

A FUNCTIONAL LIMIT THEOREM FOR INDECOMPOSABLE MULTI-TYPE BRANCHING PROCESSES WITH IMMIGRATION

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SZEGEDI TUDOMÁNYEGYETEM BOLYAI INTÉZET

2012

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1 Introduction

The main goal of this paper is to extend the limit theorem known for primitive multi-type branching processes to the class of irreducible ones in the critical case. (I. e. where the expectation matrix $\mathbf{m}_\xi$ is irreducible and the spectral radius $\rho(\mathbf{m}_\xi) = 1$.) In this section we are going to shortly review the most important concepts and the known limit theorems. We start with the notion of a branching process.

1.1. Definition. Let $(\xi_{n,k})_{n,k \in \mathbb{N}}$ be independent identically distributed (from now on, we abbreviate it as iid) random variables, all of them are nonnegative and integer valued. $(X_n)_{n \in \mathbb{N}}$ is a Galton-Watson type branching process, if

$$X_n = \sum_{k=1}^{X_{n-1}} \xi_{n,k}, \quad n \in \{1, 2, \dots\}.$$

Let $(\varepsilon_n)_{n \in \mathbb{N}}$ be iid nonnegative random variables independent from $(\xi_{n,k})_{n,k \in \mathbb{N}}$. Then $(X_n)_{n \in \mathbb{N}}$ is a Galton-Watson type branching process with immigration, if

$$X_n = \sum_{k=1}^{X_{n-1}} \xi_{n,k} + \varepsilon_n, \quad n \in \{1, 2, \dots\}.$$

The process is called critical, if $m_\xi := \mathbb{E}[\xi_{1,1}] = 1$.

The quantity X_n often denotes the number of individuals in the n -th generation of some population. Wei and Winnicki proved the following important theorem in [14]:

1.2. Theorem. Let $(X_n)_{n=0}^\infty$ be a critical Galton-Watson type branching process with $X_0 = 0$. Then

$$\mathcal{X}^{(n)} \xrightarrow{\mathcal{D}} \mathcal{X} \quad (n \rightarrow \infty)$$

in the Skorokhod space, where $\mathcal{X}_t^{(n)} := n^{-1}X_{[nt]}$ is a continuous time stochastic process and $\mathcal{X}$ is the unique strong solution of

$$d\mathcal{X}_t = m_\varepsilon dt + \sqrt{V_\xi \mathcal{X}_t^+} d\mathcal{W}_t, \quad t \in \mathbb{R}_+, \quad \mathcal{X}_0 = 0,$$

where $(\mathcal{W}_t)_{t \in \mathbb{R}_+}$ is a standard Wiener process and $V_\xi := \text{Var}[\xi_{1,1}]$.

1.3. Remark. From now on, when stating a theorem about convergence of processes in distribution, we always mean convergence in the Skorokhod space.

When dealing with multi-type branching processes, things get more complicated. In that case, if we go along with the population analogue, the population is not homogeneous, but heterogeneous. First, let us introduce the precise definition of a multi-type branching process.

1.4. Definition. Let $p \in \{1, 2, \dots\}$, and for any fixed $i, j \in \{1, 2, \dots, p\}$ let $(\xi_{n,k,i,j})_{n,k \in \mathbb{N}}$ be integer valued nonnegative iid random variables, and finally let $(\varepsilon_{n,k})_{n,k \in \mathbb{N}}$ be integer valued nonnegative iid random variables independent from $(\xi_{n,k,i,j})_{n,k \in \mathbb{N}}$. ($\xi_{n,k,i,j}$ denotes the number of j -type offsprings from the k -th i -type individuals in the n -th generation and $\varepsilon_{n,k}$ denotes the number of immigrating k -type individuals in the n -th generation.) With the introduction of the following notation:

$$\xi_{n,k,i} = \begin{bmatrix} \xi_{n,k,i,1} \\ \xi_{n,k,i,2} \\ \vdots \\ \xi_{n,k,i,p} \end{bmatrix}, \quad \varepsilon_n = \begin{bmatrix} \varepsilon_{n,1} \\ \varepsilon_{n,2} \\ \vdots \\ \varepsilon_{n,p} \end{bmatrix}, \quad \mathbf{X}_n = \begin{bmatrix} X_{n,1} \\ X_{n,2} \\ \vdots \\ X_{n,p} \end{bmatrix},$$

where

$$\mathbf{X}_n = \sum_{i=1}^p \sum_{k=1}^{X_{n-1,i}} \xi_{n,k,i} \in \mathbb{R}^p, \quad \mathbf{X}_0 = \mathbf{0} \in \mathbb{R}^p,$$

we call $\mathbf{X}_n$ a p-type branching process. If

$$\mathbf{X}_n = \sum_{i=1}^p \sum_{k=1}^{X_{n-1,i}} \xi_{n,k,i} + \varepsilon_n \in \mathbb{R}^p, \quad \mathbf{X}_0 = \mathbf{0} \in \mathbb{R}^p,$$

we call $\mathbf{X}_n$ a p-type branching process with immigration. We denote

$$\begin{aligned} \mathbf{m}_{\xi_i} &:= \mathbb{E}[\xi_{1,1,i}] \in \mathbb{R}_+^p, \quad \mathbf{m}_\xi := [\mathbf{m}_{\xi_1}, \mathbf{m}_{\xi_2}, \dots, \mathbf{m}_{\xi_p}] \in \mathbb{R}_+^{p \times p}, \\ \mathbf{V}_{\xi_i} &:= \text{Var}(\xi_{1,1,i}) \in \mathbb{R}_+^{p \times p}, \quad \mathbf{m}_\varepsilon := \mathbb{E}[\varepsilon_1] \in \mathbb{R}_+^p, \end{aligned}$$

and we call $\mathbf{m}_\xi$ the offspring expectation matrix. We say that the process is critical, if $\rho(\mathbf{m}_\xi) = 1$, where $\rho(\mathbf{m}_\xi)$ denotes the spectral radius of $\mathbf{m}_\xi$.

Similarly to the single-type process, the expectation of the offspring distribution plays a crucial role. With multi-type processes, instead of a real number, we have the matrix $\mathbf{m}_\xi$. Depending on the structure of $\mathbf{m}_\xi$, we can say different things about the limit distributions. The next definition (as seen in [3]) covers the most important ones.

Definition 1.5. Let $A \in \mathbb{R}^{p \times p}$ be a nonnegative matrix. We say that

- (i) A is primitive, if there exists an $m \in \mathbb{N}$ such that A^m is positive.
- (ii) A is reducible, if either $p = 1$ and $A = 0$ or $p \geq 2$ and there exists a $P \in \mathbb{R}^{p \times p}$ permutation matrix and $k \in \mathbb{N}$ such that

$$P^T A P = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix},$$

where $B \in \mathbb{R}^{k \times k}$, $C \in \mathbb{R}^{k \times p-k}$, $D \in \mathbb{R}^{p-k \times p-k}$, $0 \in \mathbb{R}^{p-k \times k}$.

- (iii) A is irreducible, if it is not reducible.

In the case where $\mathbf{m}_\xi$ is primitive or irreducible, we call the corresponding processes primitive and indecomposable, respectively. The following theorem has been proven by Ispány and Pap in [5]:

1.6. Theorem. Let $(\mathbf{X}_n)_{n \in \mathbb{N}}$ be a critical p-type primitive branching process with immigration and with initial value $\mathbf{X}_0 = \mathbf{0}$. Let us assume that for all $i \in \{1, 2, \dots, p\}$ the following conditions are true: $\mathbb{E}[\|\xi_{1,1,i}\|^2] < \infty$ and $\mathbb{E}[\|\varepsilon_1\|^2] < \infty$. Then

$$\mathcal{X}^{(n)} \xrightarrow{\mathcal{D}} \mathcal{X}_{\mathbf{u}_{\mathbf{m}_\xi}} \quad (n \rightarrow \infty)$$

where $\mathcal{X}_t^{(n)} := n^{-1} \mathbf{X}_{\lfloor nt \rfloor}$ is a continuous time stochastic process and $\mathcal{X}_t$ is the unique strong solution of

$$d\mathcal{X}_t = \mathbf{v}_{\mathbf{m}_\xi}^T \mathbf{m}_\varepsilon dt + \sqrt{\mathbf{v}_{\mathbf{m}_\xi}^T (\mathbf{u}_{\mathbf{m}_\xi} \odot \mathbf{V}_\xi) \mathbf{v}_{\mathbf{m}_\xi}} \mathcal{X}_t^+ d\mathcal{W}_t$$

with initial value $\mathcal{X}_0 = 0$, where for every $\mathbf{x} = (x_1, x_2, \dots, x_p)^T \in \mathbb{R}^p$, the $\odot$ denotes the operation $\mathbf{x} \odot \mathbf{V}_\xi = \sum_{i=1}^p x_i \mathbf{V}_{\xi_i}$ and $\mathbf{v}_{\mathbf{m}_\xi}$ and $\mathbf{u}_{\mathbf{m}_\xi}$ is the unique left and right Perron-vector of $\mathbf{m}_\xi$, respectively. (I. e. $\mathbf{v}_{\mathbf{m}_\xi}$ and $\mathbf{u}_{\mathbf{m}_\xi}$ are respectively left and right positive eigenvectors of $\mathbf{m}_\xi$ with maximum modulus and $\mathbf{u}_{\mathbf{m}_\xi}^T \mathbf{1} = 1$, $\mathbf{u}_{\mathbf{m}_\xi}^T \mathbf{v}_{\mathbf{m}_\xi} = 1$.)

One interesting property of the limit distribution of $\mathcal{X}^{(n)}$ is that it is concentrated to the ray generated by the right Perron-vector. We will see that this property also holds for the indecomposable case. It is important to note that the unique left and right Perron-vectors mentioned above do exist. It is also worth mentioning that several limit theorems have been proven by Kesten and Stigum in the supercritical case as can be seen in [11, 12, 13]. Our main goal is to prove an analogue of the preceding theorem to the case where $\mathbf{m}_\xi$ is irreducible.

2 Preparation

In this section we will review the tools that we are going to use for stating and proving the main result.

2.1 Structure of primitive and irreducible matrices

The purpose of this subsection is to highlight the differences between the primitive and the irreducible cases and to introduce the necessary tools for handling the offspring expectation matrix. For that, first we shall define the directed graph corresponding to a matrix.

2.1. Definition. Let $A = (a_{i,j})_{i,j=1}^p \in \mathbb{R}^{p \times p}$ be a nonnegative matrix. We call $\Gamma(A) = (V, E)$ the directed graph of A , if $V = \{P_1, P_2, \dots, P_p\}$ and $(P_i, P_j) \in E$ if and only if $a_{i,j} > 0$.

If we have a multi-type branching process, looking at the directed graph of the offspring expectation matrix can give us a lot of useful information. When studying the dynamics of such a process, the directed paths and cycles in this graph are especially important, because they represent the genealogy of a given specimen. One can easily prove the following theorem (or look up the proof in [3], Theorem 6.2.16):

2.2. Theorem. Let $A \in \mathbb{R}^{p \times p}$ be a nonnegative matrix and let P_i, P_j be two vertices from $\Gamma(A)$. Then there exists a directed path of length $m \in \mathbb{N}$ in $\Gamma(A)$ from P_i to P_j if and only if $(A^m)_{i,j} > 0$.

If $A \in \mathbb{R}^{p \times p}$ is a primitive matrix with $A^k > 0$, this means that for any two P_i, P_j vertices of $\Gamma(A)$, there is a k steps long directed path from P_i to P_j . There is no such thing for irreducible matrices. Regarding them, we can only say that if $B \in \mathbb{R}^{p \times p}$ is an irreducible matrix, then $(I + B)^{p-1} > 0$. (For proof, see [3], Theorem 6.2.23.) In other words, $\Gamma(B)$ is strongly connected, i. e. for any two P_i, P_j vertices of $\Gamma(B)$, there is a directed path from P_i to P_j . We cannot say anything about the length of this path. If we examine the spectral structure of such matrices, we see where this difference leads to. First, let us state the famous Perron-Frobenius theorem for irreducible matrices. (For proof see [3], Theorem 8.4.4.)

2.3. Theorem. Let $A \in \mathbb{R}^{p \times p}$ be a nonnegative irreducible matrix. Then the following statements are true:

- (i) $\rho(A) > 0$.
- (ii) $\rho(A)$ is an eigenvalue of A .
- (iii) There is a positive vector x such that $Ax = \rho(A)x$.
- (iv) $\rho(A)$ is a simple eigenvalue.

