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**Comparing dealing methods with repeating cards**

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# Comparing dealing methods with repeating cards

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## Abstract

In this paper we compare dealing methods and we generalize the results of Conger and Howald. They compared three well known dealing methods like the ordered dealing, the cyclic dealing and the back and forth dealing after an  $a$  shuffle in the case when there are 52 distinct cards in the deck and four players. In this paper we analyzed how fast the different dealing methods randomize the deck in the case when there are repeating cards. The main generalization of our work is that we compared these methods for those cases when the deck contains repeated cards of an arbitrary number of different types and only the hands dealt to players are of interest, moreover the number of cards in the deck and the number of players are also arbitrary. For the problem we used the variation distance what we stripped in series with using the results of Conger and Howald and we analyzed the coefficient of the first term of the Taylor series for different dealing methods. We get a formula which gives the coefficient of  $a^{-1}$  for any possible dealing method, when the size of the deck, the number of different types of cards and the number of players are arbitrary. With the help of the formula we proved for arbitrary initial deck that the coefficient of  $a^{-1}$  differs by factors of  $s$  when comparing the ordered and cyclic dealing, differs by factors of  $s$  when comparing the ordered, cyclic and back and forth dealing when  $s$  is an odd number, while the coefficient of  $a^{-1}$  is 0 for the back and forth dealing when  $s$  is an even number; here  $s$  is the number of cards each player gets after the dealing methods. At the same time this proves that using the back and forth dealing instead the cyclic dealing we can save  $\log_2 s$  riffle shuffle for large  $a$ . At the end of the work we computed the coefficients for some special cases.

## 1 Introduction

In card-games a very important requirement is that, after the mixing procedure, every hand dealt to players should have approximately the same probability. The main question is: “how fast can the riffing and shuffling method randomize the deck?” To examine this question, we need some definitions.

In 1955 Gilbert and Shannon [5] used the riffle shuffling by examining the shuffling method. In the 1980s Reeds [6] and Aldous [1] added the assumption that every possible cut/riffle combinations have equal chances using the riffle shuffling, and that has become known as the Gilbert-Shannon Reeds or GSR model of card shuffling. Riffle shuffling is the most fashionable way to randomize the initial deck. First we have to cut the deck into two packets. If the first packet contains  $k$  and the second packet  $n - k$  cards then the probability of this cut is exactly  $\frac{\binom{n}{k}}{2^n}$ . After the cut we have to mix the packets together such that the cards of each packet maintain their relative order. We suppose that each possible interleaving is equally likely. There are  $\binom{n}{k}$  possible interleavings (we only have to choose the  $k$  places among  $n$  places for the first packet, then the other  $n - k$  places are determined). So we get that any possible shuffling where  $k$  is the size of the cut has  $\frac{\binom{n}{k}}{2^n} \cdot \frac{1}{\binom{n}{k}} = \frac{1}{2^n}$  probability. This value does not depend on  $k$ , so every pair of a cut and interleaving are equally

likely. In 1992, Bayer and Diaconis [2] analyzed a generalization of the riffle shuffling. They introduce the  $a$ -shuffle to make the mixing problem easier. First we cut the deck into  $a$  packets. If the sizes of the packets are  $p_1, p_2, \dots, p_k$  then the probability of this particular cut is  $\frac{\binom{n}{p_1, p_2, \dots, p_k}}{a^n}$ . After this cut we mix the cards together such that the cards of each packet maintain their relative order. If we imagine an  $a$ -handed dealer, who can interleave the cards like a two-handed dealer, we can see what  $a$ -shuffle means. Making a randomly selected  $a$ -shuffle and then making a randomly selected  $b$ -shuffle is equivalent to making a randomly selected  $a \cdot b$  shuffle, if we add the assumption as before that every possible cut/riffle combinations are equally likely. So we get that a sequence of  $i$  riffle shuffles is equivalent to a single  $2^i$ -shuffle. Bayer and Diaconis used the variation distance to analyze the card mixing problem. We can define this distance as:

$$\|\mathbb{P}_a - U\| := \frac{1}{2} \sum_{\Pi \in S_n} |\mathbb{P}_a(\Pi) - U(\Pi)|,$$

where  $\mathbb{P}_a(\Pi)$  is the probability of a particular permutation  $\Pi$  after an  $a$ -shuffle,  $S_n$  is the symmetric group of degree  $n$ ,  $U$  represents the uniform distribution on permutations ( $U(\Pi) = \frac{1}{n!}$  for all  $\Pi \in S_n$ ), each card is distinct. Initially, the deck is ordered: if we label the cards by  $1, 2, \dots, n$ , then in the initial deck card  $i$  will be in position  $i$ . Bayer and Diaconis found an explicit formula for  $\mathbb{P}_a$ :

$$\mathbb{P}_a(\Pi) = \frac{1}{a^n} \binom{a + n - \text{des}(\Pi) - 1}{n},$$

where  $n$  is the size of the deck and

$$\text{des}(\Pi) := \#\{i : \Pi(i) > \Pi(i+1)\}.$$

In our case we consider a permutation from  $1, 2, \dots, n$ , to itself, so if we apply  $\Pi$  to a sequence of objects, then the object in position  $i$  goes to position  $\Pi(i)$ . For example, the permutation  $\Pi_1 = [42513]$  changes our initial ordering to 42513, as well as rearranging 25431 to 35124, and 53412 to 13254. We can easily check that  $\text{des}(\Pi_1) = 2$ .

We can consider another case when we allow two cards to have the same value. This makes our problem more complicated because decks (ordered sequence of cards) and transformations cannot be identified with permutations. On the other hand there is a set of permutations for each pair which transform the first into the second. Another problem is that the initial order of deck affects how fast the distribution approaches uniform. If  $D'$  is some rearrangement of  $D$ , let  $S(D, D')$  be the set of permutation which transform  $D$  into  $D'$ . Thus the transition probability between  $D$  and  $D'$  is

$$\mathbb{P}_a(D \rightarrow D') := \sum_{\Pi \in S(D, D')} \mathbb{P}_a(\Pi).$$

If we consider our explicit formula, we get the following:

$$\mathbb{P}_a(D \rightarrow D') = \frac{1}{a^n} \sum_d b_d \binom{a + n - d - 1}{n},$$

where  $b_d$  is the number of permutations in  $S(D, D')$  with  $d$  descents.

Conger and Viswanath [4] proved that the calculation of transition probabilities is a #P-complete problem. Most people believe that #P-complete problems do not admit efficient solutions, so a possible way to examine this question is to approximate this probability. Conger and Howald [3] proved a theorem, which allows us to approximate transition probabilities, when  $a$  is large, given some simple information about the decks. To understand the theorem, we have to make some definitions.

If  $a$  and  $b$  are card values, we say that  $D$  has an  $a - b$  digraph at  $i$  if  $D(i) = a$  and  $D(i+1) = b$ . We say that  $D$  has an  $a - b$  pair at  $(i, j)$  if  $i < j$ ,  $D(i) = a$ , and  $D(j) = b$ . Let

$$W(D, a, b) := \#\{a - b \text{ digraphs in } D\} - \#\{b - a \text{ digraphs in } D\},$$

$$Z(D, a, b) := \#\{a - b \text{ pairs in } D\} - \#\{b - a \text{ pairs in } D\}.$$

For example, the following deck, that consists of red ( $R$ ) and black ( $B$ ) cards, has 1  $R - B$  digraph, 2  $B - R$  digraphs, 12  $R - B$  pairs, 13  $B - R$  pairs:

$$D := BRRRB BBBRR$$

So  $W(D, B, R) = 2 - 1 = 1$ ,  $Z(D, B, R) = 13 - 12 = 1$ .

