



Interaction between runaway electrons and whistler waves in tokamak plasmas

Student Research Paper

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Abstract

In plasmas there exists a critical electric field, above which the accelerating force exceeds the friction force for high-energy electrons, thus *runaway electrons* are produced [1]. In tokamak plasmas, the generated runaway electron beam can be so energetic (up to 10 MeV) and represent such a high current (1 MA) that it can cause substantial damage to the vacuum chamber [1].

In the present work I investigated if a relativistic runaway population can interact with plasma waves. During the course of my previous investigations only the ultrarelativistic case was considered [2]. In the present work I derived the more complicated formulas of the general case and demonstrated that the ultrarelativistic case can be derived from these as a high-momentum limit. I also investigated whether the use of the ultrarelativistic approximation was justified.

Furthermore, based on analytical considerations, I deduced that out of the three plasma waves investigated before [2], runaway electron-wave interaction is possible solely with whistler waves. These results are general enough, so that they are valid for the high electric field case of the disruptions as well as for the near-critical field case, relevant for runaway electrons generated in sawtooth crashes.

Based on numerical calculations I deduced that the growth rate, which describes the electron-wave interaction, has its maximum below the range of validity of the previous approximation in disruptions and also in the near-critical case. Therefore, I continued my investigation in a lower frequency range. As a result, I determined the most unstable wave, which in both cases is a magnetosonic-whistler wave moving quasi-perpendicular to the magnetic field. However, its wave number differs in the two cases: in disruptions it is nearly an order of magnitude higher than in the near-critical field.

In case of disruptions, the stability threshold is already known [3], therefore the subject of my investigations narrowed down to runaway electrons generated in near-critical field. In order to determine if in this case the runaway electrons can destabilize a magnetosonic-whistler wave, I compared the growth rate of the most unstable wave to the damping rates, determining a stability threshold.

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1 Introduction

In the future, nuclear fusion could be a significant source of energy for humanity. The principle of nuclear fusion is that when light nuclei collide, their minimal distance might reach a small enough value where tunneling effect can occur, resulting in the fusion of the nuclei. In order to accomplish this, the nuclei must have high enough energy and collide with a high enough frequency. To achieve positive energy balance a thermal temperature of approximately 100 million K and a density of 10^{20} m^{-3} is required with a confinement time of $\mathcal{O}(1 \text{ s})$. However, at such a temperature all known substances are already in plasma state, therefore the issue of confinement has to be solved. In the present, the most commonly used method is magnetic confinement.

1.1 Magnetic confined fusion

The principle of magnetic confinement is that the Lorentz-force, acting in the presence of a magnetic field influences the orbits of charged plasma particles (i.e. ions and electrons). These particles move along the magnetic field lines, gyrating on a so-called *Larmor orbit*, with the magnetic field line acting as the guiding center. If the field lines exit the confined volume, particles start to leak at the open ends, so to inhibit this effect, the two ends are connected, forming a torus [4] (Fig. 1.1).

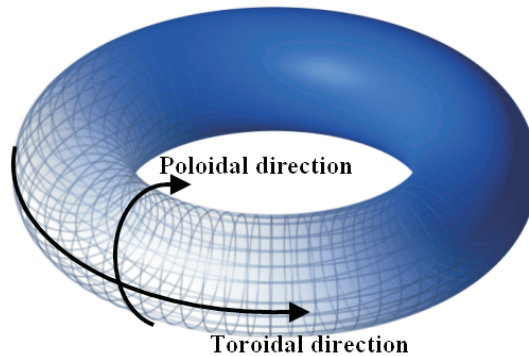


Figure 1.1: Toroidal geometry.

In the subsequent toroidal magnetic field the field line curvature is positive, therefore drifts occur in the plasma. Part of the drifts can be countered by the helical twisting of the magnetic field lines. The two known realizations of this are the stellarator and the tokamak. In a tokamak, a toroidal plasma current is induced, which in turn gives rise to a poloidal magnetic field. Together with the toroidal field, this results in a helically twisted magnetic field structure [5]. The outline of a tokamak can be seen on Fig. 1.2

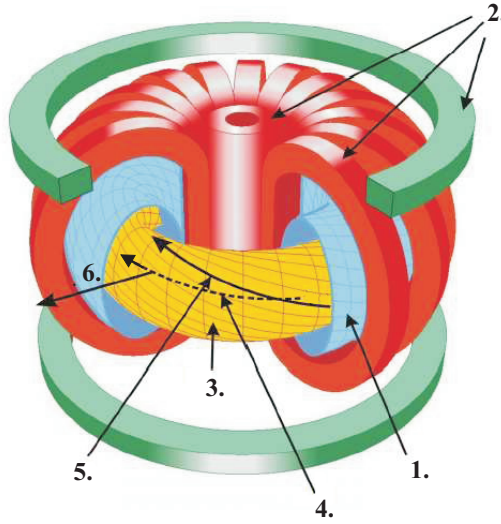


Figure 1.2: Outline of a tokamak. 1. vacuum chamber, 2. magnetic coils, 3. plasma, 4. plasma current, 5. helically twisted magnetic field line, 6. radial direction.

1.2 Runaway electrons

Friction among plasma particles originates from Coulomb collisions, therefore it is highly dependent on their energy. The calculation of drag force affecting a charged particle in a Maxwellian background can be found in [6], and results in the Chandrasekhar function, seen on Fig. 1.3. The function is non-monotonous, so there is an acceleration force which balances the friction force for particles with certain energy and above. This means that above this critical energy all electrons will accelerate infinitely, thus the name *runaway electrons*.

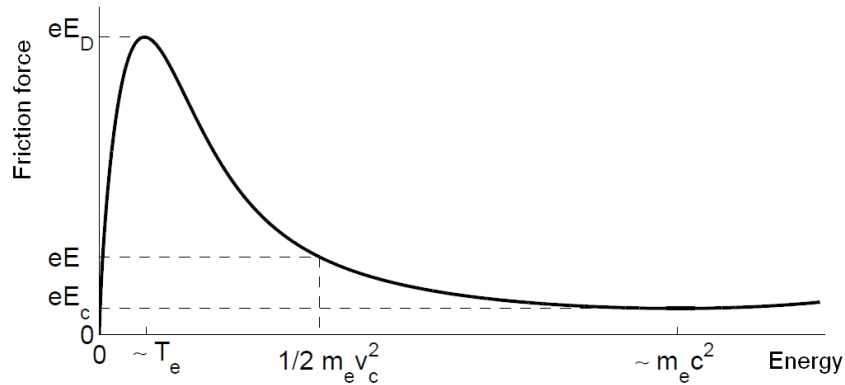


Figure 1.3: Friction force affecting an electron in a Maxwellian ion background. Because of relativistic effects, the tail of the Chandrasekhar function is slightly elevated. An electric field exceeding the critical field and the corresponding critical velocity is denoted with E and v_c .

Because of relativistic effects, the friction force increases for the high-energy electrons in the tail of the Chandrasekhar function. This means there is a minimum in the function, as shown on Fig. 1.3, and if the accelerating electric field does not exceed this critical value (*critical field*) no runaway electron production occurs. The value of the critical field is [6]:

$$E_c = \frac{n_e e^3 \ln \Lambda}{4\pi \epsilon_0^2 m_e c^2}, \quad (1.1)$$

where n_e is the thermal electron density, e the elementary charge, $\ln \Lambda$ the Coulomb-logarithm (typical value is 10-20 [6]), ϵ_0 the vacuum permittivity, m_e the electron rest mass, c the speed of light.

The electric field corresponding to the maximum of the Chandrasekhar function is called the *Dreicer field*:

$$E_D = \frac{n_e e^3 \ln \Lambda}{4\pi \epsilon_0^2 T_e}, \quad (1.2)$$

where T_e is the electron temperature (in electronvolts).

If the electric field exceeds the Dreicer field, the accelerating force is greater than the friction force for all electrons, thus the whole electron population 'runs away'.

In tokamak plasmas, the electric field usually doesn't exceed the Dreicer field, however, for the existence of runaway electrons it is sufficient to have an electric field exceeding only the critical field. In this case, electrons above the critical velocity will start to accelerate – this is called the *primary generation* of runaway electrons.

If there are already a finite number of runaway electrons in the plasma, these can generate more runaway electrons with the so-called *secondary generation*. When a runaway electron collides with a thermal electron (having subcritical velocity) energy exchange occurs. The thermal electron's energy might increase so that it exceeds the critical limit and becomes a runaway, while the energetic electron still remains a runaway electron [1].

The generation of runaway electrons – if the electric field exceeds the critical value – can result in a runaway beam, causing serious damage to the wall of the vacuum chamber of tokamaks [1].

1.2.1 Runaway electron generation during plasma disruptions

Runaway electrons are usually present in plasma disruptions. The term disruption means the sudden cooling of the plasma, which causes its resistance to increase, thus a toroidal electric field is induced to maintain the plasma current (Lenz's law) [1]. When the electric field exceeds the critical field, runaway electrons are produced.

It is documented, that the generation of runaway electrons can be observed only if the toroidal magnetic field exceeds a certain threshold (2.2 T [3]). An attempt was made in [7] to explain this phenomenon, resulting in the findings that below 2.2 T runaway electrons might destabilize a plasma wave, which could interact with the runaway distribution, thus quenching the runaway generation.

In the present work, I extended the validity of this investigation by introducing a general method of calculation. I deduced the formulas valid not only in the ultrarelativistic limit, but in the general case as well, and did the calculations of the electron – wave interaction with these more complex formulas.

1.2.2 Runaway electron generation during sawtooth crashes

During magnetic reconnections in a plasma (e.g. when plasma volumes with differently aligned magnetic fields come to contact) the magnetic field loses energy due to the topological change created by the field line reconnection. A certain fraction of this energy is given to high-energy electrons, causing them to accelerate further [8].

Sawtooth crashes occurring in tokamaks are caused by magnetic reconnections. They are named after the soft x-ray signals emitted by the plasma, whose temporal characteristic is like sawteeth [9]. Temperature and density in the plasma also suffer sawteeth-like changes during magnetic reconnections.

The toroidal electric field during sawtooth crashes in a tokamak exceeds the critical value [8], and it is documented that suprathermal electrons are generated [9, 10]. As the electric field remains slightly supercritical even between the crashes, the runaway population possibly formed during a sawtooth crash doesn't die out after it.

1.3 Plasma waves

In my work I used the *homogeneous, magnetized plasma approximation*. Thus, the discussion of the plasma waves begins with the wave equation, which can be derived from Maxwell's equations:

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) + \frac{\omega^2}{c^2} \boldsymbol{\epsilon} \cdot \mathbf{E} = \mathbf{0} , \quad (1.3)$$

where ω is the frequency of the wave, $\mathbf{k}$ is its wave number, c is the speed of light, $\mathbf{E}$ is the fluctuating electric field and $\boldsymbol{\epsilon}$ is the dielectric tensor of the plasma.

By introducing the vector $\mathbf{n} = \frac{\mathbf{k}c}{\omega}$, this wave equation yields

$$\mathbf{n} \times (\mathbf{n} \times \mathbf{E}) + \boldsymbol{\epsilon} \cdot \mathbf{E} = \mathbf{0} . \quad (1.4)$$

By assigning axis z to be the direction of the static, homogeneous magnetic field, the dielectric tensor takes the following form:

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ \epsilon_{21} & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix} . \quad (1.5)$$

Let's assign axis x so that the vector $\mathbf{n}$ is in the xz plane. If we denote the angle of $\mathbf{n}$ and axis z by θ , which is the angle of propagation, the (1.4) wave equation yields

$$\begin{bmatrix} \epsilon_{11} - n^2 \cos^2 \theta & \epsilon_{12} & n^2 \cos \theta \sin \theta \\ \epsilon_{21} & \epsilon_{22} - n^2 & 0 \\ n^2 \cos \theta \sin \theta & 0 & \epsilon_{33} - n^2 \sin^2 \theta \end{bmatrix} \cdot \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \mathbf{0} . \quad (1.6)$$

This equation is fulfilled for all $\mathbf{E}$ only if the determinant of the matrix of the equation

equals 0. If we use the generally accepted $\epsilon_{33} \gg n^2 \cos \theta \sin \theta$ electromagnetic approximation, it is justified to narrow down this condition to a 2×2 determinant:

$$\left(\epsilon_{11} - k_{\parallel}^2 \frac{c^2}{\omega^2} \right) \left(\epsilon_{22} - k_{\perp}^2 \frac{c^2}{\omega^2} \right) - \epsilon_{12} \epsilon_{21} = 0 . \quad (1.7)$$

It should be noted that the $\epsilon_{33} \gg n^2 \cos \theta \sin \theta$ condition is trivially met if $\theta = 0$ or $\theta = \pi/2$, but in other cases the validity of the mentioned simplification is not trivial.

