

OTDK-DOLGOZAT

Anna Zseleva
MSc

2013

**Unawareness in standard space models –
a new approach**

**Unawareness sztenderd modellekben –
egy új megközelítés**

Anna Zseleva

Contents

1	Introduction	1
2	Impossibility theorem	2
3	A model with unawareness	5
3.1	The model	5
3.2	Mutual and common knowledge	7
3.3	The belief, ignorance and unawareness operators	12
3.4	Example	22
4	Conclusions	25

Chapter 1

Introduction

This paper is about modeling unawareness. Unawareness seems like a notion, which can arise in our everyday life. In a realistic situation we may often not consider all possibilities. Unawareness can be looked at as a special kind of lack of information, an unaware player is not uncertain about an event, but does not even think about it. It is plausible to suggest that players may be unaware of an event of which others are not, this may give place to interesting situations.

If a player is unaware of an event, we think that she does not have any idea about that event. We do not think that the player is uncertain, or has an unclear image about the event. We think that the player does not even consider that event, cannot form any belief about the event on any level. By this we mean, that the player does not form a belief about it, and does not form a belief about not forming a belief, and so on. Interestingly enough, we have to take into account other players' beliefs as well. This means, that an unaware player should not think about other players' opinion about the event. By this, unawareness seems like a fascinating notion, and our intention is to take a closer look at it.

The aim of this paper is to create a model, which can handle unawareness. The uniqueness of this model could stand in the fact that the unawareness operator is not an axiom, but a derived operator. According to (Dekel et al, 1998) a nontrivial unawareness operator cannot be introduced in a standard state space model. The first chapter will be about showing their findings, and arguing why they do not apply on our model. After that, we will introduce our model. Obviously, we will start with the axioms, and work our way through the derived operators till we get to the unawareness operator, then we will show some of its properties. Because it is a derived operator, we cannot directly assume its properties, but have to investigate them. Usually, in other models, because of the unawareness operator's axiom nature, they assume its properties. Finally, we will show an example, in which we will check the possible properties of unawareness. We will argue why some of them are more plausible than others.

Chapter 2

Impossibility theorem

In this section we present a result, which asserts that a standard state space model does not allow nontrivial unawareness (Dekel et al, 1998). It would seem that this contradicts our result, however, this is not true, because the conditions in this and in our model are different.

Dekel et al (1998) bring forward a theorem, which shows that unawareness cannot be illustrated in a standard state space model. They allege that in a standard state space model an acceptable unawareness operator could only mean that a player is aware of everything or knows nothing.

As every theorem, this also depends on assumptions. Two of these assumptions are properties of the knowledge operator. These properties are commonly agreed upon, so no problem arises from their usage. The theorem also has assumptions of the unawareness operator. Some of the shown properties are more acceptable than others. They prefer to define unawareness by multiple levels of not knowing. This assumption has great bearing on the proof. So the allegation is that their proof is correct, but it does not contradict the definition presented later on in this paper.

The following definitions are needed for discussing Dekel et al (1998)'s result.

Definition 2.1. *Mapping $P : \Omega \rightarrow \mathcal{P}(\Omega)$ is a possibility correspondence. $P(\omega)$ consists of the states, which the player considers possible, when she is at state ω .*

Definition 2.2. *Knowledge operator $K : \mathcal{P}(\Omega) \rightarrow \mathcal{P}(\Omega)$, $K(E) = \{\omega \in \Omega | P(\omega) \subseteq E\}$. $K(E)$ consists of the states, at which the player knows event E .*

After these definitions they introduce some usually applied properties of the knowledge operator. After that they demonstrate that these contradict the (according to them) reasonable notion of unawareness.

Properties of knowledge operator K :

- Necessitation: $K(\Omega) = \Omega$,
- Monotonicity: $E \subseteq F \Rightarrow K(E) \subseteq K(F)$.

Necessitation means that the player knows all tautologies. Monotonicity implies, that if E implies F , then the knowledge of E implies the knowledge of F . These properties are commonly used in the literature.

They introduce an unawareness operator as well, and they show that, if we assume some reasonable properties about that operator, then it will be trivial in a standard state space model.

Definition 2.3. *Unawareness operator $U : \mathcal{P}(\Omega) \rightarrow \mathcal{P}(\Omega)$ is such that $U(E)$ contains the states, where the player is unaware of event E .*

Properties of unawareness operator U :

- KU introspection: $K(U(E)) = \emptyset$,
- AU introspection: $U(E) \subseteq U(U(E))$,
- (Knowledge) plausibility: $U(E) \subseteq \mathbb{C}K(E) \cap \mathbb{C}K(\mathbb{C}K(E))$.

KU introspection means that a player cannot know in any state that she is unaware of event E . They assume $K(U(E)) \subseteq U(E)$, based on a kind of nondelusion, which suggests that a player cannot be deluded about own knowledge. Therefore, if a player knows that she is unaware of an event, it must be so. But $K(U(E)) \subseteq \mathbb{C}U(E)$ should hold also, because, if a player knows she is unaware of an event, then it would make her aware of that event. The preceding two statements can hold simultaneously only if $K(U(E)) = \emptyset$. Intuitively this is easy to see, because knowledge of own unawareness contradicts the intuition behind unawareness.

AU introspection means that a player cannot be aware of being unaware of an event. If she was aware of her unawareness of that event, then after all she is not unaware of that particular event. That is why it seems reasonable to assume that she has to be unaware of her own unawareness. To sum up, awareness of own unawareness does not seem compatible with the notion of unawareness.

Knowledge plausibility is the property, where this model and our model, discussed later on, differ. The infallibility of the knowledge operator suggests that it should not be associated in any form with the unawareness operator. Knowledge plausibility is essential in the proof. This means that this is the property why this theorem does not apply on the subsequent model.

Dekel et al (1998) show that in an (Ω, K, U) state space model with the preceding properties, there cannot be a nontrivial U operator.

Theorem 2.4 (Dekel et al (1998)). *If (Ω, K, U) is plausible and satisfies KU and AU introspection, then the following statements hold:*

1. *If K satisfies necessitation, then $U(E) = \emptyset$, $E \subseteq \Omega$.*
2. *If K satisfies monotonicity, then $U(E) \subseteq \mathbb{C}K(F)$, $E, F \subseteq \Omega$.*

The first part of the theorem means that necessitation implies no one ever being unaware of anything. In this case the U operator would have no sense in any model, because there would be no state in which the player is unaware of any event. The second part of the theorem means that if monotonicity holds and the player is unaware of any event, then she cannot know any event. The latter means that unawareness of anything implies the knowledge of nothing.

Proof. If we apply plausibility on event $U(E)$, then

$$U(U(E)) \subseteq \mathbb{C}K(\mathbb{C}K(U(E)))$$

holds. Because of AU introspection:

$$U(E) \subseteq U(U(E)) \subseteq \mathbb{C}K(\mathbb{C}K(U(E))) .$$

KU introspection implies $K(U(E)) = \emptyset$, which is equivalent to $\mathbb{C}K(U(E)) = \Omega$. This implies the following

$$U(E) \subseteq \mathbb{C}K(\Omega) . \tag{2.1}$$

Necessitation is equivalent to $K(\Omega) = \Omega$, so by (2.1) this implies $U(E) = \emptyset$, $E \subseteq \Omega$. Monotonicity is equivalent to $E \subseteq F \Rightarrow K(E) \subseteq K(F)$, so $K(F) \subseteq K(\Omega)$, $F \subseteq \Omega$, holds. This means

$$\mathbb{C}K(\Omega) \subseteq \mathbb{C}K(F) , \quad F \subseteq \Omega .$$

By (2.1) $U(E) \subseteq \mathbb{C}K(F)$. □

It is important to see that the proof breaks down without (2.1), which relies on knowledge plausibility. So changing that property would mean that the theorem could not be applied.

The problem arises from inspecting the conditions. If we define unawareness in a different way, then not all conditions hold, and the theorem breaks down. In any case our model does not contradict this theorem because of the different definition of unawareness.