For positive or primitive matrices, there can be only one eigenvalue with maximum modulus. However, this is not true for irreducible matrices. If an irreducible matrix has r eigenvalues with maximum modulus, we can state the following (see [2], Theorem 8.2.4):

2.4. Theorem. Let $A \in \mathbb{R}^{p \times p}$ be an irreducible matrix with r different eigenvalues of maximum modulus. Then there exists a permutation matrix P such that

$$P^T A P = \begin{bmatrix} 0 & A_{1,2} & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_{r-1,r} \\ A_{r,1} & 0 & \dots & 0 \end{bmatrix}$$

where $A_{i,i+1} \in \mathbb{R}^{d_i \times d_{i+1}}$ for some d_i -s.

Regarding the directed graph $\Gamma(A)$, this means that we can partition its nodes into sets $D_1, D_2, \dots, D_r$ with $|D_i| = d_i$ such that there is a directed edge from P_i to P_j if and only if $j = i + 1$ modulo r . Examining this quantity r can give us a useful way to look at irreducible matrices.

2.5. Definition. Let $A \in \mathbb{R}^{p \times p}$ be an irreducible matrix with its directed graph $\Gamma(A)$. The greatest common divisor of the lengths of the cycles in $\Gamma(A)$ is called the index of imprimitivity of A .

This index of imprimitivity can be easily connected with the number of eigenvalues of maximum modulus. In fact, the following theorem is true (for details, see [2], Theorem 8.3.4 (iv)):

2.6. Theorem. Let $A \in \mathbb{R}^{p \times p}$ be an irreducible matrix with index of imprimitivity r and spectral radius $\rho(A)$. Then A has exactly r eigenvalues of maximum modulus and these eigenvalues are the roots of the equation $\lambda^r - \rho(A)^r = 0$, i. e., the eigenvalues of maximum modulus are

$$\rho(A)e^{2\pi i j/r} \quad (j \in \{0, 1, \dots, r-1\})$$

Also, we can characterize the primitive matrices with this concept.

2.7. Proposition. Let $A \in \mathbb{R}^{p \times p}$ be an irreducible matrix. Then A is primitive if and only if its index of imprimitivity is 1.

In other words, if $A \in \mathbb{R}^{p \times p}$ is an irreducible matrix with index of imprimitivity $r > 1$, then there is some kind of periodicity present with period r and it will heavily affect the limit distributions.

2.2 Limit theorems and convergence

Before saying anything about convergence, we shall precisely define what kind of objects converge in what sense. Since every trajectory of the step process created from a branching process is right continuous with left limits, it makes sense to consider the space of càdlàg functions. The $D(\mathbb{R}_+, \mathbb{R}^p)$ Skorokhod space is defined by the space of all càdlàg functions from $\mathbb{R}_+$ to $\mathbb{R}^p$ and a metric, which makes the space complete and separable. Let us define

$$\Lambda := \left\{ \lambda \in C(\mathbb{R}_+, \mathbb{R}_+) : \lambda \text{ is a bijection} \right\}$$

and

$$k_N(t) := \begin{cases} 1 & \text{if } t \leq N \\ N+1-t & \text{if } N < t < N+1 \\ 0 & \text{if } N+1 \leq t \end{cases}$$

and for every $\lambda \in \Lambda$,

$$\|\lambda\| := \sup_{0 \leq s < t} \left| \log \frac{\lambda(t) - \lambda(s)}{t - s} \right|.$$

The functions in Λ are sometimes called "wiggles in time". Then for every $\alpha, \beta \in D(\mathbb{R}_+, \mathbb{R}^p)$, define

$$\delta_N(\alpha, \beta) := \inf_{\lambda \in \Lambda} \left(\|\lambda\| + \|(k_N \alpha) \circ \lambda - k_N \beta\|_\infty \right)$$

and

$$\delta(\alpha, \beta) := \sum_{N=1}^{\infty} 2^{-N} \min\{1, \delta_N(\alpha, \beta)\}. \quad (2.1)$$

Then $D(\mathbb{R}_+, \mathbb{R}^p)$ is a complete metric space with δ . (For details, see [8], Theorem 1.14.) Let $\mathcal{D}_\infty(\mathbb{R}_+, \mathbb{R}^p)$ denote the Borel sets by this metric. If $(\mathcal{Y}_t)_{t \in \mathbb{R}_+}$ and $(\mathcal{Y}_t^{(n)})_{t \in \mathbb{R}_+}$ are $\mathbb{R}^p$ -valued

stochastic processes with càdlàg paths, then we say that $\mathbf{y}_t^{(n)} \xrightarrow{\mathcal{D}} \mathbf{y}_t$ if and only if the distributions of $\mathbf{y}_t^{(n)}$ converge weakly to the distribution of $\mathbf{y}_t$ in $(\mathcal{D}(\mathbb{R}_+, \mathbb{R}^p), \mathcal{D}_\infty(\mathbb{R}_+, \mathbb{R}^p))$.

For proving the desired limit theorem for indecomposable multi-type branching processes, we need an important result. The following theorem is the frame which we will build upon. (See [4], Corollary 2.2 for proof.)

2.8. Theorem. *Let $\beta : \mathbb{R}_+ \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ and $\gamma : \mathbb{R}_+ \times \mathbb{R}^p \rightarrow \mathbb{R}^{p \times q}$ be continuous functions and assume that for the SDE*

$$d\mathbf{u}_t = \beta(t, \mathbf{u}_t)dt + \gamma(t, \mathbf{u}_t)d\mathbf{W}_t,$$

where $(\mathbf{W}_t)_{t \in \mathbb{R}_+}$ is a standard q -dimensional Wiener process, has a unique weak solution for every initial value $\mathbf{u}_0 = \mathbf{u}_0$. Assume that the process $(\mathbf{u}_t)_{t \in \mathbb{R}_+}$ is the solution of the SDE above with initial value $\mathbf{u}_0 = \mathbf{0}$. For each fixed $n \in \mathbb{N}$, let $(\mathbf{U}_k^{(n)})_{k \in \mathbb{N}}$ a sequence of random variables with values in $\mathbb{R}^p$ adapted to a filtration $(\mathcal{F}_k^{(n)})_{k \in \mathbb{N}}$ and suppose that for all $n, k \in \mathbb{N}$ $\mathbb{E}[\|\mathbf{U}_k^{(n)}\|^2] < \infty$. Let

$$\mathbf{u}_t^{(n)} = \sum_{k=1}^{\lfloor nt \rfloor} \mathbf{U}_k^{(n)}, \quad t \in \mathbb{R}_+, \quad n \in \mathbb{N}$$

If for every $T > 0$ the following are true:

- (i) $\sup_{t \in [0, T]} \left\| \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}[\mathbf{U}_k^{(n)} | \mathcal{F}_{k-1}^{(n)}] - \int_0^t \beta(s, \mathbf{u}_s^{(n)}) ds \right\| \xrightarrow{P} 0$,
- (ii) $\sup_{t \in [0, T]} \left\| \sum_{k=1}^{\lfloor nt \rfloor} \text{Var}[\mathbf{U}_k^{(n)} | \mathcal{F}_{k-1}^{(n)}] - \int_0^t \gamma(s, \mathbf{u}_s^{(n)}) \gamma(s, \mathbf{u}_s^{(n)})^T ds \right\| \xrightarrow{P} 0$,
- (iii) $\sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}[\|\mathbf{U}_k^{(n)}\|^2 \chi_{\{\|\mathbf{U}_k^{(n)}\| > \theta\}} | \mathcal{F}_{k-1}^{(n)}] \xrightarrow{P} 0$ for every $\theta > 0$,

then $\mathbf{u}^{(n)} \xrightarrow{\mathcal{D}} \mathbf{u}$ as $n \rightarrow \infty$.

In case of martingale differences, this theorem becomes simpler. The following corollary easily follows from the properties of martingale differences:

2.9. Corollary. *Let $\gamma : \mathbb{R}_+ \times \mathbb{R}^p \rightarrow \mathbb{R}^{p \times q}$ be a continuous function and assume that the SDE*

$$d\mathbf{u}_t = \gamma(t, \mathbf{u}_t)d\mathbf{W}_t,$$

where $(\mathbf{W}_t)_{t \in \mathbb{R}_+}$ is a standard q -dimensional Wiener process, has a unique weak solution for every initial value $\mathbf{u}_0 = \mathbf{u}_0$. Assume that the process $(\mathbf{u}_t)_{t \in \mathbb{R}_+}$ is the solution of the SDE above with initial value $\mathbf{u}_0 = \mathbf{0}$. Let $(\mathbf{U}_k^{(n)})_{k \in \mathbb{N}}$ be a sequence of p -dimensional martingale differences with respect to the filtration $(\mathcal{F}_k)_{k \in \mathbb{N}}$ and suppose that for all $n, k \in \mathbb{N}$, $\mathbb{E}[\|\mathbf{U}_k^{(n)}\|^2] \leq \infty$. Let

$$\mathbf{u}_t^{(n)} = \sum_{k=1}^{\lfloor nt \rfloor} \mathbf{U}_k^{(n)}, \quad t \in \mathbb{R}_+, \quad n \in \mathbb{N}.$$

If for each $T \in \mathbb{R}_{++}$ the following are true:

- (i) $\sup_{t \in [0, T]} \left\| \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}[\mathbf{U}_k^{(n)} (\mathbf{U}_k^{(n)})^T | \mathcal{F}_{k-1}] - \int_0^t \gamma(s, \mathbf{u}_s^{(n)}) \gamma(s, \mathbf{u}_s^{(n)})^T ds \right\| \xrightarrow{P} 0$,
- (ii) $\sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}[\|\mathbf{U}_k^{(n)}\|^2 \chi_{\{\|\mathbf{U}_k^{(n)}\| > \theta\}} | \mathcal{F}_{k-1}] \xrightarrow{P} 0$ for every $\theta > 0$,

then $\mathbf{u}^{(n)} \xrightarrow{\mathcal{D}} \mathbf{u}$ as $n \rightarrow \infty$.

3 The main result

Let $(\mathbf{X}_k)_{k \in \mathbb{N}}$ be a critical p-type branching process with immigration and initial value $\mathbf{X}_0 = \mathbf{0}$. Suppose that $\mathbf{m}_\xi$ is irreducible with index of imprimitivity r and the types are partitioned into subsets $D_1, D_2, \dots, D_r$ with $|D_k| = d_k$ such that $\xi_{1,1,i,j} \neq 0$ if and only if there exists a $k \in \mathbb{N}$ with $i \in D_k, j \in D_{k-1}$, all subscripts considered modulo r .

3.1. Remark. From now on, subscripts are considered modulo the subscript's range.

Because of this, we can assume by enumerating the types in an order according to this partition that $\mathbf{m}_\xi$ has the following form:

$$\mathbf{m}_\xi = \begin{bmatrix} 0 & \mathbf{m}_{1,2} & 0 & \dots & 0 \\ 0 & 0 & \mathbf{m}_{2,3} & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mathbf{m}_{r-1,r} \\ \mathbf{m}_{r,1} & 0 & 0 & \dots & 0 \end{bmatrix},$$

where $\mathbf{m}_{k,k+1} \in \mathbb{R}^{d_k \times d_{k+1}}$ and $\sum_{k=1}^r d_k = p$. With the notation $\tilde{\mathbf{m}}_{i,j} := \mathbf{m}_{i,i+1} \dots \mathbf{m}_{j-1,j}$, for every $k \in \{1, \dots, r-1\}$, the k -th power of $\mathbf{m}_\xi$ has the following form:

$$\mathbf{m}_\xi^k = \begin{bmatrix} 0 & \dots & 0 & \tilde{\mathbf{m}}_{1,k+1} & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \tilde{\mathbf{m}}_{2,k+2} & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \tilde{\mathbf{m}}_{r-k+1,r+1} & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & \tilde{\mathbf{m}}_{r,r+k} & 0 & 0 & \dots & 0 \end{bmatrix}. \quad (3.1)$$

When $k = r$, we get

$$\mathbf{m}_\xi^r = \begin{bmatrix} \tilde{\mathbf{m}}_{1,r+1} & 0 & \dots & 0 \\ 0 & \tilde{\mathbf{m}}_{2,r+2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \vdots & \tilde{\mathbf{m}}_{r,2r} \end{bmatrix} = \tilde{\mathbf{m}}_{1,r+1} \oplus \dots \oplus \tilde{\mathbf{m}}_{r,2r},$$

where $\oplus$ denotes the orthogonal sum of matrices. From the Perron-Frobenius theory for irreducible matrices, we know that there exists two unique positive vectors $\mathbf{v}, \mathbf{u} \in \mathbb{R}^p$ such that they are respectively left and right eigenvectors of $\mathbf{m}_\xi$ for the eigenvalue $\rho(\mathbf{m}_\xi) = 1$ with $\mathbf{u}^T \mathbf{1} = 1$ and $\mathbf{u}^T \mathbf{v} = 1$. (From now on, we abbreviate them as the left and right Perron-vectors of $\mathbf{m}_\xi$.) If we partition $\mathbf{u}$ and $\mathbf{v}$ according the subsets $D_1, D_2, \dots, D_r$ and introduce the notation

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_r \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_r \end{bmatrix}, \quad \mathbf{u}_i, \mathbf{v}_i \in \mathbb{R}^{d_i},$$

by the Perron-Frobenius theory, we have a few important observations, which we summarize next. (The matrix-theoretic facts are not discussed in the most general setting. For details, see [2] and [3].)

