

Stable supply chains

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Abstract

We generalize stability to supply chains with cycles. Similarly to stable matchings, a supply chain is stable when there is no blocking walk, where the firms would like to buy or sell more, and possibly dropping some of their current contracts. We show that if the choices of the firms satisfy some natural properties, there always exist a stable supply chain. We introduce four models: the choices of the agents can differ if there is money in the market or not, and they can have discrete or continuous quantities of traded goods. In the case without money we lean on Tarski's fixed point theorem, in the case with money, the solution is related to the maximum weighted subsets.

1 Introduction

We study an abstract model of supply chains. Consider a directed graph, where the nodes are the trading companies, and the edges correspond to the possible trades. In an article [Fleiner (2009)] every edge has a maximal capacity, the traded volumes form a flow, except from the source (manufacturer) and sink (final consumer), everyone buys as much as they sell. This model differs from the traditional flows in such way that, all agents have a preference order over his input and output edges, and he wants to trade with his preferred partners.

In the stable marriage problem, an edge is blocking, if both participants are single or prefer the other to his/her current partner. Similarly an edge in the trading network is blocking, if they prefer to sell/buy from the other, and they have free capacity for it, or decrease other trades for its sake. But now we define blocking coalitions as walks, where the first and last agent would like to make new trades, and the intermediate companies pass on the sufficient quantities. We allow cycles in the graph.

In the article [Ostrovsky (2008)] the agents don't need to satisfy the Kirchhoff law, but if they buy more, then they want to sell more, and they choose the best possible set of edges. These conditions are described with the so called *same side substitutable*, *cross side complementary* and *path independent* properties. In this second model, we have an upstream-downstream market, there are basic inputs, intermediates and final outputs, so the graph is acyclic. Each edge is associated with a finite set of possible prices. The edge-price pairs are called *contracts*, they can be represented as parallel edges in the graph.

In this present work, in Section 3, we generalize Ostrovsky's theorem to the case of non-acyclic graphs, first with the assumption that each agent's choice is discrete, whether he chooses to trade over an edge or not. Then we consider continuous choices, where they can decide how much quantity they trade. A supply chain is stable if there

is no blocking walk where all participants would like to sell/forward/buy extra volume. These three models' common background is the Tarski fixed point theorem. The fixed points can be assigned to the stable sets.

A further generalization of the model is described in the article [Hatfield et al. (2011)]. Here the prices of the trades are continuous, every $p \in \mathbb{R}$ is allowed. The price is added to the utility function, so if a firm prefers selling a to b , its choice can be reversed if b will pay more.

Thus the players can compensate each other for a less useful selection, so the problem is related to maximum weight matchings instead of / in addition to the stable matchings.

In this article [Hatfield et al. (2011)] they define the *fully substitutable* property, which is, because of the prices, a stronger requirement than the SSS and CSC used in the third chapter. This stronger requirement leads to a stronger claim for the existence of a stable solution, not only where no blocking walks, but there is no blocking subset of the edges.

The authors prove the existence of competitive equilibrium first, then they show that all equilibrium solutions are stable. However, they do not use the Tarski fixed-point theorem, they reduce the problem to the many-to-one stable matchings by [Kelso and Crawford (1982)].

Our goal is to generalize it to the case of continuous quantity trading.

In Section 2, we introduce basic definitions in the theory of stable matchings.

In Section 3, we solve the no money case with discrete or continuous quantities using Tarski fix point theorem.

In Section 4, we search for equilibrium and stable sets in a market with money.

2 Preliminaries

In the original stable marriage model by [Gale-Shapley (1962)], the players have a strict preference ordering over the possible partners. From a given set of marriages, they choose the best. But in many-to-one matchings, for example for firms and workers, or universities and applicants, one player can choose a bigger set from his opportunities. This can be defined with choice functions.

Definition 1. Set function $F : 2^E \rightarrow 2^E$ is called a *choice function* if $F(A) \subseteq A$ holds for any subset A of ground set E .

For example, if a player wants to choose between two trades a and b , and prefers a (we use the notation $a > b$), his choice function will be $F : 2^{\{a,b\}} \rightarrow 2^{\{a,b\}}$ satisfying

$$F(\emptyset) = \emptyset$$

$$F(\{a\}) = \{a\}$$

$$F(\{b\}) = \{b\}$$

$$F(\{a, b\}) = \{a\}$$

As we will define later, this easy choice function is comonotone and path independent.

Definition 2. Let $\overline{F}(A) = A \setminus F(A)$ denote the set of unselected elements.

Definition 3. A set function $F : 2^E \rightarrow 2^E$ is *monotone* if $F(A) \subseteq F(B)$ whenever $A \subseteq B \subseteq E$ holds.

Definition 4. A choice function $F : 2^E \rightarrow 2^E$ is *comonotone* (sometimes it is called *substitutable*) if $\overline{F}(A) \subseteq \overline{F}(B)$ for any $A \subseteq B$. (That is, if $\overline{F}$ is a monotone function.)

The stable marriage problem can be represented with a $G=(M, W, E)$ bipartite graph, where M is the set of the men, W is the set of women, and the set of edges E represents possible marriages between them. Let F be the common choice function for men over the edges of the bipartite graph, where every man chooses the best of the edges he is incident with (i.e. the marriages he can participate in). Let G be the similar choice function for women. A set of marriages $S \subseteq E$ is stable, if any not married pair $mw = e$, $e \notin S$, m or w do not want to choose e so $e \notin F(S + e)$ (we say e is F -dominated) or $e \notin G(S + e)$ (e is G -dominated). Let A_0 be the set of F -dominated edges, and B_0 is the set of G -dominated edges. Now A_0 , B_0 and S are disjoint sets, $S \cup A_0 \cup B_0 = E$, so for $A = A_0 \cup S$, $B = B_0 \cup S$, we get $A \cup B = E$ and $A \cap B = S$. If F and G are comonotone, the edges dominated one by one will be dominated altogether, so $F(A) = S, G(B) = S$. We can generalize this stability to any kind of choice functions.

Definition 5. Subset S of E is *(three-part) stable*, if $F(S) = S = G(S)$, and there exists subsets A and B of E , such that $F(A) = S = G(B)$ and $A \cup B = E$, $A \cap B = S$. Pair (A, B) with this property is called an *FG-stable pair*, and S is an *FG-stable core*. We can define a partial order on *FG-stable pairs*: $(A', B') \leq (A, B)$, if $A' \subseteq A$ and $B' \supseteq B$.

The following theorem is the key to the Gale-Shapley algorithm that finds a stable matching. Recall that a *lattice* is a partially ordered set L with the property that any two elements x, y of L have a greatest lower bound $x \wedge y$ and a least upper bound $x \vee y$.

A lattice L is *complete* if any subset X of L has a greatest lower bound $\bigwedge X$ and a least upper bound $\bigvee X$. Function $f : L \rightarrow L'$ from poset L to poset L' is *monotone* if $x \leq y$ implies $f(x) \leq f(y)$ for any elements x, y of L .