Maybe Dekel et al (1998)'s conclusion could be changed in a way that, not that the unawareness cannot be modeled in a standard state space model, but if we assume certain properties of unawareness, then it cannot be done. If we examine their assumptions, then it might seem obvious that, if you want to weaken some of them, you cannot weaken the properties of knowledge, because they are commonly accepted. So the only way to weaken the assumptions is to change the properties of the unawareness operator.

In conclusion, the article was about certain properties of unawareness, which they find reasonable, and what we can derive from them. If we do not assume the same properties, then their conclusion does not apply.

Chapter 3

A model with unawareness

This chapter is about presenting our model, and its various properties. The main goal is to introduce a proper unawareness operator. This will be done in a different way than in Dekel et al (1998), because it will not be based on knowledge, but on belief. Unawareness will be presented as an operator, which contains different kinds of states, where belief is not formed about the event. In this model we will assume belief plausibility, and not knowledge plausibility.

The following model has many predecessors. It all began with Aumann (1999a), who compares the semantic and syntactic approach. Aumann (1999a) is about a knowledge space, Aumann (1999b) in turn is about a knowledge-belief space. Meier (2008) can be considered as the next level, because it shows the existence of a universal knowledge-belief space.

The introduction of types can be traced back to the renowned Harsányi (1967-68) article. Heifetz and Samet (1998) show that a universal type space exists in a purely measurable setting. On the other hand Pintér (2010) shows that a universal topological type space does not exist. Pintér (2012) proves that the Harsányi program stands in a purely measurably setting. The previous results suggest that an appropriate model would be in a purely measurable setting.

We can take a look at those kind of predecessors of this model, which allow unawareness in them, for example Heifetz et al (2011). Heinsalu (2011) creates a universal type space with propositional unawareness in a purely measurable setting. Pintér and Udvari (2011) create a generalized type space, which allows unawareness.

3.1 The model

This section shows the axioms of our model. In other models the unawareness operator is usually an axiom. In this case it will be a derived operator. This can be considered as the novelty of our model. The following definition will show the axioms of our model, the definitions presented later on, will be derived from these axioms.

Definition 3.1. *An extended knowledge-belief space based on parameter space S is a tuple $(S, \{\Omega, \mathcal{M}_i\}_{i \in N_0}, \{K_i\}_{i \in N}, \{A_i\}_{i \in N}, g, \{f_i\}_{i \in N})$ such that*

A S is the non-empty parameter set,

B Ω is the non-empty set of the states of the world,

C $(\Omega, \mathcal{M}_i)$ is a measurable space, $i \in N_0$,

D $K_i : \mathcal{M} \rightarrow \mathcal{M}_i$ is Player i 's knowledge operator, $i \in N$, such that for each events $A, A_n \in \mathcal{M}$:

D1 $K_i(A) \subseteq A$,

D2 $K_i(\cap_n A_n) = \cap_n K_i(A_n)$,

D3 $K_i(A) \subseteq K_i(K_i(A))$,

D4 $K_i(\Omega) = \Omega$,

E $g : \Omega \rightarrow S$ is $\mathcal{M}_0$ -measurable,

F $f_i : \Omega \rightarrow \Delta(\Omega, \mathfrak{M})$, Player i 's type function, is such that

F1 For each $p \in [0, 1]$: $\{\omega \in \Omega : f_i(\omega)(A) \geq p\} \in \mathcal{M}_i$,

F2 For each $A \in \mathcal{M}$, if $\omega \in K_i(A)$, then $f_i(\omega)(A) = 1$,

F3 For each $A \in \mathcal{M}$, if $\omega' \in \{\omega \in \Omega : f_i(\omega)(A) \geq 0\}$, then $f_i(\omega')(\{\omega \in \Omega : f_i(\omega)(A) \geq 0\}) \geq 0$,

F4 For each $A_n \in \mathcal{M}$, $\cup_n \{\omega \in \Omega : f_i(\omega)(A_n) \geq 0\} = \{\omega \in \Omega : f_i(\omega)(\cap_n A_n) \geq 0\}$,

where N is the finite players set, $N_0 = N \cup \{0\}$, $\mathcal{M} = \vee_{i \in N_0} \mathcal{M}_i$, $\mathfrak{M}$ is a class of σ -fields on set Ω , and $\Delta(\Omega, \mathfrak{M})$ is a class of probability measures on the σ -fields of $\mathfrak{M}$.

The difference between the sets S and Ω is that the set S consists of states of the nature, which contain parameters that cannot be associated with any player. But the states of the world, the point of Ω , contain not only the parameters, but such properties that can be associated with players.

It was argued earlier why the purely measurable setting might be a more appropriate model setting. The σ -algebras are similar to the partitions in Aumann (1999a), there a player cannot distinguish two states of the world if they are in the same atom of their partition. In this case two states of the world are indistinguishable if they are indistinguishable in the player's σ -field. An $\mathcal{M}_i$ σ -field is assigned to every player, and $\mathcal{M}$ is their coarsest refinement. Practically, $\mathcal{M}$ contains all available information in the model.

Property **D1** of the knowledge operator is nondelusion. If a player knows an event, then it is true. Property **D2** means that knowing some events is equivalent to knowing their intersection. Property **D3** is positive introspection. If a player knows an event, then she knows her own knowledge. Property **D4** is necessitation, the players know the universal event Ω at every state of the world.

In this model negative introspection ($\mathcal{C}K_i(A) \subseteq K_i(\mathcal{C}K_i(A))$) does not hold. Negative introspection basically means that if a player does not know an event, then she knows that she does not know it. This would mean that in every case, when a player does not know an event, for example she is unaware, she would know that she does not know that event. This holding in every case would seem odd in a model, where our goal is to model unawareness.

Property **F** defines the type function. Property **F1** defines the partitions of all players, by making f_i $\mathcal{M}_i$ -measurable. Property **F2** means that if a player knows an event, then she believes it with probability 1. Property **F3** means that if at a state of the world a player assigns probability at least 0 to an event, then she assigns probability at least 0 to assigning probability at least 0 to that event. Intuitively, if a player forms a belief about an event, then she forms a belief about forming a belief. Property **F4** means that at a state of the world assigning probability at least 0 to minimum one event of a sequence of events is equivalent to assigning probability at least 0 to their intersection.

Proposition 3.2. If $A \subseteq B$, then $K_i(A) \subseteq K_i(B)$, $i \in N$.

Proof. $A \subseteq B$ implies $K_i(A) = K_i(A \cap B)$, and by Property **D2** $K_i(A \cap B) = K_i(A) \cap K_i(B) \subseteq K_i(B)$. $\square$

If event A implies B , then the knowledge of A implies the knowledge of B .

3.2 Mutual and common knowledge

In this section we present some derived operators and their properties. Mutual and common knowledge are usually applied operators in a knowledge-belief space. The following operators will be derived from the knowledge operator.

Definition 3.3. *The 1st order mutual knowledge operator $K^1 : \mathcal{M} \rightarrow \mathcal{M}$ is such that for each event $A \in \mathcal{M}$: $K^1(A) = \cap_{i \in N} K_i(A)$.*

If at a state of the world an event is 1st order mutually known, then all players know that event.

Definition 3.4. *$K^n : \mathcal{M} \rightarrow \mathcal{M}$ ($n \geq 2$) is the n th order mutual knowledge operator, such that for each event $A \in \mathcal{M}$: $K^n(A) = K^1(K^{n-1}(A))$.*

At a state of the world an event is n th order mutually known, if it is 1st order mutually known that it is $(n - 1)$ th order mutually known.

Proposition 3.5. *Properties of n th order mutual knowledge, for all $n \in \mathbb{N}$, $A, B, A_m \in \mathcal{M}$:*

1. $K^n(A) \subseteq A$,
2. $K^n(\cap_m A_m) = \cap_m K^n(A_m)$,
3. If $A \subseteq B$, then $K^n(A) \subseteq K^n(B)$,
4. $K^n(\Omega) = \Omega$.

