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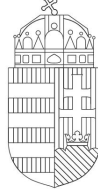
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A GENERALIZATION OF BARABÁSI PRIORITY MODEL

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Nemzeti
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In this paper we generalize the A.-L. Barabási priority queuing model, in which tasks are executed according to their priorities. In his work, A. Vázquez obtained exact results for the Barabási model. First, we add some new results related to the expected waiting time using the heuristics of Vázquez. Further, we give a probabilistic interpretation of that result of him, which determines the waiting time distribution of active tasks when the waiting list has length two. On the other hand, we introduce a natural generalization of the Barabási model, and calculate the expected waiting time at stationarity. Furthermore, we consider an example of the generalization and calculate numerically the distribution of active tasks with some fixed parameters.

1. Introduction. Nowadays, many human-driven phenomena has become momentous in different areas of life, for example in economics and in social sciences. Classical models of human dynamics are based on Poisson processes, so thus in these models the elapsed time between two consecutive events, the inter-event time is exponentially distributed. On the contrary, in many real cases the inter-event time has a heavy-tailed distribution, which distribution decreases in power-law. These heavy tailed distributions give relatively large weight to long inter-event times, and there are bursts of high activity between these inactive intervals. This is in sharp contrast with the exponential distribution, which gives low weight to the longer inter-event times.

Albert-László Barabási introduced a model [1] which exhibited the bursty nature of the arrival times of events in systems determined by decisions of some humans. In his model, somebody has a todo list which always consists of exactly L items, each of which has a non-negative priority. In every discrete time step $(1, 2, \dots)$ two things happen:

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- A task is selected for execution according to a selection protocol described later and leaves the system,
- another task arrives to the list.

The priority of all arriving task including the tasks at the 0-th step are i.i.d. random variables. The selection protocol in Barabási's model is as follows: In each step independently, the task with the highest priority is selected for execution and leaves the system with probability $0 \leq p \leq 1$. Otherwise, a task selected uniformly from the list leaves the system. In particular, if $p = 1$, then we use the highest priority first selection protocol, and if $p = 0$, then the selection is completely uniform. Using computer simulations, Barabási found that if $p \rightarrow 1$ then the waiting time τ of a task has a power law tail with exponent 1. These asymptotics hold not only for long lists but even if the list consists of two items, that is for $L = 2$. In this case Vázquez [9] proved analytically that

$$\lim_{p \rightarrow 1} \mathbb{P}(\tau = k) = \begin{cases} 1 + O\left(\frac{1-p}{2} \ln(1-p)\right) & \text{if } k = 1, \\ O\left(\frac{1-p}{2}\right) \frac{1}{k-1} & \text{if } k > 1. \end{cases}$$

In [9] the expected value of the waiting time was also investigated. For the case $L = 2$ Vázquez proved analytically that

$$\mathbb{E}(\tau) = \begin{cases} 2, & \text{if } p < 1; \\ 1, & p = 1 \end{cases}$$

in stationarity. If $p = 1$ this means that there is a task stuck in the queue with priority 0. For the case $L > 2$ Vázquez conjectured that

CONJECTURE 1.1 (Vázquez).

$$\mathbb{E}(\tau) = \begin{cases} L, & \text{if } p < 1; \\ 1, & p = 1 \end{cases}$$

in stationarity. If $p = 1$ stationarity means that there are $L - 1$ tasks stuck.

Our work is divided into two main parts.

- We extend some of the heuristic arguments of [9] related to the Barabási model to analytic proofs. We distinguish two cases according to $p < 1$ or $p = 1$.

First we assume that $0 \leq p < 1$. We extend the completely correct heuristic argument of Vázquez to an analytic proof.

Then we consider the case when $p = 1$, that is at every time step the task with the highest priority is selected. Although the relevant part of Vázquez's Conjecture 1.1 holds but the stationary case will not be achieved, since there will never be any tasks with priority 0 almost surely. Note that in this case at time t the L items on the list are as follows: Beside the newly added item we have the $L - 1$ lowest priority items among all items which have ever added to the list. Therefore we can use the theory

of the process of records. It is studied for $L = 2$ (Section 2.3), where all tasks remain on the list for one time step except for the records. An item is called lower record if it has the lowest priority at the time when it is added to the list. When a record arrives, then it remains on the list until a new record arrives. The time elapsed between the n -th and $n + 1$ -th records called the n -th inter-record time. It is well known [1] that the expected value of the inter-record times grow exponentially with n . Therefore simulations may indicate the false conclusion that one task remains indefinitely in the queue [9] [1].

- Then we study the Barabási-model for general priority-based selection protocols in the case $L = 2$. We analyze the two main characteristics of this model at stationarity, namely the distribution of the tasks priorities on the list and the waiting time τ a task spends on the list before execution. To explain these with more detail, we need to introduce some new notation. At time $t + 1$ there are two items on the list. The one which has just arrived to the list we call *new*, and the one which was already on the list at time t we call it *old* task. The distribution of the new task is given. Our goal is to understand the distribution of the old task. The selection protocol is described by the function v in the following way:

$$\mathbb{P}(\text{new is chosen} \mid \text{new has priority } x, \text{ old has } y) = v(x, y),$$

where $v : \text{supp}(r)^2 \subseteq \mathbb{R}^2 \rightarrow [0, 1]$, $\forall y \ v(., y)$ is increasing and $\forall x \ v(x, .)$ is decreasing. We use the notation of Vázquez, and also refer to the results in [9] for the *Barabási case*, when $v(x, y) = p\mathbb{1}_{x>y} + \frac{1-p}{2}$ (the model introduced by Barabási [1]). An other natural example is as follows.

EXAMPLE 1.2. *Let v be $v(x, y) = p\frac{x}{x+y} + \frac{1-p}{2}$, $0 \leq p \leq 1$ and the priority of the new task is uniformly distributed on $[c, 1]$, $0 < c < 1$. That is, we chose a task proportionally to their priorities with probability p , and uniformly with probability $1 - p$. This model takes into account not only the order of priorities, but also the proportion of those.*

Let $R_1(x, t)$ be the probability that the priority of the old task at time t is less than x . That is $R_1(x, t)$ is the cumulative distribution function (CDF) of the priority of the old task at time t . We write $r_1(x, t)$ for the corresponding density function (DF), that is $r_1(x, t) = \frac{d}{dx}R_1(x, t)$. Similarly the CDF and the DF of the priority of the new task is denoted by $R(x)$ and $r(x)$, respectively.

Very useful notations are the following ones. The probability that the new task is selected given the old task has priority s is

$$(1.1) \quad q(s) = \int_0^\infty v(y, s) dR(y),$$

and the probability that the new task is selected given the old task has priority s is

$$q_1(s, t) = \int_0^\infty (1 - v(s, y)) dR_1(y, t).$$

With these one can easily describe the evolution of $R(x, t)$. Assume that at time $t + 1$ the old task is T^* . Then either

- (a) T^* was the old task at time t , or
- (b) T^* was the new task at time t .