Theorem 1 (Tarski's fixed point theorem [Tarski (1955)]). *Let L be a complete lattice, and $f : L \rightarrow L$ be a monotone function on L . Then L_f is a nonempty, complete lattice on the restricted partial order where $L_f = \{x \in L : f(x) = x\}$ is the set of fixed points of f .*

Theorem 1 implies the following corollary.

Theorem 2. [Fleiner (2003)] *If $F, G : 2^E \rightarrow 2^E$ are comonotone choice functions, then the FG -stable pairs form a nonempty complete lattice for partial order $(\leq)$.*

Proof. Define function $f : 2^E \times 2^E \rightarrow 2^E \times 2^E$ by

$$f(A, B) := (E \setminus (\overline{G}(B)), E \setminus (\overline{F}(A))) = (E \setminus (B \setminus G(B)), E \setminus (A \setminus F(A)))$$

The FG -stable pairs are exactly the fixed points of f . Using Tarski's fixed point theorem, FG -stable pairs form a lattice. $\square$

Definition 6. A choice function F is *path independent*, if it has the property

$$F(A) \subseteq B \subseteq A \Rightarrow F(A) = F(B)$$

for every A, B subsets of E .

Sometimes path independence is defined with: $F(A \cup B) = F(F(A) \cup F(B))$ for every A, B .

Theorem 3. [Fleiner (2002)] *If F is a comonotone choice function, these two definitions are equivalent.*

Proof. Suppose that $F(A \cup B) = F(F(A) \cup F(B))$ for every A, B .

Then $F(A) = F(A \cup A) = F(F(A) \cup F(A)) = F(F(A))$.

If $F(A) \subseteq B \subseteq A$ then $F(B) = F(F(A) \cup B) = F(F(F(A) \cup F(B))) = F(F(A) \cup F(B)) = F(A \cup B) = F(A)$.

For the opposite direction: A, B are arbitrary subsets of E , now

$$F(A \cup B) = F(A \cup B) \cap (A \cup B) = (F(A \cup B) \cap A) \cup (F(A \cup B) \cap B) \subseteq F(A) \cup F(B) \subseteq A \cup B \\ F(A \cup B) \subseteq F(A) \cup F(B) \subseteq A \cup B \Rightarrow F(F(A) \cup F(B)) = F(A \cup B) \quad \square$$

If F is not comonotone, we will use the first definition.

Theorem 4. *If there exists a well-ordering of all subsets of E , such that $F(A)$ is the best choice of subsets of A , the choice function F is path independent (with definition $F(A) \subseteq B \subseteq A \Rightarrow F(A) = F(B)$).*

Note that a well-ordering means every set has a smallest element. Because we defined preferences the way that $a > b$ if a is better than b , what we need is every set has a biggest element, so an opposite well-ordering.

Proof. If there is a well-ordering above all subsets, $F(A) \subseteq B \subseteq A$ means that the best of all $X \subseteq A$ is also in B . Since B is smaller than A , all subsets of B are also subsets of A , so B cannot have a better subset, $F(A)$ is also the best in B , so $F(A) = F(B)$.

□

3 Supply chains without money

Consider a directed graph $G = (V, E)$, where vertices represent people, and the arcs are trades between suppliers and buyers. In this way, G is a supply chain. Everybody in the market has some preference order on whom he would like to sell to or buy from.

A model is described in [Fleiner (2009)], where every node has a strict ordering over the incoming and outgoing arcs. For two arcs vu and vw , we say $vu \geq_v vw$, if node v prefers selling to u instead of w . A *network* is a quadruple (G, s, t, c) , where G is a directed graph, s and t are different nodes of G and $c : E \rightarrow \mathbb{R}_+$ is a function that determines the capacity $c(a)$ of arc a .

The special nodes s and t are terminal points: source and target. Just like in the usual setting, we allow arcs to enter s or leave t .

A *flow* is a function $f : E \rightarrow \mathbb{R}$ such that $0 \leq f(a) \leq c(a)$ and every vertex different from s and t satisfies Kirchhoff's law: $\sum_{uv \in E} f(uv) = \sum_{vw \in E} f(vw)$ The amount of the incoming flow equals the amount of outgoing flow for v .

We call an arc a *f-unsaturated*, if $f(a) < c(a)$.

A *blocking walk* of flow f is an alternating sequence of incident vertices and arcs $P = (v_1, a_1, \dots, v_k)$ such that all the following properties hold:

arc a_i points from v_i to v_{i+1}

P has no terminal inner point

each arc a_i of P is *f-unsaturated*

$v_1 = s$ or $v_1 = t$, or there is an arc $a' = v_1 u$ such that $f(a') > 0$ and $a_1 >_{v_1} a'$ (a_1

dominates a' in v_1)

$v_k = s$ or $v_k = t$, or there is an arc $a'' = wv_k$ such that $f(a'') > 0$ and $a_{k-1} >_{v_k} a''$ (a_{k-1} dominates a'' in v_k)

A flow f is *stable* if no blocking walk to f exists.

Theorem 5 ([Fleiner (2009)]). *If network (G, s, t, c) and preference orders $<_v$ describe a stable flow problem then there always exist a stable flow f . If capacity function c is integral then there exists an integral stable flow.*

In another model from [Ostrovsky (2008)], every vertex v has a Ch_v choice function over the arcs adjacent with v . Let $D(v)$ be the set of arcs starting from v (contracts where v is the seller), and $U(v)$ is the set of arcs ending at v (where v is the buyer).

Let s_c denotes the seller and b_c the buyer of arc c , then

$$D(v) = \{c \in E | v = s_c\} \text{ and } U(v) = \{c \in E | v = b_c\}$$

For a set of edges X incident with v , let $D(X) = D(v) \cap X$, $U(X) = U(v) \cap X$

Definition 7. [Ostrovsky (2008)] Preferences of agent a are *same-side substitutable* (SSS) if for any two sets of contracts X and Y such that $D(X) = D(Y)$ and $U(X) \subseteq U(Y)$, $U(X) \setminus U(Ch(X)) \subseteq U(Y) \setminus U(Ch(Y))$, and for any two sets X and Y such that $U(X) = U(Y)$ and $D(X) \subseteq D(Y)$, $D(X) \setminus D(Ch(X)) \subseteq D(Y) \setminus D(Ch(Y))$. That is, preferences are same-side substitutable if, choosing from a bigger set of contracts on one side, the agent does not accept any contracts on that side that he rejected when he was choosing from the smaller set.

Preferences of agent a are *cross-side complementary* (CSC) if for any two sets of contracts X and Y such that $D(X) = D(Y)$ and $U(X) \subseteq U(Y)$, $D(Ch(X)) \subseteq D(Ch(Y))$, and for any two sets X and Y such that $U(X) = U(Y)$ and $D(X) \subseteq D(Y)$, $U(Ch(X)) \subseteq U(Ch(Y))$. That is, preferences are cross-side complementary if, when presented with a bigger set of contracts on one side, an agent does not reject any contract on the other side that he accepted before.