- (i) For every $i \in \{1, 2, \dots, r\}$, the matrix $\tilde{\mathbf{m}}_{i,i+r} \in \mathbb{R}^{d_i \times d_i}$ is primitive.
- (ii) $\tilde{\mathbf{u}}_i = r\mathbf{u}_i$ and $\tilde{\mathbf{v}}_i = \mathbf{v}_i$ are the left and right Perron-vectors of $\tilde{\mathbf{m}}_{i,i+r}$

(iii) For $\mathbf{\Pi}_i = \tilde{\mathbf{u}}_i \tilde{\mathbf{v}}_i^T$ we have

$$\tilde{\mathbf{m}}_{i,i+r}^n \rightarrow \mathbf{\Pi}_i \quad \text{as } n \rightarrow \infty.$$

(iv) Because of (iii), for $\mathbf{\Pi} = \mathbf{\Pi}_1 \oplus \cdots \oplus \mathbf{\Pi}_r$ we have

$$\mathbf{m}_\xi^{nr} \rightarrow \mathbf{\Pi} \quad \text{as } n \rightarrow \infty.$$

(v) There are constants $c \in \mathbb{R}_{++}$ and $\kappa \in (0, 1)$ such that for all n we have

$$\|\mathbf{m}_\xi^{nr} - \mathbf{\Pi}\| \leq c\kappa^n, \quad (3.2)$$

where $\|A\|$ denotes the operator norm for every $A \in \mathbb{R}^{p \times p}$.

(vi) The matrix $\mathbf{\Pi}$ has some important properties. In fact, for every $l \in \{0, 1, \dots, r\}$, we have the following identities:

$$\mathbf{m}_\xi \mathbf{\Pi} = \mathbf{\Pi} \mathbf{m}_\xi, \quad (3.3)$$

$$\mathbf{m}_\xi^r \mathbf{\Pi} = \lim_{n \rightarrow \infty} \mathbf{m}_\xi^{(n+1)r} = \mathbf{\Pi}, \quad (3.4)$$

$$\mathbf{\Pi} \mathbf{m}_\xi^l \sum_{j=1}^r \mathbf{m}_\xi^{j-1} = \mathbf{\Pi} \sum_{j=1}^r \mathbf{m}_\xi^{j-1}. \quad (3.5)$$

(vii) Since $\mathbf{m}_\xi^{nr} \rightarrow \mathbf{\Pi}$ as $n \rightarrow \infty$, the norms of $\mathbf{m}_\xi^j$ are bounded, because if $j = rk + j'$, where $j' \in \{0, 1, \dots, r-1\}$ and $c^* = \sup_{k \in \mathbb{N}} \|\mathbf{m}_\xi^{rk}\| \leq \|\mathbf{\Pi}\| + \sup_{k \in \mathbb{N}} \|\mathbf{m}_\xi^{rk} - \mathbf{\Pi}\| < \infty$, then

$$\|\mathbf{m}_\xi^j\| \leq \|\mathbf{m}_\xi^{rk}\| \|\mathbf{m}_\xi^{j'}\| \leq c^* \max\{\|\mathbf{m}_\xi^1\|, \|\mathbf{m}_\xi^2\|, \dots, \|\mathbf{m}_\xi^{r-1}\|\} =: c' < \infty. \quad (3.6)$$

(viii) About the covariance matrices $\mathbf{V}_{\xi_j} = \text{Var}(\xi_{1,1,j})$, we can say that

$$\mathbf{V}_{\xi_j} = \begin{cases} \mathbf{0} \oplus \mathbf{0} \oplus \cdots \oplus \mathbf{0} \oplus \mathbf{V}_{1,j} & \text{if } j \in D_1 \\ \mathbf{V}_{2,j} \oplus \mathbf{0} \oplus \cdots \oplus \mathbf{0} \oplus \mathbf{0} & \text{if } j \in D_2 \\ \vdots & \\ \mathbf{0} \oplus \mathbf{0} \oplus \cdots \oplus \mathbf{V}_{r,j} \oplus \mathbf{0} & \text{if } j \in D_r \end{cases}, \quad (3.7)$$

where $\mathbf{V}_{i,j} \in \mathbb{R}^{d_{i-1} \times d_{i-1}}$ denotes the covariance matrix of the $(\xi_{1,1,j,l})_{l \in D_{i-1}}$. For convenience, we introduce the $\odot$ operator: for every $\mathbf{x} = (x_1, x_2, \dots, x_p)^T \in \mathbb{R}_+^p$, define $\mathbf{x} \odot \mathbf{V}_\xi := \sum_{j=1}^p x_j \mathbf{V}_{\xi_j}$ and $\mathbf{x} \odot \mathbf{V}_i := \sum_{j \in D_i} x_j \mathbf{V}_{i,j}$.

Our main purpose is to prove the following limit theorem.

3.2. Theorem. *Let $(\mathbf{X}_k)_{k \in \mathbb{N}}$ be a critical p -type indecomposable branching process with immigration, index of imprimitivity r and initial value $\mathbf{X}_0 = \mathbf{0}$. Suppose that for every $i \in \{1, 2, \dots, p\}$, $\mathbb{E}(\|\xi_{1,1,i}\|^4) < \infty$ and $\mathbb{E}(\|\varepsilon_1\|^4) < \infty$. For every $n \in \mathbb{N}$, define the rp -dimensional random step process $(\mathbf{x}_t^{(n)})_{t \in \mathbb{R}_+}$ with*

$$\mathbf{x}_t^{(n)} = \frac{1}{n} \begin{bmatrix} \mathbf{X}_{r[nt]} \\ \mathbf{X}_{r[nt]-1} \\ \vdots \\ \mathbf{X}_{r[nt]-r+1} \end{bmatrix}. \quad (3.8)$$

Then

$$\mathbf{x}^{(n)} \xrightarrow{\mathcal{D}} \mathbf{x}, \quad (3.9)$$

where, for every $t \in \mathbb{R}_+$,

$$\mathbf{x}_t = \begin{bmatrix} \mathbf{y}_t \\ \mathbf{m}_\xi^{r-1} \mathbf{y}_t \\ \vdots \\ \mathbf{m}_\xi \mathbf{y}_t \end{bmatrix}, \quad \mathbf{y}_t = \begin{bmatrix} \mathbf{y}_{t,1} \\ \mathbf{y}_{t,2} \\ \vdots \\ \mathbf{y}_{t,r} \end{bmatrix},$$

where, for every $i \in \{1, 2, \dots, r\}$ and $t \in \mathbb{R}_+$, we have $\mathbf{y}_{t,i} = Z_{t,i} \mathbf{u}_i \in \mathbb{R}^{d_i}$, where $(Z_{t,i})_{t \in \mathbb{R}_+}$ is a real valued stochastic process and given by the unique strong solution of the SDE

$$dZ_{t,i} = r \mathbf{v}_i^T \sum_{l=i}^{i+r-1} \tilde{\mathbf{m}}_{i,l} \mathbf{m}_{\varepsilon,l} dt + r \sqrt{\mathbf{v}_i^T \sum_{j=1}^r \tilde{\mathbf{m}}_{i,i+j-1} \left[(\tilde{\mathbf{m}}_{i+j,i+r} \mathbf{u}_i) \odot \mathbf{V}_{i+j} \right] \tilde{\mathbf{m}}_{i,i+j-1}^T \mathbf{v}_i Z_{t,i}^+} d\mathcal{W}_{t,i}.$$

Moreover, for every $j \in \{0, 1, \dots, r-1\}$, we have $(\mathbf{y}_t)_{t \in \mathbb{R}_+} \stackrel{\mathcal{D}}{=} (\mathbf{m}_{\xi}^j \mathbf{y}_t)_{t \in \mathbb{R}_+}$.

3.3. Remark. The moment conditions in the theorem can be weakened. If we only assume finite second moments, the theorem remains true, but checking condition (ii) in Corollary 2.9 gets much more complicated. In applications, it is reasonable to assume finite fourth moments.

It would be impossible to apply Theorem 2.8 directly. Our strategy is to introduce a new process formed by a sum of martingale differences, apply Corollary 2.9 to this new process and deduce the convergence of $\mathcal{X}_t^{(n)}$ by the continuous mapping theorem. Let us define

$$\mathbf{M}_0 := \mathbf{0}, \quad \mathbf{M}_k = \begin{bmatrix} \mathbf{M}_{k,1} \\ \mathbf{M}_{k,2} \\ \vdots \\ \mathbf{M}_{k,r} \end{bmatrix} := \begin{bmatrix} \mathbf{X}_{rk} - \mathbb{E}[\mathbf{X}_{rk} | \mathcal{F}_{rk-1}] \\ \mathbf{X}_{rk-1} - \mathbb{E}[\mathbf{X}_{rk-1} | \mathcal{F}_{rk-2}] \\ \vdots \\ \mathbf{X}_{rk-r+1} - \mathbb{E}[\mathbf{X}_{rk-r+1} | \mathcal{F}_{rk-r}] \end{bmatrix}, \quad (3.10)$$

where $(\mathcal{F}_k)_{k \in \mathbb{N}}$ is the natural filtration of our process $(\mathbf{X}_k)_{k \in \mathbb{N}}$. Note that

$$\mathbb{E}[\mathbf{X}_k | \mathcal{F}_{k-1}] = \mathbf{m}_{\xi} \mathbf{X}_{k-1} + \mathbf{m}_{\varepsilon}. \quad (3.11)$$

From this, it follows that

$$\mathbf{M}_k = \begin{bmatrix} \mathbf{X}_{rk} - \mathbf{m}_{\xi} \mathbf{X}_{rk-1} - \mathbf{m}_{\varepsilon} \\ \mathbf{X}_{rk-1} - \mathbf{m}_{\xi} \mathbf{X}_{rk-2} - \mathbf{m}_{\varepsilon} \\ \vdots \\ \mathbf{X}_{rk-r+1} - \mathbf{m}_{\xi} \mathbf{X}_{rk-r} - \mathbf{m}_{\varepsilon} \end{bmatrix}. \quad (3.12)$$

The process $(\mathbf{X}_n)_{n \in \mathbb{N}}$ can be easily reconstructed from this. With the application of (3.12), we get

$$\mathbf{X}_{rk-i+1} = \mathbf{m}_{\xi}^r \mathbf{X}_{rk-r-i+1} + \sum_{l=i}^r \mathbf{m}_{\xi}^{l-i} (\mathbf{M}_{k,l} + \mathbf{m}_{\varepsilon}) + \sum_{l=1}^{i-1} \mathbf{m}_{\xi}^{l-i+r} (\mathbf{M}_{k-1,l} + \mathbf{m}_{\varepsilon}) \quad (3.13)$$

and hence

$$\mathbf{X}_{rk-i+1} = \sum_{j=1}^k \mathbf{m}_{\xi}^{(k-j)r} \left[\sum_{l=i}^r \mathbf{m}_{\xi}^{l-i} (\mathbf{M}_{j,l} + \mathbf{m}_{\varepsilon}) + \sum_{l=1}^{i-1} \mathbf{m}_{\xi}^{l-i+r} (\mathbf{M}_{j-1,l} + \mathbf{m}_{\varepsilon}) \right] \quad (3.14)$$

for every $k \in \mathbb{N}$ and $i \in \{1, 2, \dots, r\}$. Now we examine the limit process of $(\mathbf{M}_k)_{k \in \mathbb{N}}$. The next theorem is an important stepping stone for proving Theorem 3.2.