Hence

$$R_1(x, t) = \underbrace{\int_0^x r_1(s, t)q(s)ds}_{\text{case(a)}} + \underbrace{\int_0^x r(s)q_1(s, t)ds}_{\text{case(b)}}.$$

Assuming there is stationarity, the stationary probability that the old task is selected given the new task has priority s is

$$(1.2) \quad q_1(s) = \int_0^\infty (1 - v(s, y))dR_1(y),$$

and for the stationary CDF $R_1(x)$ and DF $r_1(x)$ of the priority of the old task we obtain the stationary equation

$$(1.3) \quad R_1(x) = \int_0^x r_1(s)q(s)ds + \int_0^x r(s)q_1(s)ds.$$

In the very special *Barabási case* Vázquez [9] determined explicitly the distribution $R_1(x)$ of the priority of the old task at stationarity (2.1). However, in the general case the function $R_1(x)$ cannot be expressed by a closed formulae. We will use Hilbert-Schmidt operator techniques to find an approximation of $R_1(x)$.

As far as the waiting time is concerned, the distribution of τ is:

$$(1.4) \quad \mathbb{P}(\tau = k) = \begin{cases} \int_0^1 (1 - q_1(x))dR(x) & \text{if } k = 1, \\ \int_0^1 q_1(x)(1 - q(x))q(x)^{k-2}dR(x) & \text{if } k > 1. \end{cases}$$

If $k = 1$, then the task has to be executed immediately, so the probability of this is exactly $1 - q_1(x)$ if the task has priority x . Otherwise, if $k > 1$, then the event that a task with priority x stays exactly k steps on the waiting list is the intersection of k independent events. When the task is added to the list, it must not be executed, this happens with probability $q_1(x)$, then it becomes old and has to stay $k - 2$ more steps on the list, the probability of this event is $q(x)^{k-2}$, and finally, the task has to be executed, which event has probability $1 - q(x)$.

In the *Barabási case* these integrals can be explicitly computed [9] (2.2). In the general case only the expected waiting time can be explicitly computed, for which we show in this paper that $\mathbb{E}(\tau) = 2$ for all remarkable selection protocols (Section(3.2)).

2. Exact results.

2.1. *The results of Vázquez.* In this section, we review some results of A. Vázquez [9] obtained in the case the waiting list has length $L = 2$. We will later see that the system

has a stationary distribution for the old task's priority if $p < 1$. In this case the stationary equation (1.3) is

$$R_1(x) = \int_0^x r_1(s)q(s)ds + \int_0^x r(s)q_1(s)ds,$$

where *old*'s DF and CDF are $r_1(x)$ and $R_1(x)$, furthermore $q_1(s) = p(1 - R_1(s)) + \frac{1-p}{2}$. Using the definitions of $q(s)$ and $q_1(s)$, $R_1(x)$ can be exactly determined in the following way:

$$\begin{aligned} R_1(x) &= \int_0^x \frac{1+p}{2}(r_1(s) + r(s)) - p(r_1(s)R(s) + r(s)R_1(s))ds \\ &= \frac{1+p}{2}(R_1(x) + R(x)) - p \int_0^x (R_1(s)R(s))'ds \\ &= \frac{1+p}{2}(R_1(x) + R(x)) - pR_1(x)R(x). \end{aligned}$$

From here $R_1(x)$ can be easily obtained:

$$(2.1) \quad R_1(x) = \frac{(1+p)R(x)}{1-p+2pR(x)} = \frac{1+p}{2p} \left(1 - \frac{1}{1 + \frac{2p}{1-p}R(x)} \right).$$

This $R_1(x)$ is absolutely continuous and satisfies the stationary equation, so the system has a stationary distribution for the old task's priority if $p < 1$.

If we consider a task arriving into the system, it is our main interest to compute its waiting time till execution. The distribution of the waiting time is given by (1.4):

$$\mathbb{P}(\tau = k) = \begin{cases} \int_0^1 (1 - q_1(x))dR(x) & \text{if } k = 1, \\ \int_0^1 q_1(x)(1 - q(x))q(x)^{k-2}dR(x) & \text{if } k > 1. \end{cases}$$

It turns out that these integrals can be explicitly computed, hence we obtain

$$(2.2) \quad \mathbb{P}(\tau = k) = \begin{cases} 1 - \frac{1-p^2}{4p} \ln \frac{1+p}{1-p} & \text{if } k = 1, \\ \frac{1-p^2}{4p} \left(\left(\frac{1+p}{2} \right)^{k-1} - \left(\frac{1-p}{2} \right)^{k-1} \right) \frac{1}{k-1} & \text{if } k > 1. \end{cases}$$

As it was expected, the waiting time is independent of $R(x)$. This phenomenon can be explained by the fact that the waiting time of a task depends only on its relative priority respect to the others, so in the case of absolute continuous distributions, with a single change of variables we can transform every other DF back to the uniform case.

From equation (2.2) we can derive a few more consequences:

1. If $p = 0$, i.e. the protocol is completely random, which we also got back in the limit:

$$\lim_{p \rightarrow 0} \mathbb{P}(\tau = k) = \left(\frac{1}{2} \right)^k.$$

2. On the other hand, if $p \rightarrow 1$, then we select only according to priority. After some simple analytical calculations we obtain:

$$\lim_{p \rightarrow 1} \mathbb{P}(\tau = k) = \begin{cases} 1 + O\left(\frac{1-p}{2} \ln(1-p)\right) & \text{if } k = 1, \\ O\left(\frac{1-p}{2}\right) \frac{1}{k-1} & \text{if } k > 1. \end{cases}$$

Thus we see that the waiting time essentially follows a power law distribution, which was also expected by the numerical simulations of A.-L. Barabási [1].

3. When $0 \leq p < 1$ is fixed and $k \rightarrow \infty$

$$\mathbb{P}(\tau = k) \sim \frac{1-p^2}{4} \frac{1}{k} e^{(-\frac{k}{k_0})},$$

where

$$k_0 = \left(\ln \frac{2}{1+p} \right)^{-1},$$

and $a_n \sim b_n$ means that $\frac{a_n}{b_n} \rightarrow 1$ as $n \rightarrow \infty$.

4. Finally, the expectation of τ , when $0 \leq p < 1$:

$$\mathbb{E}(\tau) = \sum_{i=1}^{\infty} i \mathbb{P}(\tau = i) = 2.$$

2.2. Probabilistic interpretation. Consider the selection protocol defined in Section 2.1 with $p < 1$. Now, we give a probabilistic interpretation of this process, and we determine the CDF $R_1(x)$ of the priority of the old task in the stationary case in this manner. Finally, this reasoning gives back the result obtained by Vázquez [9].

We are looking back in time and count how many other tasks had an old task to compete with in order to stay in the list. First, if the selection protocol turns out to be the random choice and a new task is executed, it leaves the list immediately. Thus, the old task does not have to compete with its priority at all, so we do not have to count these cases. This event R_n happens in each step with probability $\frac{1-p}{2}$. Secondly, if random protocol is chosen, and the old task is selected (event R_o), then the arriving new task becomes old and the old task before it leaves the system. Since the priority distribution of the new task is just $R(x)$, considering the old task's priority distribution, the system restarts at each event R_o . The third event which can happen at each step is just priority selection, P . Thus the process can be coded by sequences of type $\{R_n, R_o, P\}^{\mathbb{N}}$.