Proof. 1.: By Property **D1** of Definition 3.1 $K_i(A) \subseteq A$, $i \in N$, so

$$\cap_{i \in N} K_i(A) \subseteq A ,$$

and by Definition 3.3

$$K^1(A) = \cap_{i \in N} K_i(A) \subseteq A .$$

By Part 1. of Proposition 3.2

$$K_i(K^1(A)) \subseteq K_i(A) , \quad i \in N .$$

So

$$\cap_{i \in N} K_i(K^1(A)) \subseteq \cap_{i \in N} K_i(A) ,$$

by Definitions 3.3 and 3.4

$$K^2(A) \subseteq K^1(A) \subseteq K^1(A) \subseteq A .$$

If we go on, we get

$$K^n(A) \subseteq A .$$

2.: By Property **D2** of Definition 3.1

$$K_i(\cap_m A_m) = \cap_m K_i(A_m) , \quad i \in N ,$$

so by taking their intersection by i

$$\cap_{i \in N} K_i(\cap_m A_m) = \cap_{i \in N} (\cap_m K_i(A_m)) = \cap_m (\cap_{i \in N} K_i(A_m)) .$$

Because of Definition 3.3 this means

$$K^1(\cap_m A_m) = \cap_m K^1(A_m) .$$

$$K_i(K^1(\cap_m A_m)) = K_i(\cap_m K^1(A_m)) , \quad i \in N .$$

By Part **D2** of Definition 3.1

$$K_i(\cap_m K^1(A_m)) = \cap_m K_i(K^1(A_m)) , \quad i \in N .$$

Taking their intersection by i

$$\cap_{i \in N} K_i(K^1(\cap_m A_m)) = \cap_{i \in N} \cap_m K_i(K^1(A_m)) .$$

So

$$K^2(\cap_m A_m) = \cap_m K^2(A_m) .$$

We can go on and get

$$K^n(\cap_m A_m) = \cap_m K^n(A_m) .$$

3.: By Proposition 3.2 if $A \subseteq B$, then $K_i(A) \subseteq K_i(B)$, $i \in N$. So

$$\cap_{i \in N} K_i(A) \subseteq \cap_{i \in N} K_i(B) ,$$

and because of Definition 3.3 this is equivalent to

$$K^1(A) \subseteq K^1(B) .$$

Again by Proposition 3.2

$$K_i(K^1(A)) \subseteq K_i(K^1(B)) , \quad i \in N .$$

So

$$\cap_{i \in N} K_i(K^1(A)) \subseteq \cap_{i \in N} K_i(K^1(B)) ,$$

by Definition 3.4

$$K^2(A) \subseteq K^2(B) .$$

And so forth

$$K^n(A) \subseteq K^n(B) .$$

4.: By Property **D4** of Definition 3.1

$$K_i(\Omega) = \Omega , \quad i \in N .$$

So

$$\cap_{i \in N} K_i(\Omega) = \Omega ,$$

by Definition 3.3

$$K^1(A) = \cap_{i \in N} K_i(\Omega) = \Omega .$$

Therefore

$$K_i(K^1(\Omega)) = K_i(\Omega) = \Omega , \quad i \in N ,$$

hence

$$\cap_{i \in N} K_i(K^1(\Omega)) = \Omega ,$$

by Definition 3.3 this is equivalent to

$$K^1(K^1(\Omega)) = \Omega .$$

Using Definition 3.4

$$K^2(\Omega) = \Omega .$$

If we go on

$$K^n(\Omega) = \Omega , \quad n \in \mathbb{N} .$$

□

All these properties come from the properties of the knowledge operator. If a player knows an event, then it is true, so if an event is n th order mutually known, then it is also true. The n th order mutual knowledge of a sequence of events is equivalent to the mutual knowledge of their intersection. Moreover, if event A implies event B , then the n th order mutual knowledge of A implies the n th order mutual knowledge of event B . Finally, the universal event Ω is n th order mutually known at every state of the world.

Definition 3.6. *The common knowledge operator $CK : \mathcal{M} \rightarrow \mathcal{M}$ is such that for each event $A \in \mathcal{M}$: $CK(A) = \cap_{n=1}^{\infty} K^n(A)$.*

$CK(A)$ consists of those states of the world, where all players know event A , and all players know that all players know event A , and so on to infinity.

Proposition 3.7. *Properties of the common knowledge operator, for all $n \in \mathbb{N}$, for all $A, B, A_m \in \mathcal{M}$:*

1. $CK(A) \subseteq A$,
2. $CK(\cap_m A_m) = \cap_m CK(A_m)$,
3. If $A \subseteq B$, then $CK(A) \subseteq CK(B)$,
4. $CK(A) \subseteq K_i(CK(A))$,
5. $CK(A) = CK(CK(A))$,
6. $CK(\Omega) = \Omega$.

Proof. 1.: By Definitions 3.3, 3.6 and Property **D1** of Definition 3.1

$$CK(A) \subseteq K^1(A) \subseteq K_i(A) \subseteq A .$$

2.: By Part 2. of Proposition 3.5

$$K^n(\cap_m A_m) = \cap_m K^n(A_m) , \quad n \in \mathbb{N} .$$

So

$$\cap_{n=1}^{\infty} K^n(\cap_m A_m) = \cap_{n=1}^{\infty} (\cap_m K^n(A_m)) = \cap_m (\cap_{n=1}^{\infty} K^n(A_m)) . \quad (3.1)$$

By Definition 3.6 (3.1) is equivalent to

$$CK(\cap_m A_m) = \cap_m CK(A_m) .$$

3.: By Part 3. of Proposition 3.5 if $A \subseteq B$, then $K^n(A) \subseteq K^n(B)$, $n \in \mathbb{N}$. So

$$\cap_{n=1}^{\infty} K^n(A) \subseteq \cap_{n=1}^{\infty} K^n(B) ,$$

by Definition 3.6 this is equivalent to

$$CK(A) \subseteq CK(B) .$$

4.: By Definitions 3.3 and 3.4

$$K^{n+1}(A) = K^1(K^n(A)) = \cap_{i \in N} K_i(K^n(A)) \subseteq K_i(K^n(A)) .$$

Taking their intersection by n

$$\cap_{n=1}^{\infty} K^{n+1}(A) \subseteq \cap_{n=1}^{\infty} K_i(K^n(A)) .$$

By Property **D2** of Definition 3.1

$$\cap_{n=1}^{\infty} K^{n+1}(A) \subseteq K_i(\cap_{n=1}^{\infty} K^n(A)) . \quad (3.2)$$

By Definition 3.6,

$$CK(A) = \cap_{n=1}^{\infty} K^n(A) \subseteq \cap_{n=1}^{\infty} K^{n+1}(A) .$$

Putting the last equation and (3.2) together, we get

$$CK(A) \subseteq K_i(\cap_{n=1}^{\infty} K^n(A)) .$$

By Definition 3.6 the last equation translates to

$$CK(A) \subseteq K_i(CK(A)) .$$

5.: We will first show the $CK(A) \subseteq CK(CK(A))$ direction.

By 4. we get

$$CK(A) \subseteq K_i(CK(A)), i \in \mathbb{N} .$$

Taking their intersection by i and Definition 3.3 we get

$$CK(A) = \cap_{i \in \mathbb{N}} CK(A) \subseteq \cap_{i \in \mathbb{N}} K_i(CK(A)) = K^1(CK(A)) . \quad (3.3)$$

If we apply Proposition 3.2 on (3.3)

$$K_j(CK(A)) \subseteq K_j(K^1(CK(A))), j \in \mathbb{N} .$$

Taking their intersection by j

$$\cap_j K_j(CK(A)) \subseteq \cap_j K_j(K^1(CK(A))) .$$

The previous is equivalent (by Definitions 3.3 and 3.4) to

$$K^1(CK(A)) \subseteq K^1(K^1(CK(A))) = K^2(CK(A)) . \quad (3.4)$$

By (3.3) and (3.4)

$$CK(A) \subseteq K^2(CK(A)) . \quad (3.5)$$

By applying Point 3. of Proposition 3.5 on (3.5), and then Definition 3.4

$$K^1(CK(A)) \subseteq K^1(K^2(CK(A))) = K^3(CK(A)) ,$$

so by the previous equation and (3.3)

$$CK(A) \subseteq K^3(CK(A)) .$$

If we go on this way, we get

$$CK(A) \subseteq K^n(CK(A)), n \in \mathbb{N} .$$

Taking their intersection by n gives

$$CK(A) \subseteq \cap_n K^n(CK(A)) = CK(CK(A)) .$$

By this, one direction is proved, and now we have to show the the other one, $CK(CK(A)) \subseteq CK(A)$.