We can easily see that  $W$  and  $Z$  are antisymmetric in  $a$  and  $b$ :

$$W(D, a, b) = -W(D, b, a), \quad Z(D, a, b) = -Z(D, b, a).$$

Conger and Howald [3] proved the following theorem:

$$(1) \quad \mathbb{P}_a(D \rightarrow D') = \frac{1}{N} + c_1(D, D')a^{-1} + O(a^{-2}),$$

where  $N$  is the number of reorderings of  $D$  and

$$(2) \quad c_1(D, D') = \frac{n}{2N} \sum_{a < b} \frac{W(D, a, b)Z(D', a, b)}{n_a n_b},$$

where  $n_a$  is the number of cards, which have value  $a$ ,  $n_b$  is the number of cards, which have value  $b$ .

In this paper we will analyze dealing methods in the case when there are repeated cards in the deck and only the hands dealt to players are of interest. Conger and Howald [3] proved that switching from ordered dealing to cyclic dealing improves the randomness by a factor of 13, and switching from cyclic dealing to back and forth dealing also improves the randomness by a factor of 13, when there are 4 players and 52 distinct cards. In section 2 we describe the case when there can be repeated cards in the deck. In section 3 we analyze arbitrary dealing methods with repeated cards, in section 4 we analyze those cases when there are more players. We will write up a formula which shows us the effectiveness of a dealing method in the case when only the hands dealt to players are of interest, and we will prove from this formula, that switching from ordered dealing to cyclic dealing improves the randomness by a factor of  $s$ , and switching from cyclic dealing to back and forth dealing improves the randomness by a factor of  $s$ , when  $s$  is an odd number, thus switching from cyclic dealing to back and forth dealing improves the randomness in a huge amount, when  $s$  is an even number. Here  $s$  is the number of cards each player receives. In section 5 we analyze cases when the initial deck is arbitrary and we prove the former results also for these cases. Section 6 considers explicit computations when there are only two or three types of cards.

## 2 The basics of our model

First we consider the following case. Our deck contains  $4s$  cards and there are  $k$  different types of cards:  $P_1, P_2, \dots, P_k$ . In the initial deck the first  $p_1$  cards are  $P_1$  coloured, the next  $p_2$  cards are  $P_2$  coloured,  $\dots$ , and the last  $p_k$  cards are  $P_k$  coloured ( $\sum_{i=1}^k p_i = 4s$ ). We  $a$ -shuffle the deck then we deal it out to the four players. The four players are North(N), East(E), South(S) and West(W). In order to analyze the progress of a shuffler toward a well-mixed deck, we use the variation distance:

$$||\mathbb{P}_a - \Pi|| := \frac{1}{2} \sum_{\omega \in \Omega} |\mathbb{P}_a(\omega) - \Pi(\omega)|,$$

where  $\Omega$  is the set of possible configurations.

$$\Omega = \{\bar{p}_1, \bar{p}_2, \dots, \bar{p}_k : p_{1,N}, p_{1,E}, p_{1,S}, p_{1,W} \geq 0; p_{2,N}, p_{2,E}, p_{2,S}, p_{2,W} \geq 0; \dots; p_{k,N}, p_{k,E}, p_{k,S}, p_{k,W} \geq 0;$$

$$p_{1,N} + p_{1,E} + p_{1,S} + p_{1,W} = p_1; p_{2,N} + p_{2,E} + p_{2,S} + p_{2,W} = p_2; \dots; p_{k,N} + p_{k,E} + p_{k,S} + p_{k,W} = p_k;$$

$$\sum_{i=1}^k p_{i,N} = s; \sum_{i=1}^k p_{i,E} = s; \sum_{i=1}^k p_{i,S} = s; \sum_{i=1}^k p_{i,W} = s\},$$

where  $p_{i,N}, p_{i,E}, p_{i,S}, p_{i,W}$  is the number of  $P_i$  coloured cards which North, East, South, West has respectively after the dealing method,  $\bar{p}_i = (p_{i,N}, p_{i,E}, p_{i,S}, p_{i,W})$ .

$\Pi(\omega)$  represents the stationary distribution, which supposes that each permutation of the  $4s$  cards is equally likely.

$$\Pi(\omega) = \frac{s!^4}{(4s)!} \prod_{i=1}^k \frac{p_i!}{p_{i,N}! \cdot p_{i,E}! \cdot p_{i,S}! \cdot p_{i,W}!}.$$

$$\|\mathbb{P}_a - \Pi\| := \frac{1}{2} \sum_{\omega \in \Omega} |\mathbb{P}_a(\omega) - \Pi(\omega)| =$$

$$\frac{1}{2} \sum_{\omega \in \Omega} \left| \sum_{D': (\bar{p}_1, \bar{p}_2, \dots, \bar{p}_k) = \omega} \mathbb{P}_a(D \rightarrow D') - \Pi(\omega) \right|.$$

Here we suppose that deck  $D$  is the initial sequence of cards, and  $D'$  is some rearrangement of  $D$ .

For computing  $\mathbb{P}_a(D \rightarrow D')$  we use (1):

$$\mathbb{P}_a(D \rightarrow D') = \frac{1}{N} + c_1(D, D')a^{-1} + O(a^{-2}).$$

In our case  $N = \binom{4s}{p_1 p_2 \dots p_k} = \frac{(4s)!}{p_1! p_2! \dots p_k!}$ . We assume an implicit order on the values according to the initial deck.

We easily get the following:

$$(3) \quad \sum_{D': (\bar{p}_1, \bar{p}_2, \dots, \bar{p}_k) = \omega} 1 = s!^4 \prod_{i=1}^k \frac{1}{p_{i,N}! \cdot p_{i,E}! \cdot p_{i,S}! \cdot p_{i,W}!}$$

Using these connections we get: (see also (2))

$$\begin{aligned} \sum_{D': (\bar{p}_1, \bar{p}_2, \dots, \bar{p}_k) = \omega} \mathbb{P}_a(D \rightarrow D') &= \sum_{D': (\bar{p}_1, \bar{p}_2, \dots, \bar{p}_k) = \omega} \frac{1}{N} + a^{-1} \sum_{D': (\bar{p}_1, \bar{p}_2, \dots, \bar{p}_k) = \omega} c_1(D, D') + \sum_{D': (\bar{p}_1, \bar{p}_2, \dots, \bar{p}_k) = \omega} O(a^{-2}) \\ &= \Pi(\omega) + a^{-1} \sum_{D': (\bar{p}_1, \bar{p}_2, \dots, \bar{p}_k) = \omega} \frac{2s}{(4s)!} \left( \prod_{j=1}^k p_j! \right) \left( \sum_{a < b} \frac{W(D, a, b) Z(D', a, b)}{n_a n_b} \right) + O(a^{-2}) \\ &= \Pi(\omega) + O(a^{-2}) + a^{-1} \frac{2s}{(4s)!} \left( \prod_{j=1}^k p_j! \right) \left( \sum_{i=1}^{k-1} \frac{\sum_{D': (\bar{p}_1, \bar{p}_2, \dots, \bar{p}_k) = \omega} Z(D', P_i, P_{i+1})}{p_i \cdot p_{i+1}} \right), \end{aligned}$$

because  $W(D, P_i, P_{i+1}) = 1$  ( $i = 1, \dots, k-1$ ), and  $W(D, A, B) = 0$  if  $B \neq (A+1)$  by virtue of the initial deck.