The dielectric tensor can be written as a sum of the plasma particle susceptibilities (ion and electron terms):

$$\boldsymbol{\epsilon} = \mathbf{1} + \boldsymbol{\chi}^i + \boldsymbol{\chi}^e + \boldsymbol{\chi}^r , \quad (1.8)$$

where $\boldsymbol{\chi}^s$ is the susceptibility of plasma species s , where i denotes the ion, e the thermal electron and r the runaway electron populations, $\mathbf{1}$ is the dyadic unit.

1.3.1 Hot plasma susceptibilities

The ion and thermal electron populations in a plasma are Maxwellian. The formulas giving the temperature-dependent susceptibilities (i.e. hot plasma susceptibilities) of a Maxwellian population are given in [11]. The formulas have the same form for any Maxwellian population, s :

$$\chi_{11}^s = \frac{\omega_{ps}^2}{\omega} \sum_{n=-\infty}^{\infty} e^{-\lambda_s} \frac{n^2 I_n(\lambda_s)}{\lambda_s} \frac{1}{k_{\parallel} v_{Ts}} Z_0(\xi_{ns}) , \quad (1.9)$$

$$\chi_{22}^s = \frac{\omega_{ps}^2}{\omega} \sum_{n=-\infty}^{\infty} e^{-\lambda_s} \left(\frac{n^2}{\lambda_s} I_n(\lambda_s) + 2\lambda_s I_n(\lambda_s) - 2\lambda_s I'_n(\lambda_s) \right) \frac{1}{k_{\parallel} v_{Ts}} Z_0(\xi_{ns}) , \quad (1.10)$$

$$\chi_{12}^s = \frac{\omega_{ps}^2}{\omega} \sum_{n=-\infty}^{\infty} e^{-\lambda_s} (-i) n (I_n(\lambda_s) - I'_n(\lambda_s)) \frac{1}{k_{\parallel} v_{Ts}} Z_0(\xi_{ns}) , \quad (1.11)$$

where I_n is the first order modified Bessel function and $Z_0(\xi)$ the plasma dispersion function:

$$Z_0(\xi) = \begin{cases} Z(\xi) & \text{if } k_{\parallel} > 0, \\ -Z(-\xi) & \text{if } k_{\parallel} < 0, \end{cases} \quad (1.12)$$

$$Z(\xi) = 2ie^{-\xi^2} \int_{-\infty}^{i\xi} dz e^{-z^2} . \quad (1.13)$$

Here ξ_{ns} and λ_s are the following:

$$\xi_{ns} = \frac{\omega - n\omega_{cs}}{k_{\parallel} v_{Ts}} , \quad (1.14)$$

$$\lambda_s = \frac{k_{\perp}^2 v_{Ts}^2}{2\omega_{cs}^2} , \quad (1.15)$$

and ω_{ps} , ω_{cs} are the plasma frequency and gyrofrequency of particle population s , v_{Ts} is their thermal velocity, ω is the frequency of the wave, $k_{\parallel}$ and $k_{\perp}$ are the two components of the wave number: parallel and perpendicular to the electric field.

1.3.2 Cold plasma approximation

In the present work, simplified forms of the hot plasma susceptibilities are used, called the *cold plasma approximation*. The approximation is valid if

$$\lambda_i = \frac{k_{\perp}^2 v_{Ti}^2}{2\omega_{ci}^2} \ll 1, \quad (1.16)$$

$$\lambda_e = \frac{k_{\perp}^2 v_{Te}^2}{2\omega_{ce}^2} \ll 1, \quad (1.17)$$

are met, so if the square of the $k_{\perp} v_{Ts}$ characteristic frequency of the wave is negligible compared to the square of the gyrofrequency ω_{cs} . This is possible if the thermal velocities are sufficiently small.

With these approximations the cold plasma susceptibilities have the following form [11]:

$$\chi_{11}^s = \chi_{22}^s = \frac{\chi_s^+ + \chi_s^-}{2}, \quad (1.18)$$

$$\chi_{12}^s = -\chi_{21}^s = \frac{i(\chi_s^+ - \chi_s^-)}{2}, \quad (1.19)$$

where

$$\chi_s^+ = -\frac{\omega_{ps}^2}{\omega(\omega - \omega_{cs})}, \quad (1.20)$$

$$\chi_s^- = -\frac{\omega_{ps}^2}{\omega(\omega + \omega_{cs})}. \quad (1.21)$$

I investigated the validity of these approximations during the course of my previous work [2], and found that up to ion and electron temperatures of 20 keV, the use of the cold plasma approximation is justified even for wave numbers as high as $3 - 4 \cdot 10^3 \text{ m}^{-1}$.

1.4 Previous results regarding the electron-wave interaction

Since in [2] I concentrated mainly on the theoretical foundations of the present work, therefore a brief summary of the results is presented here, along with some of the results of [12], in order to fully understand the following calculations.

1.4.1 Dispersion relation of the high frequency electron waves

The plasma waves considered in [2] were high frequency electromagnetic electron waves, studied in the homogeneous magnetized cylindrical plasma approximation. In order to calculate the elements of the dielectric tensor the following approximations were used beside the cold plasma approximation:

$$\omega_{ci} \ll \omega \sim \omega_{ce}, \quad (1.22)$$

$$\frac{\omega_{ce}}{45} \ll \omega. \quad (1.23)$$

In this calculation the ion susceptibilities could be neglected because of condition (1.23):

$$\epsilon_{11} = \epsilon_{22} \approx 1 - \frac{\omega_{pe}^2}{\omega^2 - \omega_{ce}^2}, \quad (1.24)$$

$$\epsilon_{12} = -\epsilon_{21} \approx \frac{-i\omega_{pe}^2\omega_{ce}}{\omega(\omega^2 - \omega_{ce}^2)}. \quad (1.25)$$

Substituting these expressions into Eq. (1.7) yields the dispersion relation of high frequency plasma waves:

$$F(\omega) = \omega^6 - \omega^4 (2\omega_{pe}^2 + \omega_{ce}^2 + (k^2 + k_{\parallel}^2)c^2) + \omega^2 (\omega_{pe}^4 + (k^2 + k_{\parallel}^2)c^2(\omega_{pe}^2 + \omega_{ce}^2) + k^2k_{\parallel}^2c^4) - k^2k_{\parallel}^2c^4\omega_{ce}^2 = 0, \quad (1.26)$$

and its solution (i.e. the explicit dispersion relation) can be denoted by $\omega_0(k, \theta)$.

This dispersion relation defines three different plasma waves, among them two optical waves which satisfy the relation $k_{\parallel}c > \omega_0(k, \theta)$ and the 'electron-whistler' wave, which satisfies $k_{\parallel}c < \omega_0(k, \theta)$ for all k wave numbers and θ propagation angles [2]. This is an important property and will be used in the present work.

The $\theta = 0$ and $\theta = \pi/2$ limits of these plasma waves can be derived using the Altar-Appleton-Hartree approximation presented in [11], which was studied in detail in [2].

1.4.2 Perturbative analysis of the runaway electron – wave interaction

The presence of the runaway electron population can be taken into account with their susceptibility (χ^r) added to the dielectric tensor elements. Because of this perturbative term, the frequency of the wave changes with a small $\delta\omega$ term. Thus, a first order perturbative analysis can be done on Eq. (1.26), and the result is the following explicit form for $\delta\omega$:

$$\delta\omega = \frac{\omega_0^3(\omega_0^2 - \omega_{ce}^2) \left[\chi_{11}^r \left(\frac{k^2c^2}{\omega_0^2} - \epsilon_{22}^t \right) + \chi_{22}^r \left(\frac{k_{\parallel}^2c^2}{\omega_0^2} - \epsilon_{11}^t \right) - 2\epsilon_{12}^t\chi_{12}^r \right]}{2 \left\{ 3\omega_0^4 - 2\omega_0^2[2\omega_{pe}^2 + \omega_{ce}^2 + (k^2 + k_{\parallel}^2)c^2] + \omega_{pe}^4 + (k^2 + k_{\parallel}^2)c^2(\omega_{pe}^2 + \omega_{ce}^2) + k^2k_{\parallel}^2c^4 \right\}} \quad (1.27)$$

The growth rate (γ_i), defined as the imaginary part of $\delta\omega$ gives the information whether the runaway electrons give rise to a plasma wave (positive growth rate) or cause the decay of the wave (negative growth rate). The growth rate based on the (1.27) $\delta\omega$ is:

$$\gamma_i = \frac{\omega_0^3(\omega_0^2 - \omega_{ce}^2) \left[\text{Im}(\chi_{11}^r) \left(\frac{k^2c^2}{\omega_0^2} - \epsilon_{22}^t \right) + \text{Im}(\chi_{22}^r) \left(\frac{k_{\parallel}^2c^2}{\omega_0^2} - \epsilon_{11}^t \right) - 2\text{Im}(\epsilon_{12}^t)\text{Re}(\chi_{12}^r) \right]}{2 \left\{ 3\omega_0^4 - 2\omega_0^2[2\omega_{pe}^2 + \omega_{ce}^2 + (k^2 + k_{\parallel}^2)c^2] + \omega_{pe}^4 + (k^2 + k_{\parallel}^2)c^2(\omega_{pe}^2 + \omega_{ce}^2) + k^2k_{\parallel}^2c^4 \right\}}, \quad (1.28)$$

where ϵ_{ij}^t are the elements of the dielectric tensor defined by Eqs. (1.24) and (1.25), where the $\omega = \omega_0(k, \theta)$ substitution was made. $\text{Im}(\chi_{ij}^r)$ are the imaginary parts of the runaway susceptibilities χ_{11}^r, χ_{22}^r . However, as the ϵ_{12} term, defined by Eq. (1.25), is imaginary, the real part of the χ_{12}^r term is needed in order to calculate the imaginary part of $\delta\omega$.

All quantities in Eq. (1.28) are known from the previous analysis, except the runaway susceptibilities. If the distribution function of the runaway population is known, the susceptibilities can be calculated based on the following formula:

$$\mathbf{j} = q \int d^3\mathbf{p} \mathbf{v} f_1(\mathbf{r}, \mathbf{p}, t) = -i\omega \boldsymbol{\chi} \cdot \mathbf{E} . \quad (1.29)$$

Here $f_1(\mathbf{r}, \mathbf{p}, t) \ll 1$ is a small term, added to the f_0 equilibrium distribution function as a result of a perturbative electric field (with the amplitude of $\mathbf{E}$ and frequency of ω). The current density represented by this population is $\mathbf{j}$, and can be given as the function of the f_1 term as well as the function of the perturbative field and the susceptibilities of the population.

The exact calculations can be found in [11], here I only present the results:

$$\chi^r(\mathbf{k}, \omega) = \frac{\omega_{pr}^2}{\omega \omega_{ce}} \sum \int_0^\infty 2\pi p_\perp dp_\perp \int_{-\infty}^\infty dp_\parallel \frac{\Omega \mathbf{S}_n}{\omega - k_\parallel v_\parallel - n\Omega} , \quad (1.30)$$

where

$$\mathbf{S}_n(\mathbf{p}, \mathbf{k}, \omega) = \begin{bmatrix} \frac{n^2 J_n^2}{z^2} p_\perp U & in \frac{J_n J'_n}{z} p_\perp U \\ -in \frac{J_n J'_n}{z} p_\perp U & (J'_n)^2 p_\perp U \end{bmatrix} , \quad (1.31)$$

$$U(\mathbf{p}, k_\parallel, \omega) = \frac{\partial f_r}{\partial p_\perp} + \frac{k_\parallel}{\omega} \left(v_\perp \frac{\partial f_r}{\partial p_\parallel} - v_\parallel \frac{\partial f_r}{\partial p_\perp} \right) , \quad (1.32)$$

$\Omega = \frac{\omega_{ce}}{\gamma}$ is the relativistic cyclotron frequency of runaway electrons,

$J_n(z)$ is the Bessel function of the first kind,

$$z = \frac{k_\perp v_\perp}{\Omega} = \frac{k_\perp c p_\perp}{\omega_{ce}} ,$$

$p = \frac{\gamma v}{c}$ is the normalized relativistic momentum,

$\gamma(p_\parallel, p_\perp) = \sqrt{1 + p^2}$ is the relativistic factor and

$f_r(p_\parallel, p_\perp) = \frac{f(p_\parallel, p_\perp)}{n_r}$ is the normalized runaway distribution function.