By Point 1. of Proposition 3.7 $CK(B) \subseteq B$ holds. If $B = CK(A)$, then it translates to $CK(CK(A)) \subseteq CK(A)$, which is exactly what we wanted to show.

6.: By Point 4. of Proposition 3.5

$$K^n(\Omega) = \Omega , \quad n \in \mathbb{N} .$$

Taking the intersection of both sides means

$$\cap_{n=1}^{\infty} K^n(\Omega) = \Omega ,$$

finally by Definition 3.6

$$CK(\Omega) = \Omega .$$

□

The properties of the common knowledge operator are similar to those of the knowledge operator. The difference is that the knowledge operator is assigned to players, but the common knowledge operator is linked to all players, and not one in particular. If an event is commonly known, then it is true. The common knowledge of a sequence of events is equivalent to the common knowledge of their intersection. If event A implies event B , then the common knowledge of event A implies the common knowledge of event B . Furthermore, the common knowledge of an event is equivalent to the common knowledge of the common knowledge of that same event. The previous statement implies positive introspection ($CK(A) \subseteq CK(CK(A))$). This property holds for the knowledge and common knowledge operators, but does not hold for the mutual knowledge operator. Lastly, there is no state of the world, where the universal event Ω is not common knowledge.

3.3 The belief, ignorance and unawareness operators

This section is about introducing some more operators. In this section we finally reach the unawareness operator. First we introduce the belief operator, which is a derived operator as well. From the belief operator we will derive the ignorance operator. Finally, we will derive the unawareness operator from the ignorance operator. After introducing these operators, we will prove some properties concerning the unawareness operator. This is important, because we would like our unawareness operator to relate to its intuitive notion.

The belief operator will be derived from the type function, so all of these operators will be derived, in a sense, from the type function, in contrast with the previous ones, which were derived from the knowledge operator.

Definition 3.8. $B_i^0 : \mathcal{M} \rightarrow \mathcal{M}_i$ is Player i 's belief operator, $i \in N$, such that for each event $A \in \mathcal{M}$, $B_i^0(A) = \{\omega \in \Omega : f_i(\omega)(A) \geq 0\}$.

This definition is derived from Property **F** of Definition 3.1. The following properties of the belief operator come, of course, from the same place.

Proposition 3.9 (Properties of the belief operator). *For all $n \in \mathbb{N}$, $A, A_n \in \mathcal{M}$:*

1. $B_i^0(A) \subseteq B_i^0(B_i^0(A))$,
2. $\cup_n B_i^0(A_n) = B_i^0(\cap_n A_n)$,
3. $B_i^0(A) = B_i^0(\mathbb{C}A)$.

Proof. 1.: It comes directly from Part **F3** of Definition 3.1.

2.: It comes directly from Part **F4** of Definition 3.1.

3.: If event A is an element of a σ -field, then its complement is also an element of that σ -field. □

The belief operator shows the states of the world in which the player believes the event with probability at least 0. So if a state of the world is not in $B_i^0(A)$, then in that state of the world Player i does not form any beliefs about event A . Usually in a model the players form beliefs about all events. This means that every state of the world would be an element of $B_i^0(A)$, for all players and all events. In this model this will not stand, because our goal is to model unawareness. We have to provide the possibility for being outside of the set $B_i^0(A)$. If we did not provide this possibility, unawareness could not be plausibly introduced in our model. So, in this model the type function maps into $\Delta(\Omega, \mathfrak{M})$, which is a class of probability measures on the σ -fields of $\mathfrak{M}$. $\mathfrak{M}$ in turn is a class of σ -fields on set Ω . Thus if an event is not in a σ -algebra, then the type function may not be defined on that certain event. This is what mathematically allows the specific situation of a player not assigning probability at least 0 to an event. In other models $\mathfrak{M}$ would contain only the finest σ -algebra on Ω , which would mean that the type function is defined on every possible event, hence all players would form beliefs about all events.

The first property of the belief operator means, that if Player i has any kind of belief of an event, then she has a belief about her own belief about that event. The second property represents the notion that a player having beliefs about at least one event of a series of events is equivalent to having a belief about their intersection. The third property means that having a belief about an event means having a belief about its complement.

$B_i^0(A)$ contains the states of the world, where Player i believes with probability at least 0 that event A is true. So the player may think this event has probability 0, hence she might not even think it is true. In fact, in these states the player has a belief about the event. If a state of the world is not an element of this set, then in that particular state of the world the player has no beliefs about the event. The player does not even assign probability 0 to it. We will call these special kinds of states ignorance of event A by Player i . When a player ignores an event, she has no beliefs about it. This is important in our model, because our goal is to model unawareness. We will try to model it, in a sense, as the fix point of the ignorance operator. This shows that in our model unawareness is a derived operator, and is not just introduced as an axiom. The goal is to model unawareness as a kind of operator, which contains a lot of levels of not forming beliefs. This seems more plausible, then modeling it with knowledge. The infallibility of knowledge makes it unrealistic for being a modeling device for unawareness. The subjectivity of beliefs make the belief operator a much more reasonable modeling device.

So as previously stated, the ignorance operator will be derived from the belief operator.

Definition 3.10. *Ignorance operator $I_i : \mathcal{M} \rightarrow \mathcal{M}_i$ is defined as for each event $A \in \mathcal{M}$:*

$$I_i(A) = \mathbb{C}B_i^0(A) .$$

The definition of ignorance suggests, that ignorance of an event means no kind of belief is formed about that event. This will characterize the ignorance operator.

Proposition 3.11. *Player $i \in N$'s ignorance operator I_i meets the following points:*

- 1 $I_i(I_i(A)) \subseteq I_i(A)$,
- 2 $\cap_n I_i(A_n) = I_i(\cap_n A_n)$,
- 3 $I_i(A) = I_i(\mathbb{C}A)$.

Because ignorance is defined as the complement of the belief operator, all these properties come from Definition 3.8 and Proposition 3.9. The first property says, that if a player is ignorant of her

own ignorance of an event, then she ignores that event. The second property suggests, that if a player is ignorant of some events, it is equivalent to her ignorance of their intersection. The third property of ignorance is symmetry. This means, that if a player is ignorant of an event, then she is ignorant of the complement of that event as well.

Definition 3.12. $I^1, I^n : \mathcal{M} \rightarrow \mathcal{M}$ are mutual ignorance operators, such that for each event $A \in \mathcal{M}$:

- 1 $I^1(A) = \cap_{i \in N} I_i(A)$, 1st order mutual ignorance,
- 2 $I^n(A) = I^1(I^{n-1}(A))$, nth order mutual ignorance.

An event is 1st order mutually ignored if all players ignore that event. By definition the n th order mutual ignorance means that all players ignore that the event is $(n - 1)$ th order mutually ignored.

Proposition 3.13.

$$I_i(B_j^0(A)) = I_i(I_j(A)), \text{ for all } j \in N \setminus \{i\}$$

Proof. This is because of Definition 3.10 and Part 3. of Proposition 3.11. $\square$

This means that ignoring an other player's ignorance is the same as ignoring their belief about that event. This seems intuitively plausible, because if a player ignores the belief of an other player, it should not make a difference if the other player forms a belief about that particular event or not.

Here comes the operator, which was the purpose of this model. Unawareness will be derived from the ignorance operator. For this, we will introduce the following notations: $I_i^2(A) = I_i(I_i(A))$, and $I_i^k(A) = I_i(I_i^{k-1}(A))$.