3.4. Theorem. For every $n \in \mathbb{N}$, let $(\mathcal{M}_t^{(n)})_{t \in \mathbb{R}_+}$ be the rp -dimensional continuous time random step process with the following definition:

$$\mathcal{M}_t^{(n)} = \frac{1}{n} \sum_{k=1}^{\lfloor nt \rfloor} \mathbf{M}_k. \quad (3.15)$$

Then we have

$$\mathcal{M}^{(n)} \xrightarrow{\mathcal{D}} \mathcal{M}, \quad (3.16)$$

where for every $t \in \mathbb{R}_+$

$$\mathcal{M}_t = \begin{bmatrix} \mathcal{M}_{t,1} \\ \mathcal{M}_{t,2} \\ \vdots \\ \mathcal{M}_{t,r} \end{bmatrix},$$

is the unique strong solution of the SDE

$$d\mathcal{M}_{t,i} = \sqrt{\left[\mathbf{m}_\xi^{r-i} \Pi \sum_{j=1}^r \mathbf{m}_\xi^{j-1} (\mathcal{M}_{t,j} + t \mathbf{m}_\varepsilon) \right]^+ \odot \mathbf{V}_\xi d\mathcal{W}_{t,i}} \quad (3.17)$$

with initial value $\mathcal{M}_{0,i} = \mathbf{0}$, where $(\mathcal{W}_{t,i})_{t \in \mathbb{R}_+}$, $i \in \{1, 2, \dots, r\}$, are independent p -dimensional Wiener processes.

Proof. The proof consists of two parts. In the first part, we will prove that the SDE (3.17) has a unique strong solution $(\mathcal{M}_t^{(\mathbf{y}_0)})_{t \in \mathbb{R}_+}$ with initial value $\mathcal{M}_0^{(\mathbf{y}_0)} = \mathbf{y}_0$ for every $\mathbf{y}_0 \in \mathbb{R}^p$, as required in Corollary 2.9. We will prove the actual convergence in the second part.

Handling the SDE-s (3.17) separately is difficult, since they all depend on each other. For solving this issue, we introduce the p -dimensional process

$$\mathcal{N}_t := \mathcal{M}_{t,1} + \mathbf{m}_\xi \mathcal{M}_{t,2} + \dots + \mathbf{m}_\xi^{r-1} \mathcal{M}_{t,r}, \quad t \in \mathbb{R}_+. \quad (3.18)$$

The corresponding SDE of the $(\mathcal{N}_t)_{t \in \mathbb{R}_+}$ process is

$$d\mathcal{N}_t = \sum_{i=1}^r \mathbf{m}_\xi^{i-1} \sqrt{\left[\mathbf{m}_\xi^{r-i} \Pi \left(\mathcal{N}_t + t \sum_{l=1}^r \mathbf{m}_\xi^{l-1} \mathbf{m}_\varepsilon \right) \right]^+ \odot \mathbf{V}_\xi d\mathcal{W}_{t,i}}. \quad (3.19)$$

In this $(\mathcal{N}_t)_{t \in \mathbb{R}_+}$ process we have all the information about the $\mathcal{M}_{t,i}$ -s "rolled up" into one process. This information can be extracted easily: if $(\mathcal{N}_t)_{t \in \mathbb{R}_+}$ is the unique strong solution of (3.19) with initial value $\mathcal{N}_0 = \mathbf{0}$, then (3.17) takes the form

$$d\mathcal{M}_{t,i}^{(\mathbf{y}_0)} = \sqrt{\left[\mathbf{m}_\xi^{r-i} \Pi \left(\mathcal{N}_t + t \sum_{j=1}^r \mathbf{m}_\xi^{j-1} \mathbf{m}_\varepsilon \right) \right]^+ \odot \mathbf{V}_\xi d\mathcal{W}_{t,i}}.$$

Since the right side does not depend on $\mathcal{M}_t^{(\mathbf{y}_0)}$, the solution can be written in the form of

$$\mathcal{M}_{t,i}^{(\mathbf{y}_0)} = \mathbf{y}_0 + \int_0^t \sqrt{\left[\mathbf{m}_\xi^{r-i} \Pi \left(\mathcal{N}_s + s \sum_{j=1}^r \mathbf{m}_\xi^{j-1} \mathbf{m}_\varepsilon \right) \right]^+ \odot \mathbf{V}_\xi d\mathcal{W}_{s,i}}.$$

Thus it follows that it is enough to prove the existence of a unique strong solution to the SDE (3.19) with initial value $\mathcal{N}_0 = \mathbf{0}$. Using the special forms of $\mathbf{m}_\xi$ and $\mathbf{V}_{\xi_i}$ (see (3.1) and (3.7)), we can simplify (3.19). Since this is not trivial, we state it in a separate lemma.

3.5. Lemma. *For the above defined $(\mathcal{N}_t)_{t \in \mathbb{R}_+}$ process with $\mathcal{N}_t = [\mathcal{N}_{t,1}, \dots, \mathcal{N}_{t,r}]^T$, $\mathcal{N}_{t,i} \in \mathbb{R}^{d_i}$, we have*

$$d\mathcal{N}_{t,i} = \sqrt{r \left[\mathbf{v}_i^T \left(\mathcal{N}_{t,i} + t \sum_{l=i}^{i+r-1} \tilde{\mathbf{m}}_{i,l} \mathbf{m}_{\varepsilon,l} \right) \right]^+ d\widetilde{\mathcal{W}}_{t,i}}, \quad (3.20)$$

where $(\widetilde{\mathcal{W}}_{t,i})_{t \in \mathbb{R}_+}$ are a d_i -dimensional Wiener process.

Proof. The proof is based on two observations:

(i) If $\mathbf{u}, \mathbf{v}, \mathbf{x} \in \mathbb{R}_+^p$, $\mathbf{M} \in \mathbb{R}_+^{p \times p}$, then

$$\sqrt{\mathbf{M}\mathbf{u}\mathbf{v}^T\mathbf{x} \odot \mathbf{V}_i} = \sqrt{\mathbf{v}^T\mathbf{x}}\sqrt{\mathbf{M}\mathbf{u} \odot \mathbf{V}_i}. \quad (3.21)$$

(ii) If $\mathbf{A} = (\mathbf{A}_{l,m})_{l,m=1}^r \in \mathbb{R}_+^{p \times p}$, where the $\mathbf{A}_{l,m}$ -s are suitably sized blocks, then for every $j \in \{1, 2, \dots, r\}$, we have

$$\mathbf{m}_\xi^{j-1}\mathbf{A} = (\tilde{\mathbf{m}}_{l,l+j-1}\mathbf{A}_{l+j-1,m})_{l,m=1}^r.$$

From this, we get

$$\sum_{j=1}^r \mathbf{m}_\xi^{j-1}\mathbf{A} = \left(\sum_{j=1}^r \tilde{\mathbf{m}}_{l,l+j-1}\mathbf{A}_{l+j-1,m} \right)_{l,m=1}^r. \quad (3.22)$$

With the choice

$$\mathbf{A} = \sqrt{\left[\mathbf{m}_\xi^{r-i}\mathbf{\Pi} \left(\mathcal{N}_t + t \sum_{j=1}^r \mathbf{m}_\xi^{j-1}\mathbf{m}_\varepsilon \right) \right]^+ \odot \mathbf{V}_\xi},$$

we see that

$$\mathbf{A} = \bigoplus_{i=r-j+2}^{2r-j+1} \sqrt{\left[\tilde{\mathbf{m}}_{i-r+j,i}\mathbf{\Pi}_i \left(\mathcal{N}_{t,i} + t \sum_{l=i}^{r+i-1} \tilde{\mathbf{m}}_{i,l}\mathbf{m}_{\varepsilon,l} \right) \right]^+ \odot \mathbf{V}_{i-r+j}}. \quad (3.23)$$

Combining (3.22) and (3.23), we can conclude that

$$d\mathcal{N}_{t,i} = \sum_{j=1}^r \tilde{\mathbf{m}}_{i,i+j} \sqrt{\left[\tilde{\mathbf{m}}_{i-r+j,i}\mathbf{\Pi}_i \left(\mathcal{N}_{t,i} + t \sum_{l=i}^{i+r-1} \tilde{\mathbf{m}}_{i,l}\mathbf{m}_{\varepsilon,l} \right) \right]^+ \odot \mathbf{V}_{i-r+j}} d\mathcal{W}_{t,j,i+j}.$$

Using $\mathbf{\Pi}_i = r\mathbf{u}_i\mathbf{v}_i^T$, with the application of (3.21) the previous equation can be written in the form

$$d\mathcal{N}_{t,i} = \sqrt{r \left[\mathbf{v}_i^T \left(\mathcal{N}_{t,i} + t \sum_{l=i}^{i+r-1} \tilde{\mathbf{m}}_{i,l}\mathbf{m}_{\varepsilon,l} \right) \right]^+} d\widetilde{\mathcal{W}}_{t,i},$$

where

$$\begin{aligned} \widetilde{\mathcal{W}}_{t,i} &:= \sum_{j=1}^r \tilde{\mathbf{m}}_{i,i+j-1} \sqrt{(\tilde{\mathbf{m}}_{i-r+j,i}\mathbf{u}_i) \odot \mathbf{V}_{i-r+j}} \mathcal{W}_{t,j,i+j} \\ &\quad \sqrt{\sum_{j=1}^r \tilde{\mathbf{m}}_{i,i+j-1} [(\tilde{\mathbf{m}}_{i-r+j,i}\mathbf{u}_i) \odot \mathbf{V}_{i-r+j}] \tilde{\mathbf{m}}_{i,i+j-1}^T} \widetilde{\mathcal{W}}_{t,i}, \end{aligned}$$

where $(\widetilde{\mathcal{W}}_{t,i})_{t \in \mathbb{R}_+}$ are d_i -dimensional independent standard Wiener processes. This is what we had to prove. $\square$

Now let us define the process $(\mathcal{P}_{t,i}, \mathcal{Q}_{t,i})_{t \in \mathbb{R}_+}$ for every $i \in \{1, 2, \dots, r\}$ with

$$\mathcal{P}_{t,i} := r\mathbf{v}_i^T \left(\mathcal{N}_{t,i} + t \sum_{l=i}^{i+r-1} \tilde{\mathbf{m}}_{i,l}\mathbf{m}_{\varepsilon,l} \right), \quad \mathcal{Q}_{t,i} := \mathcal{N}_{t,i} - \mathcal{P}_{t,i}\mathbf{u}_i.$$

With the application of Itô's lemma, we see that this process is a strong solution of the SDE

$$\begin{cases} d\mathcal{P}_{t,i} &= r\mathbf{v}_i^T \sum_{l=1}^{i+r-1} \tilde{\mathbf{m}}_{i,l} \mathbf{m}_{\varepsilon,l} dt \\ &+ \sqrt{\mathcal{P}_{t,i}^+ r \mathbf{v}_i^T \sum_{j=1}^r \tilde{\mathbf{m}}_{i,i+j-1} [(\tilde{\mathbf{m}}_{i+j,i+r} \mathbf{u}_i) \odot \mathbf{V}_{i+j}] \tilde{\mathbf{m}}_{i,i+j-1}^T} d\tilde{\mathcal{W}}_{t,i} \\ d\mathcal{Q}_{t,i} &= -\mathbf{\Pi}_i \sum_{l=1}^{i+r-1} \tilde{\mathbf{m}}_{i,l} \mathbf{m}_{\varepsilon,l} dt \\ &+ \sqrt{\mathcal{P}_{t,i}^+ (\mathbf{I} - \mathbf{\Pi}_i) \sum_{j=1}^r \tilde{\mathbf{m}}_{i,i+j-1} [(\tilde{\mathbf{m}}_{i+j,i+r} \mathbf{u}_i) \odot \mathbf{V}_{i+j}] \tilde{\mathbf{m}}_{i,i+j-1}^T} d\tilde{\mathcal{W}}_{t,i} \end{cases} \quad (3.24)$$

with initial value $(\mathcal{P}_{0,i}, \mathcal{Q}_{0,i}) = (0, \mathbf{0})$. With the choice

$$b_i := r\mathbf{v}_i^T \sum_{l=i}^{i+r-1} \tilde{\mathbf{m}}_{i,l} \mathbf{m}_{\varepsilon,l} \in \mathbb{R}_+,$$

$$\tilde{\mathcal{W}}_{t,i} := r \sqrt{\mathbf{v}_i^T \sum_{j=1}^r \tilde{\mathbf{m}}_{i,i+j-1} [(\tilde{\mathbf{m}}_{i+j,i+r} \mathbf{u}_i) \odot \mathbf{V}_{i+j}] \tilde{\mathbf{m}}_{i,i+j-1}^T} \tilde{\mathcal{W}}_{t,i} \in \mathbb{R}_+,$$

we can write the first equation in the form

$$d\mathcal{P}_{t,i} = b_i dt + \sqrt{\mathcal{P}_{t,i}^+} d\tilde{\mathcal{W}}_{t,i}. \quad (3.25)$$