Let X_t be the t -th element of this sequence. Then the renewal in every R_o event can be formalized as follows. Using the fact that the sequence $\{X_t\}_{t=1}^{\infty}$ is independent of the priorities, for every $x \in \mathbb{R}$

$$\mathbb{P}(O_{t+1} < x \mid X_t = R_o) = \mathbb{P}(N_t < x \mid X_t = R_o) = \mathbb{P}(N_t < x) = R(x),$$

where O_t and N_t is the priority of the *old* and the *new* task at time step t .

Since the priority distribution of the old task is renewed at every event R_o , in order to determine the priority distribution of the old task in the system for a big enough T , we have to count how many P -s we had after the last R_o event.

It is easy to see that the arrival time of the last R_o is just $T - \text{GEO}\left(\frac{1-p}{2}\right) + 1$. Similarly, if we only want to count the number of P -s after the last R_o , we have to re-normalize the probabilities to exclude cases of R_n , so it's distribution is

$$\text{GEO}\left(\frac{\frac{1-p}{2}}{p + \frac{1-p}{2}}\right) - 1 = \text{GEO}\left(\frac{1-p}{1+p}\right) - 1.$$

Thus, the old task in the system at time T had to compete with the other tasks at each event P and had to "win", i.e its priority had to be less than all of them, in order to stay in. So, its priority had to "defeat" $GEO\left(\frac{1-p}{1+p}\right) - 1$ many other tasks to stay in the list so far. Hence the distribution we were looking for is the minimum of $X \sim GEO\left(\frac{1-p}{1+p}\right)$ many independent random variables $(\{Y_i\}_{i=1}^X)$, of distribution $R(x)$. Let us compute it's CDF using the law of total probability:

$$\begin{aligned}
 F(x) &= \mathbb{P}(\min_{i=1 \dots X} Y_i < x) = \mathbb{P}(\exists i \leq X : Y_i \leq x) \\
 &= \sum_{k=1}^{\infty} \mathbb{P}(\exists i \leq X : Y_i \leq x \mid X = k) \mathbb{P}(X = k) \\
 &= \frac{1-p}{1+p} \sum_{k=1}^{\infty} \left(\frac{2p}{1+p}\right)^{k-1} (1 - (1 - R(x))^k) \\
 &= 1 - (1 - R(x)) \frac{1-p}{1+p} \sum_{k=1}^{\infty} \left(\frac{2p(1 - R(x))}{1+p}\right)^{k-1} \\
 &= \frac{1+p}{2} \left(1 - \frac{1}{1 + \frac{2p}{1-p} R(x)}\right),
 \end{aligned}$$

as in the paper [9], since the summation converges if $p < 1$.

2.3. Records and expected waiting time. We have seen that the system has a stationary distribution for the old task's priority if $p < 1$. We have also seen that the expected waiting time of a task in the stationary system is 2. Using the arguments introduced by Vázquez [9] let us denote by τ_i the waiting time of the task executed at the i -th step, and by $\tau'_{1,t}$ the resident time of the task that is still in the buffer at the t step. Since the total buffer time is $2t$ and in each time step exactly one task leaves the system, we have that

$$\sum_{i=1}^t \tau_i + \tau'_{1,t} = 2t.$$

Hence

$$(2.3) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=1}^t \tau_i = 2 - \lim_{t \rightarrow \infty} \frac{\tau'_{1,t}}{t}$$

if these limits exist. If $p < 1$, then we have shown that $\tau'_{1,t}$ has a stationary distribution given by formulas (2.2). So $\frac{\tau'_{1,t}}{t} \rightarrow 0$ as $t \rightarrow \infty$ holds almost surely (a.s.) and in L_1 .

As a consequence we get that the ergodic theorem also holds in this case, that is,

$$(2.4) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=1}^t \tau_i = \mathbb{E}\tau = 2. \quad \text{a.s. and in } L_1.$$

Let us introduce the notation $\{M_t\}_{t=1}^{\infty}$ of the old tasks' priorities. Clearly, the sequence is an aperiodic Markov-chain on the state space $[0, 1]$. When $p < 1$, irreducibility of the chain is the consequence of the argument in Section 2.2: the system restarts at each event

R_0 , which happens infinitely often a.s.. Thus the Markov-chain is ergodic, and this fact immediately implies that (2.4) holds.

This reasoning verifies rigorously the heuristics of Vázquez [9], and finds the expectation of τ in the stationary case. These arguments can be extended to the case when the buffer length L is greater than 2. In this case, there is no analytic formula for the distribution of waiting time yet, and there is no rigorous argument showing that its expectation is $\mathbb{E}_L(\tau) = L$.

THEOREM 2.1. [Ergodic theorem for $L \geq 2, p < 1$.] *The priority queueing system described in (1.3) is ergodic for any $L > 2$ and $p < 1$ and the following limit holds:*

$$(2.5) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=1}^t \tau_i = \mathbb{E}_L(\tau) = L. \quad \text{a.s. and in } L_1.$$

PROOF. Set $p < 1$ and $L > 2$. As before, denote by τ_i the waiting time of the task executed at the i -th step, and by $\tau'_{l,t}$ ($l = 1, \dots, L-1$) the resident times of the tasks that are still in the buffer at the t -th step. Then the analogue of equation (2.3) is

$$(2.6) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=1}^t \tau_i = L - \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{l=1}^{L-1} \tau'_{l,t}$$

if these limits exist. Similarly to the $L = 2$ case, the process renews whenever there are $L - 1$ successive R_0 events, where R_0 means that the oldest task is selected for execution because of the random selection. For every t the probability that there is a renewal section of the process starting at the step t is $\left(\frac{1-p}{L}\right)^{L-1}$. In addition, two renewal sections starting at time t_1 and t_2 are independent if $|t_1 - t_2| \geq L - 1$. So the system renews infinitely often a.s., implying that it has a stationary distribution for the $L - 1$ not newly added task's priority. Then, as before, it follows that

$$(2.7) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{l=1}^{L-1} \tau'_{l,t} = 0.$$

There remains the proof of ergodicity. Let us denote by $\underline{M}_t = (M_t^1, \dots, M_t^{L-1})$ the $L - 1$ dimensional vector of the oldest elements in the buffer at time t , such that M_t^1 is the oldest, M_t^i is the i -th oldest task in the system. Then clearly the sequence $\{\underline{M}_t\}_{t=1}^\infty$ is Markovian with state space $[0, 1]^{L-1}$. It is clearly aperiodic, and since the system renews, it is irreducible. Thus ergodicity and equation (2.6) and (2.7) immediately implies (2.5) as it was suggested by the heuristic arguments of Vázquez [9]. $\square$

2.3.1. Records. In this section we investigate the case $p = 1$, i.e. the case when the selection protocol has no randomness and always picks the task in the system with higher priority. We show that the system achieves no stationary distribution in this case.