Statement 6. *If all nodes have strict preferences over arcs, the capacities are integers, and they choose the best buys and sells such that it satisfies Kirchhoff's law, this defines a SSS and CSC choice function.*

We say that a network is chain stable if there is no sequence of agents who could become better off by forming new contracts among themselves and possibly dropping some of their current contracts.

We call a collection of bilateral relationships between the nodes in a market a network, i.e., a network is simply a set of contracts. Let $M(a)$ denote the set of contracts involving vertex a in network M . Network M is *individually rational* if, for any agent a , $Ch_a(M(a)) = M(a)$ so no agent would like to unilaterally drop any of his contracts.

A chain is a sequence of contracts, $c_1, \dots, c_n$, such that for any $i < n$, $b_{c_i} = s_{c_{i+1}}$, i.e., the buyer in contract c_i is the same node as the seller in contract c_{i+1} .

For notational convenience, let $b_i = b_{c_i}$ and $s_i = s_{c_i}$. For a network M , a *chain block* is a chain $c_1, \dots, c_n$ such that

$$\begin{aligned} &\forall i \leq n, c_i \notin M, \\ &c_1 \in Ch_{s_1}(M(s_1) \cup c_1), \\ &c_n \in Ch_{b_n}(M(b_n) \cup c_n), \\ &\text{and } \forall i < n, \{c_i, c_{i+1}\} \subseteq Ch_{b_i=s_{i+1}}(M(b_i) \cup c_i \cup c_{i+1}). \end{aligned}$$

A network is *chain stable* if it is individually rational and has no chain blocks.

Ostrovsky defined the chain stability only for acyclic graphs: upstream-downstream networks, where there are suppliers of basic inputs (sources), intermediaries and consumers of final outputs (targets).

Theorem 7 ([Ostrovsky (2008)]). *If every vertex's choice function is SSS, CSC and path independent, and G is acyclic, then there exists a chain stable network.*

Now we allow circles in the graph.

Definition 8. A *blocking walk* of network M is an alternating sequence of incident vertices and arcs $P = (v_1, a_1, \dots, a_{k-1}, v_k)$ such that all the following properties hold:

arc a_i points from v_i to v_{i+1}

each arc $a_i \notin M$

$$a_1 \in Ch_{v_1}(M(v_1) \cup a_1),$$

$$a_{k-1} \in Ch_{v_k}(M(v_k) \cup a_{k-1}),$$

$$\text{and } \forall i < n, \{a_i, a_{i+1}\} \subseteq Ch_{v_{i+1}}(M(v_{i+1}) \cup a_i \cup a_{i+1}).$$

Nodes $v_1, \dots, v_k$ do not have to be different.

A network is *chain stable* if it is individually rational and has no blocking walks.

Theorem 8. (*Generalization of Ostrovsky's theorem*) *If every vertex's choice function is SSS, CSC and path independent, then there exists a chain stable network.*

Proof. Transform $G = (V, E)$ to a bipartite graph such way that we divide each arc to two arcs by a new vertex. From arc $e = uv$ we get $e_1 = ux$ and $e_2 = xv$. Denote the new graph $G' = (V', E')$. Let be set of the beginning of the arcs D , $D = \{ux | u \in V\}$, the end of the arcs is U . $U = \{xv | v \in V\}$ Therefore $E' = D \cup U$.

We call the original edges full edges, and the new ones half edges.

Every new vertex has the following choice function:

The x node has exactly two arcs, $e_1 = ux$ és $e_2 = xv$.

$$G(\{e_1, e_2\}) = \{e_1, e_2\}$$

$$G(\{e_1\}) = \emptyset$$

$$G(\{e_2\}) = \emptyset$$

$$G(\emptyset) = \emptyset$$

Note that this choice function is also SSS, CSC and path independent (the subset ordering $\{e_1, e_2\} > \emptyset > \text{all other sets}$ defines it).

Define a new partial ordering over the subsets of E' : for $X, Y \subseteq E'$, we say $X \preceq Y$ if $X \cap D \subseteq Y \cap D$ and $X \cap U \supseteq Y \cap U$.

Lemma 9. *If choice function F is SSS and CSC, then F is comonotone for the new partial ordering, that is $\bar{F}$ is monotone: $X \preceq Y \Rightarrow \bar{F}(X) \preceq \bar{F}(Y)$.*

If $X \preceq Y$, let $Z = (Y \cap D) \cup (X \cap U)$. Now $X \preceq Z \preceq Y$. The X and Z set has the same intersection with U , and $X \cap D \subseteq Z \cap D = Y \cap D$, so from the SSS property $D(X) \setminus D(F(X)) \subseteq D(Z) \setminus D(F(Z))$, from the CSC property $U(F(X)) \subseteq U(F(Z))$, and because $U(X) = U(Z)$, we get $U(X) \setminus U(F(X)) \supseteq U(Z) \setminus U(F(Z))$. Therefore $F(X) \preceq F(Z)$.

In the similar way, Z and Y has the same intersection with D , so we get the same statements with opposite roles of U and D . $F(X) \preceq F(Z)$, $F(Z) \preceq F(Y)$ so $F(X) \preceq F(Y)$.

Since the original vertices have disjoint arcs, we can define a common choice function. Let $F(M) = \bigcup Ch_a(M(a))$.

And let G be a common choice function for the new vertices.

Functions F and G are comonotone for the new $\preceq$ ordering. So using Theorem 2 there exist a three-part stable solution, i.e. $\exists A, B$ for which $F(A) = S = G(B)$, and $A \cap B = S$, $A \cup B = E$.

Lemma 10. *If subset S of E is three-part stable, then it is chain stable.*

Suppose that S three-part stable. Since $G(S) = S$, S contains only full edges. From the edges of $E \setminus S$, set $B \setminus S$ cannot contain a full edge, because then G would choose it. Therefore $A \setminus S$ has at least one half from every edges in $E \setminus S$.

Color the half edges in $A \setminus S$ to red.

Consider an arbitrary $P = (e_1, \dots, e_{2k})$ directed walk (where e_i -s are half edges) outside of S . ($P \cap S = \emptyset$). A directed path on the original edges is also a directed path on the new edges. If the first or last half-edge of P is red, then P can't be a blocking walk, since $e_1 \in A \setminus S$, so $S \subseteq S \cup e_1 \subseteq A$, and from the path independent property $F(S \cup e_1) = S$ which means e_1 is dominated. The same applies to e_{2k} .

Suppose that neither the first, nor the last half-edge is red. Since every edge has at least one red half, there exists an original vertex, where two red half-edges meet. Call them e_{2j} and e_{2j+1} . From $S \subseteq S \cup \{e_{2j}, e_{2j+1}\} \subseteq A$, we get $F(S \cup \{e_{2j}, e_{2j+1}\}) = S$. Therefore P is not a blocking walk.

We showed that no walk can block, so S is chain stable.