The SDE (3.25) is the so-called Cox-Ingersoll-Ross model, which is used extensively in mathematical finance for modelling interest rates. This equation has a strong solution according to [10], Chapter 5.2, Theorem 2.9. Moreover, from a theorem of Yamada and Watanabe (see [15], Theorem 1 for this), it follows that this solution is also unique. It is also worth to note that applying a comparison theorem due to Ikeda and Watanabe (see [7], Theorem 1.1) yields that if a solution of (3.25) has positive initial value, it is positive everywhere. This is not unexpected, since our Galton-Watson process is positive. The second equation in (3.24) does not depend on $\mathcal{Q}_{t,i}$ and can be solved if $\mathcal{P}_{t,i}$ is known. Therefore, because $\mathcal{N}_{t,i} = \mathcal{Q}_{t,i} + \mathcal{P}_{t,i} \mathbf{u}_i$, we have proven that the strong solution of (3.17) exists.

Now we wish to apply Corollary 2.6 with $\mathbf{u} = \mathcal{M}$ and $\mathbf{u}^{(n)} = \mathcal{M}^{(n)}$ with filtration $(\mathcal{F}_{rk})_{k \in \mathbb{N}}$. For this, we have to check conditions (i) and (ii). It is easy to see that $\mathbb{E}[\mathcal{M}_t^{(n)}] = \mathbf{0}$ and $\mathcal{M}_t^{(n)}$ is a martingale with respect to $(\mathcal{F}_{rk})_{k \in \mathbb{N}}$. For checking condition (i), first we prove two lemmas.

3.6. Lemma. *For all $k \in \mathbb{N}$, we have*

$$\mathbb{E}[\mathbf{M}_k (\mathbf{M}_k)^T | \mathcal{F}_{rk-r}] = \bigoplus_{l=1}^r \left\{ \left[\mathbf{m}_{\xi}^{r-l} \mathbf{X}_{rk-r} + \sum_{j=1}^{r-l} \mathbf{m}_{\xi}^{j-1} \mathbf{m}_{\varepsilon} \right] \odot \mathbf{V}_{\xi} + \mathbf{V}_{\varepsilon} \right\}.$$

Consequently, we have

$$\frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}[\mathbf{M}_k (\mathbf{M}_k)^T | \mathcal{F}_{rk-r}] = \bigoplus_{l=1}^r \left\{ \frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} \left[\mathbf{m}_{\xi}^{r-l} \mathbf{X}_{rk-r} + \sum_{j=1}^{r-l} \mathbf{m}_{\xi}^{j-1} \mathbf{m}_{\varepsilon} \right] \odot \mathbf{V}_{\xi} + \frac{\lfloor nt \rfloor}{n^2} \mathbf{V}_{\varepsilon} \right\}.$$

Proof. From the definition of $\mathbf{M}_k$, it easily follows that

$$\mathbf{M}_{k,l} = \mathbf{X}_{rk-l+1} - \mathbf{m}_{\xi} \mathbf{X}_{rk-l} - \mathbf{m}_{\varepsilon} = \left(\varepsilon_{rk-l+1} - \mathbb{E}[\varepsilon_{rk-l+1}] \right) + \sum_{i=1}^p \sum_{j=1}^{X_{rk-l,i}} \left(\xi_{rk-l,j,i} - \mathbb{E}[\xi_{rk-l,j,i}] \right). \quad (3.26)$$

Since, by definition, the random variables $\{\xi_{rk-l+1,j,i}, \varepsilon_k : k, j \in \mathbb{N}, i \in \{1, 2, \dots, p\}\}$ are independent from each other and from $\mathcal{F}_{rk-l}$, we have

$$\mathbb{E}[\mathbf{M}_{k,i} (\mathbf{M}_{k,i})^T | \mathcal{F}_{rk-l}] = \sum_{i=1}^p \mathbf{X}_{rk-l,i} \mathbf{V}_{\xi_i} + \mathbf{V}_{\varepsilon} = (\mathbf{X}_{rk-l} \odot \mathbf{V}_{\xi}) + \mathbf{V}_{\varepsilon}$$

With the repeated application of the equality (3.11) and the tower rule, we have

$$\begin{aligned}\mathbb{E}[\mathbf{M}_{k,i}(\mathbf{M}_{k,i})^T|\mathcal{F}_{rk-r}] &= \left(\mathbb{E}[\mathbf{X}_{rk-l}|\mathcal{F}_{rk-r}] \odot \mathbf{V}_\xi\right) + \mathbf{V}_\varepsilon \\ &= \left[\mathbf{m}_\xi^{r-l}\mathbf{X}_{rk-r} + \sum_{j=1}^{r-l}\mathbf{m}_\xi^{j-1}\mathbf{m}_\varepsilon\right] \odot \mathbf{V}_\xi + \mathbf{V}_\varepsilon.\end{aligned}$$

For $i \neq j$, we have

$$\mathbb{E}[\mathbf{M}_{k,i}(\mathbf{M}_{k,j})^T|\mathcal{F}_{rk-r}] = \mathbb{E}[\mathbf{M}_{k,i}(\underbrace{\mathbb{E}[\mathbf{M}_{k,j}|\mathcal{F}_{rk-j}]}_{=\mathbf{0}})^T|\mathcal{F}_{rk-r}] = \mathbf{0}.$$

Combining these two observations, we can conclude our proof. $\square$

3.7. Lemma. *Regarding the order of the absolute moments of the processes $(\mathbf{X}_k)_{k \in \mathbb{N}}$ and $(\mathbf{M}_k)_{k \in \mathbb{N}}$, we have*

$$\mathbb{E}[\|\mathbf{X}_k\|] = O(k) \quad (3.27)$$

$$\mathbb{E}[\|\mathbf{X}_k\|^2] = O(k^2) \quad (3.28)$$

$$\mathbb{E}[\|\mathbf{M}_k\|] = O(k^{1/2}) \quad (3.29)$$

$$\mathbb{E}[\|\mathbf{M}_k\|^4] = O(k^2) \quad (3.30)$$

Proof. From (3.11), we see that $\mathbb{E}[\mathbf{X}_k] = \mathbf{m}_\xi \mathbb{E}[\mathbf{X}_{k-1}] + \mathbf{m}_\varepsilon$. With the repeated application of this identity and the fact that $\mathbb{E}[\mathbf{X}_0] = \mathbf{0}$ we get $\mathbb{E}[\mathbf{X}_k] = \sum_{j=0}^{k-1} \mathbf{m}_\xi^j \mathbf{m}_\varepsilon$. Using this, we have

$$\mathbb{E}[\|\mathbf{X}_k\|] \leq \|\mathbf{m}_\varepsilon\| \sum_{j=0}^{k-1} \|\mathbf{m}_\xi^j\|$$

Because the $\|\mathbf{m}_\xi^j\|$ norms are bounded (see (3.6)), we get $\mathbb{E}[\|\mathbf{X}_k\|] \leq kc' = O(k)$.

The proof of (3.28) can be found in [6]. We proceed to prove (3.30). It is enough to show that for every $l \in \{1, 2, \dots, r\}$, $\mathbb{E}[\|\mathbf{M}_{k,l}\|^4] = O(k^2)$. From (3.26), it is clear that

$$\|\mathbf{M}_{k,l}\| \leq \left\| \varepsilon_{rk-l+1} - \mathbb{E}[\varepsilon_{rk-l+1}] \right\| + \sum_{i=1}^p \left\| \sum_{j=1}^{X_{rk-l,i}} \left(\xi_{rk-l,j,i} - \mathbb{E}[\xi_{rk-l,j,i}] \right) \right\|,$$

so by using the generalized power mean inequality

$$\left(\sum_{k=1}^p \frac{1}{p} a_k^\alpha \right)^{1/\alpha} \leq \left(\sum_{k=1}^p \frac{1}{p} a_k^\beta \right)^{1/\beta} \quad (0 < \alpha < \beta)$$

with $\alpha = 1, \beta = 4$, we get

$$\begin{aligned}\mathbb{E}[\|\mathbf{M}_{k,l}\|^4] &\leq (p+1)^3 \mathbb{E}[\|\varepsilon_{rk-l+1} - \mathbb{E}[\varepsilon_{rk-l+1}]\|^4] + \\ &\quad (p+1)^3 \sum_{i=1}^p \mathbb{E} \left[\left\| \sum_{j=1}^{X_{rk-l,i}} \xi_{rk-l,j,i} - \mathbb{E}[\xi_{rk-l,j,i}] \right\|^4 \right].\end{aligned}$$

It is enough to show that the last part is $O(k^2)$. For that, we have

$$\begin{aligned}\mathbb{E} \left[\left\| \sum_{j=1}^{X_{rk-l,i}} \xi_{rk-l,j,i} - \mathbb{E}[\xi_{rk-l,j,i}] \right\|^4 \right] &= \mathbb{E} \left[\left(\sum_{m=1}^p \left(\sum_{j=1}^{X_{rk-l,i}} \xi_{rk-l,j,i,m} - \mathbb{E}[\xi_{rk-l,j,i,m}] \right)^2 \right)^2 \right] \\ &\leq p \sum_{m=1}^p \mathbb{E} \left[\left(\sum_{j=1}^{X_{rk-l,i}} \xi_{rk-l,j,i,m} - \mathbb{E}[\xi_{rk-l,j,i,m}] \right)^4 \right].\end{aligned}$$

Since

$$\begin{aligned}
& \mathbb{E} \left[\left(\sum_{j=1}^{X_{rk-l,i}} \xi_{rk-l,j,i,m} - \mathbb{E}[\xi_{rk-l,j,i,m}] \right)^4 \middle| \mathcal{F}_{kr-l} \right] \\
&= X_{rk-l,i} \mathbb{E}[(\xi_{1,1,i,m} - \mathbb{E}[\xi_{1,1,i,m}])^4] + X_{rk-l,i}(X_{rk-l,i} - 1) \left(\mathbb{E}[(\xi_{1,1,i,m} - \mathbb{E}[\xi_{1,1,i,m}])^2]^2 \right) \\
&\leq (X_{rk-l,i})^2 \mathbb{E}[(\xi_{1,1,i,m} - \mathbb{E}[\xi_{1,1,i,m}])^4],
\end{aligned}$$

we have

$$\mathbb{E} \left[\left(\sum_{j=1}^{X_{rk-l,i}} \xi_{rk-l,j,i,m} - \mathbb{E}[\xi_{rk-l,j,i,m}] \right)^4 \right] \leq \mathbb{E}[(\xi_{1,1,i,m} - \mathbb{E}[\xi_{1,1,i,m}])^4] \mathbb{E}[(X_{rk-l,i})^2].$$

Because $\mathbb{E}[\|\mathbf{X}_k\|^2] = O(k^2)$, it follows easily that $\mathbb{E}[\|\mathbf{M}_k\|^4] = O(k^2)$ and thus we have concluded our proof. $\square$

For every $n \in \mathbb{N}$, let us define the process $(\mathcal{R}_t^{(n)})_{t \in \mathbb{R}_+}$ with

$$\mathcal{R}_t^{(n)} = \bigoplus_{l=1}^r \left\{ \left[\mathbf{m}_\xi^{r-l} \Pi \sum_{j=1}^r \mathbf{m}_\xi^{j-1} (\mathcal{M}_{t,j}^{(n)} + t \mathbf{m}_\varepsilon) \right] \odot \mathbf{V}_\xi \right\}.$$