It is easy to see that when a task arrives whose priority is less than all the priorities before, then the old task in the system will be chosen for execution and this task will

remain in the buffer. This implies that $R_1(x, t)$ (the distribution of the old task's priority at time t) is the distribution of the minimum, i.e. the lower record value of the first $t + 2$ tasks' priority. That is, we need to consider the minimum of $t + 2$ i.i.d. random variables with distribution $R(x)$.

On the other hand, a stationary distribution $R_1^s(x)$ should satisfy equation (1.3), which has only a degenerate solution $R_1(x) = 0$ a.s. if $p = 1$. Thus, the stationary distribution $R_1^s(t)$ will never be achieved, since

$$\mathbb{P}(\forall t \ R_1(x, t) \neq 0) = 1.$$

Hence, the properties of the waiting time τ are related to the process of records.

In order to introduce some classical theorems of record theory, we will need some definitions.

DEFINITION 2.2. *Let $\{X_t\}_{t=1}^\infty$ be i.i.d. random variables with absolutely continuous CDF F . Then we define the followings:*

1. *The sequence $\{T_k\}_{k=1}^\infty$ of lower record times is defined by*

$$\begin{aligned} T_1 &= 1, \text{ and for } k > 1 \\ T_k &= \min\{t \mid X_t < X_{T_{k-1}}\}. \end{aligned}$$

2. *The sequence $\{\Delta_k\}_{k=2}^\infty$ of inter-record times is defined by*

$$\Delta_k = T_k - T_{k-1}.$$

3. *The sequence $\{I_t\}_{t=1}^\infty$ of record indicators is defined by*

$$\begin{aligned} I_1 &= 1, \text{ and for } t > 1 \\ I_t &= \mathbb{1}\{X_t < \min\{X_1, \dots, X_{t-1}\}\}. \end{aligned}$$

4. *The sequence $\{x_k\}_{k=1}^\infty$ of lower record values is defined by*

$$x_k = X_{T_k}.$$

If we do not care about the record values, without loss of generality one can assume that the distribution of the X_i -s is $U[0, 1]$. Thus one can easily compute the joined distribution of the I_t -s. Let $n, 1 = t_1 < t_2 < \dots < t_n$ be arbitrary positive integers. Then

$$\begin{aligned} &\mathbb{P}(I_{t_1} = 1, \dots, I_{t_n} = 1) \\ &= \int_{0 < x_1 < \dots < x_n < 1} \mathbb{P}(I_{t_1} = 1, \dots, I_{t_n} = 1 \mid X_{t_1} = x_1, \dots, X_{t_n} = x_n) dx_1 \dots dx_n \\ &= \int_{0 < x_1 < \dots < x_n < 1} x_1^{t_2-1} x_2^{t_3-t_2-1} \dots x_n^{t_n-t_{n-1}-1} dx_1 \dots dx_n \\ &= \frac{1}{t_1} \frac{1}{t_2} \dots \frac{1}{t_n}. \end{aligned}$$

Since the integers $n, 1 = t_1 < t_2 < \dots < t_n$ were arbitrary, and the I_t -s are discrete random variables, it is evident that these random variables are independent, and $I_t \sim \text{BER}(\frac{1}{t})$.

Now consider the ratio $\frac{\Delta_k}{T_k}$. The distribution of $\frac{\Delta_k}{T_k}$ conditioned on the value of T_{k-1} can be determined by the reasoning of Tata [8]:

$$\begin{aligned} \mathbb{P}\left(\frac{\Delta_k}{T_k} > x \mid T_{k-1} = t\right) &= \mathbb{P}\left(\frac{\Delta_k}{t + \Delta_k} > x \mid T_{k-1} = t\right) \\ &= \mathbb{P}\left(\Delta_k > \frac{x}{1-x}t \mid T_{k-1} = t\right) \\ &= \mathbb{P}(I_{t+1} = 0, \dots, I_{\lfloor t/(1-x) \rfloor} = 0) \\ &= \frac{t}{t+1} \frac{t+1}{t+2} \dots \frac{\lfloor t/(1-x) \rfloor - 1}{\lfloor t/(1-x) \rfloor} \\ &= \frac{t}{\lfloor t/(1-x) \rfloor}. \end{aligned}$$

Since $T_k \rightarrow \infty$ a.s. as $k \rightarrow \infty$, hence

$$\lim_{k \rightarrow \infty} \mathbb{P}\left(\frac{\Delta_k}{T_k} > x\right) = 1 - x,$$

so thus $\frac{\Delta_k}{T_k}$ and $\frac{T_{k-1}}{T_k}$ are asymptotically $U[0, 1]$.

Now we turn back to the Barabási model in the case $p = 1$ and $L = 2$. Also in this situation, equation (2.3) holds:

$$\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=1}^t \tau_i = 2 - \lim_{t \rightarrow \infty} \frac{\tau'_{1,t}}{t}$$

if these limits exist. But now, with the help of record theory one can prove that the limit on the right hand side does not exist. On the one hand if we choose $t = T_k + 1$ then we obtain:

$$\liminf_{t \rightarrow \infty} \frac{\tau'_{1,t}}{t} = 0 \text{ a.s.},$$

since the sequence of records is infinite a.s. (This is ensured by the continuity of the CDF F), so thus $\tau'_{1,t} = 0$ will hold for infinitely many t -s. On the other hand the *limsup* is taken if $t = T_k$ (we look the subsequence of records):

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\tau'_{1,t}}{t} &= \lim_{k \rightarrow \infty} \frac{\Delta_{k-1}}{T_k} = \lim_{k \rightarrow \infty} \frac{T_k - T_{k-1}}{T_k} \\ &= 1 - \lim_{k \rightarrow \infty} \frac{T_{k-1}}{T_k} = 1 - Z, \end{aligned}$$

where T_k and Δ_k are the k -th record and inter-record times, and Z is a $U[0, 1]$ random variable.

Thus, the heuristic of Vázquez does not work in this case. Note that the dynamics is not even ergodic, since the irreducibility does not hold, as the priority of the old task decreases monotonically.

Further, we can state precise asymptotic results about the waiting time of the records in the queue. It was proven by Holmes and Strawderman [5] that the Strong Law of Large

Numbers (SLLN), by Neuts [6] that the Central Limit Theorem (CLT) and by Strawderman and Holmes [7] that the Law of Iterated Logarithm (LIL) hold for the series of $\ln T_k$ and $\ln \Delta_k$ as well:

THEOREM 2.3. *Let z_k be either Δ_k or T_k . Then*

1. [SLLN][5]

$$\frac{\ln z_k}{k} \xrightarrow{a.s.} 1 \text{ as } k \rightarrow \infty.$$

2. [CLT][6]

$$\frac{\ln z_k - k}{\sqrt{k}} \Rightarrow N(0, 1) \text{ as } k \rightarrow \infty.$$

3. [LIL][7]

$$\begin{aligned} \limsup_{k \rightarrow \infty} \left\{ \frac{\ln z_k - k}{\sqrt{2k \ln \ln k}} \right\} &= 1 \text{ a.s.}, \\ \liminf_{k \rightarrow \infty} \left\{ \frac{\ln z_k - k}{\sqrt{2k \ln \ln k}} \right\} &= -1 \text{ a.s.} \end{aligned}$$

Thus we see that the k -th record spends roughly e^k time in the buffer.