There always exists a three-part stable solution, and every three-part stable is chain stable, so there exists a chain stable solution. □

Theorem 11. *The opposite direction is also true: every chain stable S is three-part stable with the new ordering $\prec$.*

Proof. Color a half edge $e \notin S$ outside of S to red if $e \notin F(S + e)$, so S dominates them with the F choice function. Color $e \notin S$ to blue, if $e \in F(S + e)$, i.e. it is not dominated. If there exists a blocking path, the first and last half edge must be blue, hence the players would choose them.

Since there is no blocking edge, both halves of an edge cannot be blue, so every edge has at least one red part. Therefore $G(S \cup \text{blue edges}) = S$.

It would be simple to place the red edges in $A \setminus S$, and the blue ones in $B \setminus S$. But it may occur that a and b half edges are red, yet $\{a, b\} \subseteq F(S \cup \{a, b\})$. Because of the path independent property, from a and b the choice $F(S \cup \{a, b\})$ contains both or none, since if $a \notin F(S \cup \{a, b\})$, then $F(S \cup \{a, b\}) \subseteq S \cup b \subseteq S \cup \{a, b\}$ so $F(S \cup \{a, b\}) = F(S \cup b)$ and $b \notin F(S \cup b)$, therefore $b \notin F(S \cup \{a, b\})$

If $\{a, b\} \subseteq F(S \cup a, b)$, we draw a new edge from the beginning of a to the end of b . We color these new edges to green.

The set of the middle vertices is V_h . Let D_1 be set of vertices in V_h who has a blue incoming edge, and U_1 is the set of V_h -vertices with blue outgoing edge. As we said before, there is no blue full edge, hence $D_1 \cap U_1 = \emptyset$.

If there is a directed walk on only green edges from D_1 to U_1 , there exist a blocking walk, the first and last vertices choose the blue edge, and the middle vertices choose the two red edges connected with the green edge.

Since S is chain stable, there is no such walk, we cannot go from D_1 to U_1 . Let Z_1 be the set of V_h -vertices available from D_1 on green edges, and let $Z_2 = V_h \setminus Z_1$.

If $v \in Z_1$, no blue edge can leave it. Recolor the incoming red edges to blue. If $v \in Z_2$, no blue edge can go into it. Recolor the outgoing red edges to blue. With this method, there is still no full blue edge.

Now put the red edges into $A \setminus S$ and the blue ones into $B \setminus S$.

Every edge outside of S has at least one red half, so $G(B) = S$

For the original vertices, if $v \in V$, let $a = xv$ be an incoming half edge, and $b = vy$ is an outgoing half edge, which were originally red.

If $\{a, b\} \subseteq F(S \cup ab)$, then one of a and b was changed to blue. If it did not, then x changes the edge before it, so $x \in V_1$, and y changes the edge after it, so $y \in V_2$. Moreover there is a green edge from x to y . This contradicts the definition of V_1, V_2 .

Consider the set of remaining red edges incident with v . By the definition of red edges,

$F(S \cup a) = S$, and from the former argument, for every incoming-outgoing pair, $F(S \cup ab) = S$.

Lemma 12. *If the choice function F of vertex v is SSS, CSC and path independent, and the incoming edges are $a_1, \dots, a_k$, the leaving edges are $b_1 \dots b_l$, where for all a_i, b_j pairs, $F(S \cup \{a_i, b_j\}) = S$, then $F(S \cup \{a_1, \dots, a_k, b_1 \dots b_l\}) = S$.*

Since $F(S \cup \{a_i, b_j\})$ refuses b_j , from the SSS property $F(S \cup \{a_i, b_1, \dots, b_l\})$ refuses every b_j . Since $F(S \cup \{a_i, b_1, \dots, b_l\}) \subseteq S \cup \{a_i\} \subseteq S \cup \{a_i, b_1, \dots, b_l\}$ from path independence $F(S \cup \{a_i, b_1, \dots, b_l\}) = S$ for every i .

The same can be said for the opposite side, so $F(S \cup \{a_1, \dots, a_k, b_j\}) = S$ for every j . Using SSS again, $F(S \cup \{a_i, b_1, \dots, b_l\})$ rejects a_i , so $F(S \cup \{a_1, \dots, a_k, b_1 \dots b_l\})$ also reject a_i -t, for every i . $F(S \cup \{a_1, \dots, a_k, b_j\}) = S$ rejects b_j , so $F(S \cup \{a_1, \dots, a_k, b_1 \dots b_l\})$

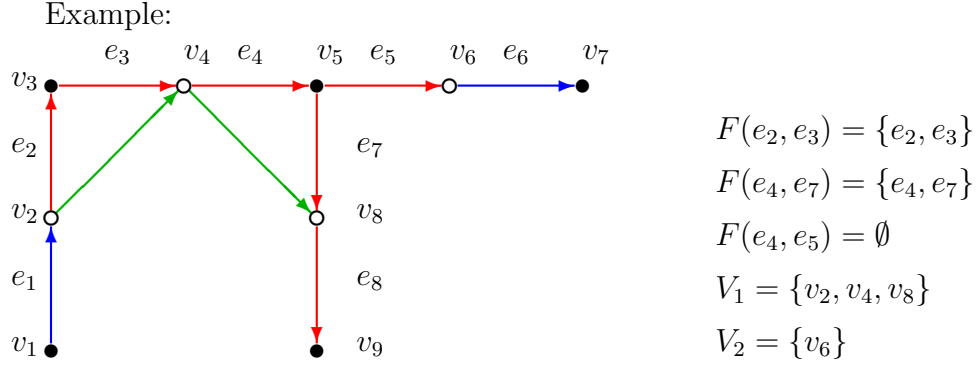
rejects b_j -t, for every j .

Therefore $F(S \cup \{a_1, \dots, a_k, b_1 \dots b_l\}) \subseteq S \subset S \cup \{a_1, \dots, a_k, b_1 \dots b_l\}$

so $F(S \cup \{a_1, \dots, a_k, b_1 \dots b_l\}) = F(S) = S$.

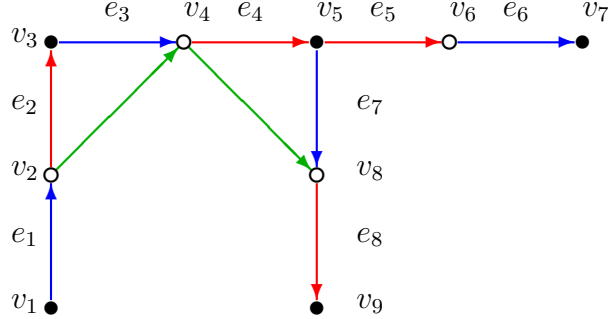
Using this lemma to the red edges, $F(A) = S$, and from the construction $G(B) = S$ and $A \cap B = S, A \cup B = E$, therefore S is three-part stable.

□



Here $S = \emptyset$ is stable, there is no blocking walk.

After the color changing:



In this model, we only know that there are no blocking walks, where first and last edge are not chosen by the players separately. But if the walk is a circle, it can happen that $v_1 = v_k$ would like to choose e_1 and e_{k-1} together.