Using (3.3), (3.4), (3.5)

$$\begin{aligned}
\Pi \sum_{j=1}^r \mathbf{m}_\xi^{j-1} (\mathcal{M}_{t,j}^{(n)} + t \mathbf{m}_\varepsilon) &= \Pi \sum_{j=1}^r \mathbf{m}_\xi^{j-1} \left(\frac{1}{n} \sum_{k=1}^{\lfloor nt \rfloor} (\mathbf{X}_{rk-j+1} - \mathbf{m}_\xi \mathbf{X}_{rk-j} - \mathbf{m}_\varepsilon) + t \mathbf{m}_\varepsilon \right) \\
&= \frac{1}{n} \sum_{k=1}^{\lfloor nt \rfloor} \Pi \left(\sum_{j=1}^r (\mathbf{m}_\xi^{j-1} \mathbf{X}_{rk-j+1} - \mathbf{m}_\xi^j \mathbf{X}_{rk-j}) - \sum_{j=1}^r \mathbf{m}_\xi^{j-1} \mathbf{m}_\varepsilon \right) \\
&\quad + t \Pi \sum_{j=1}^r \mathbf{m}_\xi^{j-1} \mathbf{m}_\varepsilon \\
&= \frac{1}{n} \sum_{k=1}^{\lfloor nt \rfloor} \Pi \left(\mathbf{X}_{rk} - \mathbf{m}_\xi^r \mathbf{X}_{rk-r} - \sum_{j=1}^r \mathbf{m}_\xi^{j-1} \mathbf{m}_\varepsilon \right) + t \Pi \sum_{j=1}^r \mathbf{m}_\xi^{j-1} \mathbf{m}_\varepsilon \\
&= \frac{1}{n} \Pi \mathbf{X}_{r \lfloor nt \rfloor} + \left(t - \frac{\lfloor nt \rfloor}{n} \right) \Pi \sum_{j=1}^r \mathbf{m}_\xi^{j-1} \mathbf{m}_\varepsilon.
\end{aligned}$$

From this, we get

$$\mathcal{R}_t^{(n)} = \bigoplus_{l=1}^r \left\{ \left[\frac{1}{n} \mathbf{m}_\xi^{r-l} \Pi \mathbf{X}_{r \lfloor nt \rfloor} + \left(t - \frac{\lfloor nt \rfloor}{n} \right) \Pi \sum_{j=1}^r \mathbf{m}_\xi^{j-1} \mathbf{m}_\varepsilon \right] \odot \mathbf{V}_\xi \right\}.$$

It immediately follows that $(\mathcal{R}_t^{(n)})^+ = \mathcal{R}_t^{(n)}$. Since

$$\int_0^t \mathbf{X}_{r \lfloor ns \rfloor} ds = \frac{1}{n} \int_0^{nt} \mathbf{X}_{r \lfloor u \rfloor} du = \frac{1}{n} \left(\mathbf{X}_{r \lfloor nt \rfloor} (nt - \lfloor nt \rfloor) + \sum_{k=0}^{\lfloor nt \rfloor} \mathbf{X}_{rk} \right),$$

and

$$\int_0^t \left(s - \frac{\lfloor ns \rfloor}{n} \right) ds = \frac{t^2}{2} - \frac{1}{n^2} \left(\lfloor nt \rfloor (nt - \lfloor nt \rfloor) + \sum_{k=0}^{\lfloor nt \rfloor - 1} k \right) = \frac{\lfloor nt \rfloor + (nt - \lfloor nt \rfloor)^2}{2n^2},$$

we have

$$\int_0^t (\mathcal{R}_s^{(n)})^+ ds = \int_0^t \mathcal{R}_s^{(n)} ds = \bigoplus_{l=1}^r \left\{ \left[\frac{1}{n^2} \mathbf{m}_\xi^{r-l} \mathbf{\Pi} \sum_{k=0}^{\lfloor nt \rfloor - 1} \mathbf{X}_{rk} + \frac{nt - \lfloor nt \rfloor}{n^2} \mathbf{m}_\xi^{r-l} \mathbf{\Pi} \mathbf{X}_{r\lfloor nt \rfloor} + \frac{\lfloor nt \rfloor + (nt - \lfloor nt \rfloor)^2}{2n^2} \mathbf{\Pi} \sum_{j=1}^r \mathbf{m}_\xi^{j-1} \mathbf{m}_\varepsilon \right] \odot \mathbf{V}_\xi \right\}.$$

Now we proceed to check condition (i), i.e. for every $T \in \mathbb{R}_+$,

$$\sup_{0 \leq t \leq T} \left\| \frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}[\mathbf{M}_k (\mathbf{M}_k)^T | \mathcal{F}_{rk-r}] - \int_0^t (\mathcal{R}_s^{(n)})^+ ds \right\| \xrightarrow{P} 0 \quad (n \rightarrow \infty).$$

We see it immediately that for any $l \in \{1, 2, \dots, r\}$,

$$\begin{aligned} \sup_{0 \leq t \leq T} \frac{\lfloor nt \rfloor}{n^2} \|\mathbf{V}_\varepsilon\| &\xrightarrow{P} 0, \quad \sup_{0 \leq t \leq T} \frac{\lfloor nt \rfloor}{n^2} \left\| \sum_{j=1}^{r-l} \mathbf{m}_\xi^{j-1} \mathbf{m}_\varepsilon \right\| \xrightarrow{P} 0, \\ \sup_{0 \leq t \leq T} \frac{\lfloor nt \rfloor + (nt - \lfloor nt \rfloor)^2}{n^2} \left\| \mathbf{\Pi} \sum_{j=1}^{r-l} \mathbf{m}_\xi^{j-1} \mathbf{m}_\varepsilon \right\| &\xrightarrow{P} 0, \end{aligned}$$

as $n \rightarrow \infty$. It remains to show that for any $l \in \{1, 2, \dots, r\}$,

$$\sup_{0 \leq t \leq T} \frac{1}{n^2} \left\| \mathbf{m}_\xi^{r-l} \sum_{k=1}^{\lfloor nt \rfloor} \mathbf{X}_{rk-r} - \mathbf{m}_\xi^{r-l} \mathbf{\Pi} \sum_{k=1}^{\lfloor nt \rfloor} \mathbf{X}_{rk} \right\| \xrightarrow{P} 0 \quad (3.31)$$

and

$$\sup_{0 \leq t \leq T} \left\| \frac{nt - \lfloor nt \rfloor}{n^2} \mathbf{m}_\xi^{r-l} \mathbf{\Pi} \mathbf{X}_{r\lfloor nt \rfloor} \right\| \xrightarrow{P} 0 \quad (3.32)$$

as $n \rightarrow \infty$.

3.8. Lemma. *For any fixed $T > 0$, we have*

$$\sup_{0 \leq t \leq T} \frac{1}{n^2} \sum_{k=0}^{\lfloor nt \rfloor} \|(\mathbf{I} - \mathbf{\Pi}) \mathbf{X}_{rk}\| \xrightarrow{P} 0 \quad (3.33)$$

and

$$\sup_{0 \leq t \leq T} \frac{1}{n^2} \|\mathbf{X}_{r\lfloor nt \rfloor}\| \xrightarrow{P} 0, \quad (3.34)$$

where $\mathbf{I} \in \mathbb{R}^{p \times p}$ is the unit matrix.

Proof. We note immediately that the supremum in (3.33) is attained at $t = T$. Using (3.13), we see that

$$\mathbf{X}_{rk} = \mathbf{m}_\xi^r \mathbf{X}_{rk-r} + \sum_{j=1}^r \mathbf{m}_\xi^{j-1} (\mathbf{M}_{k,j} + \mathbf{m}_\varepsilon).$$

Applying this recursion for $(\mathbf{X}_{rk})_{k \in \mathbb{N}}$, we get

$$\mathbf{X}_{rk} = \sum_{i=1}^k \mathbf{m}_\xi^{(k-i)r} \sum_{j=1}^r \mathbf{m}_\xi^{j-1} (\mathbf{M}_{i,j} + \mathbf{m}_\varepsilon).$$

With (3.4), we have

$$(\mathbf{I} - \mathbf{\Pi})\mathbf{X}_{rk} = \sum_{i=1}^k (\mathbf{m}_{\xi}^{(k-i)r} - \mathbf{\Pi}) \sum_{j=1}^r \mathbf{m}_{\xi}^{j-1} (\mathbf{M}_{i,j} + \mathbf{m}_{\varepsilon}).$$

Using this and (3.2) with the boundedness of the $\|\mathbf{m}_{\xi}^j\|$ norms, we get

$$\begin{aligned} \sum_{k=1}^{\lfloor nT \rfloor} \|(\mathbf{I} - \mathbf{\Pi})\mathbf{X}_{rk}\| &\leq \sum_{k=1}^{\lfloor nT \rfloor} \sum_{i=1}^k c\kappa^{k-i} \sum_{j=1}^r \|\mathbf{m}_{\xi}^{j-1}\| (\|\mathbf{M}_{i,j}\| + \|\mathbf{m}_{\varepsilon}\|) \\ &\leq cc' \left(\lfloor nT \rfloor \frac{r}{1-\kappa} + \sum_{i=1}^{\lfloor nT \rfloor} \sum_{j=1}^r \|\mathbf{M}_{i,j}\| \underbrace{\sum_{k=i}^{\lfloor nT \rfloor} \kappa^{k-j}}_{\leq \frac{1}{1-\kappa}} \right). \end{aligned}$$

Because of Lemma 3.7, we see that

$$\frac{1}{n^2} \sum_{j=1}^{\lfloor nT \rfloor} \|\mathbf{M}_{i,j}\| \xrightarrow{P} 0.$$

In fact, we have convergence in L^1 , but stochastic convergence is enough for us. Combining these observations, we note that we proved (3.33). From this, (3.34) immediately follows. $\square$

Applying our previous lemma, we see easily that (3.31) and (3.32) is true, so we can conclude that condition (i) holds.

Checking condition (ii) is simple in the light of Lemma 3.7. Applying basic inequalities, we get

$$\frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}[\|\mathbf{M}_k\|^2 \chi_{\{\|\mathbf{M}_k\| > n\theta\}} | \mathcal{F}_{k-1}] \leq \frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} \frac{\mathbb{E}[\|\mathbf{M}_k\|^4 | \mathcal{F}_{k-1}]}{n^2 \theta} = \frac{1}{n^4 \theta} \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}[\|\mathbf{M}_k\|^4 | \mathcal{F}_{k-1}].$$

Because of $\mathbb{E}[\|\mathbf{M}_k\|^4] = O(k^2)$, we have $\frac{1}{n^4 \theta} \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}[\|\mathbf{M}_k\|^4 | \mathcal{F}_{k-1}] \xrightarrow{P} 0$. Thus we have concluded the proof of Theorem 3.4. $\square$

Now we can prove Theorem 3.2, our main theorem. To deduce this from Theorem 3.4, we wish to apply the theorem of continuous mappings, which we state below. (For proof, see [9], Theorem 3.27.)