In our case the variation distance will be the following:

$$(4) \quad \frac{1}{2} \sum_{\omega \in \Omega} |\mathbb{P}_a(\omega) - \Pi(\omega)| = a^{-1} \frac{s}{(4s)!} \left( \prod_{j=1}^k p_j! \right) \sum_{\omega \in \Omega} \left| \sum_{i=1}^{k-1} \frac{\sum_{D': (\bar{p}_1, \bar{p}_2, \dots, \bar{p}_k) = \omega} Z(D', P_i, P_{i+1})}{p_i \cdot p_{i+1}} \right| + O(a^{-2}).$$

### 3 Dealing methods

Now we consider the dealing methods. The dealer  $a$ -shuffles the deck and then deals it out to the four players. We can describe the dealing method function as a function  $f$  from  $1, 2, \dots, 4s$  to  $N, E, S, W$ .

For example:

$$\begin{aligned} f(1) &= f(2) = f(3) = \dots = f(s-1) = f(s) = N, \\ f(s+1) &= f(s+2) = f(s+3) = \dots = f(2s-1) = f(2s) = E, \\ f(2s+1) &= f(2s+2) = f(2s+3) = \dots = f(3s-1) = f(3s) = S, \\ f(3s+1) &= f(3s+2) = f(3s+3) = \dots = f(4s-1) = f(4s) = W, \end{aligned}$$

represents “ordered dealing”, where the first  $s$  cards go to North, the next  $s$  cards go to East, the next  $s$  cards go to South and the last  $s$  cards go to West. We can represent the dealing method with a sequence of  $N, E, S$  and  $W$ . In this case the representing sequence is:

$$NNN \dots NN EEE \dots EE SSS \dots SS WWW \dots WW.$$

Another dealing method is the cyclic dealing:

$$\begin{aligned} f(1) &= f(5) = f(9) = \dots = f(4s-7) = f(4s-3) = N, \\ f(2) &= f(6) = f(10) = \dots = f(4s-6) = f(4s-2) = E, \\ f(3) &= f(7) = f(11) = \dots = f(4s-5) = f(4s-1) = S, \\ f(4) &= f(8) = f(12) = \dots = f(4s-4) = f(4s) = W, \end{aligned}$$

so the top card goes to North, the second goes to East, the third goes to West, etc. The representing sequence is:

$$NESWNESWNESW \dots NESWNESW.$$

The back and forth dealing is the following for even  $s$  numbers:

$$\begin{aligned} f(1) &= f(8) = f(9) = f(16) = f(17) = f(24) = f(25) = f(32) = \dots = f(4s) = N, \\ f(2) &= f(7) = f(10) = f(15) = f(18) = f(23) = f(26) = f(31) = \dots = f(4s-1) = E, \\ f(3) &= f(6) = f(11) = f(14) = f(19) = f(22) = f(27) = f(30) = \dots = f(4s-2) = S, \\ f(4) &= f(5) = f(12) = f(13) = f(20) = f(21) = f(28) = f(29) = \dots = f(4s-3) = W. \end{aligned}$$

The representing sequence is the following in this case:

$$NESWWSENNESWWSEN \dots NESWWSEN.$$

The back and forth dealing is the following for odd  $s$  numbers:

$$\begin{aligned} f(1) &= f(8) = f(9) = f(16) = f(17) = f(24) = f(25) = f(32) = \dots = f(4s-3) = N, \\ f(2) &= f(7) = f(10) = f(15) = f(18) = f(23) = f(26) = f(31) = \dots = f(4s-2) = E, \\ f(3) &= f(6) = f(11) = f(14) = f(19) = f(22) = f(27) = f(30) = \dots = f(4s-1) = S, \\ f(4) &= f(5) = f(12) = f(13) = f(20) = f(21) = f(28) = f(29) = \dots = f(4s) = W. \end{aligned}$$

The representing sequence is the following in this case:

$$NESWWSENNESWWSEN \dots NESW.$$

We will write up a formula which allows us to compute the coefficient of  $a^{-1}$  for any dealing method, if we know the dealing method function. Here we suppose that in the initial deck the first type is black (B),

the second type is red (R). In this case  $b$  is the number of black cards in the deck,  $r$  is the number of red cards in the deck and  $p$  is the number of non-red and non-black cards in the deck.

Let  $N_1, E_1, S_1, W_1$  be the first position which goes to North, East, South, West respectively in the dealing method. Let  $N_2, E_2, S_2, W_2$  be the second position which goes to North, East, South, West respectively in the dealing method, etc. Let  $p_N, p_E, p_S, p_W$  be the number of non-red and non-black cards which North, East, South, West has respectively after the dealing method. Let  $b_N, b_E, b_S, b_W$  be the number of black cards which North, East, South, West has respectively after the dealing method.

**Theorem 3.1.**

$$\begin{aligned}
& \sum_{D': (\bar{b}, \bar{r}, \bar{p}_3, \bar{p}_4, \dots, \bar{p}_k) = \omega} Z(D', B, R) \\
&= s!^4 \left( \prod_{i=1}^k \frac{1}{p_{i,N}! \cdot p_{i,E}! \cdot p_{i,S}! \cdot p_{i,W}!} \right) \left( (4s+1)b + \frac{b_N}{s} \left( \frac{p_E \cdot Z(E, N) + p_S \cdot Z(S, N) + p_W \cdot Z(W, N)}{s} - 2 \sum_{p=1}^s N_p \right) \right. \\
& \quad + \frac{b_E}{s} \left( \frac{p_N \cdot Z(N, E) + p_S \cdot Z(S, E) + p_W \cdot Z(W, E)}{s} - 2 \sum_{p=1}^s E_p \right) \\
& \quad + \frac{b_S}{s} \left( \frac{p_N \cdot Z(N, S) + p_E \cdot Z(E, S) + p_W \cdot Z(W, S)}{s} - 2 \sum_{p=1}^s S_p \right) \\
& \quad \left. + \frac{b_W}{s} \left( \frac{p_N \cdot Z(N, W) + p_E \cdot Z(E, W) + p_S \cdot Z(S, W)}{s} - 2 \sum_{p=1}^s W_p \right) \right),
\end{aligned}$$

where

$$\begin{aligned}
Z(i, j) &= \# \{i - j \text{ pairs in the representing sequence of the dealing method}\} \\
&\quad - \# \{j - i \text{ pairs in the representing sequence of the dealing method}\}.
\end{aligned}$$

*Proof.* First we will prove that for a particular permutation  $D'$ :

(5)

$$\begin{aligned}
Z(D', B, R) &= \sum_{i=1}^{4s} (4s+1 - 2i + (\text{the number of non-red and non-black cards in } D' \text{ before the } i^{\text{th}} \text{ position}) \\
&\quad - (\text{the number of non-red and non-black cards in } D' \text{ after the } i^{\text{th}} \text{ position})) \cdot \\
&\quad \mathbf{1} \{ \text{in the } i^{\text{th}} \text{ position there is a black card in permutation } D' \}.
\end{aligned}$$