Using this expression we can formulate the runaway susceptibilities:

$$\begin{aligned} \text{Im} \chi_{11}^r(\mathbf{k}, \omega_0) &= -\frac{2\pi^2 \omega_{pr}^2 \omega_{ce}^2}{\omega_0^2 k_\perp^2 c^2 (k_\parallel c - \omega_0)} \int_0^\infty dp_\perp \sum_{n=-\infty}^\infty n^2 J_n^2(z) \times \\ &\quad \left[\left(\frac{\partial f_r(\mathbf{p})}{\partial p_\perp} n \omega_{ce} + \frac{\partial f_r(\mathbf{p})}{\partial p_\parallel} k_\parallel c p_\perp \right) \frac{1}{p_\parallel} \right]_{p_\parallel = p_{res}} , \end{aligned} \quad (1.33)$$

$$\begin{aligned} \text{Im} \chi_{22}^r(\mathbf{k}, \omega_0) &= -\frac{2\pi^2 \omega_{pr}^2}{\omega_0^2 (k_\parallel c - \omega_0)} \int_0^\infty p_\perp^2 dp_\perp \sum_{n=-\infty}^\infty (J'_n(z))^2 \times \\ &\quad \left[\left(\frac{\partial f_r(\mathbf{p})}{\partial p_\perp} n \omega_{ce} + \frac{\partial f_r(\mathbf{p})}{\partial p_\parallel} k_\parallel c p_\perp \right) \frac{1}{p_\perp} \right]_{p_\parallel = p_{res}} , \end{aligned} \quad (1.34)$$

$$\begin{aligned}
-\text{Re}\chi_{12}^r(\mathbf{k}, \omega_0) &= -\frac{2\pi^2\omega_{pr}^2\omega_{ce}}{\omega_0^2 k_{\perp} c(k_{\parallel} c - \omega_0)} \int_0^{\infty} p_{\perp} dp_{\perp} \sum_{n=-\infty}^{\infty} n J_n(z) J'_n(z) \times \\
&\quad \left[\left(\frac{\partial f_r(\mathbf{p})}{\partial p_{\perp}} n\omega_{ce} + \frac{\partial f_r(\mathbf{p})}{\partial p_{\parallel}} k_{\parallel} c p_{\perp} \right) \frac{1}{p_{\parallel}} \right]_{p_{\parallel}=p_{res}}. \tag{1.35}
\end{aligned}$$

Here, the general (and implicit) condition for the resonant momentum

$$p_{\parallel} = \frac{\omega_0 \gamma - n\omega_{ce}}{k_{\parallel} c} \tag{1.36}$$

was simplified in the ultrarelativistic limit:

$$\gamma(p_{\parallel}, p_{\perp}) = \sqrt{1 + p_{\perp}^2 + p_{\parallel}^2} \approx |p_{\parallel}| \tag{1.37}$$

and the resonant momentum yielded

$$p_{res} = \frac{-n\omega_{ce}}{k_{\parallel} c - \omega_0}. \tag{1.38}$$

The growth rate is most significant for small values of n (i.e. $n = 1, 0, -1$), thus these were the values investigated. The growth rate was calculated for the high frequency electron-whistler wave and these calculations yielded positive results, meaning the runaway electron – whistler wave interaction might be possible. Also, the results indicated that the most unstable wave is in the region of low wave numbers, where the electron-whistler approximation is no longer valid.

1.4.3 Investigation of the interaction in high electric fields for the magnetosonic-whistler wave

In previous works [3, 7] the interaction between runaway electrons and the magnetosonic whistler wave was investigated in the high electric field case of disruptions. As the magnetosonic-whistler wave and its growth rate will be important in this work, I briefly introduce them.

In the dispersion relation of the magnetosonic-whistler wave, as opposed to the (1.26) dispersion relation of the electron waves, the ion terms play a significant part. The main approximation used is $\omega_{ci} \ll \omega \ll \omega_{ce}$, therefore the susceptibility terms are first simplified to

$$\chi_{11}^i = \chi_{22}^i = -\frac{\omega_{pi}^2}{\omega^2}, \tag{1.39}$$

$$\chi_{12}^i = -i \frac{\omega_{pi}^2 \omega_{ci}}{\omega^3} \text{ and} \tag{1.40}$$

$$\chi_{11}^e = \chi_{22}^e = \frac{\omega_{pe}^2}{\omega_{ce}^2}, \tag{1.41}$$

$$\chi_{12}^e = i \frac{\omega_{pe}^2}{\omega \omega_{ce}}. \tag{1.42}$$

Then, further approximations are applied:

- $|k| \gg |k_{\parallel}|$ (meaning near-perpendicular propagation),

- $k^2 c^2 \ll \omega_{pe}^2$ (valid for low wave numbers).

Thus, the following dispersion relation is derived for the magnetosonic-whistler wave:

$$\omega_0 = k v_A \sqrt{1 + \frac{k_{\parallel}^2 c^2}{\omega_{pi}^2}}, \quad (1.43)$$

where $v_A = c \omega_{ci} / \omega_{pi}$ is the Alfvén-velocity.

The magnetosonic-whistler and the electron-whistler wave are compared on Fig. 1.4. The magnetosonic-whistler is a lower frequency limit and is not valid for high wave numbers. In turn the electron-whistler wave is only valid for high wave numbers, thus the slight difference in the dispersions for lower values of k .

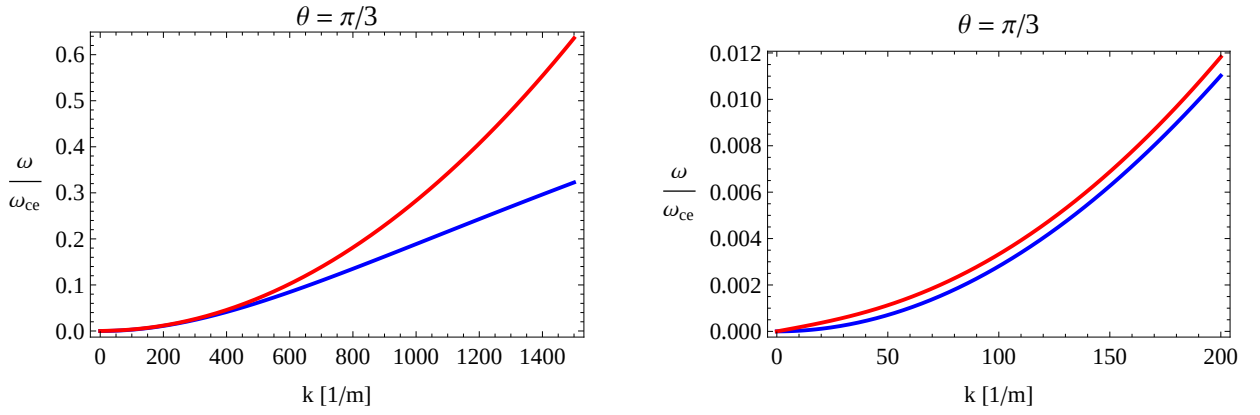


Figure 1.4: Dispersion relation of the magnetosonic-whistler wave (red) compared to the electron-whistler wave (blue).

The growth rate can be derived for the magnetosonic-whistler wave as a first order perturbation of Eq. (1.43) by the runaway electrons:

$$k^2 v_A^2 \left(1 + \frac{k_{\parallel}^2 c^2}{\omega_{pi}^2} \right) - \omega^2 = \omega^2 \frac{k^2 v_A^2}{\omega_{pi}^2} \chi_{11}^r, \quad (1.44)$$

where further terms were neglected to simplify the right side of this equation. These approximations were based on the following two assumptions: the effect of perturbation on the wave frequency is small and $\text{Im}\chi_{11}^r$, $\text{Im}\chi_{22}^r$ and $\text{Re}\chi_{12}^r$ are of the same order of magnitude.

Thus, the growth rate for the magnetosonic-whistler wave takes the following simple form

$$\gamma_i = -\omega_0 \frac{k^2 v_A^2}{2\omega_{pi}^2} \text{Im}(\chi_{12}^r), \quad (1.45)$$

where again $\text{Im}(\chi_{11}^r)$ is the imaginary part of the (11) component of the runaway susceptibility defined by Eq. (1.33).

In the previous results the (1.45) formula yielded a maximum of the growth rate for the wave number $k \approx 800 \text{ m}^{-1}$ and for near-perpendicular propagation, but as this maximum was near the limit where the magnetosonic-whistler approximation loses its validity it would be

useful to map the growth rate for high wave numbers as well. Therefore, in the present work I will calculate the growth rate for the electron-whistler wave, valid for higher wave numbers, with the general resonance condition.

The structure of the present work is the following: in Section 2.1, more generalized formulas will be derived for the resonant momentum and the runaway susceptibilities. The growth rates calculated with the general formulas will be compared to the results calculated previously in the ultrarelativistic case. Furthermore, in Section 2.2 a detailed analysis will be given, regarding the possibility of each plasma wave interacting with the runaway beam, resulting in the conclusion that only the whistler wave can interact with the runaway electrons. In Section 2.3 and 2.4 the growth rates of the disruption case and the near-critical case will be presented, and the most unstable waves will be determined. Finally, in Section 3 a stability threshold will be mapped in the near-critical case.

2 Growth rates

In the present paper the possibility of runaway electron – plasma wave interaction is investigated in the case of near-critical electric field as well as high electric field cases, manifesting in plasma disruptions. The existence of runaway electrons in these two cases have been mentioned before in Section 1.2.

The theory of the interaction is based on the kinetic theory, where the particle – runaway electron – distribution function is assumed to be known. Based on this distribution function, the susceptibility of the runaway electron population can be calculated along the lines demonstrated in Section 1.4. The presence of the runaway electrons can be taken into consideration as a first-order perturbation of the dispersion relation of the corresponding wave, each presented in Section 1.4.1 and 1.4.3.

The aim is to determine – based on the results presented in [2] – which plasma waves are the most unstable in each cases, and to estimate the order of magnitude of their growth rate.

2.1 Runaway electron – plasma wave interaction in the general, relativistic case

The runaway electron distribution function in the near-critical case and in the case of disruptions can be found in [13] and [3] respectively. The plasma waves and their dispersion relations are presented in Sections 1.4.1 and 1.4.3 of the present work.

In these works the interaction was handled in the ultrarelativistic limit, which meant the formula for the resonant momentum could be simplified, therefore the runaway susceptibilities had a relatively simple form. In this section I present the more complex formulas of the general case and investigate if the ultrarelativistic handling of the problem is justified.

2.1.1 Runaway susceptibilities without approximations

In order to calculate the growth rate, a first order perturbation on the wave equation was performed, by using the runaway electron susceptibilities, as mentioned in Section 1.4.

In previous works [2, 3], the (1.36) implicit resonance condition was simplified in the ultrarelativistic limit ($\gamma(p_{\parallel}, p_{\perp}) \approx |p_{\parallel}|$) and the formula

$$p_{res} = \frac{-n\omega_{ce}}{k_{\parallel}c - \omega_0}$$

was derived.

However, with slightly more effort, the general case can be derived as well. We can substitute the expression $\gamma = \sqrt{1 + p_{\perp}^2 + p_{\parallel}^2}$ into Eq. (1.36), and get the following explicit formula for the resonant momentum:

$$p_{res}(p_{\perp}, k_{\parallel}, \omega_0) = \frac{-k_{\parallel}cn\omega_{ce} \pm \omega_0 \sqrt{(k_{\parallel}^2c^2 - \omega_0^2)(1 + p_{\perp}^2) + n^2\omega_{ce}^2}}{k_{\parallel}^2c^2 - \omega_0^2}. \quad (2.1)$$

By using this general resonance condition, the expressions giving the imaginary parts of the runaway susceptibilities, thus the growth rate, also become more complicated. The calculation goes the same as in [2], only the resonant momentum, therefore the evaluation of the integrals in $p_{\parallel}$ is different.

In the general case the susceptibilities have the following form:

$$\begin{aligned} \text{Im}\chi_{11}^r(\mathbf{k}, \omega_0) = & -\frac{2\pi^2\omega_{pr}^2\omega_{ce}^2}{\omega_0^2 k_{\perp}^2 c^2} \int_0^{\infty} dp_{\perp} \int_{-\infty}^{\infty} dp_{\parallel} \sum n^2 J_n^2(z) \times \\ & \left(\frac{\partial f_r(\mathbf{p})}{\partial p_{\perp}} \left(\frac{n\omega_{ce}}{\gamma} \right) + \frac{\partial f_r(\mathbf{p})}{\partial p_{\parallel}} \frac{k_{\parallel} c p_{\perp}}{\gamma} \right) \frac{1}{\gamma} \cdot \delta \left(\omega_0 - \frac{k_{\parallel} c p_{\parallel}}{\gamma} - \frac{n\omega_{ce}}{\gamma} \right), \end{aligned} \quad (2.2)$$

and the $\text{Im}\chi_{22}^r$ and $\text{Im}\chi_{12}^r$ terms only differ in a few constants, and will be presented later.