Definition 9. A *blocking circle* of network M is an alternating sequence of incident vertices and arcs $C = (v_1, a_1, \dots, a_{k-1}, v_k)$ such that all the following properties hold:

$$v_1 = v_k$$

arc a_i points from v_i to v_{i+1}

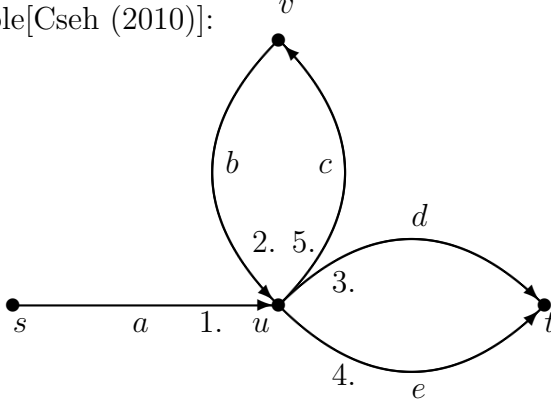
each arc $a_i \notin M$

$$\{a_1, a_{k-1}\} \subseteq Ch_{v_1}(M(v_1) \cup a_1 \cup a_{k-1}),$$

$$\forall i < n, \{a_i, a_{i+1}\} \subseteq Ch_{v_{i+1}}(M(v_{i+1}) \cup a_i \cup a_{i+1}).$$

A network is *fully stable* if it is individually rational, and there is no blocking circle and no blocking walk.

Unfortunately, a fully stable network not always exist. Consider the following example[Cseh (2010)]:



This model is described as in [Fleiner (2009)]. The agents u and v want to satisfy the Kirchoff law, and u 's preference ordering is $a > b > d > e > c$, he want to buy and sell equal quantities to his best possible partners. Agent v buys and sells both b and c or none of them.

The source is s , the target is t , they want to trade everything they can.

With choice functions, $Ch_s(a) = a$

$Ch_v(b) = \emptyset, Ch_v(c) = \emptyset, Ch_v(b, c) = b, c$

$Ch_t(d) = d, Ch_t(e) = e, Ch_t(d, e) = d, e$

$Ch_u(a) = \emptyset, Ch_u(b) = \emptyset, \dots Ch_u(a, b, c, d, e) = a, b, d, e$

The network $\{a, d\}$ is chain stable, since adding one of b and c is blocked at v , adding the b, c circle, we can decide where to start it. If $v_1 = u = v_3, v_2 = v$ it is blocked at u , because $Ch_u(a, b, d) = a, d, Ch_u(a, c, d) = a, d$. If $v_1 = v = v_3, v_2 = u$ it is blocked at u , because $Ch_v(b) = \emptyset, Ch_v(c) = \emptyset$.

If we try e , the edge d is better, so $Ch_u(a, d, e) = a, d$

If we add b, c, e together, $Ch_u(a, b, d) = a, d, Ch_u(a, c, d, e) = a, d$, so it is not blocking.

But b, c is a blocking circle, $Ch_u(a, b, c, d) = a, b, c, d, Ch_v(b, c) = b, c$ so a, d is not fully stable. Adding this circle to the network, a, b, c, d is not stable, since e edge is blocking: $Ch_u(a, b, c, d, e) = a, b, d, e$ and $Ch_t(d, e) = d, e$.

3.1 Continuous quantities

In the previous model, one edge is sold or not, but we didn't talk about how much they trade. Now, assign a volume to every trade, $q : E \rightarrow \mathbb{R}$ describes the supply chain. If E is finite, we can consider it as a vector in $\mathbb{R}^{|E|}$. Instead of Kirchoff's law, every node makes some transformation on the goods. (For example: a firm v buys 100 kg wood, and sells 20 chairs) For every vertex, an SSS, CSC choice function describes the preferences.

In the continuous case, a choice function is $F : \mathbb{R}^{|E|} \rightarrow \mathbb{R}^{|E|}$ for which $F(A) \leq A$ for every $A \in \mathbb{R}^{|E|}$. Here $\leq$ means $F(A)(e) \leq A(e)$ for every edge. If for a subset A of the edges, $A(e) = 1$ if $e \in A$ and $A(e) = 0$ if $e \notin A$, we get the original definition.

Denote the set of unselected elements with $\overline{F}(A) = A - F(A)$.

A choice function F is comonotone, if for every $A \leq B$, $A - F(A) \leq B - F(B)$, so $\overline{F}(A) \leq \overline{F}(B)$.

Cut each edge in half, E' is the set of the half edges. We can generalize the $q(e')$ quantity of products for every $e' \in E'$, now the function q maps from E' to $\mathbb{R}$. The edge set $E' = D \cup U$ where D is the set of the beginner half of the edges, and U is the ending half of the edges. Let q_D be the value of function q , on the half edges in D , and q_U , is the value of q restricted to U .

The F choice function is SSS and CSC, if for any two quantity functions q and q' , if $q_D = q'_D$, $q_U \leq q'_U$ then $\overline{F}(q_U) \leq \overline{F}(q'_U)$ and $F(q_D) \leq F(q'_D)$. And similarly, for every $q_U = q'_U$, $q_D \leq q'_D$ we get $\overline{F}(q_D) \leq \overline{F}(q'_D)$ and $F(q_U) \leq F(q'_U)$.

In the new partial order $q \preceq q'$, if $q_U \leq q'_U$, $q_D \geq q'_D$. As we have seen in the discrete case, the choice function F is SSS and CSC if and only if F is comonotone for this partial order, i.e. $q \preceq q' \Rightarrow \overline{F}(q) \preceq \overline{F}(q')$.

Each edge has a capacity $c(e)$. Let L be a set of nonnegative mappings $l : E \rightarrow \mathbb{R}$ such that $0 \leq l(e) \leq c(e)$ holds for any edge e of E . Observe that L forms a complete lattice under partial order $\leq$. If $l_i : E \rightarrow \mathbb{R}$ are elements of L for $i \in I$ then $\bigvee_{i \in I} l(e) = \sup\{l_i(e) : i \in I\}$ and $\bigwedge_{i \in I} l(e) = \inf\{l_i(e) : i \in I\}$, since the $l_i(e)$ elements are bounded, infimum and supremum always exist.

We define two choice functions on L , for the two sides of the market, now F for the original nodes, and G for the middle nodes.

Define mappings on L by $F^*(l) = c - \overline{F}(l)$ and $G^*(l) = c - \overline{G}(l)$ that is for each edge e , $F^*(l)(e) = c(e) - \overline{F}(l)(e)$ and $G^*(l)(e) = c(e) - \overline{G}(l)(e)$. For any element l of L , we have $F^*(l), G^*(l) \in L$ and since $\overline{F}$ is monotone for the ordering $\preceq$, if $l \preceq l'$, then $F^*(l) \succeq F^*(l')$ and $G^*(l) \succeq G^*(l')$. So if we compose these maps, then we get a monotone function $H = F^* \circ G^*$

$H(l)(e) = c(e) - \overline{F}(c - \overline{G}(l))(e)$. From Tarski's fixed point theorem, there is some function l of L such that

$$H(l)(e) = c(e) - \overline{F}(c - \overline{G}(l))(e) = l(e)$$

Define $s = G(l)$, $s_G = l - s$, $s_F = c - l$, so $s = s_G + s + s_F$.