If we change in the position  $i$  the value from Red to Black in a deck  $D'$ , then we can count the change of the value  $Z(D', B, R)$ . Put the case that in the first  $i-1$  cards there are  $A$  Black,  $C$  Red cards and  $G$  non-red and non-black cards; the card in the position  $i$  is Red coloured; and in the last  $4s-i$  cards there are  $D$  Black,  $F$  Red cards and  $H$  non-red and non-black cards. So  $A + C + G = i-1$  and  $D + F + H = 4s-i$ . In this case the card in the position  $i$  stands in  $\{B-R \text{ pairs}\}$  with those  $A$  cards in the first  $i-1$  cards which are Black coloured; and stands in  $\{R-B \text{ pairs}\}$  with those  $D$  cards in the last  $4s-i$  cards which are Black coloured. If we change in the position  $i$  the value from Red to Black we get that the change of the value  $Z(D', B, R)$  is:  $-A - C + D + F = -(i-1) + G + 4s - i - H = 4s+1 - 2i + (\text{the number of non-red and non-black cards in } D' \text{ before the } i^{\text{th}} \text{ position}) - (\text{the number of non-red and non-black cards in } D' \text{ after the } i^{\text{th}} \text{ position})$ . We note that we only have to count the change between the card in the position  $i$  and the remaining cards. The  $A \{B-R \text{ pairs}\}$  and the  $D \{R-B \text{ pairs}\}$  vanish and there will be  $C$  new  $\{R-B \text{ pairs}\}$  and  $F$  new  $\{B-R \text{ pairs}\}$ , so the change of the value  $Z(D', B, R)$  is indeed:  $-A - C + D + F$ . Let the initial deck have all its cards Red colored, so in the initial deck the value  $Z(D', B, R)$  is 0. First we change one by one in those positions  $i$  the value from red to another card value in the initial deck in which the card value is non-black and non-red in the deck. After these changes the value  $Z(D', B, R)$  will be still 0 in this revised deck. After this procedure, we have

to change one by one in those positions  $i$  the value from red to black in this revised deck in which the card value is black in the deck  $D'$ , and then we can count one by one the change of the value  $Z(D', B, R)$ . If we sum up these changes we get the value  $Z(D', B, R)$ , because in the initial deck and in the revised deck the value  $Z(D', B, R)$  is 0. So we proved (5).

$$\sum_{D':(\bar{b}, \bar{r}, \bar{p}_3, \bar{p}_4, \dots, \bar{p}_k) = \omega} Z(D', B, R) = \sum_{D':(\bar{b}, \bar{r}, \bar{p}_3, \bar{p}_4, \dots, \bar{p}_k) = \omega} (4s+1)b$$

$$+ \sum_{D':(\bar{b}, \bar{r}, \bar{p}_3, \bar{p}_4, \dots, \bar{p}_k) = \omega} \sum_{i=1}^{4s} (U(i)) \mathbf{1}\{\text{in the } i^{\text{th}} \text{ position there is a black card in permutation } D'\},$$

where  $U(i) = (\text{the number of non-red and non-black cards in } D' \text{ before the } i^{\text{th}} \text{ position} - \text{the number of non-red and non-black cards in } D' \text{ after the } i^{\text{th}} \text{ position} - 2i)$ .

In order to compute  $\sum_{D'} (U(i))$ , we introduce an auxiliary probability measure. According to this measure each permutation is equally likely. With the help of this measure we handle the sum as a conditional expectation. In this probability measure  $U(i)$  is a function of the permutation and, as such, becomes a random variable in our auxiliary probability measure, so we write this sum as a conditional expectation.

$$\sum_{D':(\bar{b}, \bar{r}, \bar{p}_3, \bar{p}_4, \dots, \bar{p}_k) = \omega} (U(i)) \mathbf{1}\{\text{in the } i^{\text{th}} \text{ position there is a black card in permutation } D'\}$$

$$= \left( |D' : (\bar{b}, \bar{r}, \bar{p}_3, \bar{p}_4, \dots, \bar{p}_k) = \omega, \text{ in the } i^{\text{th}} \text{ position there is a black card}| \right.$$

$$\cdot \mathbf{E}(U(i) | \text{in the } i^{\text{th}} \text{ position there is a black card in permutation } D', (\bar{b}, \bar{r}, \bar{p}_3, \bar{p}_4, \dots, \bar{p}_k) = \omega) \Big),$$

for  $i = 1, \dots, 4s$ .

The reason of the last equality: for each permutation  $D'$ , where  $(\bar{b}, \bar{r}, \bar{p}_3, \bar{p}_4, \dots, \bar{p}_k) = \omega$  realized we sum up the value of  $U(i)$ , if position  $i$  is black in permutation  $D'$ .

In  $\mathbf{E}(U(i) | \text{in position } i \text{ there is a black card in permutation } D', (\bar{b}, \bar{r}, \bar{p}_3, \bar{p}_4, \dots, \bar{p}_k) = \omega)$ , we also sum up these terms, but we divide each term with  $|D' : (\bar{b}, \bar{r}, \bar{p}_3, \bar{p}_4, \dots, \bar{p}_k) = \omega, \text{ in position } i \text{ there is a black card}|$ .

Using these results we get the following:

$$\sum_{D':(\bar{b}, \bar{r}, \bar{p}_3, \bar{p}_4, \dots, \bar{p}_k) = \omega} Z(D', B, R) = |D' : (\bar{b}, \bar{r}, \bar{p}_3, \bar{p}_4, \dots, \bar{p}_k) = \omega| (4s+1)b$$

$$+ \left( \sum_{i=1}^{4s} |D' : (\bar{b}, \bar{r}, \bar{p}_3, \bar{p}_4, \dots, \bar{p}_k) = \omega, \text{ in the } i^{\text{th}} \text{ position there is a black card}| \right.$$

$$\cdot \mathbf{E}(U(i) | \text{in the } i^{\text{th}} \text{ position there is a black card in permutation } D', (\bar{b}, \bar{r}, \bar{p}_3, \bar{p}_4, \dots, \bar{p}_k) = \omega) \Big).$$

If we know the dealing method, we also know which player position  $i$  belongs to. Suppose that position  $i$  belongs to player North and in the  $i^{\text{th}}$  position there is a black card. In this case:

$$\sum_{D':(\bar{b}, \bar{r}, \bar{p}_3, \bar{p}_4, \dots, \bar{p}_k) = \omega} 1 = b_N \cdot (s-1)! \cdot s!^3 \prod_{j=1}^k \frac{1}{p_{j,N}! \cdot p_{j,E}! \cdot p_{j,S}! \cdot p_{j,W}!}.$$

Suppose that position  $i$  belongs to player East and in the  $i^{\text{th}}$  position there is a black card. In this case:

$$\sum_{D':(\bar{b}, \bar{r}, \bar{p}_3, \bar{p}_4, \dots, \bar{p}_k) = \omega} 1 = b_E \cdot (s-1)! \cdot s!^3 \prod_{j=1}^k \frac{1}{p_{j,N}! \cdot p_{j,E}! \cdot p_{j,S}! \cdot p_{j,W}!}.$$

Suppose that position  $i$  belongs to player South and in the  $i^{\text{th}}$  position there is a black card. In this case:

$$\sum_{D': (\bar{b}, \bar{r}, \bar{p}_3, \bar{p}_4, \dots, \bar{p}_k) = \omega} 1 = b_S \cdot (s-1)! \cdot s!^3 \prod_{j=1}^k \frac{1}{p_{j,N}! \cdot p_{j,E}! \cdot p_{j,S}! \cdot p_{j,W}!}.$$