To solve the integral in $p_{\parallel}$, first we should analyze a simpler form of it:

$$\int dp_{\parallel} \cdot \delta \left(\omega_0 - \frac{k_{\parallel} c p_{\parallel}}{\gamma} - \frac{n\omega_{ce}}{\gamma} \right) g(p_{\parallel}) = \int dx \cdot \delta \left(A - \frac{Bx}{\sqrt{C+x^2}} + \frac{D}{\sqrt{C+x^2}} \right) g(x). \quad (2.3)$$

Here I made the substitutions $x = p_{\parallel}$, $A = \omega_0$, $B = k_{\parallel} c$, $C = 1 + p_{\perp}^2$ and $D = -n\omega_{ce}$.

By a change in the variables

$$\begin{aligned} y &= \frac{Bx - D}{\sqrt{C+x^2}}, \\ dx &= \frac{(C+x^2)^{3/2}}{BC + xD} dy, \\ x &= \frac{BD \pm y\sqrt{(B^2 - y^2)C + D^2}}{B^2 - y^2}, \end{aligned}$$

Eq. (2.3) yields

$$\begin{aligned} \int dy \frac{(C+x^2)^{3/2}}{BC + xD} \delta(A - y) \cdot g \left(\frac{BD \pm y\sqrt{(B^2 - y^2)C + D^2}}{B^2 - y^2} \right) = \\ \frac{\left(AD \pm B\sqrt{(B^2 - A^2)C + D^2} \right)^3}{(B^2 - A^2)^3 \cdot \left[BC + \frac{D}{B^2 - A^2} \left(BD \pm A\sqrt{(B^2 - A^2)C + D^2} \right) \right]} \cdot g \left(\frac{BD \pm A\sqrt{(B^2 - A^2)C + D^2}}{B^2 - A^2} \right). \end{aligned} \quad (2.4)$$

We can use this result to solve the integrals present in the runaway susceptibilities, and arrive at the following formulas.

$$\begin{aligned} \text{Im}\chi_{11}^r(\mathbf{k}, \omega_0) = & -\frac{2\pi^2\omega_{pr}^2\omega_{ce}^2}{\omega_0^2 k_{\perp}^2 c^2} \int_0^{\infty} dp_{\perp} \sum n^2 J_n^2(z) \times \\ & \left[\left(\frac{\partial f_r(\mathbf{p})}{\partial p_{\perp}} \left(\frac{n\omega_{ce}}{\gamma} \right) + \frac{\partial f_r(\mathbf{p})}{\partial p_{\parallel}} \frac{k_{\parallel} c p_{\perp}}{\gamma} \right) \right]_{p_{\parallel}=p_{res}} \cdot \frac{h(p_{\perp}, k_{\parallel}, \omega_0)}{\gamma}, \end{aligned} \quad (2.5)$$

where

$$h(p_{\perp}, k_{\parallel}, \omega_0) = \frac{1}{(k_{\parallel}^2 c^2 - \omega_0^2)^3} \times \frac{\left(-\omega_0 n \omega_{ce} \pm k_{\parallel} c \sqrt{(k_{\parallel}^2 c^2 - \omega_0^2)(1 + p_{\perp}^2) + n^2 \omega_{ce}^2}\right)^3}{\left[k_{\parallel} c (1 + p_{\perp}^2) - \frac{n \omega_{ce}}{k_{\parallel}^2 c^2 - \omega_0^2} \left(-k_{\parallel} c n \omega_{ce} \pm \omega_0 \sqrt{(k_{\parallel}^2 c^2 - \omega_0^2)(1 + p_{\perp}^2) + n^2 \omega_{ce}^2}\right)\right]} . \quad (2.6)$$

Similarly the other two terms of the runaway susceptibility:

$$\text{Im} \chi_{22}^r(\mathbf{k}, \omega_0) = -\frac{2\pi^2 \omega_{pr}^2}{\omega_0^2} \int_0^{\infty} p_{\perp}^2 dp_{\perp} \sum (J'_n(z))^2 \times \left[\left(\frac{\partial f_r(\mathbf{p})}{\partial p_{\perp}} \left(\frac{n \omega_{ce}}{\gamma} \right) + \frac{\partial f_r(\mathbf{p})}{\partial p_{\parallel}} \frac{k_{\parallel} c p_{\perp}}{\gamma} \right) \right]_{p_{\parallel}=p_{res}} \cdot \frac{h(p_{\perp}, k_{\parallel}, \omega_0)}{\gamma} , \quad (2.7)$$

$$-\text{Re} \chi_{12}^r(\mathbf{k}, \omega_0) = -\frac{2\pi^2 \omega_{pr}^2 \omega_{ce}}{\omega_0^2 k_{\perp} c} \int_0^{\infty} p_{\perp} dp_{\perp} \sum n J_n(z) J'_n(z) \times \left[\left(\frac{\partial f_r(\mathbf{p})}{\partial p_{\perp}} \left(\frac{n \omega_{ce}}{\gamma} \right) + \frac{\partial f_r(\mathbf{p})}{\partial p_{\parallel}} \frac{k_{\parallel} c p_{\perp}}{\gamma} \right) \right]_{p_{\parallel}=p_{res}} \cdot \frac{h(p_{\perp}, k_{\parallel}, \omega_0)}{\gamma} . \quad (2.8)$$

By substituting these runaway susceptibilities into the expression (1.28) or (1.45) of the growth rate, the calculations yield results in the general, relativistic case regarding the runaway electrons interacting with the corresponding wave.

2.1.2 Ultrarelativistic limit of the general resonance condition

The resonant momentum in the general case has the form, as presented in Eq. (2.1):

$$p_{\parallel} = p_{res}(p_{\perp}, k_{\parallel}, \omega_0) = \frac{-k_{\parallel} c n \omega_{ce} \pm \omega_0 \sqrt{(k_{\parallel}^2 c^2 - \omega_0^2)(1 + p_{\perp}^2) + n^2 \omega_{ce}^2}}{k_{\parallel}^2 c^2 - \omega_0^2} .$$

Now we can validate the previous results by taking the ultrarelativistic limit of this formula. This means the term $(1 + p_{\perp}^2)$ can be neglected compared to the other term, $p_{\parallel}^2$ in the relativistic factor γ :

$$p_{res} = \frac{-k_{\parallel} c n \omega_{ce} \pm \omega_0 \sqrt{(k_{\parallel}^2 c^2 - \omega_0^2)(1 + p_{\perp}^2) + n^2 \omega_{ce}^2}}{k_{\parallel}^2 c^2 - \omega_0^2} \rightarrow \frac{-k_{\parallel} c n \omega_{ce} \pm \omega_0 |n \omega_{ce}|}{k_{\parallel}^2 c^2 - \omega_0^2} \quad (2.9)$$

$$= \begin{cases} \frac{-n \omega_{ce}}{k_{\parallel} c - \omega_0} & \text{for } n < 0 \text{ if the plus sign is used,} \\ \frac{-n \omega_{ce}}{k_{\parallel} c + \omega_0} & \text{for } n < 0 \text{ if the minus sign is used.} \end{cases}$$

We can deduce that the corresponding expression of the ultrarelativistic case, Eq. (1.38) (for $n = -1$ in the former calculations in both [2] and [3]) can be derived by using the plus sign

in expression (2.9).

We can also confirm this statement by reminding ourselves how the ultrarelativistic expression of the resonant momentum was derived. The most general form is the following:

$$p_{\parallel} k_{\parallel} c + n \omega_{ce} - \omega_0 \gamma = 0 , \quad (2.10)$$

then the $\gamma \approx p_{\parallel}$ approximation was used in the case of $p_{\parallel} > 0$. Mathematically speaking, we can also assume $p_{\parallel} < 0$. Then, the ultrarelativistic resonant momentum will be the limit of Eq. (2.9) with the minus sign.

The physically relevant formula is the one derived using the $p_{\parallel} > 0$ condition, thus the one with the plus sign. The reason why this is the physical solution is that the runaway electrons have their parallel momentum in the direction of the electric field, which was chosen to be the positive direction.

As the ultrarelativistic condition is much simpler, it would be useful if that could be used instead of the general one. The question is whether this approximation is justified and for what ranges of k and θ .

This question will be answered in Sections 2.3 and 2.4 by comparing the growth rates calculated with the two resonance conditions in different cases. However, even now it is apparent, that for higher wave numbers, the ultrarelativistic resonance condition can not be used. This is due to the fact that in the region of high wave numbers $(k_{\parallel}^2 c^2 - \omega_0^2)$ becomes large, therefore the relation $(k_{\parallel}^2 c^2 - \omega_0^2)(1 + p_{\perp}^2) \ll n^2 \omega_{ce}^2$ does not hold anymore, not even for small values of $p_{\perp}$.

2.2 Investigation of the resonance condition

The dispersion relation of the studied electromagnetic electron waves was presented in Section 1.4. Studying the resonance condition in the runaway susceptibilities, presented in Eq. (2.1), I could narrow down my research to the electron-whistler wave only. Physically only the $p_{res} > 0$ interval is relevant, therefore we have a condition for the resonant momentum, which won't be satisfied for all the presented plasma waves.

The condition is the following:

$$p_{res}(n, p_{\perp}, k_{\parallel}, \omega_0) = \frac{-k_{\parallel} c n \omega_{ce} + \omega_0 \sqrt{(k_{\parallel}^2 c^2 - \omega_0^2)(1 + p_{\perp}^2) + n^2 \omega_{ce}^2}}{k_{\parallel}^2 c^2 - \omega_0^2} > 0 . \quad (2.11)$$

I studied this condition in the different cases defined by the sign of n and the relation between $k_{\parallel} c$ and $\omega_0(k, \theta)$.

Doppler resonance For Doppler resonance (the case of $n > 0$) the (2.11) condition, defining the physically relevant region of the p_{res} resonant momentum, becomes

$$\frac{-|k_{\parallel}|c|n|\omega_{ce} + \omega_0\sqrt{(k_{\parallel}^2c^2 - \omega_0^2)(1 + p_{\perp}^2)} + n^2\omega_{ce}^2}{k_{\parallel}^2c^2 - \omega_0^2} > 0. \quad (2.12)$$

- In the case of $k_{\parallel}^2c^2 < \omega_0^2(k, \theta)$ the denominator of this expression is negative, but the sign of the numerator is undecided. (The reason for the latter is that the discriminant is smaller than $n^2\omega_{ce}^2$, but the factor before it, ω_0 is greater than $|k_{\parallel}|c$.)
- In the case of $k_{\parallel}^2c^2 > \omega_0^2(k, \theta)$ the denominator of the expression is positive, but the sign of the numerator is undecided again.

This means that for the $n > 0$ Doppler resonance there is the possibility of a physically relevant result (if $p_{res} > 0$) for any of the plasma waves presented, as no condition was acquired for $\omega_0(k, \theta)$ yet.

However, it can be said that if for $n = 1$ the $p_{res} > 0$ condition is not met, then for all $n > 1$ it remains so. The reason is that in the relevant expression, Eq. (2.12), in the numerator there are two terms with different signs. The one with the positive sign increases with $\omega_0\omega_{ce}$ if n increases, while the other one with the negative sign decreases with $k_{\parallel}c\omega_{ce}$. Thus, the sum of these two terms – while n increases – is dependent only on the relation between $k_{\parallel}c$ and $\omega_0(k, \theta)$.

- In the case of $k_{\parallel}^2c^2 < \omega_0^2(k, \theta)$ the positive term outweighs the negative, thus the numerator of Eq. (2.12) will increase for increasing n . In addition, the denominator is always negative, therefore if for small n the expression is negative, it will continue to be negative for increasing n .
- In the case of $k_{\parallel}^2c^2 > \omega_0^2(k, \theta)$ the negative term outweighs the positive, thus the numerator of Eq. (2.12) will decrease for increasing n . In this case, the denominator is always positive, therefore if for small n the expression is negative, it will once again continue to be negative for increasing n .

The only remaining question is whether expression (2.1), giving the resonant momentum will be negative for $n = 1$. If so, the whole $n > 0$ case can be excluded when calculating the growth rate.

According to numerical evidence seen on Figs. 2.1a and 2.1b, for the two optical waves even the $[(k_{\parallel}^2c^2 - \omega_0^2)(1 + p_{\perp}^2) + n^2\omega_{ce}^2]/\omega_{ce}^2$ (normalized) determinant of expression (2.1) will be negative. Therefore p_{res} won't even be a real number in these two cases.

For the electron-whistler wave the determinant on Fig. 2.1c is positive. However, the value of p_{res} , seen on Fig. 2.2 is negative, which again means this is not a physical solution.