Definition 3.14. $U_i : \mathcal{M} \rightarrow \mathcal{M}_i$ is Player i 's unawareness operator, $i \in N$, such that for each event $A \in \mathcal{M}$:

$$U_i(A) = \cap_k I_i^k(\cap_n I^n(A)).$$

The unawareness operator is derived by applying the ignorance operator countably infinitely many times. In a sense, it is a fix point of the ignorance operator, unawareness implies ignorance on every level. It is similar, in a way, to the common knowledge operator. Only, the common knowledge operator is a fix point of the mutual knowledge operator, and not the ignorance operator. There is a difference in that way also, that common knowledge applies to all players, but unawareness is assigned to only one player.

The following proposition describes properties of the ignorance operator, which will be needed to analyze the properties of the unawareness operator.

Proposition 3.15. Player i 's ignorance operator meets the following points:

1. If $A \subseteq B$, then $I_i(A) \subseteq I_i(B)$.
2. $\cap_l I^l(A) \subseteq \cap_l I_i^l(A)$.
3. If $A \subseteq B$, then $\cap_k I_i^k(A) \subseteq \cap_k I_i^k(B)$.
4. If $A \subseteq B$, then $\cap_n I^n(A) \subseteq \cap_n I^n(B)$.

Proof. 1.: Because of Part 2. of Proposition 3.11 and $A \subseteq B$,

$$I_i(A) = I_i(A \cap B) = I_i(A) \cap I_i(B) \subseteq I_i(B) .$$

2.: Because of Part 1. of Definition 3.12, $I^1(A) \subseteq I_i(A)$. From Part 1. of Proposition 3.15

$$I_i(I^1(A)) \subseteq I_i(I_i(A)) = I_i^2(A) ,$$

by Parts 1. and 2. of Definition 3.12

$$I^2(A) = I^1(I^1(A)) \subseteq I_i(I^1(A)) \subseteq I_i^2(A)$$

and so on:

$$I^l(A) \subseteq I_i^l(A) .$$

Hence

$$\cap_l I^l(A) \subseteq \cap_l I_i^l(A) .$$

3.: Because $A \subseteq B$ and Part 1. of Proposition 3.15 stand,

$$I_i(A) \subseteq I_i(B) .$$

Using the same once more

$$I_i(I_i(A)) \subseteq I_i(I_i(B)) ,$$

this is equivalent to

$$I_i^2(A) \subseteq I_i^2(B) ,$$

and so on:

$$I_i^k(A) \subseteq I_i^k(B) ,$$

so

$$\cap_k I_i^k(A) \subseteq \cap_k I_i^k(B) .$$

4.: By $A \subseteq B$ and Part 1. of Proposition 3.15

$$I_i(A) \subseteq I_i(B) , \quad i \in N ,$$

therefore

$$\cap_{i \in N} I_i(A) \subseteq \cap_{i \in N} I_i(B) .$$

Because of Part 1. of Definition 3.12 this is equivalent to

$$I^1(A) \subseteq I^1(B) .$$

Using Part 1. of Proposition 3.15

$$I_i(I^1(A)) \subseteq I_i(I^1(B)) , \quad i \in N ,$$

so

$$\cap_{i \in N} I_i(I^1(A)) \subseteq \cap_{i \in N} I_i(I^1(B)) .$$

By Part 1. of Definition 3.12

$$I^1(I^1(A)) \subseteq I^1(I^1(B)) ,$$

by Part 2. of Definition 3.12

$$I^2(A) \subseteq I^2(B) ,$$

and so on:

$$I^n(A) \subseteq I^n(B) ,$$

so

$$\cap_n I^n(A) \subseteq \cap_n I^n(B) .$$

In conclusion if $A \subseteq B$, then

$$\cap_n I^n(A) \subseteq \cap_n I^n(B) .$$

□

The second part of the proposition means, that mutual ignorance of an event up till level l implies the ignorance of that same event by Player i , up till the same level. If event A implies event B , then the ignorance of event A implies the ignorance of event B . Also if event A implies event B , then Player i 's n th level of ignorance of event A implies the same level of ignorance of event B . Lastly, if event A implies event B , then the n th order mutual ignorance of event A implies the n th order mutual of ignorance of event B .

The following property constitutes of unawareness of own unawareness. This is an intuitively required property of unawareness. If a player was aware of her own unawareness, we would think that she could not be unaware after all. This is a property that is usually required from unawareness.

In short, we will prove, that if a player is unaware of an event in a state of the world, then she is also unaware of her unawareness of that same event.

Lemma 3.16. *AU introspection:* $U_i(A) \subseteq U_i(U_i(A))$.

Proof. Because of Definition 3.14

$$U_i(U_i(A)) = \cap_k I_i^k(\cap_n I^n(U_i(A))) = \cap_k I_i^k \left(\cap_n I^n \left(\cap_l I_i^l(\cap_m I^m(A)) \right) \right) .$$

If we apply Point 2. of Proposition 3.15 on $\cap_m I^m(A)$, then

$$\cap_l I^l(\cap_m I^m(A)) \subseteq \cap_l I_i^l(\cap_m I^m(A)) . \quad (3.6)$$

By Point 2. of Proposition 3.11

$$\cap_l \cap_m I^{l+m}(A) = \cap_l I^l(\cap_m I^m(A)) . \quad (3.7)$$

If we put (3.6) and (3.7) together, then we get

$$\cap_l \cap_m I^{l+m}(A) \subseteq \cap_l I_i^l(\cap_m I^m(A)) . \quad (3.8)$$

If we apply Part 4. of Proposition 3.15 on (3.8), then

$$\cap_n I^n(\cap_l \cap_m I^{l+m}(A)) \subseteq \cap_n I^n \left(\cap_l I_i^l(\cap_m I^m(A)) \right) ,$$

by Part 2. of Proposition 3.11

$$\cap_n I^n(\cap_l \cap_m I^{l+m}(A)) = \cap_l \cap_m \cap_n I^{l+m+n}(A) .$$

To sum up

$$\cap_l \cap_m \cap_n I^{l+m+n}(A) \subseteq \cap_n I^n \left(\cap_l I_i^l(\cap_m I^m(A)) \right) . \quad (3.9)$$

If we apply Part 3. of Proposition 3.15 on (3.9), then

$$\cap_k I_i^k(\cap_l \cap_m \cap_n I^{l+m+n}(A)) \subseteq \cap_k I_i^k \left(\cap_n I^n \left(\cap_l I_i^l(\cap_m I^m(A)) \right) \right) .$$

By Definition 3.14 and Part 1. of Proposition 3.11

$$U_i(A) \subseteq U_i(U_i(A)) .$$

□

The following properties may clarify, why the previously given definition of unawareness seems appropriate.

Proposition 3.17. *The basic properties of U_i :*

1. $U_i(A) \subseteq \cap_k I_i^k(A)$,
2. $U_i(A) \subseteq \cap_k I_i^k(B_j^0(A))$, $j \in N$,
3. $U_i(A) \subseteq \cap_n \cap_j \cap_k I_i^n(B_j^0(B_k^0(A)))$.

Proof. 1.: Because of Part 2. of Proposition 3.15

$$\cap_l I^l(A) \subseteq \cap_l I_i^l(A) .$$

Using Part 3. of Proposition 3.15

$$\cap_k I_i^k(\cap_l I^l(A)) \subseteq \cap_k I_i^k(\cap_l I_i^l(A)) .$$

By Property 2. of Proposition 3.11

$$\cap_k I_i^k(\cap_l I_i^l(A)) = \cap_k \cap_l I_i^{k+l}(A) .$$

Because of Definition 3.14 and Part 1. of Proposition 3.11

$$U_i(A) \subseteq \cap_n I_i^n(A) .$$

2.: By Definition 3.12

$$\cap_n I^n(A) \subseteq I^1(A) \subseteq I_j(A) .$$

Because of Part 3. of Proposition 3.15

$$\cap_k I_i^k(\cap_n I^n(A)) \subseteq \cap_k I_i^k(I_j(A)) . \quad (3.10)$$

The left side of (3.10) is equivalent to $U_i(A)$, by Definition 3.14. By Part 3. of Proposition 3.11 the right side of (3.10) is

$$\cap_k I_i^{k-1}(I_i(I_j(A))) = \cap_k I_i^{k-1}(I_i(\mathbb{G}I_j(A))) .$$

Finally, by Definition 3.10

$$\cap_k I_i^{k-1}(I_i(\mathbb{G}I_j(A))) = \cap_k I_i^{k-1}(I_i(B_j^0(A))) .$$

3.: It goes as Part 2. of Proposition 3.17. □

The first property means that if a player is unaware of an event, then she ignores it on every level. The second property shows, that it is irrelevant whether the other player has a belief of that event or ignores it, the unaware player ignores these alternatives. So if a player is unaware of an event, then she does not just ignore it, but ignores others ignorance or belief. The fourth property means that if a player is unaware of an event, then she ignores other players' belief about other players' beliefs of that event.