Suppose that position  $i$  belongs to player West and in the  $i^{\text{th}}$  position there is a black card. In this case:

$$\sum_{D': (\bar{b}, \bar{r}, \bar{p}_3, \bar{p}_4, \dots, \bar{p}_k) = \omega} 1 = b_W \cdot (s-1)! \cdot s!^3 \prod_{j=1}^k \frac{1}{p_{j,N}! \cdot p_{j,E}! \cdot p_{j,S}! \cdot p_{j,W}!}.$$

With these connections and with (3) we get the following:

$$\begin{aligned} & |D' : (\bar{b}, \bar{r}, \bar{p}_3, \bar{p}_4, \dots, \bar{p}_k) = \omega, \text{ in the } i^{\text{th}} \text{ position there is a black card}| \\ &= |D' : (\bar{b}, \bar{r}, \bar{p}_3, \bar{p}_4, \dots, \bar{p}_k) = \omega| \cdot \left( \frac{b_N}{s} \mathbf{1}\{\text{position } i \text{ belongs to player North in the dealing method}\} \right. \\ &\quad + \frac{b_E}{s} \mathbf{1}\{\text{position } i \text{ belongs to player East in the dealing method}\} \\ &\quad + \frac{b_S}{s} \mathbf{1}\{\text{position } i \text{ belongs to player South in the dealing method}\} \\ &\quad \left. + \frac{b_W}{s} \mathbf{1}\{\text{position } i \text{ belongs to player West in the dealing method}\} \right). \end{aligned}$$

So the most important question is which position belongs to player North, player East, player South and player West, and we have to analyze these positions by groups.

First we analyze those positions which belong to player North in the dealing method. There are  $s$  such positions. Let position  $i$  be one of these  $s$  positions.

$$\begin{aligned} & \mathbf{E}(U(i) | \text{in the } i^{\text{th}} \text{ position there is a black card in permutation } D', (\bar{b}, \bar{r}, \bar{p}_3, \bar{p}_4, \dots, \bar{p}_k) = \omega) \\ &= \mathbf{P}(\text{first position is non-red and non-black} | \text{in the } i^{\text{th}} \text{ position there is a black card in permutation } D', \\ &\quad (\bar{b}, \bar{r}, \bar{p}_3, \bar{p}_4, \dots, \bar{p}_k) = \omega) \\ &+ \mathbf{P}(\text{second position is non-red and non-black} | \text{in the } i^{\text{th}} \text{ position there is a black card in permutation } D', \\ &\quad (\bar{b}, \bar{r}, \bar{p}_3, \bar{p}_4, \dots, \bar{p}_k) = \omega) \\ &+ \dots + \mathbf{P}(\text{position } (i-1) \text{ is non-red and non-black} | \text{in the } i^{\text{th}} \text{ position there is a black card in permutation } D', \\ &\quad (\bar{b}, \bar{r}, \bar{p}_3, \bar{p}_4, \dots, \bar{p}_k) = \omega) \\ &- \mathbf{P}(\text{position } (i+1) \text{ is non-red and non-black} | \text{in the } i^{\text{th}} \text{ position there is a black card in permutation } D', \\ &\quad (\bar{b}, \bar{r}, \bar{p}_3, \bar{p}_4, \dots, \bar{p}_k) = \omega) \\ &- \mathbf{P}(\text{position } (i+2) \text{ is non-red and non-black} | \text{in the } i^{\text{th}} \text{ position there is a black card in permutation } D', \\ &\quad (\bar{b}, \bar{r}, \bar{p}_3, \bar{p}_4, \dots, \bar{p}_k) = \omega) \\ &- \dots - \mathbf{P}((\text{position } (4s) \text{ is non-red and non-black} | \text{in the } i^{\text{th}} \text{ position there is a black card in permutation } D', \\ &\quad (\bar{b}, \bar{r}, \bar{p}_3, \bar{p}_4, \dots, \bar{p}_k) = \omega) - 2i). \end{aligned}$$

We can compute one of these probabilities in the following way:

$$\mathbf{P}(\text{position } j \text{ is non-red and non-black} | \text{in the } i^{\text{th}} \text{ position there is a black card in permutation } D',$$

$$\begin{aligned}
& (\bar{b}, \bar{r}, \bar{p}_3, \bar{p}_4, \dots, \bar{p}_k) = \omega) \\
&= \frac{p_N}{s-1} \mathbf{1}\{\text{position } j \text{ belongs to player North in the dealing method}\} \\
&+ \frac{p_E}{s} \mathbf{1}\{\text{position } j \text{ belongs to player East in the dealing method}\} \\
&+ \frac{p_S}{s} \mathbf{1}\{\text{position } j \text{ belongs to player South in the dealing method}\} \\
&+ \frac{p_W}{s} \mathbf{1}\{\text{position } j \text{ belongs to player West in the dealing method}\}.
\end{aligned}$$

When calculating the conditional expectations this probability has a positive sign if  $j < i$ , and a negative sign if  $i < j$ . This is true for every position  $j \neq i$ . If we go over every position  $i$  which belongs to player North this will still be true for every position  $j \neq i$ .

If we analyze just those positions  $j$  which belong to player East, this probability will have a positive sign whenever  $j < i$ , and a negative sign whenever  $i < j$ . This is what we called  $Z(E, N)$ .

If we analyze just those positions  $j$  which belong to player South, this probability will have a positive sign whenever  $j < i$ , and a negative sign whenever  $i < j$ . This is what we called  $Z(S, N)$ .

If we analyze just those positions  $j$  which belong to player West, this probability will have a positive sign whenever  $j < i$ , and a negative sign whenever  $i < j$ . This is what we called  $Z(W, N)$ .

The value  $Z(N, N)$  is trivially 0 for each dealing method.

Using these results we get:

$$\left( \sum_{i=1}^s |D' : (\bar{b}, \bar{r}, \bar{p}_3, \bar{p}_4, \dots, \bar{p}_k) = \omega,$$

in the  $i^{\text{th}}$  position there is a black card in permutation  $D'$  and position  $i$  belongs to player North|

$\cdot \mathbf{E}(U(i)|$  in the  $i^{\text{th}}$  position there is a black card in permutation  $D'$  and position  $i$  belongs to player North,

$$\begin{aligned}
& (\bar{b}, \bar{r}, \bar{p}_3, \bar{p}_4, \dots, \bar{p}_k) = \omega) \Big) \\
&= |D' : (\bar{b}, \bar{r}, \bar{p}_3, \bar{p}_4, \dots, \bar{p}_k) = \omega| \left( \frac{b_N}{s} \left( \frac{p_E \cdot Z(E, N) + p_S \cdot Z(S, N) + p_W \cdot Z(W, N)}{s} - 2 \sum_{p=1}^s N_p \right) \right).
\end{aligned}$$

So here we analyzed just those positions which belong to player North.

If we analyze every position, we get the following: (see also (3))

$$\begin{aligned}
& \sum_{D' : (\bar{b}, \bar{r}, \bar{p}_3, \bar{p}_4, \dots, \bar{p}_k) = \omega} Z(D', B, R) \\
&= s!^4 \left( \prod_{i=1}^k \frac{1}{p_{i,N}! \cdot p_{i,E}! \cdot p_{i,S}! \cdot p_{i,W}!} \right) \left( (4s+1)b + \frac{b_N}{s} \left( \frac{p_E \cdot Z(E, N) + p_S \cdot Z(S, N) + p_W \cdot Z(W, N)}{s} - 2 \sum_{p=1}^s N_p \right) \right. \\
&\quad + \frac{b_E}{s} \left( \frac{p_N \cdot Z(N, E) + p_S \cdot Z(S, E) + p_W \cdot Z(W, E)}{s} - 2 \sum_{p=1}^s E_p \right) \\
&\quad + \frac{b_S}{s} \left( \frac{p_N \cdot Z(N, S) + p_E \cdot Z(E, S) + p_W \cdot Z(W, S)}{s} - 2 \sum_{p=1}^s S_p \right) \\
&\quad \left. + \frac{b_W}{s} \left( \frac{p_N \cdot Z(N, W) + p_E \cdot Z(E, W) + p_S \cdot Z(S, W)}{s} - 2 \sum_{p=1}^s W_p \right) \right),
\end{aligned}$$

so the proof is complete. □

## 4 The case of more players

Consider the following case: there are  $\ell$  players and, in the deck, there are  $\ell \cdot s$  cards and there are  $k$  different card values.