3.9. Theorem. *Let S and T be two metric spaces and let $\xi, \xi_1, \xi_2, \dots$ be random variables in S with $\xi_n \xrightarrow{\mathcal{D}} \xi$. Let $f, f_1, f_2, \dots : S \rightarrow T$ be measurable mapping with a measurable set $C \subset S$ with $P(\xi \in C) = 1$ such that $f_n(s_n) \rightarrow f(s)$ if $s_n \rightarrow s$. Then $f_n(\xi_n) \xrightarrow{\mathcal{D}} f(\xi)$.*

Now we define the mappings $\Psi, (\Psi^{(n)})_{n \in \mathbb{N}}$ where

$$\Psi^{(n)} = \begin{bmatrix} \Psi_1^{(n)} \\ \Psi_2^{(n)} \\ \vdots \\ \Psi_r^{(n)} \end{bmatrix} : \mathcal{D}(\mathbb{R}_+, (\mathbb{R}^p)^r) \rightarrow \mathcal{D}(\mathbb{R}_+, (\mathbb{R}^p)^r) \quad (3.35)$$

is defined by

$$\begin{aligned}\Psi_i^{(n)}(f)(t) := & \mathbf{m}_\xi^{\lfloor nt \rfloor r} \left[\sum_{l=i}^r \mathbf{m}_\xi^{l-i} f_l(0) + \sum_{l=1}^{i-1} \mathbf{m}_\xi^{l-i+r} f_l(0) \right] \\ & + \sum_{j=1}^{\lfloor nt \rfloor} \mathbf{m}_\xi^{(\lfloor nt \rfloor - j)r} \left[\sum_{l=i}^r \mathbf{m}_\xi^{l-i} \left(f_l\left(\frac{j}{n}\right) - f_l\left(\frac{j-1}{n}\right) + \frac{1}{n} \mathbf{m}_\varepsilon \right) \right. \\ & \left. + \sum_{l=1}^{i-1} \mathbf{m}_\xi^{l-i+r} \left(f_l\left(\frac{j-1}{n}\right) - f_l\left(\frac{j-2}{n}\right) + \frac{1}{n} \mathbf{m}_\varepsilon \right) \right]\end{aligned}$$

for every

$$f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_r \end{bmatrix} \in \mathcal{D}(\mathbb{R}_+, (\mathbb{R}^p)^r), \quad t \in \mathbb{R}_+, \quad n \in \mathbb{N}, \quad i \in \{1, 2, \dots, r\}.$$

The mapping Ψ is defined by

$$\Psi(f)(t) := \begin{bmatrix} \Pi \sum_{l=1}^r \mathbf{m}_\xi^{l-1} (f_l(t) + t \mathbf{m}_\varepsilon) \\ \Pi \sum_{l=1}^r \mathbf{m}_\xi^{l+r-2} (f_l(t) + t \mathbf{m}_\varepsilon) \\ \vdots \\ \Pi \sum_{l=1}^r \mathbf{m}_\xi^l (f_l(t) + t \mathbf{m}_\varepsilon) \end{bmatrix}, \quad f \in \mathcal{D}(\mathbb{R}_+, (\mathbb{R}^p)^r), \quad t \in \mathbb{R}_+. \quad (3.36)$$

Using (3.14), one can easily check that $\Psi^{(n)}(\mathcal{M}^{(n)}) = \mathcal{X}^{(n)}$. From (3.14) and (3.16) we have $\Psi(\mathcal{M}) \stackrel{\mathcal{D}}{=} \mathcal{X}$. The mappings $\Psi, (\Psi^{(n)})_{n \in \mathbb{N}}$ are measurable, as it can be seen in Lemma 5.1 at the Appendix. Now we define the sets $C, C_{\Psi, (\Psi^{(n)})_{n \in \mathbb{N}}}$ with

$$C_{\Psi, (\Psi^{(n)})_{n \in \mathbb{N}}} := \{f \in \mathcal{D}(\mathbb{R}_+, (\mathbb{R}^p)^r) : \Psi_n(f_n) \xrightarrow{lu} \Psi(f) \text{ whenever } f_n \xrightarrow{lu} f\},$$

$$C := \left\{ f \in \mathcal{C}(\mathbb{R}_+, (\mathbb{R}^p)^r) : (\mathbf{I} - \Pi) \left[\sum_{l=i}^r \mathbf{m}_\xi^{l-i} f_l(0) + \sum_{l=1}^{i-1} \mathbf{m}_\xi^{l-i+r} f_l(0) \right] = \mathbf{0}, i \in \{1, 2, \dots, r\} \right\},$$

where $f_n \xrightarrow{lu} f$ denotes the locally uniform convergence. The C set is measurable, as it can be seen in Lemma 5.2 at the Appendix. We must show that $C \in \mathcal{D}_\infty(\mathbb{R}_+, (\mathbb{R}^p)^r)$, $C \subset C_{\Psi, (\Psi^{(n)})_{n \in \mathbb{N}}}$ and $P(\mathcal{M} \in C) = 1$.

3.10. Lemma. $C \subset C_{\Psi, (\Psi^{(n)})_{n \in \mathbb{N}}}$.

Proof. Let $f \in C$, and let $(f^{(n)})_{n \in \mathbb{N}}$ a sequence in $\mathcal{D}_\infty(\mathbb{R}_+, (\mathbb{R}^p)^r)$ such that $f^{(n)} \xrightarrow{lu} f$. We will

show that $\psi_n(f^{(n)}) \xrightarrow{lo} \psi(f)$. Adding and subtracting the same member, we can write

$$\begin{aligned}\Psi_i^{(n)}(f^{(n)})(t) &= \mathbf{\Pi} \sum_{l=i}^r \mathbf{m}_\xi^{l-i} \left[f_l^{(n)} \left(\frac{\lfloor nt \rfloor}{n} \right) + \frac{\lfloor nt \rfloor}{n} \mathbf{m}_\varepsilon \right] \\ &\quad + \mathbf{\Pi} \sum_{l=1}^{i-1} \mathbf{m}_\xi^{l-i+r} \left[f_l^{(n)} \left(\frac{\lfloor nt \rfloor - 1}{n} \right) + \frac{\lfloor nt \rfloor}{n} \mathbf{m}_\varepsilon \right] \\ &\quad + \left(\mathbf{m}_\xi^{\lfloor nt \rfloor r} - \mathbf{\Pi} \right) \left[\sum_{l=i}^r \mathbf{m}_\xi^{l-i} f_l^{(n)}(0) + \sum_{l=1}^{i-1} \mathbf{m}_\xi^{l-i+r} f_l(0) \right] \\ &\quad + \sum_{j=1}^{\lfloor nt \rfloor} \left(\mathbf{m}_\xi^{(\lfloor nt \rfloor - j)r} - \mathbf{\Pi} \right) \left[\sum_{l=i}^r \mathbf{m}_\xi^{l-i} \left(f_l^{(n)} \left(\frac{j}{n} \right) - f_l^{(n)} \left(\frac{j-1}{n} \right) + \frac{1}{n} \mathbf{m}_\varepsilon \right) \right. \\ &\quad \left. + \sum_{l=1}^{i-1} \mathbf{m}_\xi^{l-i+r} \left(f_l^{(n)} \left(\frac{j-1}{n} \right) - f_l^{(n)} \left(\frac{j-2}{n} \right) + \frac{1}{n} \mathbf{m}_\varepsilon \right) \right].\end{aligned}$$

Now let $T > 0$ fixed and let $t \in [0, T]$. We will prove that $\sup_{t \in [0, T]} \|\Psi_i^{(n)}(f^{(n)})(t) - \Psi_i(f)(t)\| \rightarrow 0$.

$$\begin{aligned}\|\Psi_i^{(n)}(f^{(n)})(t) - \Psi_i(f)(t)\| &\leq c' \|\mathbf{\Pi}\| \sum_{l=1}^{r-i+1} \left(\left\| f_l^{(n)} \left(\frac{\lfloor nt \rfloor}{n} \right) - f_l(t) \right\| + \frac{1}{n} \|\mathbf{m}_\varepsilon\| \right) \\ &\quad + c' \|\mathbf{\Pi}\| \sum_{l=1}^{r-i+1} \left(\left\| f_l^{(n)} \left(\frac{\lfloor nt \rfloor - 1}{n} \right) - f_l(t) \right\| + \frac{1}{n} \|\mathbf{m}_\varepsilon\| \right) \\ &\quad + \left\| \left(\mathbf{m}_\xi^{\lfloor nt \rfloor r} - \mathbf{\Pi} \right) \left[\sum_{l=i}^r \mathbf{m}_\xi^{l-i} f_l^{(n)}(0) + \sum_{l=1}^{i-1} \mathbf{m}_\xi^{l-i+r} f_l^{(n)}(0) \right] \right\| \\ &\quad + c' \sum_{j=1}^{\lfloor nt \rfloor} \left\| \mathbf{m}_\xi^{(\lfloor nt \rfloor - j)r} - \mathbf{\Pi} \right\| \left[\left(\sum_{l=i}^r \left\| f_l^{(n)} \left(\frac{j}{n} \right) - f_l^{(n)} \left(\frac{j-1}{n} \right) \right\| + \frac{1}{n} \|\mathbf{m}_\varepsilon\| \right) \right. \\ &\quad \left. + \left(\sum_{l=1}^{i-1} \left\| f_l^{(n)} \left(\frac{j}{n} \right) - f_l^{(n)} \left(\frac{j-1}{n} \right) \right\| + \frac{1}{n} \|\mathbf{m}_\varepsilon\| \right) \right]\end{aligned}$$

With the definition $\omega_T(f, I) := \sup_{t_1, t_2 \in I} \|f(t_1) - f(t_2)\|$, we have

$$\begin{aligned}\left\| f^{(n)} \left(\frac{\lfloor nt \rfloor}{n} \right) - f(t) \right\| &\leq \left\| f^{(n)} \left(\frac{\lfloor nt \rfloor}{n} \right) - f \left(\frac{\lfloor nt \rfloor}{n} \right) \right\| + \left\| f \left(\frac{\lfloor nt \rfloor}{n} \right) - f(t) \right\| \\ &\leq \omega_T(f, [t, t + n^{-1}]) + \sup_{t \in [0, T]} \|f^{(n)}(t) - f(t)\|.\end{aligned}$$

Since f is continuous, it is known that $\omega_T(f, [t, t + n^{-1}]) \rightarrow 0$ as $n \rightarrow \infty$. Similarly,

$$\left\| f_l^{(n)} \left(\frac{j}{n} \right) - f_l^{(n)} \left(\frac{j-1}{n} \right) \right\| \leq \omega_T(f, [(j-1)n^{-1}, (j-1)n^{-1} + n^{-1}]) + 2 \sup_{t \in [0, T]} \|f^{(n)}(t) - f(t)\|.$$

With (3.2), we get that

$$c' \sum_{j=1}^{\lfloor nt \rfloor} \|\mathbf{m}_\xi^{(\lfloor nt \rfloor - j)r} - \mathbf{\Pi}\| \leq \frac{c'}{1 - \kappa}.$$

Finally, we have

$$\begin{aligned}
& \left\| \left(\mathbf{m}_\xi^{\lfloor nt \rfloor r} - \Pi \right) \left[\sum_{l=i}^r \mathbf{m}_\xi^{l-i} f_l^{(n)}(0) + \sum_{l=1}^{i-1} \mathbf{m}_\xi^{l-i+r} f_l^{(n)}(0) \right] \right\| \\
& \leq \left\| \left(\mathbf{m}_\xi^{\lfloor nt \rfloor r} - \Pi \right) \left[\sum_{l=i}^r \mathbf{m}_\xi^{l-i} (f_l^{(n)}(0) - f_l(0)) + \sum_{l=1}^{i-1} \mathbf{m}_\xi^{l-i+r} (f_l^{(n)}(0) - f_l(0)) \right] \right\| \\
& \quad + \underbrace{\left\| \left(\mathbf{m}_\xi^{\lfloor nt \rfloor r} - \Pi \right) \left[\sum_{l=i}^r \mathbf{m}_\xi^{l-i} f_l(0) + \sum_{l=1}^{i-1} \mathbf{m}_\xi^{l-i+r} f_l(0) \right] \right\|}_{=0, \text{ since } f \in C} \\
& \leq cc' \sum_{l=1}^r \|f_l^{(n)}(0) - f_l(0)\|.
\end{aligned}$$

Combining these, we have that

$$\begin{aligned}
& \|\Psi_i^{(n)}(f^{(n)})(t) - \Psi_i(f)(t)\| \\
& \leq c' \|\Pi\| (r-i+1) \left(\omega_T(f, [t, t+n^{-1}]) + \sup_{t \in [0, T]} \|f^{(n)}(t) - f(t)\| + \frac{1}{n} \mathbf{m}_\varepsilon \right) \\
& \quad + c' \|\Pi\| (r-i+1) \left(\omega_T(f, [t-n^{-1}, t]) + \sup_{t \in [0, T]} \|f^{(n)}(t) - f(t)\| + \frac{1}{n} \mathbf{m}_\varepsilon \right) \\
& \quad + cc' \sum_{l=1}^r \|f_l^{(n)}(0) - f_l(0)\| \\
& \quad + \frac{c'}{1-\kappa} r \left(\omega_T(f, [(j-1)n^{-1}, (j-1)n^{-1} + n^{-1}]) + 2 \sup_{t \in [0, T]} \|f^{(n)}(t) - f(t)\| + \frac{2}{n} \mathbf{m}_\varepsilon \right).
\end{aligned}$$