Rearranging the sides, we get $s_F = c - l = \overline{F}(c - \overline{G}(l)) = \overline{F}(s + s_F)$.

Extending the definition of three-part stability, $s = G(l)$ is three-part stable, since for $A = s + s_F$, $B = l = s + s_G$, $F(A) = s = G(B)$ and $A + B = c + s$.

An arc e is q -unsaturated, if $q(e) < c(e)$.

Definition 10. A blocking walk of network q is an alternating sequence of incident vertices and arc $P = (v_1, a_1, \dots, v_k)$ such that all the following properties hold:

arc a_i points from v_i to v_{i+1}

each arc a_i of P is q -unsaturated

there exist a ε_i for every v_i such that

$$\varepsilon_1 a_1 \leq F(q + \varepsilon_1 a_1)$$

$$\varepsilon_k a_{k-1} \leq F(q + \varepsilon_k a_{k-1})$$

$$\varepsilon_i a_{i-1} + \varepsilon_i a_i \leq F(q + \varepsilon_i a_{i-1} + \varepsilon_i a_i)$$

Definition 11. A network l is chain-stable, if it is individually rational: $F(l) = l = G(l)$, and there is no blocking walk.

Theorem 13. A network is chain stable if and only if it is three-part stable.

Theorem 14. If preferences are SSS, CSC and path independent, there always exist a chain stable solution.

Proof. Similarly to the discrete case, we cut each edge in half. For the middle points, we define choice function G . From edge $e = uv$ we created $e_1 = ux$ and $e_2 = xv$. If in a network q , the volume of sold products on the edges are $a = q(e_1)$ and $b = q(e_2)$, the choice of x is $G(a, b) = (\min(a, b), \min(a, b))$ which means x sells exactly what he buys, with the biggest possible amount.

We show that G is comonotone for $\preceq$.

If $a < b$ $G(a, b) = (a, a)$, so $\overline{G}(a, b) = (0, b - a)$. $(a, b) \preceq (a', b')$ means $a \leq a', b \geq b'$

Suppose that $a < b$.

If a decreases, b increases $\overline{G}(a', b') = (0, b' - a')$, and $b' - a' > b - a$, so $\overline{G}(a, b) \succeq \overline{G}(a', b')$.

If a increases, b decreases, and still $a' < b'$ $\overline{G}(a', b') = (0, b' - a')$, and $b' - a' < b - a$, so $\overline{G}(a, b) \preceq \overline{G}(a', b')$

If a increases, b decreases, and $a' > b'$ $\overline{G}(a', b') = (a' - b', 0)$, and $a' - b' > 0$, $0 < a - b$ so $\overline{G}(a, b) \preceq \overline{G}(a', b')$.

Choice function G is also path independent, if $(\min(a, b), \min(a, b)) \leq (a', b') \leq (a, b)$, one of a', b' must be $\min(a, b)$, so $G(a', b') = (\min(a, b), \min(a, b))$

Suppose that s is three-part stable. It mean there exist $A, B : E' \rightarrow \mathbb{R}$ such that $A = s + s_F$, $B = l = s + s_G$, $F(A) = s = G(B)$ and $A + B = c + s$. From path independency, s is also invidually rational, $F(s) = s = G(s)$.

Color an unsaturated half edge e to red, if $A(e) > s(e)$, so $A - s$ has some positive quantity of e . If $A(e) = s(e)$, color e to blue.

Since $G(s) = s$, for two halves of an original edge e_1, e_2 , $s(e_1) = s(e_2)$. If e_1, e_2 are unsaturated, $B(e_1) = s(e_1)$ or $B(e_2) = s(e_2)$ otherwise G would choose a bigger volume. So at least one of the two halves is red. This is true for every unsaturated edge. Consider an arbitrary $P = (e_1, \dots, e_{2k})$ directed walk of unsaturated half edges. If the first or last half-edge of P is red, then P can't be a blocking walk, since $\varepsilon e_1 \in A - s$, so $s \leq s + \varepsilon e_1 \leq A$, and from the path independent property $F(S + \varepsilon e_1) = S$ which means e_1 is dominated. The same applies to e_{2k} .

Suppose that neither the first, nor the last half-edge is red. Since every edge has at least one red half, there exists an original vertex, where two red half-edges meet. Call them e_{2j} and e_{2j+1} . From $s \leq s + \varepsilon_{2j} e_{2j} + \varepsilon_{2j+1} e_{2j+1} \leq A$, we get $F(s + \varepsilon_{2j} e_{2j} + \varepsilon_{2j+1} e_{2j+1}) = s$. Therefore P is not a blocking walk.

We showed that no walk can block, so S is chain stable.

There always exists a three-part stable solution, and every three-part stable is chain stable, so there exists a chain stable solution.

□

4 Supply chains with money

In the paper [Hatfield et al. (2011)], they introduce a model with continuous prices. There is a finite set $I = 1, 2, \dots, n$ of agents in the economy. They are represented with vertices. The agents can participate in bilateral trades, which correspond to the edges

of the directed graph $G = (I, E)$. So E is the set of all possible trades. Every trade e is associated with buyer $b(e)$ and a seller $s(e)$. In this model, circles in the graph are allowed.

The following definitions are all from [Hatfield et al. (2011)]:

We call a pair (e, p_e) a *contract* where e is a trade and p_e is the price at which the trade occurs. The set of available contracts is $X = E \times \mathbb{R}$.

For any set of contracts Y , the set of trades involved in Y is denoted with $\tau(Y)$, so $\tau(Y) = \{e \in E : (e, p_e) \in Y \text{ for some } p_e \in \mathbb{R}\}$. For an agent i and a set of contracts $Y \subseteq X$, denote the contracts coming into i with Y_{i-} so $Y_{i-} = \{y \in Y : i = b(y)\}$, and the set of contracts in Y leaving i is Y_{i+} so $Y_{i+} = \{y \in Y : i = s(y)\}$

Let Y_i denote the set of contracts in Y incident with i , so $Y_i = Y_{i-} \cup Y_{i+}$, and E_i the set of edges incident with i .

We say the set of contracts Y is *feasible*, if there is no trade e and prices $p_e \neq p'_e$ such that both contracts (e, p_e) and (e, p'_e) are in Y .

An *outcome* $A \subseteq X$ is a feasible set of contracts. So it is a set of edges with a price for every edge in the set, but we don't have prices for the edges outside the set.

An *arrangement* is a pair $[\Psi, p]$ where $\Psi \subseteq E$ is a set of trades and $p \in \mathbb{R}^{|E|}$ is a vector of prices for all trades in the economy.