3. A natural generalization of the model. In this section we consider an other variant of the Barabási model which were not analyzed in the literature before. Also in this new model there is a list with L tasks, the new tasks has i.i.d. non-negative priorities, and in each discrete time step one task is selected to execution and a new task arrives to it's place. The only change is that the dynamics of the system is now given by a general selection protocol. We will analyze this new model for the case $L = 2$, so we define the selection protocol to this case. In every time step we select a task according to the priorities in the following way:

$$\mathbb{P}(\text{new is chosen} \mid \text{new has priority } x, \text{ old has } y) = v(x, y),$$

where $v : \text{supp}(r)^2 \subseteq \mathbb{R}^2 \rightarrow [0, 1]$, $\forall y$ $v(\cdot, y)$ is increasing and $\forall x$ $v(x, \cdot)$ is decreasing.

Let us introduce the notation $\{M_t\}_{t=1}^\infty$ for the priorities of old tasks. Clearly, the sequence is an aperiodic Markov-chain on the state space $[0, 1]$.

LEMMA 3.1. *The Markov-chain $\{M_t\}_{t=1}^\infty$ defined above is positive recurrent if*

$$(\exists c)(\forall s) \ q(s) \leq c < 1.$$

PROOF. For any $x \in \text{supp}(r(x))$ let us denote by $T_{x,y,\epsilon}$ the hitting time from x to $(y - \epsilon, y + \epsilon)$. If $(\exists c)(\forall s) \ q(s) \leq c < 1$, then the distribution of $T_{x,y,\epsilon}$ is stochastically dominated by the geometric distribution with success probability $(1-c)(R(y+\epsilon) - R(y-\epsilon))$. Therefore

$$\mathbb{E}(T_{x,y,\epsilon}) \leq \frac{1}{(1-c)(R(y+\epsilon) - R(y-\epsilon))} < \infty.$$

□

REMARK 3.2. *As a consequence, in the Barabási case positive recurrence holds if $p < 1$.*

REMARK 3.3. *Similarly to Lemma 3.1:*

$$(\exists c)(\forall s) \ q_1(s) \geq c > 0 \implies \text{Positive recurrence.}$$

Let us consider the general case and suppose that $q_1(s) \geq c_1 > 0$ and $q(s) \leq c_2 < 1$. Then clearly stationarity holds. The stationary equation (1.3) is

$$R_1(x) = \int_0^x r_1(s)q(s)ds + \int_0^x r(s)q_1(s)ds.$$

After substituting (1.1) and (1.2) we obtain

$$R_1(x) = \int_0^x \int_0^\infty (r_1(s)r(y)v(y,s) + r(s)r_1(y)(1 - v(s,y)))dyds$$

Differentiation with respect to variable x yields the following equation for the stationary density function $r_1(x)$ of the old task's priority:

$$r_1(x) = \int_{y=0}^\infty (r_1(x)r(y)v(y,x) + r(x)r_1(y)(1 - v(x,y)))dy$$

With the constant c_2 from Remark 3.3 we can write:

$$r_1(x) \left(1 - \int_{y=0}^\infty r(y)v(y,x)dy \right) = r(x) \left(\int_{y=0}^\infty r_1(y)(1 - v(x,y))dy - c_2 \right) + c_2 r(x).$$

Easy to see that the coefficient of $r_1(x)$ on the left side cannot be zero, hence we arrive at:

$$(3.1) \quad \begin{aligned} r_1(x) &= \int_{y=0}^\infty \frac{r_1(y)r(x)(1 - v(x,y) - c_2)}{1 - \int_{z=0}^\infty r(z)v(z,x)dz} dy \\ &\quad + \frac{c_2 r(x)}{1 - \int_{y=0}^\infty r(y)v(y,x)dy}. \end{aligned}$$

For shorter formulas we introduce the following functions:

$$(3.2) \quad \begin{aligned} \alpha(x, y) &:= 1 - v(x, y) - c_2, \\ g(x) &:= \frac{r(x)}{1 - \int_{y=0}^\infty r(y)v(y, x)dy}, \\ K(x, y) &:= \alpha(x, y)g(x), \quad \text{and} \\ f(x) &:= \frac{c_2 r(x)}{1 - \int_{y=0}^\infty r(y)v(y, x)dy}. \end{aligned}$$

Let us further introduce the integral operator A associated to kernel function K , given by

$$A\varphi(s) := \int_{t=0}^\infty K(s, t)\varphi(t)dt.$$

With this notation, we see from (3.1) that the density function satisfies

$$r_1(x) = Ar_1(x) + f(x).$$

Hence, assuming that the operator $I - A$ is invertible for certain choice of the underlying distribution R and the selection protocol v , we would like to determine the function $\varphi(s)$ which satisfies

$$(3.3) \quad \varphi(s) = (I - A)^{-1} f(s)$$

If $\sum_{n=0}^{\infty} A^n f(s)$ converges we could write

$$\varphi(s) = \sum_{n=0}^{\infty} A^n f(s).$$

It turns out that A^n is also an integral operator with some kernel function K_n . Let us compute this kernel $K_n(s, t)$:

$$\begin{aligned} A^2 h(s) &= \int_0^{\infty} K(s, t) \int_0^{\infty} K(t, u) h(u) du dt = \int_0^{\infty} \underbrace{\int_0^{\infty} K(s, t) K(t, u) dt}_{K_2(s, u)} h(u) du \\ &= \int_0^{\infty} K_2(s, t) h(t) dt. \end{aligned}$$

Similarly

$$K_n(s, t) = \underbrace{\int_0^{\infty} \dots \int_0^{\infty}}_{n-1} K(s, u_1) K(u_1, u_2) \dots K(u_{n-1}, t) du_1 \dots du_{n-1}.$$

Note that we can benefit from the fact that $f(s)$ and $g(s)$ (see equation (3.2)) are a constant factor of each other, so we get

$$\begin{aligned} A^n f(s) &= A^n g(s) c = c_2 \int_0^{\infty} K_n(s, t) g(t) dt \\ &= c_2 \int_0^{\infty} \dots \int_0^{\infty} \alpha(s, u_1) g(s) \dots \alpha(u_{n-1}, t) g(u_{n-1}) g(t) du_1 \dots du_{n-1} dt \\ &= c_2 g(s) \underbrace{\int_0^{\infty} \dots \int_0^{\infty} \underbrace{\alpha(s, u_1) g(u_1) \dots \alpha(u_{n-1}, u_n) g(u_n)}_{\tilde{K}(s, u_1)} du_1 \dots du_n}_{H_n(s)}. \end{aligned}$$