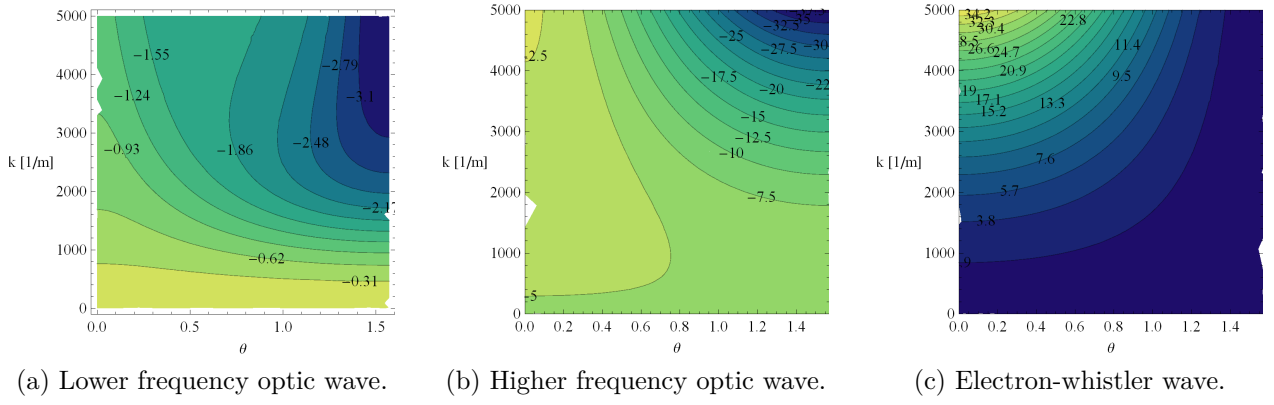


Figure 2.1: Normalized determinant of the resonance condition for $n = 1$ at $p_{\perp} = 1$. The parameters are $n_e = 5 \cdot 10^{19} \text{ m}^{-3}$ thermal electron density, $B = 2 \text{ T}$ magnetic field.

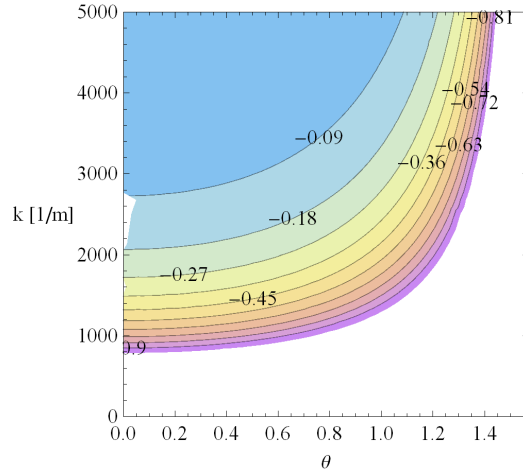


Figure 2.2: Resonance condition for the electron-whistler wave for $n = 1$ at $p_{\perp} = 0$. The parameters are $n_e = 5 \cdot 10^{19} \text{ m}^{-3}$, $B = 2 \text{ T}$.

Concluding the case of Doppler resonance, for $n = 1$ none of the waves presented can satisfy the $p_{res} > 0$ condition, which implies that neither can they satisfy it for arbitrary $n > 0$. This means there is no need to calculate the growth rate for $n > 0$, as it would not be physically relevant.

Anomalous Doppler resonance For anomalous Doppler resonance (the case of $n < 0$) the (2.11) condition, defining the physically relevant region of the p_{res} resonant momentum, is

$$\frac{+|k_{\parallel}|c|n|\omega_{ce} + \omega_0\sqrt{(k_{\parallel}^2c^2 - \omega_0^2)(1 + p_{\perp}^2) + n^2\omega_{ce}^2}}{k_{\parallel}^2c^2 - \omega_0^2} > 0. \quad (2.13)$$

The numerator of this expression is positive for all cases, thus its denominator has to be positive as well. This is satisfied only if the $k_{\parallel}^2c^2 > \omega_0^2(k, \theta)$ condition is met. This indicates the electron-whistler wave, as in Section 1.4 it was mentioned that the two optical waves satisfy the $k_{\parallel}^2c^2 < \omega_0^2(k, \theta)$ condition.

Therefore, only the growth rate for the whistler wave would yield physically relevant results in the case of $n < 0$ anomalous Doppler resonance.

Cherenkov resonance For Cherenkov resonance (the case of $n = 0$) the (2.11) condition, defining the physically relevant region of the p_{res} resonant momentum, is

$$\frac{+\omega_0 \sqrt{(k_{\parallel}^2 c^2 - \omega_0^2)(1 + p_{\perp}^2)}}{k_{\parallel}^2 c^2 - \omega_0^2} > 0 . \quad (2.14)$$

As in the case of the anomalous Doppler resonance, the numerator of this expression is positive for all cases, thus its denominator has to be positive as well. This means that the condition $k_{\parallel}^2 c^2 > \omega_0^2(k, \theta)$ has to be satisfied again, narrowing down the possible waves once again to the whistler wave.

Summarizing the results above, we can conclude that for $n \leq 0$, only the whistler wave can yield physically relevant results out of the high frequency electron waves defined by the (1.26) dispersion relation. Also, for $n > 0$, none of these waves yield physical results.

In conclusion, I can narrow down my investigation to the whistler wave, and to the $n \leq 0$ cases.

2.3 Growth rate in the high electric field case

In this chapter I am investigating the possibility of runaway electron – whistler wave interaction in high electric fields, typical for disruptions. Part of this discussion has already been done before in [12], but only in the ultrarelativistic limit. In the present work I extend the validity of the calculations by using the general resonance condition, then I compare the results with the ones valid in the ultrarelativistic case.

The aim of this investigation is to map the higher wave number region of the k -plane. As in [12] the most unstable wave was found near the limit where the magnetosonic-whistler approximations used were no longer valid, it would be interesting to see the characteristics of the growth rate for higher wave numbers, outside the previous region of validity. Therefore, I calculated the growth rate first for the high frequency electron-whistler wave, and then for the magnetosonic-whistler wave.

2.3.1 Runaway electron distribution function in disruptions

For runaway electrons present in plasma disruptions, in large tokamaks the secondary runaway generation becomes dominant. Therefore if we restrict our attention to large devices, we can follow the method outlined in [12].

The Boltzmann equation, describing a population with the particle mass m and charge q , moving in a static electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$, is the following:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \mathbf{v}} = C(f) , \quad (2.15)$$

where $C(f)$ is the collision operator and f the distribution function of mentioned population.

It can be assumed that $E \gg 1$, which is a valid assumption in disruptions and $p_\perp \ll p_\parallel \approx p$, which is the case for a beam-like runaway distribution. Then the Lorentz scattering operator, thus the collision operator can be simplified and the Boltzmann equation can be written in a simpler form.

The exact method can be found in [12], here I only present the resulting distribution function, valid for runaway electrons in disruptions:

$$f(p_\parallel, p_\perp, t) = \frac{n_{r0}a}{2\pi c_Z p_\parallel} \exp \left(\frac{(E-1)t/\tau_c - p_\parallel}{c_Z} - \frac{ap_\perp^2}{2p_\parallel} \right), \quad (2.16)$$

where

n_{r0} is the initial ($t = 0$) density of the runaway population,

$$a = \frac{E-1}{Z+1},$$

$$c_Z = \sqrt{\frac{3(Z+5)}{\pi}} \ln \Lambda,$$

$E = \frac{E_\parallel}{E_c}$ is the electric field parallel to the magnetic field, normalized to the critical value,

$$\tau_c = \frac{4\pi\epsilon_0^2 m_e^2 c^3}{n_e e^4 \ln \Lambda} \text{ is the relativistic electron collision time,}$$

Z is the effective ion charge,

n_e is the density of the thermal electron population.

In the numerical calculations, I used the following normalized distribution function:

$$f_r(p_\parallel, p_\perp) = \frac{a}{2\pi c_Z p_\parallel} \exp \left(\frac{-p_\parallel}{c_Z} - \frac{ap_\perp^2}{2p_\parallel} \right). \quad (2.17)$$

This distribution function is shown on Fig. 2.3. It is beam-like, as expected of a runaway distribution function.

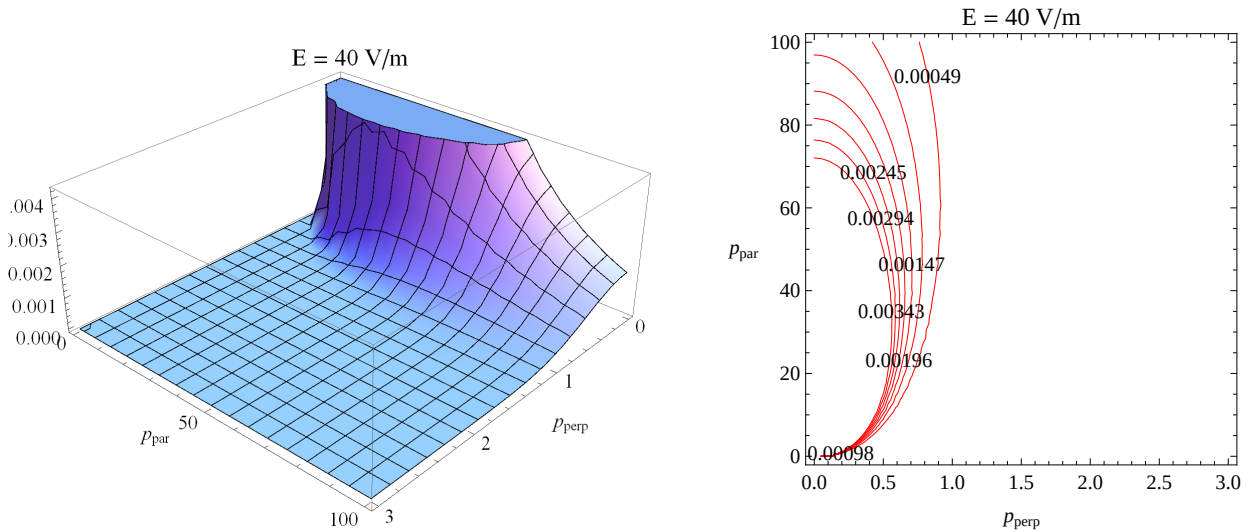


Figure 2.3: Normalized runaway electron distribution function in disruptions. The thermal electron density is $n_e = 5 \cdot 10^{19} \text{ m}^{-3}$.

2.3.2 Growth rate for the electron-whistler wave – high frequency range

As the candidate for interaction with the runaway population, I used the whistler wave, as it has been previously proven in Section 2.2 that none of the other electron waves can satisfy the resonance condition.

The fact that it is sufficient to calculate the growth rate only for $n = -1$ and $n = 0$ (anomalous Doppler and Cherenkov resonance) has been also analyzed thoroughly in Section 2.2. Therefore, the growth rates are calculated for these two parameters, using the (1.28) formula presented in Section 1.4.

The growth rate for typical disruption parameters can be seen on Fig. 2.4. The calculations, numerical integrations and plotting are done with routines written by me in Wolfram Mathematica 7.0 (licence provided by Chalmers University of Technology).

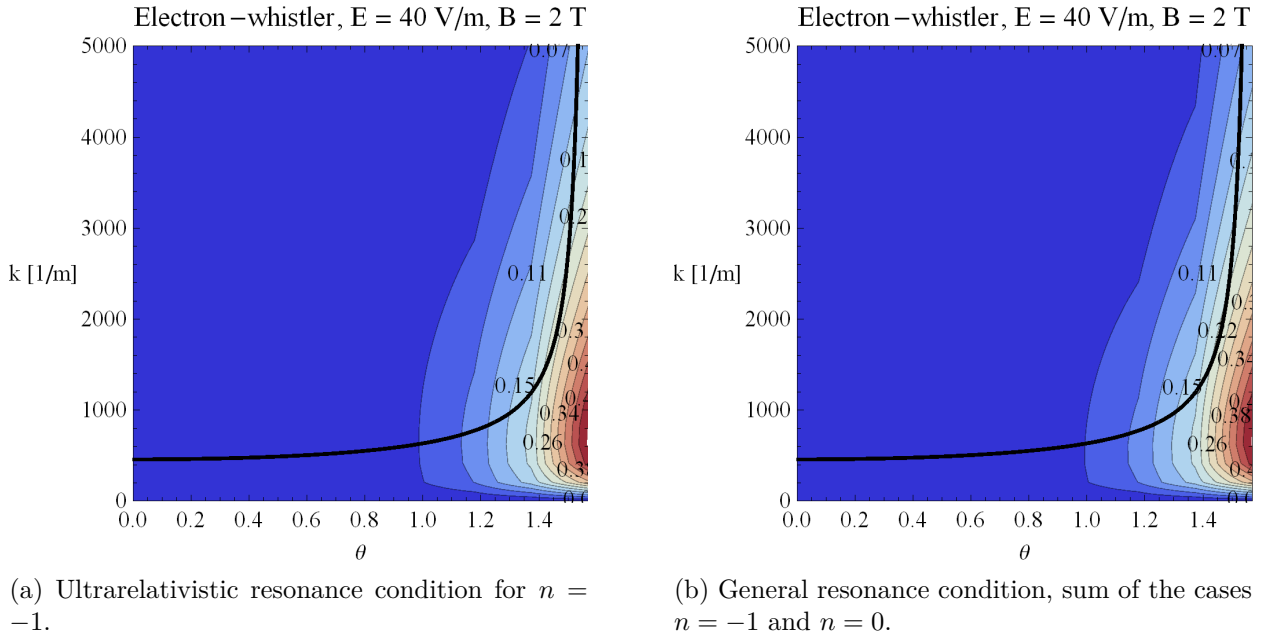


Figure 2.4: Growth rate (γ_i/ω_{ci}) in disruptions. The black line is the frequency $\omega_{ce}/10$, the electron-whistler approximation is valid in the region above it. The parameters are $n_e = 5 \cdot 10^{19} \text{ m}^{-3}$, $n_r = 3 \cdot 10^{17} \text{ m}^{-3}$, $B = 2 \text{ T}$.

