measured. The simulation in Figure 48c. was performed using the model illustrated in Figure 21., where only compartment 2 is measured. The simulation in Figure 48d. were performed using the model illustrated in Figure 43b., where only compartments 2 and 3 are measured.



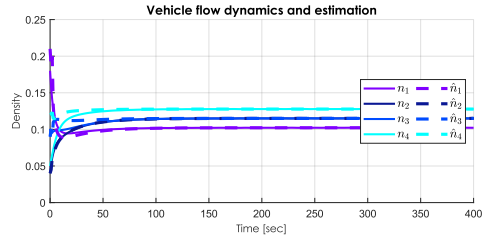
(a) Estimation when the source component is measured on the graph in Figure 28.



(b) Estimation when both source component is measured on the graph in Figure 43a.



(c) Estimation when one of the two transit component is measured on the graph in Figure 43c.



(d) Estimation when both trap component is measured on the graph in Figure 43b.

Figure 48.: Simulations of measurements for simple and multiple component graphs with inputs.

Although EKF estimation performs poorly without inputs, placing inputs in the right locations enables the estimation of all model states using EKF. Similarly, when measurements are needed for a category of components with multiple instances in the graph, it is sufficient to measure only some of them, as with the continuous state estimator.

5. Summary

I began my research by developing a general form of the differential equations for the Traffic Reaction Model. This approach was crucial because writing separate equations for multi-compartment models can be both complicated and time-consuming. Using a general representation made the process more efficient and easier to implement in Matlab. Additionally, I explored structural observability in linear compartmental models. For full observability in these linear models, the key requirement is that at least one compartment in each trap component of the compartmental graph must be measured.

5.1. Results

My main new results were achieved in the field of nonlinear observability analysis of TRMs which are the following:

- **Nonlinear observability analysis of autonomous TRMs.** The initial analysis focused on the nonlinear observability of the Traffic Reaction Model without on and off ramps. I computed the nonlinear state-dependent observability co-distributions of TRMs, and identified the measurement conditions and possible singular points, where the co-distributions lose rank. I showed that these points may correspond to states when either compartments become empty or get full. I compared the observability results to the linear case using the graph representation. I showed that in the case of the nonlinear model, it may be possible to achieve observability by measuring within any strongly connected component, given certain conditions. From this, it follows that if the compartmental graph is strongly connected, full observability is achieved by starting from any nonnegative initial value and measuring any compartment.
- **Nonlinear observability analysis of controlled TRMs.** The observability conditions of TRMs with inputs (on and/or off ramps) can be milder than in the autonomous case. If we put an on-ramp in each source strongly connected component and an off-ramp in each trap strongly connected component, then complete observability is achieved due to the continuous flow of vehicles. Observability does not depend on the initial values, and when measurements are required in a category of components with multiple instances in the graph, it is not necessary to measure all of them.

- **Measurement conditions for complete observability.** I studied the problem of selecting a preferably minimal subset of compartments to be measured which ensures the complete observability of the system. In the strongly connected component DAG of the input-free TRM, the smallest set of nodes that covers all nodes must be identified. I showed that this is a minimum set cover problem which is known to be NP-hard. Moreover, it is important to account for the number of compartments that are measured, can reach the measured compartments, or are reachable from them. The sum of the initial values must be determined based on the number of compartments in these categories.
- **Observer design for TRMs.** I designed two types of state estimators and tested them in simulations. I studied a continuous-time nonlinear state estimator and the EKF state estimator. The continuous-time nonlinear state estimator always worked well when the observability conditions were fulfilled, while the EKF estimator did not always converge.

5.2. Future plans

A future plan is to combine traffic reaction model state estimation with control strategies. This would help the system better predict traffic behavior and make smarter control decisions at the same time. By working together, these two parts can improve how the system handles real-time traffic and make it more efficient and reliable.

Further work will be focused on the proof of the stability of the proposed nonlinear observers to ensure they work reliably and consistently in different conditions.

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