Thus we see that it is useful for us to investigate the kernel function given by

$$\tilde{K}(s, t) := \alpha(s, t) g(t),$$

and the corresponding integral operator

$$\tilde{A}h(s) := \int_0^{\infty} \tilde{K}(s, t) h(t) dt.$$

With this notations, the terms $A^n f(s)$ can be obtained as $c_2 g(s) H_n(s)$, where

$$\begin{aligned} H_n(s) &= \int_0^{\infty} \dots \int_0^{\infty} \alpha(s, u_1) g(u_1) \dots \alpha(u_{n-1}, u_n) g(u_n) du_1 \dots du_n \\ &= \int_0^{\infty} \dots \int_0^{\infty} \tilde{K}(s, u_1) \dots \tilde{K}(u_{n-1}, u_n) du_1 \dots du_n = \tilde{A}^n \mathbf{1}(s). \end{aligned}$$

So we see that (3.3) is equivalent to

$$(3.4) \quad \varphi(s) = c_2 g(s) (1 + H_1(s) + H_2(s) + \dots).$$

3.1. *An example.* In the following we consider Example 1.2. In this section we determine $r_1(x)$ numerically for Example 1.2, using the software *Mathematica*. We compute the terms in (3.4) and give estimates on $H_n(s)$. From these we can derive estimates on parameters (p, c) where the series in (3.4) converges. Before doing these, we argue shortly that shifting the distribution of the new task's priority up by c has a real relevance in the literature. To see this, note that if $X, Y \sim U[c, 1]$, and $V, W \sim U[0, 1]$, then

$$\frac{X}{X+Y} \sim \frac{V(1-c) + c}{V(1-c) + c + W(1-c) + c} \sim \frac{V + \delta}{V + W + 2\delta},$$

where $\delta = \frac{c}{1-c} \geq 0$. This protocol is similar to the growth rule of the preferential attachment model. This model was heuristically introduced by Barabási and Albert [2], the first who investigated the model rigorously were Bollobás et al [4], and the mathematically rigorous construction was done by Bollobás and Riordan [3]. The preferential attachment model produces a random graph sequence $\{PA_t(1, \delta)\}_{t=1}^\infty$, where $PA_t(1, \delta)$ has t vertices $(v_1, v_2, \dots, v_t)$ and t edges. Then $PA_{t+1}(1, \delta)$ is generated from $PA_t(1, \delta)$ by adding a new vertex with a single edge to the graph. The procedure of connecting the new edge is very similar to our execution procedure. Namely, the new edge is connected to v_i with probability proportional to $D_i(t) + \delta$, where $D_i(t)$ is the degree of the vertex v_i in $PA_t(1, \delta)$ and $\delta \geq -1$ is constant. When $\delta > -1$ the preferential attachment model has a power-law degree sequence, so it shows similar features also in this sense.

We have seen that $H_n(s) = \tilde{A}^n \mathbf{1}(s)$. So in order to prove convergence of (3.4) it is enough to give an upper bound on the L^2 norm of $H_n(s)$. This can be achieved by estimating the L^2 norm of $\tilde{A}$ or the Hilbert-Schmidt norm of the kernel $\tilde{K}$. We want to give a good estimate on $\|\tilde{A}\|_2$.

For an integral operator T with kernel $\kappa(s, t)$ and a function $f(s)$ with $\|f(s)\|_2^2 = 1$ the L_2 norm of the operator is defined as:

$$(3.5) \quad \|T\|_2 := \sup_{f: \|f\|_2=1} \|Tf\|_2$$

If the operator T is an integral operator with a kernel κ then we have the estimate

$$(3.6) \quad \begin{aligned} \|(Tf)(s)\|_2^2 &= \int_c^1 \left(\int_c^1 \kappa(s, t) f(t) dt \right)^2 ds \leq \int_c^1 \int_c^1 \kappa^2(s, t) dt \int_c^1 f^2(t) dt ds \\ &= \int_c^1 \int_c^1 \kappa^2(s, t) dt ds = \|\kappa(s, t)\|_{HS}^2. \end{aligned}$$

Since $H_n(s) = \tilde{A}^n \mathbf{1}(s)$ we have from (3.5)

$$\|H_n(s)\|_2^2 \leq \|\tilde{A}\|_2^2 \cdot \|\mathbf{1}(s)\|_2^2.$$

We apply the estimates in (3.6) on the operator $\tilde{A}$ to arrive at

$$\|\tilde{A}\|_2^2 \leq \|\tilde{K}\|_{HS}^2.$$

If $\|\tilde{K}\|_{HS}^2 < 1$ then the sum in equation (3.4) converges, so it remains to find the parameter region (p, c) for which this holds.

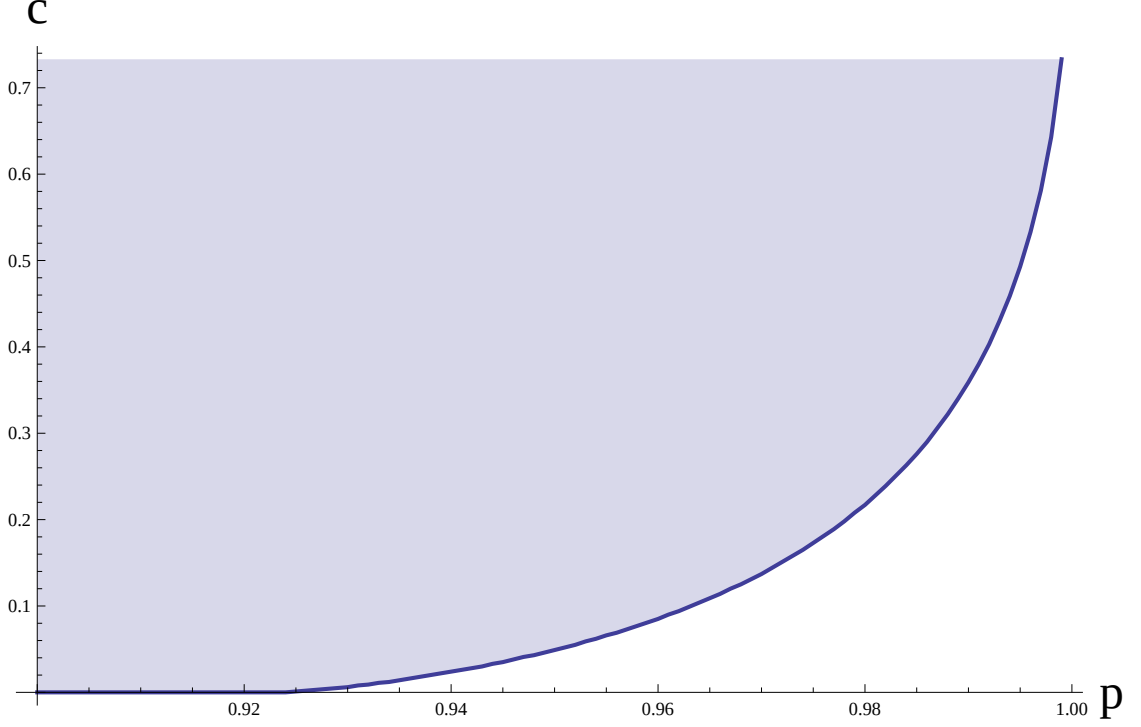


FIG 1. The pair of parameters (p, c) where $\|\tilde{K}(s, t)\|_{HS}^2 < 1$ and $p \geq 0.9$.