On Fig. 2.4 it can be seen that by using the (1.38) ultrarelativistic condition the growth rate does not differ much from the case when the (2.1) general condition is used, although it was mentioned in Section 2.1.2 that the difference of the two results is expected.

The agreement of the two results can be understood if investigating the exact statement of Section 2.1.2. The statement was that for high wave numbers the $(k_{\parallel}^2 c^2 - \omega_0^2)(1 + p_{\perp}^2) \ll n^2 \omega_{ce}^2$ condition does not hold anymore, therefore the ultrarelativistic limit should yield incorrect results for the electron-whistler wave, valid for high wave numbers. However, even for high wave numbers this condition could be satisfied for a quasi-perpendicular propagation, as $k_{\parallel}$ is the parameter appearing in the expression and not k . Thus the agreement of the results of Fig. 2.4a and 2.4b, because the important range corresponds to near-perpendicular propagation.

On Fig. 2.4 it can be seen that there is a maximum in the growth rate for low wave numbers, around $k \approx 1000 \text{ m}^{-1}$ and for perpendicular propagation. However, this maximum is below the range of validity of the electron-whistler approximation, therefore we can not be sure about both the parameters of the most unstable wave and the value of the growth rate.

2.3.3 Growth rate for the magnetosonic-whistler wave – lower frequency range

As in the previous section the maximum of the growth rate was not exact yet, it was necessary to introduce an approximation valid in the lower region of wave numbers. I used the (1.43) magnetosonic-whistler dispersion relation, as it is an approximation valid for the lower frequencies of the whistler wave.

The growth rate calculated with the corresponding (1.45) formula for typical disruption parameters can be seen on Fig. 2.5.

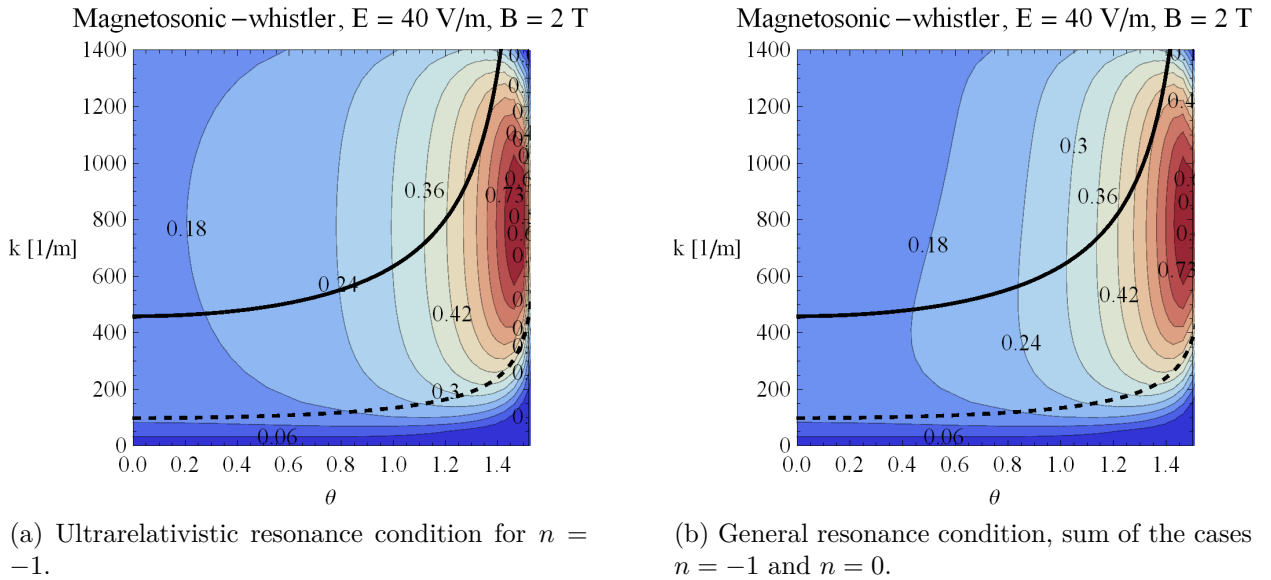


Figure 2.5: Growth rate (γ_i/ω_{ci}) in disruptions. The continuous black line is the frequency $\omega_{ce}/10$, the dashed line denotes $10\omega_{ci}$, the magnetosonic-whistler approximation is valid in the region between them. The parameters are $n_e = 5 \cdot 10^{19} \text{ m}^{-3}$, $n_r = 3 \cdot 10^{17} \text{ m}^{-3}$, $B = 2 \text{ T}$.

Of Fig. 2.5 we can deduce that by using the (1.38) ultrarelativistic condition instead of the (2.1) general one the difference in the growth rates is small. It should be noted that this difference is present for higher wave numbers, which indicates the failure of the ultrarelativistic condition for high wave numbers.

However, the growth rate in this low wave number case can be calculated with the ultrarelativistic resonance condition. The results around the maximum of the growth rate are shown on Fig. 2.6.

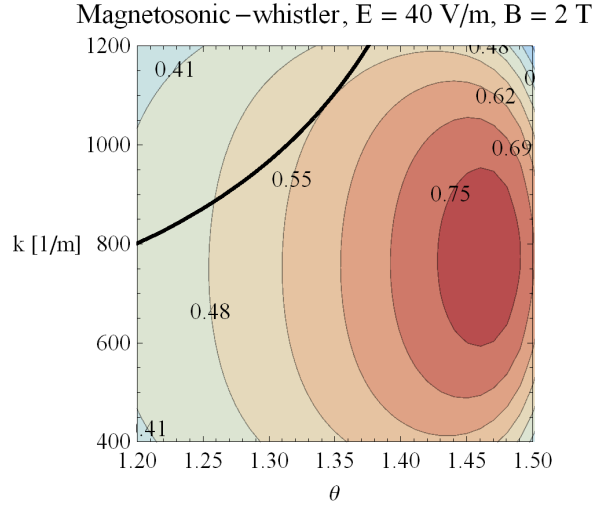


Figure 2.6: Growth rate (γ_i/ω_{ci}) in disruptions. The black line is the frequency $\omega_{ce}/10$, the magnetosonic-whistler approximation is valid in the region below it. The parameters are $n_e = 5 \cdot 10^{19} \text{ m}^{-3}$, $n_r = 3 \cdot 10^{17} \text{ m}^{-3}$, $B = 2 \text{ T}$.

On Figs. 2.5 and 2.6 it can be seen that the maximum previously present on Fig. 2.4 is indeed there. The parameters for the most unstable wave are $k \approx 800 \text{ m}^{-1}$, $\theta \approx 1.4$, thus $\omega_0 \approx 2.2 \cdot 10^{10} \text{ s}^{-1}$, which are still in the range of validity of the magnetosonic-whistler approximation. Therefore we can deduce that this result is valid, and by comparing this to the growth rate calculated before in [12], we can see that the two results are in agreement.

2.4 Growth rate in near-critical field

In this section I am investigating the possibility of runaway electrons destabilizing a whistler wave in near-critical electric field. The calculations are based on the theory presented in Sections 1.4 and 2.3. This means that the runaway electrons are taken into account with their distribution function and a first order perturbation is done to the wave equation in order to calculate the growth rate.

2.4.1 Runaway electron distribution function in near-critical electric field

The distribution function characterizing the runaway electrons in the near-critical case is presented in [13]. The method how it was acquired has been summarized before in [2], therefore here I just state the most important steps.

The equation solved was the (2.15) Boltzmann-equation, with a collision operator describing the particle collisions in near-critical field. The equation was then gyro-averaged and solved in spherical coordinates for different ranges of momentum. Asymptotic techniques were used to match the distribution functions valid in different ranges. The distribution function of the runaway tail, valid in the ultrarelativistic case ($p \gg 1$, p is the normalized momentum) then

became the following:

$$f(p_{\parallel}, p_{\perp}) = \frac{A}{p_{\parallel}^{(C_s-2)/(\alpha-1)}} \exp\left(-\frac{(\alpha+1)p_{\perp}^2}{2(1+Z)p_{\parallel}}\right) {}_1F_1\left(1 - \frac{C_s}{\alpha+1}, 1; \frac{(\alpha+1)p_{\perp}^2}{2(1+Z)p_{\parallel}}\right), \quad (2.18)$$

where

$$C_s = \alpha - \frac{(1+Z)}{4}(\alpha-2)\sqrt{\frac{\alpha}{\alpha-1}}, \quad (2.19)$$

$p_{\parallel}$ is the component of the normalized momentum which is parallel to the static electric field, $p_{\perp}$ is the momentum perpendicular to the electric field (in the perpendicular direction azimuthal symmetry is assumed), $\alpha = E/E_c$ is the electric field normalized to the critical field, Z is the ion effective charge, ${}_1F_1$ is the confluent hypergeometric (Kummer) function [14].

This distribution function, according to [13] is valid if

$$1 < \alpha < \min\left(\frac{16}{15 - Z^2 - 2Z}; 2\right), \quad (2.20)$$

so there is an upper limit to the parameter α , and means the electric field has to be near the critical value, slightly exceeding it. However, based on previous investigations [2] there is also a lower limit for α , and this limit is greater than 1:

$$1 - \frac{C_s(\alpha)}{\alpha+1} > 0. \quad (2.21)$$

This means if we wish to use the distribution function presented in Eq. (2.18), we have to be careful when choosing the value of α .

As into Eqs. (1.33)-(1.35) the normalized $f_r = f/n_r$ distribution function has to be substituted, the normalization constant of the (2.18) distribution function has to be acquired. When integrating this function in $p_{\parallel}$ and $p_{\perp}$ for the whole momentum space, the resulting integral is infinite. This is due to the fact that in near-critical electric field the primary runaway generation is dominant. Therefore if the electric field exists for long enough it can accelerate electrons to arbitrarily high energies.

However, as the existence of the electric field is finite in time, there is a maximum energy which runaway electrons can reach in real tokamak plasmas. During sawtooth crashes in the T-10 tokamak suprathermal electrons reached the energy of 20 – 100 keV [8], and according to Oleroset et al. [15] the energy of suprathermal electrons in magnetic reconnections does not exceed 300 keV, not even in the earth's magnetotail.

Based on these results, the (2.18) distribution function can be normalized, by integrating till the normalized momentum (p_{max}) corresponding to the maximum energy:

$$A_{new} = \int_0^{\infty} dp_{\perp} 2\pi p_{\perp} \int_0^{p_{max}} dp_{\parallel} f_0(p_{\parallel}, p_{\perp}), \quad (2.22)$$

where f_0 denotes the (2.18) distribution function divided by the constant A .

This integral can be easily solved numerically, and with the resulting A_{new} constant the

normalized distribution function is

$$f_r(p_{\parallel}, p_{\perp}) = \frac{f_0(p_{\parallel}, p_{\perp})}{A_{new}} . \quad (2.23)$$

The distribution function with this normalization constant is shown on Fig. 2.7. It is again a beam-like distribution function, however, if compared to the (2.17) runaway distribution function of disruptions, seen on Fig. 2.3, two main differences occur. One is that in disruptions the distribution is more beam-like, in the sense that the electrons are more localized around $p_{\perp} = 0$. The other is that in the near-critical case the distribution has an elevated tail, compared to the disruption case, which has been already mentioned before.