Each player v_i has a valuation function $u_i(A)$ over all subsets of edges $A \subseteq E$ it is incident with. The valuation u_i gives rise to an utility function U_i over sets of contracts.

$$U_i(Y) = u_i(\tau(Y)) + \sum_{(e, p_e) \in Y_{i+}} p_e - \sum_{(e, p_e) \in Y_{i-}} p_e$$

There are two kind of choice function here: The *choice correspondence* of agent i from the set of contracts $Y \subseteq X$ is

$$C_i(Y) = \underset{Z \subseteq Y, Z \text{ is feasible}}{\operatorname{argmax}} U_i(Z)$$

This C_i is a choice function over the contracts, but it differs from a regular choice function, that it can be multiple valued, if maximum is taken on more than one set. So $C_i(Y)$ is set of all best choices from Y . It may be empty-valued too, for example if $Y = \{(e_0, p), 0 < p < 1\}$.

The *demand correspondence* of agent i with a given price vector $p \in \mathbb{R}^{|E|}$ is defined

as the set of trades maximizing agent i 's utility under prices p :

$$D_i(p) = \operatorname{argmax}_{A \subseteq E} U_i([A, p])$$

The demand correspondence function chooses from the edges, not from the contracts, and always from the whole set of the edges. The optimal set is not uniquely defined, $D_i(p)$ is the set of all optimal edge-sets.

The *indirect utility function* of agent i is

$$V_i(p) = \max_{\Psi \subseteq E_i} U_i([\Psi, p])$$

Let $V(p) = \sum_{i \in I} V_i(p)$.

Let $a(Z)$ denote the of agents involved in contracts in Z as buyers or sellers.
 $a(Z) = \bigcup_{z \in Z} \{b(z), s(z)\}$

Definition 12. An outcome A is *stable* if it is:

1. *Individually rational*: $A_i \in C_i(A)$ for all i ,
2. *Unblocked*: There is no feasible nonempty blocking set of contracts $Z \subseteq X$ such that $Z \cap A = \emptyset$ and for all $i \in a(Z)$, for all of i 's choices $Y \in C_i(Z \cup A)$ we have $Z_i \subseteq Y$.

Definition 13. An agreement $[\Psi, p]$ is a *competitive equilibrium* if for all $i \in I$

$$\Psi_i \in D_i(p)$$

Based on the SSS, CSC definitons by [Ostrovsky (2008)], they introduce the full substitutable property. It is similar to the "comonotone for ordering $\prec$ " definition in Section 3, but now they define monotonicity over contracts, not edges, and we need some strictening because of the multiple valued definiton of C_i . They introduce three equivalent definitions of full substitutability: choice-language fully substitutable, demand-language fully substitutable and indicator-language fully substitutable. Here we will only need the first definition.

Definition 14. [Hatfield et al. (2011)] Agent i 's preferences are *choice-language fully substitutable*(CFS) if:

1. For all sets of contracts $Y, Z \subseteq X_i$, such that $|C_i(Z)| = |C_i(Y)| = 1$, if $Y_{i+} = Z_{i+}$ and $Y_{i-} \subseteq Z_{i-}$ then for the unique $Y^* \in C_i(Y)$ and $Z^* \in C_i(Z)$, we have $(Y_{i-} - Y_{i-}^*) \subseteq (Z_{i-} - Z_{i-}^*)$ and $Y_{i+}^* \subseteq Z_{i+}^*$
2. For all sets of contracts $Y, Z \subseteq X_i$, such that $|C_i(Z)| = |C_i(Y)| = 1$, if $Y_{i-} = Z_{i-}$ and $Y_{i+} \subseteq Z_{i+}$ then for the unique $Y^* \in C_i(Y)$ and $Z^* \in C_i(Z)$, we have $(Y_{i+} - Y_{i+}^*) \subseteq (Z_{i+} - Z_{i+}^*)$ and $Y_{i-}^* \subseteq Z_{i-}^*$

It means that when attention is restricted to sets for which C_i is single-valued, and the set of options to i on one side expands, i rejects a larger set of contracts on that side (SSS), and selects a larger set of contracts on the other side.

Theorem 15. *[Hatfield et al. (2011)] If all agents' preferences are fully substitutable, there always exists a competitive equilibrium.*

They proved this theorem by reducing it to many-to-one stable matchings in a bipartite graph, a model from [Kelso and Crawford (1982)].

For an arrangement $[\Psi, p]$, let $\kappa([\Psi, p]) = \bigcup_{e \in \Psi} \{(e, p_e)\}$ denote the set of contracts induced by the arrangement. So $\kappa([\Psi, p])$ is an outcome.

Theorem 16. *[Hatfield et al. (2011)] Suppose $[\Psi, p]$ is a competitive equilibrium. Then $\kappa([\Psi, p])$ is stable.*

These two theorems together prove that there always exists a stable solution.

Definition 15. [Hatfield et al. (2011)] The set of trades Ψ is *efficient* if $\sum_{i \in V} u_i(\Psi) \geq \sum_{i \in V} u_i(\Psi')$ for any other $\Psi' \in E$, that is where the sum of utilities is maximal.

Denote this maximum with

$$U^* = \max_{\Psi \in E} \sum_{i \in V} u_i(\Psi).$$

Therefore Ψ is efficient if and only if $U^* = \sum_{i \in I} u_i(\Psi)$

Theorem 17. *[Hatfield et al. (2011)] Suppose agents' preferences are fully substitutable. Then for any competitive equilibrium $[\Phi, p]$ and efficient set of trades Ψ , $[\Psi, p]$ is also a competitive equilibrium. Furthermore, the set of competitive equilibrium prices is a lattice.*

Lemma 18. *[Hatfield et al. (2011)] A price vector p' is a competitive equilibrium price vector if and only if $p' \in \operatorname{argmin}_p V(p)$*

They proved this lemma using the existence of competitive equilibrium.

The search for a competitive equilibrium is easier after these theorems. We only have to check the efficient edge sets, and the $V(p)$ -minimizing price vectors.

We can conclude it in a theorem, which is weaker than the former statements, but sufficient for showing the existence of a competitive equilibrium, and can be generalized to the continuous quantity case.

Theorem 19. *If $p' \in \operatorname{argmin}_p V(p)$ there exists an efficient set Ψ , where $[\Psi, p']$ is a competitive equilibrium.*

This theorem is a corollary of Theorem 15, Theroem 17 and Lemma 18.

For any subset of edges $\Phi \in E$ and arbitrary price vector p , let $Y = \kappa([\Phi, p])$ denote the set of contracts formed from the edges of Φ with corresponding prices from p . For every $i \in I$, $U_i(Y) \leq V_i(p)$ from the definition of $V_i(p)$. If we sum the utilities for all agents, each price appear twice, with different signs, so they cancel out.

$$\sum_{i \in I} U_i(Y) = \sum_{i \in I} (u_i(\tau(Y)) + \sum_{(e, p_e) \in Y_{i+}} p_e - \sum_{(e, p_e) \in Y_{i-}} p_e) = \sum_{i \in I} u_i(\tau(Y))$$

So they have the same maximum:

$$U^* = \max_{\tau(Y) \subseteq E} \sum_{i \in I} u_i(\tau(Y)) = \max_{Y \subseteq X} \sum_{i \in I} U_i(Y)$$

The indirect utility function maximizes the utilities one by one, and U^* maximizes the utilities together, so $V(p)$ is greater.

$$U^* = \max_{Y \subseteq X} \sum_{i \in I} U_i(Y) \leq \sum_{i \in I} \max_{Y \subseteq X} U_i(Y) = \sum_{i \in I} V_i(p) = V(p)$$

Let $V^* = \min_p V(p)$, that is $V^* = V(p')$ Since $U^* \leq V(p)$ for any p , $U^* \leq \min_p V(p) = V^*$.