For fixed $0 \leq c < 1$, $\|\tilde{K}\|_{HS}^2$ is a monotonously increasing function of p ($0 \leq p < 1$), since the norm is given by

$$\begin{aligned} \|\tilde{K}\|_{HS}^2 &= \int_c^1 \int_c^1 \left(\frac{t}{t+s} \frac{1}{\frac{1+p}{2p} - \left(1 - \frac{x}{1-c} \ln \frac{1+x}{c+x}\right)} \frac{1}{1-c} \right)^2 ds dt \\ &= \int_c^1 t^2 \left(\frac{1}{t+c} - \frac{1}{t+1} \right) \left(\frac{1}{\frac{1+p}{2p} - \left(1 - \frac{x}{1-c} \ln \frac{1+x}{c+x}\right)} \frac{1}{1-c} \right)^2 dt \end{aligned}$$

The expression increases in p since $\frac{1+p}{2p}$ in the denominator decreases. This implies that if we have found a pair (p, c) for which $\|\tilde{A}\|_{HS}^2 < 1$ then this inequality also holds for all pairs (q, c) where $0 < q \leq p$. The region of pair of parameters for which $\|\tilde{A}\|_{HS}^2 < 1$ can be determined numerically, and it is represented on Fig. 1. In addition, Fig. 2 and 3 illustrates the density $r_1(x)$ and the distribution of τ on a log-log plot for three different choice of parameters, respectively.

Now we give estimates on the distribution of τ when $k > 1$, which is given in equation (1.4). It is easy to see that $q'(x) \neq 0$, so we can write

$$\begin{aligned} \mathbb{P}(\tau = k) &= \int_c^1 q_1(x)(1-q(x))q(x)^{k-2}dR(x) \\ &= \int_c^1 \underbrace{\frac{1}{1-c} \frac{q_1(x)(1-q(x))}{-q'(x)}}_{K(x)} (-q'(x))q(x)^{k-2}dx. \end{aligned}$$

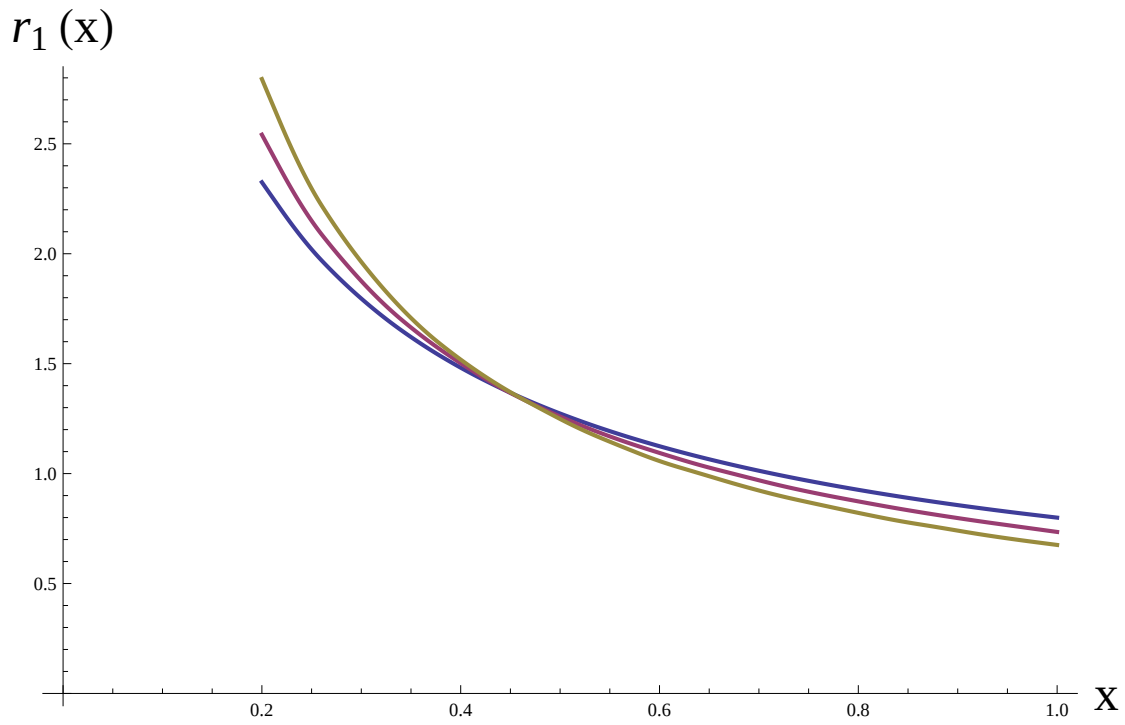


FIG 2. The DF of the old task's priority in the stationary case with $c = 0.2$ and different values of p . Blue: $p = 0.7$. Red: $p = 0.8$. Mustard: $p = 0.9$.

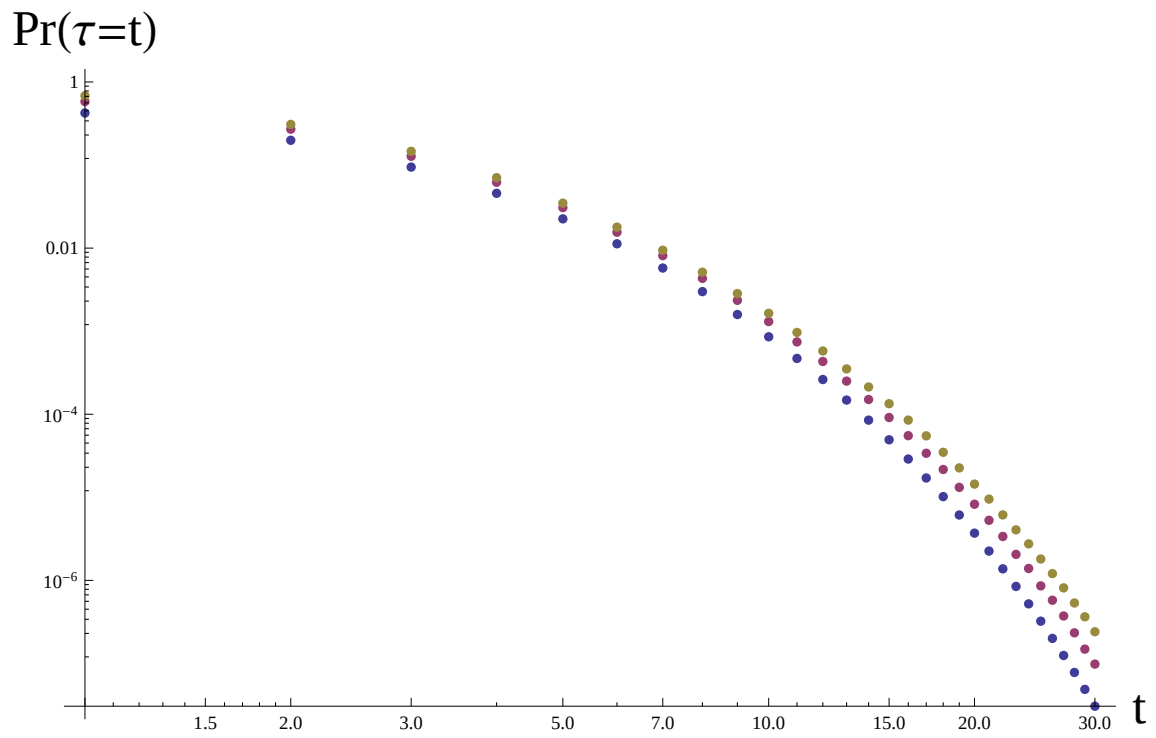


FIG 3. The distribution of τ (i.e. the waiting time of a new task in the stationary case) with $c = 0.2$ and different values of p . Blue: $p = 0.7$. Red: $p = 0.8$. Mustard: $p = 0.9$.