The variation distance will be the following:

$$\|\mathbb{P}_a - \Pi\| = a^{-1} \frac{s \cdot \left( \prod_{j=1}^k p_j! \right)}{(\ell s)!} \sum_{\omega \in \Omega} \left| \sum_{i=1}^{k-1} \frac{\sum_{D': (\bar{p}_1, \bar{p}_2, \dots, \bar{p}_k) = \omega} Z(D', P_i, P_{i+1})}{p_i \cdot p_{i+1}} \right| + O(a^{-2}),$$

where  $p_1, p_2, \dots, p_k$  are the number of cards coloured  $P_1, P_2, \dots, P_k$  in the deck.

The main question is how we can calculate  $\sum_{D': (\bar{p}_1, \bar{p}_2, \dots, \bar{p}_k) = \omega} Z(D', X, Y)$ , where  $X$  and  $Y$  are two different types. Let  $x_i$  be the number of  $X$  coloured cards which player  $i$  receives after the dealing method and  $x$  be the number of cards which have value  $X$ . Let  $p_{o,j}$  be the number of  $P_o$  coloured cards which player  $j$  receives after the dealing method.

**Theorem 4.1.**

$$\begin{aligned} & \sum_{D': (\bar{p}_1, \bar{p}_2, \dots, \bar{p}_k) = \omega} Z(D', X, Y) \\ &= |D' : (\bar{p}_1, \bar{p}_2, \dots, \bar{p}_k) = \omega| \left( (\ell s + 1)x + \frac{1}{s} \sum_{i=1}^{\ell} x_i \left( \sum_{j=1, j \neq i}^{\ell} \frac{((\sum_{o=1}^k p_{o,j}) - x_j - y_j) \cdot Z(j, i)}{s} - 2 \sum_{t=1}^s i_t \right) \right) \\ &= |D' : (\bar{p}_1, \bar{p}_2, \dots, \bar{p}_k) = \omega| \left( \sum_{i=1}^{\ell} x_i \left( \ell s + 1 - 2 \frac{\sum_{t=1}^s i_t}{s} + \sum_{j=1, j \neq i}^{\ell} \frac{((\sum_{o=1}^k p_{o,j}) - x_j - y_j) \cdot Z(j, i)}{s^2} \right) \right), \end{aligned}$$

where  $i_1$  is the first position which goes to player  $i$  in the dealing method,  $i_2$  is the second position which goes to player  $i$  in the dealing method, etc.

*Proof.* We can prove this in the same way as in the previous case, when there were only 4 players.  $\square$

**Theorem 4.2.** *The coefficient of  $a^{-1}$  is exactly  $s$  times larger in the ordered dealing than in the cyclic dealing for every possible  $k$  and  $\ell$  values, and if  $s$  is an even number then the coefficient is 0 in the back and forth dealing, if  $s$  is an odd number then the coefficient is exactly  $s$  times smaller in the back and forth dealing than in the cyclic dealing for every possible  $k$  and  $\ell$  values.*

*Proof.* In the ordered dealing suppose that player  $j$  receives all his cards before player  $i$  receives his first card. In this case in the representing sequence there are  $s^2 j - i$  pairs and 0  $i - j$  pairs. So  $Z(j, i) = s^2$  in this case.

In the representing sequence of the cyclic dealing the first position which belongs to player  $j$  stands in  $j - i$  pair with  $s$  positions which belong to player  $i$ , the second position which belongs to player  $j$  stands in  $j - i$  pair with  $s - 1$  positions which belong to player  $i$ , the last position which belongs to player  $j$  stands in  $j - i$  pair with 1 position which belongs to player  $i$ ; if  $j$  receives his first card before player  $i$ . Similarly, the first position which belongs to player  $i$  stands in  $i - j$  pair with  $s - 1$  positions which belong to player  $j$ , the second position which belongs to player  $i$  stands in  $i - j$  pair with  $s - 2$  positions which belong to player  $j$ , the last position which belongs to player  $i$  stands in  $i - j$  pair with 0 positions which belong to player  $j$ ; if  $j$  receives his first card before player  $i$ . So  $Z(j, i) = s$  in this case.

If  $s$  is an even number then the representing sequence of the back and forth dealing is symmetric, so  $Z(j, i) = 0$ .

If  $s$  is an odd number then in the representing sequence of the back and forth dealing let the first  $\ell s - \ell$  positions be the first group, the last  $\ell$  positions be the second group. The first group is symmetric, so within the first group  $Z(j, i) = 0$ . In the first group the  $s - 1$  positions which belong to player  $j$  stand in  $j - i$  pair with the position which belongs to player  $i$  in the second group. In the first group the  $s - 1$  positions which

belong to player  $i$  stand in  $i - j$  pair with the position which belongs to player  $j$  in the second group. If player  $j$  receives his first card before player  $i$  then in the second group the position which belongs to player  $j$  stands in  $j - i$  pair with the position which belongs to player  $i$  in the second group. So  $Z(j, i) = 1$  in this case.

So we proved the following:

Ordered dealing:  $Z(j, i) = s^2 \cdot I_{j, i}$ .

Cyclic dealing:  $Z(j, i) = s \cdot I_{j, i}$ ,

Back and forth:  $Z(j, i) = I_{j, i}$ , if  $s$  is an odd number.

Back and forth:  $Z(j, i) = 0$ , if  $s$  is an even number.

Here:

$$I_{j, i} = \begin{cases} 1, & \text{player } j \text{ receives his first card before player } i \\ -1, & \text{player } i \text{ receives his first card before player } j. \end{cases}$$

This is true for each pair  $(i, j)$  of players, therefore the same holds for  $\sum_{j=1, j \neq i}^{\ell} \frac{((\sum_{o=1}^k p_{o, j}) - x_j - y_j) \cdot Z(j, i)}{s^2}$ . this is  $s$  times bigger for the ordered dealing than for the cyclic dealing, and this sum is  $s$  times bigger for the cyclic dealing than for the back and forth dealing with odd  $s$  values, while the sum is zero for the back and forth dealing with even  $s$  values.

Now we analyze  $2 \frac{\sum_{t=1}^s i_t}{s}$ .