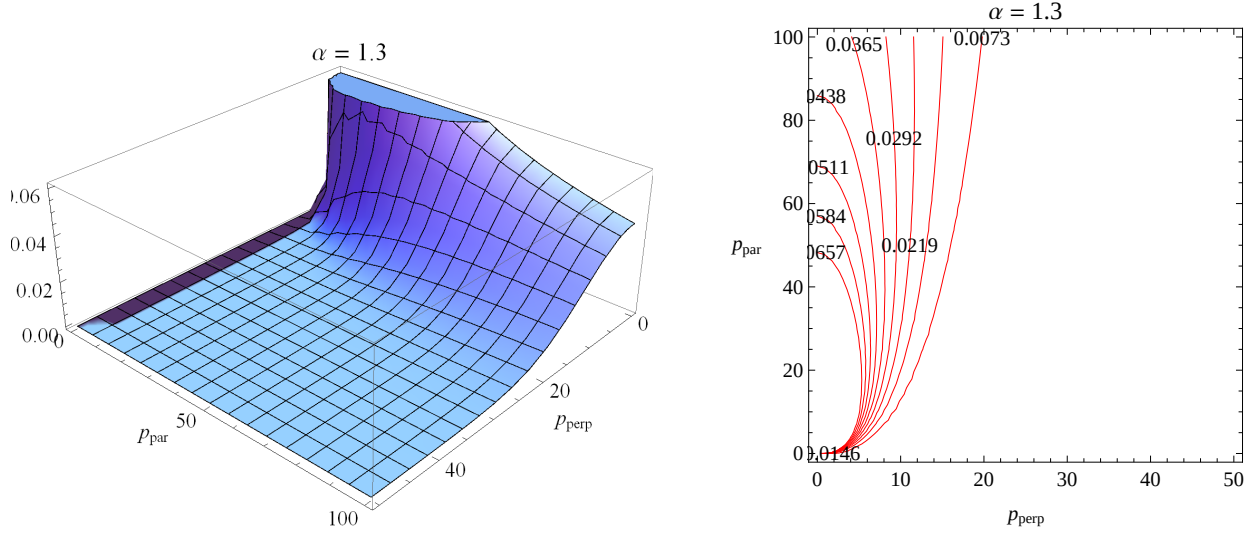


Figure 2.7: Runaway electron distribution function in near-critical field. The thermal electron density is $n_e = 5 \cdot 10^{19} \text{ m}^{-3}$.

2.4.2 Growth rate for the electron-whistler wave – high frequency range

In this section the growth rate is calculated for the runaway electron – wave interaction in a near-critical electric field with the corresponding distribution function presented in the previous section. The calculations again follow the method outlined in Section 1.4 for the high frequency electron-whistler wave. Again it is sufficient to calculate the growth rate for $n = -1$ and $n = 0$ (anomalous Doppler and Cherenkov resonance), as previously shown in Section 2.2.

The growth rate calculated for the near-critical runaway distribution with the expression (1.28) is shown on Fig. 2.8.

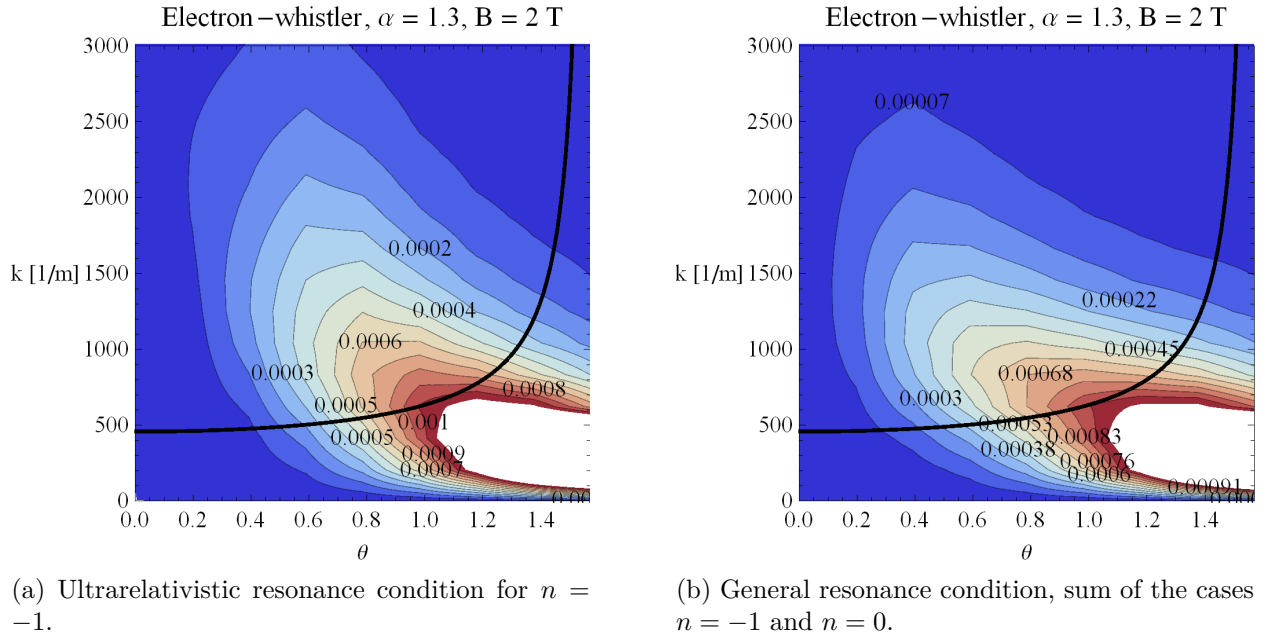


Figure 2.8: Growth rate (γ_i/ω_{ce}) in disruptions. The black line is the frequency $\omega_{ce}/10$, the electron-whistler approximation is valid in the region above it. The parameters are $n_e = 5 \cdot 10^{19} \text{ m}^{-3}$, $n_r = 3 \cdot 10^{17} \text{ m}^{-3}$, $B = 2$ T.

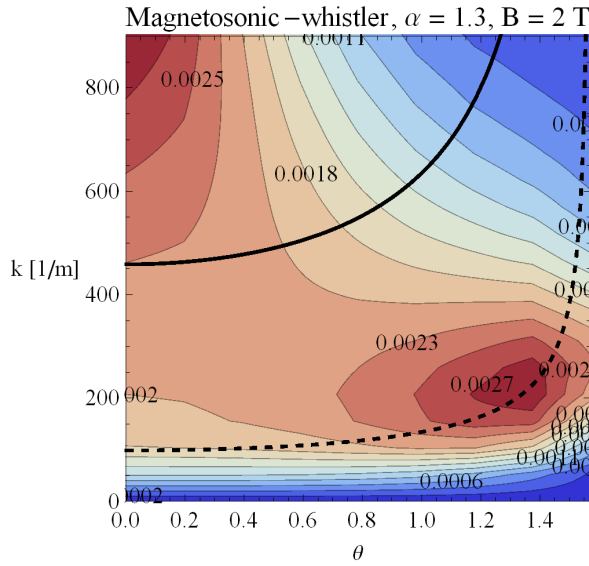
On Fig. 2.8 it can be seen that by using the (1.38) ultrarelativistic condition one gets different results than by using the (2.1) general condition. This is due to the fact that on Fig. 2.8 the region of higher wave numbers is inspected, and as it was mentioned in Section 2.1.2 the ultrarelativistic condition does not hold for high wave numbers. Therefore, in order to get correct results for the electron-whistler wave, the general resonance condition have to be used.

It can be seen on Fig. 2.8b that there is no maximum of the growth rate. As the growth rate increases for lower wave numbers and around perpendicular wave propagation, it can be assumed that there is a maximum, only it is out of the range of validity of the electron-whistler approximation.

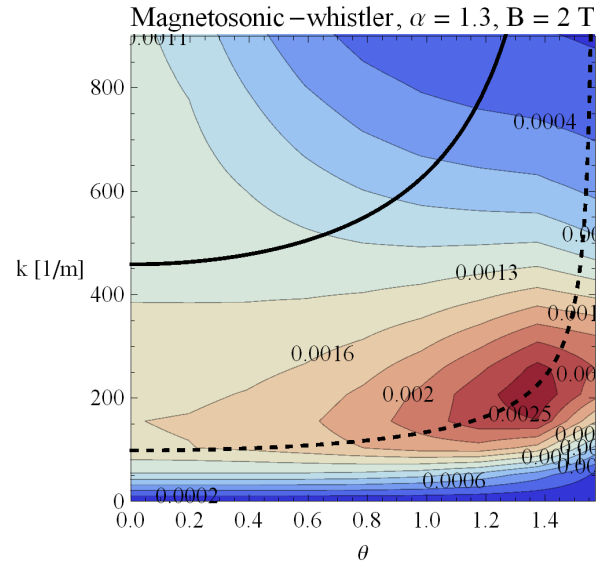
2.4.3 Growth rate for the magnetosonic-whistler wave – lower frequency range

As the results of the previous section indicated that the most unstable wave is below the frequency limit of validity of the electron-whistler approximation, it was necessary to introduce another approximation. Therefore I calculated the growth rate for the magnetosonic-whistler wave, valid for slightly lower frequencies.

The magnetosonic-whistler wave was introduced before in Section 1.4.3 and the formulas giving the runaway susceptibilities and the growth rate were also presented. By substituting the (2.18) near-critical runaway distribution, the growth rate of this case can be calculated (Fig. 2.9).



(a) Ultrarelativistic resonance condition for $n = -1$.



(b) General resonance condition, sum of the cases $n = -1$ and $n = 0$.

Figure 2.9: Growth rate (γ_i/ω_{ce}) in disruptions. The black line is the frequency $\omega_{ce}/10$, the dashed line denotes $10\omega_{ci}$, the magnetosonic-whistler approximation is valid in the region between them. The parameters are $n_e = 5 \cdot 10^{19} \text{ m}^{-3}$, $n_r = 3 \cdot 10^{17} \text{ m}^{-3}$, $B = 2 \text{ T}$.

On Fig. 2.9 it can be seen, that as suspected, the maximum of the growth rate is indeed there in the low wave number region, just inside the region of validity of the magnetosonic-whistler approximation.

Furthermore, by comparing Figs. 2.9a and 2.9b it is evident, that the (1.38) ultrarelativistic resonance condition can be used for the low wave number region of the magnetosonic-whistler wave. The growth rate calculated with the ultrarelativistic resonance condition around the maximum is shown on Fig. 2.10.

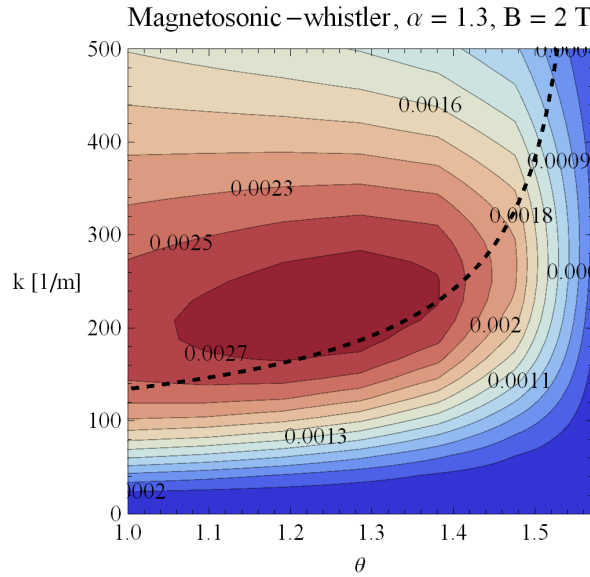


Figure 2.10: Growth rate (γ_i/ω_{ce}) in near-critical field. The dashed line is the frequency $10\omega_{ci}$, the magnetosonic-whistler approximation is valid in the region above it. The parameters are $n_e = 5 \cdot 10^{19} \text{ m}^{-3}$, $n_r = 3 \cdot 10^{17} \text{ m}^{-3}$, $B = 2 \text{ T}$.

Of Fig. 2.10, one can easily say that the most unstable wave is indeed a magnetosonic-whistler wave, propagating with approximately 200 m^{-1} wave number in near-perpendicular direction, however, this is not true. Substituting the parameters of this wave into the resonance condition (1.38) yields

$$p_{res} = \frac{-n\omega_{ce}}{k_{\parallel}c - \omega_0} \approx 20, \quad (2.24)$$

which corresponds to an energy of approximately 10 MeV. This means that first a finite number of runaway electrons have to reach the energy of 10 MeV in order to destabilize this plasma wave. As in near-critical field in tokamak plasmas this is difficult to achieve [8, 9], this wave is unlikely to be destabilized by runaway electrons.

2.4.4 Most unstable wave

As seen in Sections 2.4.2 and 2.4.3 the wave with the maximum growth rate can not be destabilized by runaway electrons in near-critical electric field. This means the most unstable wave is not defined by the global maximum of the growth rate, but as the local maximum of it inside the region defined by the maximum energy of runaway electrons. As the growth rate essentially decreases for decreasing resonant energy, the most unstable wave will be that which the highest energy electrons destabilize.