With these properties we tried to prove, that this unawareness operator relates to the notion of unawareness. An unaware player does not just ignore that event on every level, but ignores other players ignorance (or belief). To go on, the player ignores, that other players may have any beliefs about other players' beliefs about the event. It is important that the unaware player does not just ignore own ignorance, but ignores other players' belief or ignorance. If we thought, that a player did not ignore an other player's belief, then we would not think that she is unaware. The intuitive meaning of these properties is, that no one can show anything associated with the event of which the player is unaware, which is not ignored.

For some properties of unawareness, we will need the following definition, which practically gives back a subset of the belief operator.

Definition 3.18. $B_i^p : \mathcal{M} \rightarrow \mathcal{M}_i$, $i \in N$, for all $A \in \mathcal{M}$, $B_i^p(A) = \{\omega \in \Omega : f_i(\omega)(A) \geq p\}$.

Apparently, B_i^p is more restricting than B_i^0 . In the case of $B_i^0(A)$, Player i forms a belief about event A . The player may ascribe probability 0 to it, but she does consider the event. However, if $p > 0$, then $B_i^p(A)$ consists of a set of states of the world, where Player i assigns probability at least p to event A . This is, of course, more restricting, because it contains less states of the world. For example, the states of the world where Player i only forms a belief about event A , but finds it untrue, will not be in this set. On the other hand, $B_i^p(A)$ contains all states of the world, which are in $B_i^0(A)$. The cause if this is, that assigning probability at least p to an event implies assigning probability at least 0 to it.

The following properties are based on Heifetz et al (2011). These properties are found to be expected of an unawareness operator.

Proposition 3.19. *Properties of U_i :*

1. *Plausibility*: $U_i(A) \subseteq \mathbb{C}B_i^0(A) \cap \mathbb{C}B_i^0(\mathbb{C}B_i^0(A))$,
2. *Strong plausibility*: $U_i(A) \subseteq \cap_n (\mathbb{C}B_i^0)^n(A)$,
3. *Necessitation*: $B_i^1(\Omega) = \Omega$,
4. $U_i(A) \subseteq \mathbb{C}K_i(A)$,
5. *Symmetry*: $U_i(A) = U_i(\mathbb{C}A)$,
6. *U conjunction*: $\cap_n U_i(A_n) = U_i(\cap_n A_n)$,
7. *KU introspection*: $K_i(U_i(A)) = \emptyset$.

Proof. 1. and 2.: They come directly from Definition 3.10 combined with Part 1. of Proposition 3.17.

3.: By Part **D4** of Definition 3.1 $K_i(\Omega) = \Omega$, by Part **F2** of Definition 3.1, if $\omega \in K_i(A)$, then $f_i(\omega)(A) = 1$. So if $A = \Omega$, then for each $\omega \in K_i(\Omega) = \Omega$, then $f_i(\omega)(\Omega) = 1$. Because of Definition 3.18 $B_i^1(\Omega) = \Omega$.

4.: By Part 1. of Proposition 3.19 $U_i \subseteq \mathbb{C}B_i^0(A)$, which is equivalent to $B_i^0(A) \subseteq \mathbb{C}U_i(A)$.

By Property **F2** of Definition 3.1 and Definitions 3.8 and 3.18, we get

$$K_i(A) \subseteq B_i^1(A) \subseteq B_i^0(A) ,$$

putting the last two equations together means $K_i(A) \subseteq \mathbb{C}U_i(A)$, and then taking its complement means

$$U_i(A) \subseteq \mathbb{C}K_i(A) .$$

5.: From Part 3. of Proposition 3.11 $I_i(A) = I_i(\mathbb{C}A)$, $i \in N$. Thus $\cap_{i \in N} I_i(A) = \cap_{i \in N} I_i(\mathbb{C}A)$, this is equivalent to (Definition 3.12) $I^1(A) = I^1(\mathbb{C}A)$.

$$I_i(I^1(A)) = I_i(I^1(\mathbb{C}A)) , \quad i \in N ,$$

so

$$\cap_{i \in N} I_i(I^1(A)) = \cap_{i \in N} I_i(I^1(\mathbb{C}A)) .$$

By Definition 3.12 this means $I^2(A) = I^2(\mathbb{C}A)$. And going so forth

$$I^n(A) = I^n(\mathbb{C}A) , \quad n \in \mathbb{N} .$$

So

$$\cap_{n=1}^{\infty} I^n(A) = \cap_{n=1}^{\infty} I^n(\mathbb{C}A) ,$$

and

$$I_i(\cap_{n=1}^{\infty} I^n(A)) = I_i(\cap_{n=1}^{\infty} I^n(\mathbb{C}A)) ,$$

and so on

$$I_i^k(\cap_{n=1}^{\infty} I^n(A)) = I_i^k(\cap_{n=1}^{\infty} I^n(\mathbb{C}A)) .$$

Therefore

$$\cap_k I_i^k(\cap_{n=1}^\infty I^n(A)) = \cap_k I_i^k(\cap_{n=1}^\infty I^n(\mathbb{C}A)) ,$$

which is equivalent to (by Definition 3.14)

$$U_i(A) = U_i(\mathbb{C}A) .$$

6.: By Part 2. of Proposition 3.11

$$\cap_n I_i(A_n) = I_i(\cap_n A_n) , \quad i \in N .$$

So by taking their intersection by i

$$\cap_{i \in N} \cap_n I_i(A_n) = \cap_{i \in N} I_i(\cap_n A_n) ,$$

by Definition 3.12 this is equivalent to

$$\cap_n I^1(A_n) = I^1(\cap_n A_n) .$$

So

$$I_i(\cap_n I^1(A_n)) = I_i(I^1(\cap_n A_n)) , \quad i \in N .$$

By Part 2. of Proposition 3.11, the left hand side can be written in the following way

$$I_i(\cap_n I^1(A_n)) = \cap_n I_i(I^1(A_n)) .$$

Therefore

$$\cap_{i \in N} \cap_n I_i(I^1(A_n)) = \cap_{i \in N} I_i(I^1(\cap_n A_n)) ,$$

and by Definition 3.12 this is equivalent to

$$\cap_n I^2(A_n) = I^2(\cap_n A_n) .$$

And so on

$$\cap_n I^l(A_n) = I^l(\cap_n A_n) , \quad l \in \mathbb{N} .$$

Taking the intersection of both sides gives

$$\cap_l \cap_n I^l(A_n) = \cap_l I^l(\cap_n A_n) .$$

$$I_i(\cap_n \cap_l I^l(A_n)) = I_i(\cap_l I^l(\cap_n A_n)) ,$$

by Part 2. of Proposition 3.11 the left hand side is equivalent to

$$I_i(\cap_n \cap_l I^l(A_n)) = \cap_n I_i(\cap_l I^l(A_n)) .$$

And so on

$$\cap_n I_i^k(\cap_l I^l(A_n)) = I_i^k(\cap_l I^l(\cap_n A_n)) , \quad k \in \mathbb{N} .$$

Hence, taking their intersection by k

$$\cap_n \cap_k I_i^k(\cap_l I^l(A_n)) = \cap_k I_i^k(\cap_l I^l(\cap_n A_n)) .$$