Ordered dealing:

In the representing sequence the position  $(i-1)s+1, (i-1)s+2, \dots, is$  belong to the  $i^{th}$  player, so we have to sum up these positions when we compute  $\sum_{t=1}^s i_t$ .

$$2 \frac{\sum_{t=1}^s i_t}{s} = 2 \frac{((i-1)s+1 + is)s}{2s} = 2is - s + 1.$$

Cyclic dealing:

In the representing sequence the position  $i, \ell+i, 2\ell+i, (s-1)\ell+i$  belong to the  $i^{th}$  player, so we have to sum up these positions when we compute  $\sum_{t=1}^s i_t$ .

$$2 \frac{\sum_{t=1}^s i_t}{s} = 2 \frac{(2i + \ell(s-1))s}{2s} = 2i + \ell(s-1).$$

Back and forth dealing:

In the representing sequence the position  $i, 2\ell-(i-1), 2\ell+i, 4\ell-(i-1), 4\ell+i, 6\ell-(i-1), \dots, \ell s-(i-1)$  belong to the  $i^{th}$  player if  $s$  is an even number, so we have to sum up these positions when we compute  $\sum_{t=1}^s i_t$ .

$$2 \frac{\sum_{t=1}^s i_t}{s} = 2 \frac{(\ell s + 1)s}{2s} = (\ell s + 1),$$

if  $s$  is an even number.

In the representing sequence the position  $i, 2\ell-(i-1), 2\ell+i, 4\ell-(i-1), 4\ell+i, 6\ell-(i-1), \dots, \ell s-(\ell-i)$  belong to the  $i^{th}$  player if  $s$  is an odd number, so we have to sum up these positions when we compute  $\sum_{t=1}^s i_t$ .

$$2 \frac{\sum_{t=1}^s i_t}{s} = \frac{2}{s} \left( \frac{(1 + \ell(s-1))(s-1)}{2} + \ell(s-1) + i \right) = \frac{1}{s} (\ell s^2 + s - 1 + 2i - \ell),$$

if  $s$  is an odd number.

Now we can easily analyze  $\ell s + 1 - 2 \frac{\sum_{t=1}^s i_t}{s}$ .

Ordered dealing:

$$\ell s + 1 - 2 \frac{\sum_{t=1}^s i_t}{s} = s(\ell - 2i + 1).$$

Cyclic dealing:

$$\ell s + 1 - 2 \frac{\sum_{t=1}^s i_t}{s} = (\ell - 2i + 1).$$

Back and forth dealing:

$$\ell s + 1 - 2 \frac{\sum_{t=1}^s i_t}{s} = 0,$$

if  $s$  is an even number.

$$\ell s + 1 - 2 \frac{\sum_{t=1}^s i_t}{s} = \frac{1}{s}(\ell - 2i + 1),$$

if  $s$  is an odd number.

Thus we see that these terms also differ by factors of  $s$  when comparing the ordered, cyclic, and back and forth dealing methods (odd  $s$  values), while this term is also 0 for the back and forth dealing if  $s$  is an even number.

We proved the claim for each term, so it is true for  $\sum_{D':(\bar{p}_1, \bar{p}_2, \dots, \bar{p}_k)=\omega} Z(D', X, Y)$  and therefore it is also true for the coefficient of  $a^{-1}$ , so we proved the statement.  $\square$

## 5 The case of arbitrary initial deck

Now, we suppose that the initial deck is arbitrary. In this case the variation distance is the following:

$$\frac{1}{2} \sum_{\omega \in \Omega} |\mathbb{P}_a(\omega) - \Pi(\omega)| = a^{-1} \frac{s}{(\ell s)!} \left( \prod_{j=1}^k p_j! \right) \sum_{\omega \in \Omega} \left| \sum_{a < b} \frac{W(D, a, b) \sum_{D':(\bar{p}_1, \bar{p}_2, \dots, \bar{p}_k)=\omega} Z(D', a, b)}{n_a \cdot n_b} \right| + O(a^{-2}).$$

In this formula the only term which depends on the dealing method is  $\sum_{D':(\bar{p}_1, \bar{p}_2, \dots, \bar{p}_k)=\omega} Z(D', a, b)$ . In the previous section we proved for these terms the following: these terms differ by factors of  $s$  when comparing the ordered, cyclic, and back and forth dealing with odd  $s$  values, while this term is also 0 for the back and forth dealing if  $s$  is an even number. We can easily see that these terms do not depend on the initial deck, therefore for every initial deck the coefficient of  $a^{-1}$  is  $s$  times bigger for the ordered dealing than for the cyclic dealing and  $s$  times bigger for the cyclic dealing than for the back and forth dealing with odd  $s$  values, while the coefficient of  $a^{-1}$  is zero for the back and forth dealing with even  $s$  values.

## 6 The case of two or three different types of cards

### 6.1 Two different types of cards

In this section we analyze those cases when there are only two types of cards: black and red, 52 cards in the deck and four players. Using our results we know that the coefficient of  $a^{-1}$  is exactly 13 times smaller in the cyclic dealing method than in the ordered dealing method, and 13 times smaller in the back and forth dealing method than in the cyclic dealing method. Let  $b$  be the number of black cards in the deck. Using the computer and our formulas we are able to compute the coefficient of  $a^{-1}$  for any possible value  $b$ .

With using (3.1) the coefficient of  $a^{-1}$  in (4) will be the following:

$$\begin{aligned} & \frac{13}{b(52-b) \binom{52}{b}} \sum_{\omega \in \Omega} \left| \binom{13}{b_N} \binom{13}{b_E} \binom{13}{b_S} \binom{13}{b_W} \left( 53b \right. \right. \\ & \left. \left. - \frac{2b_N}{13} \sum_{p=1}^{13} N_p - \frac{2b_E}{13} \sum_{p=1}^{13} E_p - \frac{2b_S}{13} \sum_{p=1}^{13} S_p - \frac{2b_W}{13} \sum_{p=1}^{13} W_p \right) \right|. \end{aligned}$$

In this formula there are four sums: the sum of positions  $i$  where  $f(i) = N$ , the sum of positions  $i$  where  $f(i) = E$ , the sum of positions  $i$  where  $f(i) = S$  and the sum of positions  $i$  where  $f(i) = W$ . In the ordered dealing the values of these sums are the following: 91; 260; 429; 598. In the cyclic dealing the values of these sums are the following: 325; 338; 351; 364. In the back and forth dealing the values of these sums are the

following: 343; 344; 345; 346. We see that the distances of the different values of the sums are the smallest in the back and forth dealing and the largest in the ordered dealing in these methods. In this sense the best dealing of these methods is the back and forth dealing, and the worst method is the ordered dealing. We can check that in an unfeasible dealing method in which the sums are 344,5; 344,5; 344,5; 344,5 the coefficient of  $a^{-1}$  would be 0. We think that dealing method  $A$  is better than dealing method  $B$  if the following value is smaller in  $A$  than in  $B$ :  $|\sum_{p=1}^{13} N_p - 344, 5| + |\sum_{p=1}^{13} E_p - 344, 5| + |\sum_{p=1}^{13} S_p - 344, 5| + |\sum_{p=1}^{13} W_p - 344, 5|$ . Our conjecture is that the best dealing method is a dealing method in which two sums are 344 and two sums are 345. As an example, let  $g$  be the following:

$$\begin{aligned} g(2) &= g(8) = g(9) = g(16) = g(17) = g(24) = g(25) = g(32) = g(33) = g(40) \\ &= g(41) = g(48) = g(49) = N, \\ g(3) &= g(7) = g(10) = g(15) = g(18) = g(23) = g(26) = g(31) = g(34) = g(39) \\ &= g(42) = g(47) = g(50) = E, \\ g(1) &= g(6) = g(12) = g(14) = g(19) = g(22) = g(27) = g(30) = g(35) = g(38) \\ &= g(43) = g(46) = g(51) = S, \\ g(4) &= g(5) = g(11) = g(13) = g(20) = g(21) = g(28) = g(29) = g(36) = g(37) \\ &= g(44) = g(45) = g(52) = W. \end{aligned}$$

So the representing sequence is:

*SNEWWSENNEWSWSENNESWWSENNESWWSEN*

*NESWWSENNESWWSENNESW.*

If we sketch a graph we can see that this dealing method has a smaller coefficient than the back and forth dealing for each value  $b$ .