With the new notation $K(x) := -\frac{1}{1-c} \frac{q_1(x)(1-q(x))}{q'(x)}$, using the definition of $q(x)$ and $q_1(x)$ it is easy to verify that

$$\begin{aligned} \frac{1-p}{2} &\leq q_1(x) \leq 1, \\ 0 &< 1-q(c) \leq 1-q(x) \leq 1-q(1) < 1, \\ 0 &< -\frac{1}{q'(1)} \leq -\frac{1}{q'(x)} \leq -\frac{1}{q'(c)} < 1. \end{aligned}$$

Thus

$$m \leq K(x) \leq M,$$

where

$$m = -\frac{1}{1-c} \frac{1-p}{2} \frac{1-q(c)}{q'(1)}, \text{ and } M = -\frac{1}{1-c} \frac{1-q(1)}{q'(c)}.$$

With these bounds, one can easily estimate $\mathbb{P}(\tau = k)$:

$$\mathbb{P}(\tau = k) \leq M \int_c^1 q'(x) q(x)^{k-2} dR(x) = M \frac{1}{k-1} (q^{k-1}(c) - q^{k-1}(1))$$

and similarly

$$\mathbb{P}(\tau = k) \geq m \frac{1}{k-1} (q^{k-1}(c) - q^{k-1}(1)).$$

Since $q(c) > q(1)$, thus

$$(3.7) \quad \mathbb{P}(\tau = k) \sim \text{const} \cdot \frac{1}{k} q^k(c) = \text{const} \cdot \frac{1}{k} e^{-k/k_0},$$

where

$$k_0 = -\frac{1}{\ln(q(c))}.$$

This heuristically means that the probability that the waiting time is of length k behaves approximately as $\frac{1}{k}$ until $k < k_0$, i.e. it obeys a power law-like behavior. Furthermore, if $c \neq 0$ and $(p, c) \rightarrow (1, 0)$, then $k_0 \rightarrow \infty$, i.e. the exponential cutoff in equation (3.7) shifts to infinity, and thus for (p, c) values close to $(1, 0)$ the distribution of τ will be close to a power-law distribution.

3.2. Expected waiting time. In this section we will determine the average waiting time for a task at stationarity. As we have shown in section 2.3 the expected waiting time should equal 2 for any selection protocol where the system is ergodic and has some stationary distribution. We will verify this analytically for our selection protocol. Using the distribution

of the waiting time (1.4) the expected waiting time is

$$\begin{aligned}
\mathbb{E}(\tau) &= \sum_{k=1}^{\infty} k \mathbb{P}(\tau = k) = \int_0^{\infty} 1 - q_1(x) + \sum_{k=2}^{\infty} k q_1(x)(1 - q(x))q^{k-2}(x) dR(x) \\
&= \int_0^{\infty} 1 - q_1(x) + \\
&\quad + q_1(x)(1 - q(x)) \left(\sum_{k=2}^{\infty} (k-1)q^{k-2}(x) + \sum_{k=2}^{\infty} q^{k-2}(x) \right) dR(x) \\
&= \int_0^{\infty} 1 - q_1(x) + q_1(x)(1 - q(x)) \frac{2 - q(x)}{(1 - q(x))^2} dR(x) \\
&= \int_0^{\infty} \frac{(1 - q_1(x))(1 - q(x)) + q_1(x)(2 - q(x))}{1 - q(x)} dR(x) \\
&= \int_0^{\infty} \frac{q_1(x)}{1 - q(x)} dR(x) + 1.
\end{aligned}$$

Using the definitions (1.1), (1.2) and equation (3.1) one can obtain that

$$r_1(x) = \frac{q_1(x)r(x)}{1 - q(x)},$$

and hence

$$\mathbb{E}(\tau) = \int_0^{\infty} \frac{q_1(x)}{1 - q(x)} dR(x) + 1 = \int_0^{\infty} r_1(x) dx + 1 = 2.$$

Note that this calculation was valid for all choices of $q(x) < 1$ and $q_1(x)$, so it stays valid for any protocol where $q(x) < 1 \forall x$ holds. If we consider Example 1.2, this criterion is met for all $p < 1$, where also the stationarity holds. Furthermore, if $p = 1$ and $c > 0$, then clearly $\forall x \in [c, 1] \ q(x) \leq 1 - \frac{c}{1+c} < 1$. And, since the probability that a task in the buffer is selected for execution can be bounded from below with $\frac{c}{1+c} > 0$, thus there exists a stationary distribution.

4. Work in progress. There are several ways to continue this research.

1. We gave some parameters (p, c) for which (3.4) converges. It is known that (3.4) could converge with parameters for which $\|\tilde{A}\|_2^2 > 1$ as well, thus the region of parameters could be expanded.
2. The case $L \geq 3$ could be a subject of further research. In this case we have to find the fixpoint of a non-linear operator acting on 2-dim vector field.
3. The models discussed in sections 2 and 3 could be generalized to the case where $L = 2$ and the priorities of the arriving task are not i.i.d. but come from a stationary process. Similar results for this case would be useful if $L \geq 3$ as the following heuristics shows it. If $L = 3$ then the oldest and the second-oldest task as a system behaves as a slowed down version of $L = 2$ where the new priorities are from a stationary process. Thus, in order to determine the behavior of $L \geq 3$ it is of crucial importance to understand the behavior of the system if the new priorities are not i.i.d..

5. History and context of the research. Five institutes (including the Math Institute at BME) received a consortial project with OTKA NKTH no. 77778 for research of the future internet. The leader of the project is professor Károly Simon. The research group aims to state new results on the field of internet related mathematical questions. I joined them a year ago when I was taking the course Random Graphs and Complex Networks. The research areas of the group are mainly concentrated around internet-related mathematical questions, most importantly it focuses on the theory of different network models and dynamics on networks. Our first aim is to describe and present network models that show similar properties to some real networks, e.g. they have hierarchical structure combined with scale-free degree distribution, or they consist of small, well-wired communities connected by weak edges. These networks try to model complex systems such as social networks. On the other hand, a very important other aim of ours is to model the behavior of the spread of information in real networks. This is how modeling human behavior came to our interest, since it models the communication process on a single edge in social networks.

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