Because of Definition 3.14 this is equivalent to

$$\cap_n U_i(A_n) = U_i(\cap_n A_n) .$$

7.: By Property **D1** of Definition 3.1 and Point 1. of Proposition 3.19

$$K_i(U_i(A)) \subseteq U_i(A) \subseteq \mathbb{C}B_i^0(\mathbb{C}B_i^0(A)) .$$

By Property **F2** of Definition 3.1 and Point 1. of Proposition 3.19

$$K_i(U_i(A)) \subseteq B_i^1(U_i(A)) \subseteq B_i^1(\mathbb{C}B_i^0(A)) .$$

Putting the last two equations together, where $D = \mathbb{C}B_i^0(A)$, means

$$K_i(U_i(A)) \subseteq \mathbb{C}B_i^0(D) \cap B_i^1(D) = \emptyset .$$

□

Plausibility means that being unaware of event A implies forming no belief about event A , and forming no belief about forming no belief about event A . Strong plausibility is very similar to plausibility, except that it goes on to infinity. This meets the requirement, that unawareness of an event means forming no belief about it on any level. In conclusion, in this model belief, and not knowledge, plausibility holds. So when a player is unaware, she does not form a belief of the event on any level. This property was practically shown before, when we presented that unawareness implies countably infinite levels of ignorance. Moreover, we showed, that unawareness implies ignorance of other players belief.

Belief necessitation means that all players believe with probability 1 the universal event Ω at every state of the world. The fourth property suggests, that unawareness of an event implies the complement of knowing that event. If a player is unaware of an event, it would be unreasonable to suggest, that she knows it.

Symmetry stands for unawareness of one event is equivalent to the unawareness of its complement. Symmetry basically comes directly from the belief operator. If you can form a belief about an event, then you can form a belief about its complement. The sixth property implies that being unaware of a sequence of events is equivalent to unawareness of their intersection.

The last property is KU introspection. This means that the player does not ever know the state of the world, where she is unaware. If the player knew that state of the world, she would be in it, but then she would not be unaware. This is more plausible than belief introspection, because a player could form a belief about being unaware in a different state of the world.

With this we concluded the presentation of our model. Our goal was to create axioms from which we can derive an unawareness operator. The unawareness operator is indirectly derived from the type function. In other models the unawareness operator is usually an axiom, so in a way, the innovation of our model is that the unawareness operator is not an axiom.

3.4 Example

In this section we will present an example, on which we can check the properties of unawareness, and why they can be expected or not.

Example 3.20 (The knowledge plausibility does not hold). Let $S = \{*\}$, $\Omega = \{\omega_1, \omega_2\}$, $N = \{i\}$,

$$K(A) = \begin{cases} \Omega, & \text{if } A = \Omega, \\ \{\omega_2\}, & \text{if } A = \{\omega_2\}, \\ \emptyset & \text{otherwise.} \end{cases}$$

$f(\omega_1)$ is defined on $\{\emptyset, \Omega\}$,
 $f(\omega_2)$ is defined on $\mathcal{P}(\Omega)$ such that $f(\omega_2)(\{\omega_2\}) = 1$,
 $U(A) = \begin{cases} \emptyset, & \text{if } A = \Omega \text{ or } A = \emptyset, \\ \{\omega_1\}, & \text{otherwise.} \end{cases}$

In this example the player knows ω_2 at state of the world ω_2 . Also, the player is unaware at state of the world ω_1 . At state of the world ω_2 the player ascribes probability 0 to ω_1 . We will check some properties on this example. The properties come from (Dekel et al, 1998) and (Heifetz et al, 2011).

Before we check the properties, we will show a possible story, which can be put behind this example. (Dekel et al, 1998) mention a situation, where the player contracts to have a home built (including a driveway), which could be the universal event Ω . ω_2 is the state of the world, where the player is told that city regulations preclude him from placing his driveway wherever he wants to. Hence, in ω_2 the player knows ω_2 . If the player is told about that city regulations preclude him from something, then he knows that he has been told that.

In turn, ω_1 is the state of the world, where the player is not told the previous statement about city regulations. In this state of the world the player is unaware of ω_1 and ω_2 as well. The player does not form any kind of belief about these situations, because he does not think about city regulations connected to his driveway. At that state of the world the player only knows, that he is building a home. The player does not form a belief about being told or not being told, he does not consider this kind of aspect of his home building.

On the other hand, at ω_2 the player assigns probability 0 to ω_1 . This means, that when the player is told about the city regulations, he forms a belief about not being told about this information. If the player is told about something, he is able to form a belief about not being told that very same thing. Obviously, because he is told about the city regulations, he assigns exactly probability 0 to not being told about them.

Properties:

- Knowledge necessitation holds: $K(\Omega) = \Omega$,
- Belief necessitation holds: $B^1(\Omega) = \Omega$,
- Monotonicity holds: $K(\{\omega_2\}) = \{\omega_2\} \subseteq \Omega = K(\Omega)$,
- Knowledge plausibility does not hold: $U(\{\omega_1\}) = \{\omega_1\} \not\subseteq \mathbb{C}K(\{\omega_1\}) \cap \mathbb{C}K(\mathbb{C}K(\{\omega_1\})) = \Omega \cap \emptyset = \emptyset$,
- Belief plausibility holds:

A	$B^0(A)$	$\mathbb{C}B^0(A)$	$\mathbb{C}B^0\mathbb{C}B^0(A)$	$U(A)$
$\emptyset$	$\emptyset$	Ω	$\emptyset$	$\emptyset$
Ω	Ω	$\emptyset$	Ω	$\emptyset$
$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$
$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$

- KU introspection holds: $K(U(\{\omega_1\})) = K(U(\{\omega_2\})) = \emptyset$,
- BU introspection does not hold: $B^0(U(\{\omega_1\})) = B^0(U(\{\omega_2\})) = \{\omega_2\}$,
- AU introspection holds: $U(\{\omega_1\}) = U(\{\omega_2\}) \subseteq U(U(\{\omega_1\})) = U(U(\{\omega_2\}))$,

- $U(A) \subseteq \mathbb{C}B^p(A)$ holds:

A	$B^p(A)$	$\mathbb{C}B^p(A)$	$U(A)$
$\emptyset$	$\emptyset$	Ω	$\emptyset$
Ω	Ω	$\emptyset$	$\emptyset$
$\{\omega_1\}$	$\emptyset$	Ω	$\{\omega_1\}$
$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_1\}$	$\{\omega_1\}$

- $U(B^q(A)) \subseteq \mathbb{C}B^p(A)$ holds,

A	$B^q(A)$	$\mathbb{C}B^p(A)$	$U(B^q(A))$
$\emptyset$	$\emptyset$	Ω	$\emptyset$
Ω	Ω	$\emptyset$	$\emptyset$
$\{\omega_1\}$	$\emptyset$	Ω	$\emptyset$
$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_1\}$	$\{\omega_1\}$

- Symmetry holds: $U(A) = U(\mathbb{C}A)$,

A	$\mathbb{C}A$	$U(A)$	$U(\mathbb{C}A)$
$\emptyset$	Ω	$\emptyset$	$\emptyset$
Ω	$\emptyset$	$\emptyset$	$\emptyset$
$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_1\}$	$\{\omega_1\}$
$\{\omega_2\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$

- U conjunction does not hold: $U(\{\omega_1\}) \cap U(\{\omega_2\}) = \{\omega_1\} \neq U(\{\omega_1\} \cap \{\omega_2\}) = U(\emptyset) = \emptyset$,
- AB self reflection does not hold: $U(B^p(\{\omega_1\})) = U(\emptyset) = \emptyset \neq U(\{\omega_1\}) = \{\omega_1\}$,

- AA self reflection holds: $U(\mathbb{C}U(A)) = U(A)$,

A	$U(A)$	$\mathbb{C}U(A)$	$U(\mathbb{C}U(A))$
$\emptyset$	$\emptyset$	Ω	$\emptyset$
Ω	$\emptyset$	Ω	$\emptyset$
$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_1\}$
$\{\omega_2\}$	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_1\}$

- $B^p(\mathbb{C}U(A)) = \mathbb{C}U(A)$ holds:

A	$U(A)$	$\mathbb{C}U(A)$	$B^p(\mathbb{C}U(A))$
$\emptyset$	$\emptyset$	Ω	Ω
Ω	$\emptyset$	Ω	Ω
$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_2\}$
$\{\omega_2\}$	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_2\}$

Belief and knowledge necessitations hold in this example, these are commonly expected properties of any model. (Dekel et al, 1998) assume some properties of the unawareness operator, and from those, which are associated with knowledge, only one holds. KU introspection holds, but knowledge plausibility does not hold. This is not a problem in the sense, that we find belief plausibility more important, which on the other hand, holds. We found, that it seems more reasonable to model

unawareness with multiple levels of not forming a belief than multiple levels of not knowing. Belief introspection not holding is, in a sense, reasonable, because at state of the world ω_2 the player forms a belief about being unaware at state of the world ω_1 . KU introspection holds, because the player is unaware at state of the world ω_1 , and does not know that state of the world at any state of the world. So BU introspection may not hold, because the player does form a belief about ω_1 elsewhere, she just does not know that state of the world anywhere.