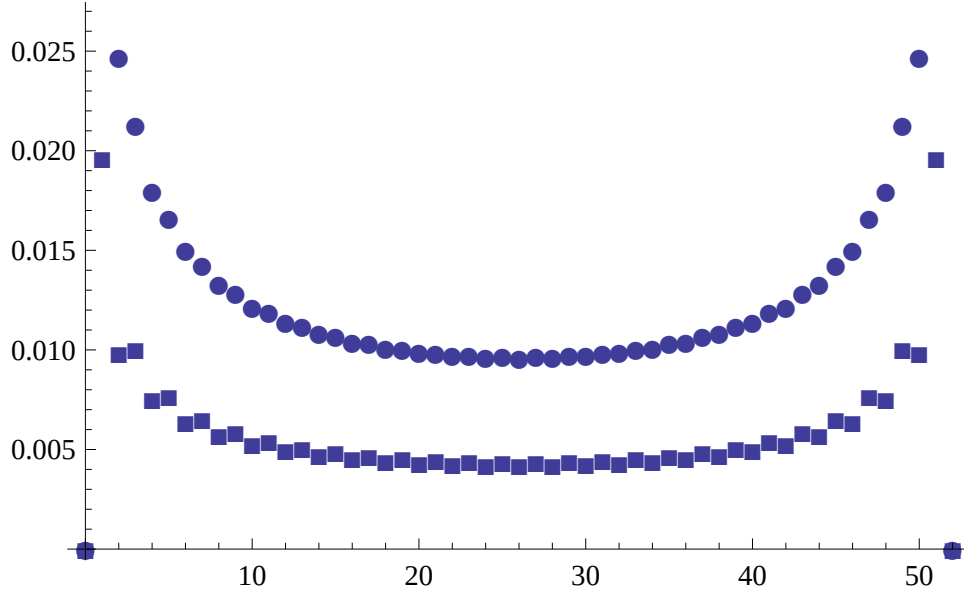


Figure 1: The coefficient of  $a^{-1}$  for each possible value  $b$  in the back and forth and in our conjectured best dealing method. The horizontal axis marks the value  $b$  and the vertical axis marks the coefficient of  $a^{-1}$ . Square plots the best dealing method and circle plots the back and forth dealing method. For better illustration we did not represent the points  $b = 1$  and  $51$  for the ordered dealing.

If we compare these coefficients and the coefficients of the back and forth dealing, we can see that these coefficients are circa twice smaller than the coefficients of the back and forth dealing. So for large  $a$  we can save circa 1 riffle shuffle if we use this dealing method instead of the back and forth dealing. We note that there are other dealing methods which have the same coefficient of  $a^{-1}$  for each value  $b$ , but our conjecture is that we cannot construct a better dealing method.

## 6.2 Three different types of cards

Now we suppose that there are three different types of cards in the deck: black, red and green. The number of cards is 52 and there are four players. By our results we know that the coefficient of  $a^{-1}$  is exactly 13 times smaller in the cyclic dealing method than in the ordered dealing method, and 13 times smaller in the back and forth dealing method than in the cyclic dealing method. Let  $b$  be the number of black cards,  $r$  be the number of red cards,  $g$  be the number of green cards in the deck. If we know  $b$  and  $r$  we also know  $g$ , because  $g = 52 - b - r$ . Using the computer and our formula we are able to compute the coefficient of  $a^{-1}$  for any possible  $b, r$  values. Figure 2 shows the coefficient of  $a^{-1}$  for each possible  $b, r$  values in the back and forth dealing method.

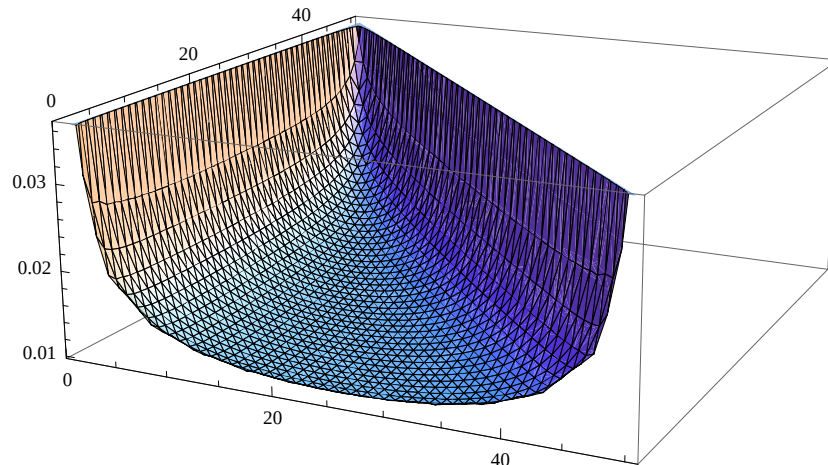


Figure 2: The coefficient of  $a^{-1}$  for each possible value  $b, r$  in the back and forth dealing method.

A main question is whether the former dealing method  $g$  in which two sums are 344 and two sums are 345 is better than the back and forth dealing in this case. The answer is: not for every configurations. For configuration:  $b = 1, r = 1$  the coefficient of  $a^{-1}$  is  $\frac{56}{1275}$  for the black and forth dealing and  $\frac{76}{1275}$  for dealing method  $g$ . So we can see that for this configuration the back and forth dealing is better than dealing method  $g$ . Figure 3 shows the coefficient of  $a^{-1}$  for each possible  $b, r$  values in dealing method  $g$ . In most cases dealing method  $g$  has a smaller coefficient than the back and forth dealing, but there are some configurations when the back and forth dealing is better than dealing method  $g$ . This proves that the conjecture what we drew up for two different types is false for this case. Hence this conjecture is not true for arbitrary different types of cards.

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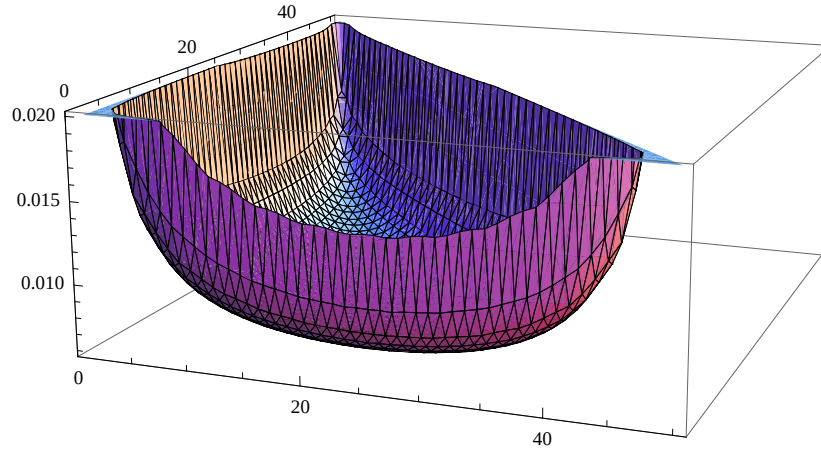


Figure 3: The coefficient of  $a^{-1}$  for each possible value  $b, r$  in dealing method  $g$ .

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