If $U^* = V^*$, then

$$U^* = \max_{Y \subseteq X} \sum_{i \in I} U_i(Y) = \sum_{i \in I} \max_{Y \subseteq X} U_i(Y) = \sum_{i \in I} V_i(p') = V(p')$$

so there is a Y^* set of contracts, where all agents have maximal utility, $U^* = \sum_{i \in I} U_i(Y^*)$ and $\max_{Y \subseteq X_i} U_i(Y) = U_i(Y_i)$, so $Y_i \in D_i(p)$, therefore $[Y^*, p']$ is a competitive equilibrium.

Conjecture 20. *If $V(p') > U^*$, there exist a better p^* price vector for which $V(p') > V(p^*)$, so $V(p')$ can't be minimal.*

4.1 Continuous prices and quantities

Denote the set of trades with E , and to every $e \in E$ edge belong an quantity q_e and a price p_e . We denote the price vector with p , the quantity vector with q . If $q(e) \leq q'(e)$ for every $e \in E$, we say $q \leq q'$.

Each edge has a maximal capacity $c(e)$, so for every i , $0 < q(e) < c(e)$. Let $T = \prod_{i \in I} [0, c_e]$, the set of possible quantity vectors. The prices can take any real value, $p \in \mathbb{R}^{|E|}$.

Now we call a quantity vector-price vector pair an *agreement*. The set of possible arrangements is $X = T \times \mathbb{R}^{|E|}$

Valuation function u_i is defined on T for every agent i . Suppose that u_i is continuous and $u_i(0) = 0$. Let E_{i+} denotes the set of edges leaving vertex i , and E_{i-} is the set of edges coming to i . A player i 's utility over the agreements is

$$U_i(q, p) = u_i(q) + \sum_{e \in E_{i+}} p_e q(e) - \sum_{e \in E_{i-}} p_e q(e).$$

We can change the definitions of *choice correspondence* and *demand correspondence* as follows:

If $Y \subseteq X$ is a set of agreements

$$C_i(Y) = \operatorname{argmax}_{q, p \in Y} U_i(q, p)$$

If in a special case where we only need to choose from a feasible set, where each edge has only one possible price, and the quantities has some upper bound, the C_i function can be written as

$$C_i(q, p) = \operatorname{argmax}_{q' \leq q} U_i(q', p)$$

$$D_i(p) = \operatorname{argmax}_{q \in T} U_i(q, p)$$

The *indirect utility function* of agent i is

$$V_i(p) = \max_{q \in T} U_i(q, p)$$

Let $V(p) = \sum_{i \in I} V_i(p)$, and $V^* = \min_p V(p)$.

Let $U^* = \max_{q \in T} \sum_{i \in V} u_i(q)$ the maximal common utility over the networks. We say q is *efficient*, if $U^* = \sum_{i \in V} u_i(q)$.

Definition 16. An agreement $[q, p]$ is a *competitive equilibrium* if for all $i \in I$

$$q \in D_i(p)$$

The continuous version of Theorem 19 is the following:

Conjecture 21. If $p' \in \operatorname{argmin}_p V(p)$ there exist an efficient quantity function q , where $[q, p]$ is a competitive equilibrium.

To prove Conjecture 21, the following lemma can be useful:

Lemma 22. *Agent i 's preferences are fully substitutable, and for agreements $[q_0, p]$, $[q_1, p]$, $[q_2, p]$ the price vector is the same, and for each $e \in E_{i-}$ $q_0(e) = q_1(e) = q_2(e)$, and for each $e \in E_{i+}$ $q_0(e) \leq q_1(e) \leq q_2(e)$. If $q_0, q_2 \in D_i(p)$, then $q_1 \in D_i(p)$. Similarly, if for each $e \in E_{i+}$ $q_0(e) = q_1(e) = q_2(e)$, and for each $e \in E_{i-}$ $q_0(e) \leq q_1(e) \leq q_2(e)$, we get $q_0, q_2 \in D_i(p) \Rightarrow q_1 \in D_i(p)$.*

Proof. We prove the first part:

Suppose that $q_1 \notin D_i(p)$, then $U_i(q_2, p) = U_i(q_0, p) > U_i(q_1, p)$, because $U(q_0, p)$ and $U(q_2, p)$ are maximal and $U(q_1, p)$ is not.

Choose an edge $e \in E_{i+}$ for which $q_2(e) > q_0(e)$. Let ε be a small positive number, such that $0 < \varepsilon q_1(e) < U(q_0, p) - U(q_1, p)$. Increase e 's price by ε , so $p'_e = p_e + \varepsilon$, $p'_f = p_f$ for any $f \neq e$ edge

Now $U_i(q_2, p') = U_i(q_2, p) + q_2(e)\varepsilon$
 $U_i(q_1, p') = U_i(q_1, p) + q_1(e)\varepsilon$
 $U_i(q_0, p') = U_i(q_0, p) + q_0(e)\varepsilon$
 Since $U_i(q_2, p) = U_i(q_0, p)$ and $q_2(e) > q_0(e)$, for the new utilities $U_i(q_2, p') \geq U_i(q_0, p')$, and ε is so small that q_1 can't reach the other's utility, so $U_i(q_1, p') \geq U_i(q_0, p')$.

$U_i(q_2, p)$ has the maximal utility of all agreements $[q, p] : q \in T, p$ is fixed, so $U_i(q_2, p')$ has the maximal utility of all agreements $[q, p] : q \in T, q(e) \leq q_2(e)p'$ is fixed. Therefore $q_2 \in C_i(q_2, p')$. If $|C_i(q_2, p)| \neq 1$, we can perturbate the prices with smaller ε_j -s in a way that the ordering of the utilities remain, but $U_i(q, p'')$ is different for all $q \in T$. So $\{q_2\} = C_i(q_2, p'')$

For q_0 , there is a smaller flow with bigger utility, so $C_i(q_0, p') \neq q_0$. It means $\overline{C}_i(q_2, p') = 0$ and $\overline{C}_i(q_0, p') > 0$ which contradicts same side substitutability.

The second part is analogous with changing the roles of E_{i+} , and E_{i-} and we decrease p_e by ε instead of increasing. □

If we prove Conjecture 21, the existence of equilibrium with continuous quantities, theorem 19 is a special case. Using theorem 16, the chosen q in a competitive equilibrium is also a stable quantity vector with the given p prices.

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