In this example AU introspection holds. This is important, because this is the property, which is a commonly expected property of unawareness. This is a property, which even (Dekel et al, 1998) demand from an unawareness operator, and the only one in their case, which is not connected to the knowledge operator. AU introspection means, that if a player is unaware of an event at a state of the world, then she is also unaware of being unaware.

$U(A) \subseteq \mathbb{C}B^p(A)$ holds, which means that being unaware of an event implies not assigning probability at least p to it. We already know that if a player is unaware of an event, then she does not even assign probability at least 0 to it. Assigning probability at least 0 to an event obviously contains more states of the world, then assigning at least probability p to it. So if we take the complement of these sets, not assigning probability at least p to an event will be the larger one. To sum up, not forming a belief about an event is more restricting, then not assigning probability at least p to it. Because we already know (belief plausibility) that unawareness implies the former, it has to imply the latter also.

$U(B^q(A)) \subseteq \mathbb{C}B^p(A)$ holds. This means that being unaware of assigning probability at least q to an event implies not assigning probability at least p to it. The intuitive meaning of this property could be that if a player is unaware of assigning probability at least q to that event, then she does not form a belief about it. Because if the player formed a belief about that event, she would form a belief about forming a belief. But in the case of unawareness that cannot happen because of belief plausibility. According to belief plausibility if a player is unaware of an event, then she cannot form a belief about that event.

Symmetry holds, which is an important property, because it is a basic property of the belief operator. We did not define a property for the type function to satisfy this property. U conjunction does not hold in this model, which is not a major problem. If U conjunction would hold, then being unaware of a sequence of events would be equivalent to being unaware of their intersection.

AB self reflection does not hold. AB self reflection would mean that being unaware of an event is equivalent to being unaware of assigning probability at least p to it. This is not unusual, if we assume that if a player forms a belief about an event, she has to form a belief about forming a belief. AA self reflection holds, which states that being unaware of an event is equivalent to being unaware of being not unaware of that same event.

$B^p(\mathbb{C}U(A)) = \mathbb{C}U(A)$ holds, which means not being unaware of an event is equivalent to assigning probability at least p to not being unaware. So if a player assigns probability at least p to being not unaware, then she is not unaware indeed. The last sentence seems plausible, because if the player formed some kind of a belief about that event, she cannot be unaware. Also, if a player is not unaware of an event, then she forms a belief about not being unaware.

Chapter 4

Conclusions

The accomplishment of this paper is the nature of the unawareness operator. We have built a special kind of model, which allows unawareness, and we introduced it as a derived operator. For this, it was important to allow the possibility in this model, that players are able not to form beliefs about an event. In models, normally, players form beliefs about all events, they may just assign probability 0 to them, if they believe them to be untrue. However, in this case we had to allow an other scenario, in which the previous statement does not hold. This is because unawareness is about not forming beliefs about an event on any level. Furthermore, the player cannot form a belief about other players' belief about the event.

It may seem as a problem, that (Dekel et al, 1998) allege that in a standard state space model a nontrivial unawareness operator cannot be introduced. We have shown, that this does not apply on our model, because we did not define unawareness as multiple levels of not knowing an event. We find that it is more plausible to define unawareness with belief. We have shown an extended knowledge-belief space, from which we derived our unawareness operator. It is an intriguing property of our model, that the unawareness operator's properties are given in that sense, that they come from the axioms. This means, that we do not assume them, as it is done in other models.

Bibliography

- Aumann RJ (1999a) Interactive epistemology i: Knowledge. *International Journal of Game Theory* 28:263–300
- Aumann RJ (1999b) Interactive epistemology ii: Probability. *International Journal of Game Theory* 28:301–314
- Dekel E, Lipman BL, Rustichini A (1998) Standard state-space models preclude unawareness. *Econometrica* 66(1):159–173
- Fagin R, Halpern JY (1988) Belief, awareness and limited reasoning. *Artificial Intelligence* 34:39–76
- Geanakoplos J (1994) Common knowledge. In: Aumann RJ, Hart S (eds) *Handbook of Game Theory with Economic Applications*, Volume 2, North Holland, pp 1437–1496
- Halpern JY (2001) Alternative semantics for unawareness. *Games and Economic Behavior* 37(2):321–339
- Halpern JY, Rego LC (2008) Interactive unawareness revisited. *Games and Economic Behavior* 62(1):232–262
- Halpern JY, Rego LC (2009) Reasoning about knowledge of unawareness. *Games and Economic Behavior* 67(2):503–525
- Harsányi J (1967-68) Games with incomplete information played by bayesian players part I., II., III. *Management Science* 14:159–182, 320–334, 486–502
- Heifetz A, Samet D (1998) Topology-free typology of beliefs. *Journal of Economic Theory* 82:324–341
- Heifetz A, Meier M, Schipper BC (2006) Interactive unawareness. *Journal of Economic Theory* 130(1):78–94
- Heifetz A, Meier M, Schipper BC (2008) A canonical model of interactive unawareness. *Games and Economic Behavior* 62(1):304–324
- Heifetz A, Meier M, Schipper BC (2011) Unawareness, beliefs, and speculative trade, URL <http://www.econ.ucdavis.edu/faculty/schipper/unawbelief.pdf>, manuscript
- Heinsalu S (2011) Universal type spaces with unawareness, URL <http://sanderheinsalu.net/files/papers/Universal%20type%20spaces%20with%20unawareness.pdf>, manuscript
- Hill B (2010) Awareness dynamics. *Journal of Philosophical Logic* 39(2):113–137
- Li J (2009) Information structures with unawareness. *Journal of Economic Theory* 144(3):977–993

- Meier M (2008) Universal knowledge-belief structures. *Games and Economic Behavior* 62:53–66
- Modica S, Rustichini A (1994) Awareness and partitional information structures. *Theory and Decision* 37(1):107–124
- Modica S, Rustichini A (1999) Unawareness and partitional information structures. *Games and Economic Behavior* 27(2):265–298
- Pintér M (2010) The non-existence of a universal topological type space. *Journal of Mathematical Economics* 46:223–229
- Pintér M (2012) Every hierarchy of beliefs is a type, URL <http://arxiv.org/abs/0805.4007>, arXiv0805.4007v3
- Pintér M, Udvari Z (2011) Generalized type spaces, mPRA Paper No. 34107
- Rantala V (1982) Quantified modal logic: Non-normal worlds and propositional attitudes. *Studia Logica* 41(1):41–65
- Sillarri G (2006) Models of awareness. In: Bonnano G, van der Hoek W, Wooldridge M (eds) LOFT06: Proceedings, Logic and the Foundations of Game and Decision Theory, vol LOFT06, pp 209–219
- Sillarri G (2008) Quantified logic of awareness and impossible possible worlds. *The Review of Symbolic Logic* 1(4):514–529
- Wansing H (1990) A general possible worlds framework for reasoning about knowledge and belief. *Studia Logica* 49(4):523–539