The highest energy runaway electrons which has been observed in magnetic reconnections are suprathermal electrons in the earth's magnetotail [15], and they have energies up to approximately 300 keV. As in tokamak plasmas the maximum energy of 20 – 100 keV was observed in the T-10 tokamak [8], it can be safely assumed that runaway electrons won't exceed the energy of 300 keV, not even in larger tokamaks.

The normalized momentum corresponding to the maximum energy of 300 keV is approximately $p_{res} = 1.23$. The corresponding wave numbers and propagation angles of the electron-whistler wave can be calculated by using the (2.1) general resonance condition and the (1.26) dispersion relation. The growth rate and the values corresponding to $p_{res} = 1.23$ are shown on Fig. 2.11.

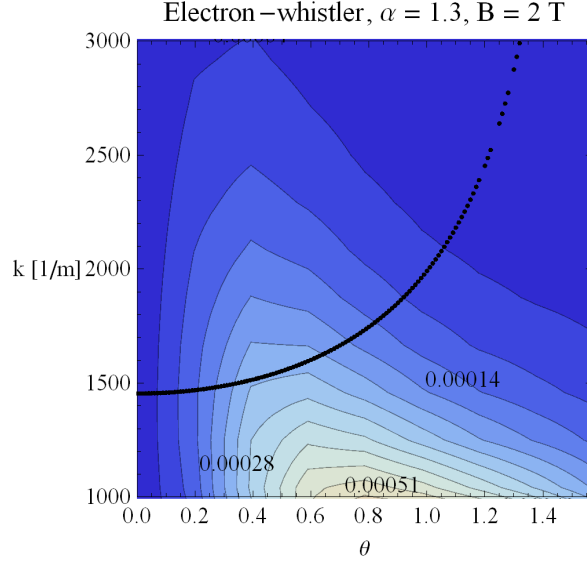


Figure 2.11: Most unstable wave in the near-critical case: maximum of the growth rate (γ_i/ω_{ce} , contour lines) on the line corresponding to the maximum runaway energy (300 keV, black dots). The parameters are $n_e = 5 \cdot 10^{19} \text{ m}^{-3}$, $n_r = 3 \cdot 10^{17} \text{ m}^{-3}$, $B = 2 \text{ T}$.

On Fig. 2.11 we can see that the most unstable wave in the near-critical case is an electron-whistler wave of the wave number of approximately 1500 m^{-1} , propagating in the direction $\theta \approx 0.4$, which is close to parallel propagation, thus this wave has a frequency of $1.7 \cdot 10^{11} \text{ s}^{-1}$.

It should be noted however, that the parameters of the most unstable wave are highly dependent on the static magnetic field. The reason is that the gyrofrequency, thus the dispersion relation of the plasma wave, and more importantly the (2.1) general resonance condition are highly dependent on the magnetic field. Because of this fact, the $p < 1.23$ condition for the momentum of the runaway electrons yields very different ranges of physical relevance on the k -plane. For example, if the static field is 4 T instead of the $B = 2 \text{ T}$ on Fig. 2.11, the $p_{res} = 1.23$ resonance condition yields $k \sim 5000 \text{ m}^{-1}$ wave numbers and higher. Therefore, by increasing the magnetic field, the wave number of the most unstable wave, thus its frequency increases, while the angle of propagation does not change significantly.

3 Linear stability

In the previous sections the growth rates were calculated and the most unstable waves were found for both the disruption and the near-critical case. The next step is to map a linear stability threshold in order to determine if the runaway electron – wave interaction can occur and what density of the runaway electrons is needed for this interaction. For the disruption case this analysis has been already done in [16], therefore from now on I investigate only the near-critical case.

To determine a threshold of runaway density, above which the wave destabilization occurs, first we should specify the various damping factors countering the growth rate. According to [7], collisional and convective damping should be considered.

3.1 Collisional damping

Collisional damping is caused by the collisions of the wave particles with other particles. The damping of high frequency cold plasma waves due to particle collisions is discussed in detail in [17]. The electron-whistler wave discussed in the present work is a high frequency wave, and the cold plasma approximations hold quite well even up to 20 keV as proven in my previous work [2], thus the discussion of Brambilla [17] is valid for this case.

As Brambilla [17] investigates the high frequency case, when the ions are essentially stationary, thus only electron-ion and electron-electron collisions are discussed. It is proven that collisions between the same species does not affect the plasma wave, therefore collisional damping in the high frequency case consists of solely the electron-ion collisions.

An exact formula is given for this damping rate:

$$\gamma_d = \frac{8}{3\sqrt{\pi}}\nu_{ei} \simeq 1.5\nu_{ei} , \quad (3.1)$$

where ν_{ei} is the electron-ion collision frequency. This damping rate is supposed to be exact for high frequency cold plasma waves, excluding the effect of resonant particles. The first two conditions are satisfied, and the conditions for the latter are:

$$|\omega_0 - n\omega_{ce}| \gg k_{\parallel}v_{Te} , \quad (3.2)$$

$$\omega_{ce} \gg k_{\perp}v_{Te} , \quad (3.3)$$

where Eq. (3.3) holds for the most unstable wave of Section 2.4.4, while Eq. (3.2) is nearly satisfied, as $|\omega_0 - n\omega_{ce}|$ is greater than $k_{\parallel}v_{Te}$ for $n = -1$ by a factor of two.

This means the (3.1) expression for the collisional damping rate should be applied with caution. However, it could be used to estimate the order of magnitude of the damping rate. For $T_e = 20$ keV electron temperature, $n_i = 5 \cdot 10^{19} \text{ m}^{-3}$ ion density and hydrogenic plasmas:

$$\nu_{ei} = \frac{1}{\tau_{ei}} \approx 930 \text{ s}^{-1} ,$$

where τ_{ei} is the electron-ion collision time, $\tau_{ei} = 3\pi^{3/2} \frac{m_e^2 v_{Te}^3 \epsilon_0^2}{n_i Z^2 e^4 \ln \Lambda}$.

3.2 Convective damping

Up to this point the homogeneous plasma approximation has been used. However, when considering the damping, we have to take into account that the destabilizing effect (i.e. the runaway beam) is finite in space. This is what causes the convective damping, and depending on the L_r radius of the runaway beam it can cause a damping rate of [7]

$$\gamma_v = \frac{\frac{\partial \omega}{\partial k_{\perp}}}{4L_r}, \quad (3.4)$$

where $\frac{\partial \omega}{\partial k_{\perp}}$ is the perpendicular group velocity of the wave. For the typical values $k \approx 2050 \text{ m}^{-1}$, $\theta \approx 0.32$ for the electron-whistler wave and $L_r = 0.2 \text{ m}$ radius this yields $\gamma_v \approx 3 \cdot 10^7 \text{ 1/s}$. This means that the rate of collisional damping, compared to convective damping, can be neglected.

The (3.4) formula is dependent on the wave number and propagation angle, therefore for different k and θ values the damping rate is different. This means that the most unstable wave found in Section 2.4.4 based solely on the growth rate is not necessarily the most unstable wave if damping is taken into account, as there can be a slight difference due to the k - and θ -dependence of the convective damping. Therefore in the following calculations the most unstable wave was determined as the maximum of

$$\gamma = \gamma_i - \gamma_v, \quad (3.5)$$

where γ_i is the linear growth rate, γ_v the rate of convective damping. Again, this maximum means not the real maximum of the function γ but the maximum *inside* the physically relevant range, considering that runaway electrons have a maximum energy of 300 keV.

Thus, the linear stability threshold was determined in the following way. For any given value of the magnetic field the growth rate is calculated (for $n = -1$, $n = 0$, then adding them for all k and θ), then the convective damping rate is subtracted from it. The parameters of the most unstable wave is then determined.

My aim was to find the value of runaway density where the growth rate induced by runaway electrons and the convective damping rate balance each other for the most unstable wave. The runaway density appears only as a scaling constant in the runaway susceptibilities (see Eqs. (2.5)-(2.8)), thus in the growth rate as well. Therefore my method was to calculate the growth rate with a known runaway density, then rescale the resulting function to balance out the convective damping.

This could be done for any values of the magnetic field. The resulting stability thresholds are shown on Fig. 3.1.

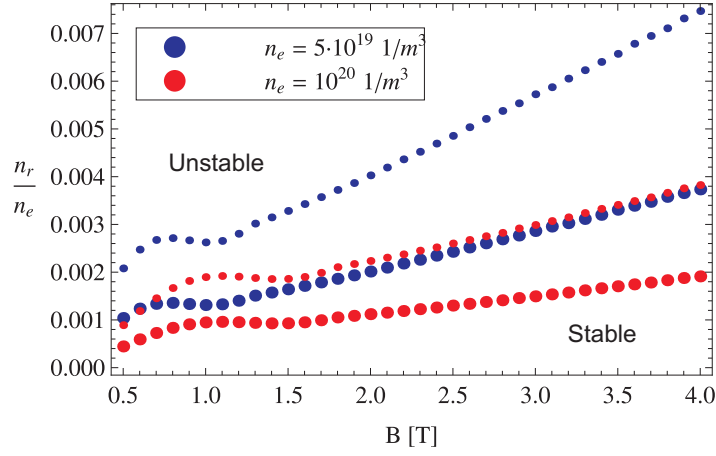


Figure 3.1: Stability thresholds for the most unstable electron-whistler wave in near-critical electric field. Small points correspond to the runaway-beam radius $L_r = 0.2$ m, large points to the radius $L_r = 0.4$ m.

On Fig. 3.1 it can be seen that the runaway density needed to counter the damping rates, therefore to destabilize an electron-whistler wave is of the order of 10^{17} m^{-3} . This means the interaction is possible, as approximately $2 \cdot 10^{17} \text{ m}^{-3}$ runaway density has already been measured in the TCV tokamak [9].

According to Fig. 3.1 for a wider runaway beam, lower runaway density is needed to destabilize the wave, which is logical, as the rate of convective damping is lower if the runaway beam is wider. For an infinitely wide runaway beam, according to these calculations, any finite density of runaway electrons would suffice. However, in this limit collisional damping would start to dominate, which has been previously neglected.

Furthermore, the stability threshold is higher for higher values of the magnetic field. This can be understood with the knowledge that for higher magnetic field, the $p_{res} < 1.23$ condition defines a higher wave number region of the k -plane, where the growth rate is smaller. This means that if the convective damping does not decrease or its decrease is slower than the decrease of the growth rate, higher runaway density is needed to balance the (nearly) same damping rate.

By comparing the stability thresholds for different thermal electron densities, we can deduce that the change in the thermal density does not affect significantly the stability threshold. By doubling the thermal electron density, the n_r/n_e ratio becomes halved for nearly all values of the magnetic field, which means the absolute value of runaway electron density needed to destabilize the wave does not change.

Finally, the order of magnitude of the runaway densities seen on Fig. 3.1 implies that the runaway electrons can indeed destabilize a high frequency electron-whistler wave in near-critical electric field. The parameters of the most unstable wave depend on the magnetic field, and typically have a value of $k > 1500 \text{ m}^{-1}$, $\theta \sim 0.4$, close to parallel propagation.

4 Summary

In the present work I continued my previous investigations [2] regarding the interaction of runaway electrons and plasma waves in tokamaks. My aim was to calculate the growth rates of this interaction, determine the most unstable wave and by comparing these results to the damping rates, determine a linear stability threshold. In this process I expanded the validity of nearly all calculations done before by introducing generalized models.

In previous works, the ultrarelativistic limit was used when calculating the runaway electron – wave interaction. I deduced the general formulas of the resonance condition and the runaway susceptibilities, and demonstrated that the usage of their ultrarelativistic limit is only justified if the investigated range of the wave numbers is low enough. As in the present work this condition was not always met, I frequently used the general formulas.

Also, based on the general resonance condition I proved that only the whistler wave can be destabilized by runaways, out of the high frequency electron waves discussed. I investigated both the case of high and near-critical electric fields with the corresponding runaway distributions, and determined the most unstable wave in both cases. In the case of high electric fields, it is a magnetosonic-whistler wave, propagating perpendicularly to the static magnetic field.

Also in near-critical field the maximum of the growth rate would be at low wave numbers and quasi-perpendicular propagation, but the runaway electrons can not reach the energy required to destabilize this wave. Nevertheless, I could determine the parameters of the most unstable wave, which turned out to be an electron-whistler wave, propagating close to the parallel direction. An important property of this wave is that its parameters are highly dependent on the magnetic field.

Finally, in the near-critical case I determined the dominant damping rates and mapped a linear stability threshold. The resulting runaway electron densities needed to destabilize the most unstable wave are of the order 10^{17} m^{-3} , essentially independent of the thermal electron density. As runaway densities this high were measured [9], this indicates that the interaction between the runaway electrons and an electron-whistler wave can indeed occur in tokamak plasmas.

Acknowledgement

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