Because the right side does not depend on t , with using $f^{(n)} \xrightarrow{lu} f$ we can conclude that $\psi_n(f^{(n)}) \xrightarrow{lu} \psi(f)$. Thus we have proved our lemma. $\square$

Now, because $\mathbf{M}$ is the unique strong solution of the SDE (3.17) with initial value $\mathbf{M}_0 = \mathbf{0}$, it is trivial that $P(\mathbf{M} \in C) = 1$. Applying Theorem 3.9 yields that

$$\mathcal{X}^{(n)} = \Psi^{(n)}(\mathcal{M}^{(n)}) \xrightarrow{\mathcal{D}} \Psi(\mathcal{X}) \stackrel{\mathcal{D}}{=} \mathcal{X}.$$

Now we prove the distributional equality $(\mathcal{Y}_t)_{t \in \mathbb{R}_+} \stackrel{\mathcal{D}}{=} (\mathbf{m}_\xi^j \mathcal{Y}_t)_{t \in \mathbb{R}_+}$. From (3.14), (3.16) and (3.18), we know that

$$\mathcal{X}_t = \begin{bmatrix} \mathcal{Y}_t \\ \mathbf{m}_\xi^{r-1} \mathcal{Y}_t \\ \vdots \\ \mathbf{m}_\xi \mathcal{Y}_t \end{bmatrix} = \begin{bmatrix} \Pi \sum_{l=1}^r \mathbf{m}_\xi^{l-1} (\mathcal{M}_{t,l} + t \mathbf{m}_\varepsilon) \\ \Pi \sum_{l=1}^r \mathbf{m}_\xi^{l+r-2} (\mathcal{M}_{t,l} + t \mathbf{m}_\varepsilon) \\ \vdots \\ \Pi \sum_{l=1}^r \mathbf{m}_\xi^l (\mathcal{M}_{t,l} + t \mathbf{m}_\varepsilon) \end{bmatrix} = \begin{bmatrix} \Pi (\mathcal{N}_t + t \sum_{l=1}^r \mathbf{m}_\xi^{l-1} \mathbf{m}_\varepsilon) \\ \mathbf{m}_\xi^{r-1} \Pi (\mathcal{N}_t + t \sum_{l=1}^r \mathbf{m}_\xi^{l-1} \mathbf{m}_\varepsilon) \\ \vdots \\ \mathbf{m}_\xi \Pi (\mathcal{N}_t + t \sum_{l=1}^r \mathbf{m}_\xi^{l-1} \mathbf{m}_\varepsilon) \end{bmatrix}.$$

From the SDE (3.19) of $\mathcal{N}_t$, we have

$$d\mathcal{Y}_t = \Pi \sum_{l=1}^r \mathbf{m}_\xi^{l-1} \mathbf{m}_\varepsilon dt + \Pi \sum_{j=1}^r \mathbf{m}_\xi^{j-1} \sqrt{[\mathbf{m}_\xi^{r-j} \mathcal{Y}_t] \odot \mathbf{V}_\xi} d\mathcal{W}_{t,i}.$$

With the transformation $\mathbf{u}_t = \mathbf{m}_\xi \mathbf{y}_t$ and with the application of the special properties (3.3), (3.4), (3.5) of Π , we have

$$\begin{aligned}
d\mathbf{u}_t &= \Pi \sum_{l=1}^r \mathbf{m}_\xi^l \mathbf{m}_\varepsilon dt + \mathbf{m}_\xi \Pi \sum_{j=1}^{r-1} \mathbf{m}_\xi^{j-1} \sqrt{[\mathbf{m}_\xi^{r-j-1} \mathbf{u}_t] \odot \mathbf{V}_\xi} d\mathbf{w}_{t,i} \\
&\quad + \mathbf{m}_\xi \Pi \mathbf{m}_\xi^{r-1} \sqrt{\mathbf{y}_t \odot \mathbf{V}_\xi} d\mathbf{w}_{t,j} \\
&= \Pi \sum_{l=1}^r \mathbf{m}_\xi^{l-1} \mathbf{m}_\varepsilon dt + \Pi \sum_{j=2}^r \mathbf{m}_\xi^{j-1} \sqrt{[\mathbf{m}_\xi^{r-j} \mathbf{u}_t] \odot \mathbf{V}_\xi} d\mathbf{w}_{t,j-1} \\
&\quad + \Pi \sqrt{[\mathbf{m}_\xi^{r-1} \mathbf{u}_t] \odot \mathbf{V}_\xi} d\mathbf{w}_{t,r} \\
&= \Pi \sum_{l=1}^r \mathbf{m}_\xi^{l-1} \mathbf{m}_\varepsilon dt + \Pi \sum_{j=0}^r \mathbf{m}_\xi^{j-1} \sqrt{[\mathbf{m}_\xi^{r-j} \mathbf{u}_t] \odot \mathbf{V}_\xi} d\mathbf{w}_{t,j-1}.
\end{aligned}$$

From this, it follows that $\mathbf{y}_t \stackrel{\mathcal{D}}{=} \mathbf{m}_\xi \mathbf{y}_t$. By induction, we have $\mathbf{y}_t \stackrel{\mathcal{D}}{=} \mathbf{m}_\xi^j \mathbf{y}_t$ for every $j \in \{1, 2, \dots, r-1\}$. This proves Theorem 3.2.

4 Conclusion

4.1 Applications

Multi-type branching processes are often used in biology to model population dynamics with genetic effects. For example, types can represent different genotypes which can interact with each other in various ways. When dealing with real experimental data, it is important to obtain some knowledge about the offspring expectation matrix $\mathbf{m}_\xi$, because it describes the underlying dynamics of the process. The limit distributions of the process can be used to estimate this matrix.

Another useful application can be found in diffusion approximations. Simulating trajectories of various stochastic differential equations can be difficult. If the solution of a given SDE can be approximated with a process that can be simulated easily, we can fix this problem. In our case, we can approximate the so-called Feller-type diffusions, which are used extensively for example, in genetics and in mathematical finance.

4.2 Research plans

After describing the limit processes for indecomposable and critical multi-type branching processes, the next step is to deal with the decomposable and critical case. We call a multi-type branching process decomposable, if $\mathbf{m}_\xi$ is reducible. This means that the directed graph $\Gamma(\mathbf{m}_\xi)$ is not strongly connected. There are several complications in this case. For one, there are no Perron-Frobenius theory for reducible matrices. This makes it difficult to handle the powers of $\mathbf{m}_\xi$. Another problem is, as suspected, the limit processes are very complex, even more than in the indecomposable case. To picture this, let us examine the structure of a reducible matrix. Let $\mathbf{m}_\xi$ be a reducible matrix. Then $\Gamma(\mathbf{m}_\xi)$ is connected, but not strongly connected. Then, by elementary graph theory, its nodes can be partitioned into strongly connected components. There are two classes of strongly connected components: those that you can exit, and those that you cannot. Let us call the former class transient and the latter class essential. It can easily be seen that if we start a population without immigration and with one specimen from a type that belongs to a transient class, after long enough time, the population will only consist of types from the essential class. But, as far as we can tell, the limit distribution is affected by the number

of transient classes the population passes through. Currently, there are only partial results in this case. So far, we have succeeded to describe the limit processes when the offspring expectation matrix has the form

$$\mathbf{m}_\xi = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}.$$

5 Appendix

5.1. Lemma. (i) Ψ is continuous, therefore measurable.

(ii) For every $n \in \mathbb{N}$, $\Psi^{(n)}$ is measurable.

Proof. (i): Let $(f_n)_{n \in \mathbb{N}}$ sequence which converges to f in $\mathbf{D}(\mathbb{R}_+, (\mathbb{R}^p)^r)$. Then it is enough to show $(\Psi(f_n))_{n \in \mathbb{N}}$ converges to $\Psi(f)$ in $\mathbf{D}(\mathbb{R}_+, (\mathbb{R}^p)^r)$. According to [8], Chapter VI Theorem 1.14 (a), the convergence can be characterized in a way that an $(f_n)_{n \in \mathbb{N}}$ sequence converges to f if and only if there is a sequence $(\lambda_n)_{n \in \mathbb{N}} \subset \Lambda$ such that

$$\begin{aligned} \sup_{s \in \mathbb{R}_+} |\lambda_n(s) - s| &\rightarrow 0, \\ \sup_{s \leq T} \|f_n \circ \lambda_n(s) - f(s)\| &\rightarrow 0, \quad \text{for every } T \in \mathbb{R}_+. \end{aligned}$$

Now we have

$$\|\Psi(f_n)(\lambda_n(s) - \Psi(f)(s))\| \leq \sum_{i=0}^{r-1} \|\Pi\| \sum_{l=1}^r \|\mathbf{m}_\xi^{l+i-1}\| \left(|\lambda_n(s) - s| \|\mathbf{m}_\varepsilon\| + \|f_{n,l}(\lambda_n(s)) - f_l(t)\| \right)$$

This implies that $\sup_{s \leq T} \|\Psi(f_n)(\lambda_n(s) - \Psi(f)(s))\| \rightarrow 0$, from which continuity follows.

(ii): To prove the measurability of $\Psi^{(n)}$, it is enough to prove that for every $N \in \mathbb{N}$, the mapping $\Psi_N^{(n)}(f)(t) := \Psi^{(n)}(f)(t \wedge N)$ is measurable, because then for every $f \in \mathbf{D}(\mathbb{R}_+, (\mathbb{R}^p)^r)$, we have $\Psi_N^{(n)}(f) \rightarrow \Psi^{(n)}(f)$ in $\mathbf{D}(\mathbb{R}_+, (\mathbb{R}^p)^r)$. Now we split up $\Psi_N^{(n)}$ into two separate mappings $\Psi_{N,1}^{(n)} : \mathbf{D}(\mathbb{R}_+, (\mathbb{R}^p)^r) \rightarrow \mathbb{R}^{rp(nN+1)}$ and $\Psi_{N,2}^{(n)} : \mathbb{R}^{rp(nN+1)} \rightarrow \mathbf{D}(\mathbb{R}_+, (\mathbb{R}^p)^r)$ such that $\Psi_N^{(n)} = \Psi_{N,2}^{(n)} \circ \Psi_{N,1}^{(n)}$. We define these with

$$\Psi_{N,1}^{(n)}(f) := \left(f(0), f(1/n), f(2/n), \dots, f(N) \right)$$

and

$$\Psi_{N,2}^{(n)} := \begin{bmatrix} \Psi_{N,2,1}^{(n)} \\ \Psi_{N,2,2}^{(n)} \\ \vdots \\ \Psi_{N,2,r}^{(n)} \end{bmatrix}, \quad \Psi_{N,2,i}^{(n)} : \mathbb{R}^{rp(nN+1)} \rightarrow \mathbf{D}(\mathbb{R}_+, (\mathbb{R}^p)^r), \quad i \in \{1, 2, \dots, r\}$$

with

$$\begin{aligned} \Psi_{N,2,i}^{(n)}(x)(t) &:= \mathbf{m}_\xi^{\lfloor n(t \wedge N) \rfloor r} \left[\sum_{l=i}^r \mathbf{m}_\xi^{l-i} x_{0,l} + \sum_{l=1}^{i-1} \mathbf{m}_\xi^{l-i+r} x_{0,l} \right] \\ &+ \sum_{j=1}^{\lfloor n(t \wedge N) \rfloor} \mathbf{m}_\xi^{(\lfloor n(t \wedge N) \rfloor - j)r} \left[\sum_{l=i}^r \mathbf{m}_\xi^{l-i} \left(x_{j,l} - x_{j-1,l} + \frac{1}{n} \mathbf{m}_\varepsilon \right) \right. \\ &\quad \left. + \sum_{l=1}^{i-1} \mathbf{m}_\xi^{l-i+r} \left(x_{j-1,l} - x_{j-2,l} + \frac{1}{n} \mathbf{m}_\varepsilon \right) \right] \end{aligned}$$

for every

$$x = (x_0, x_2, \dots, x_{nN}) \in (\mathbb{R}^{rp})^{(nN+1)}, \quad x_i = (x_{i,1}, x_{i,2}, \dots, x_{i,rp}) \in \mathbb{R}^{rp}.$$

The measurability of $\Psi_{N,1}^{(n)}$ is obvious, because the projections actually generate the σ -field $\mathcal{D}_\infty(\mathbb{R}_+, \mathbb{R}^p)$. (See [8], Chapter VI, Theorem 1.14 for details.) It can also be seen easily that for every $i \in \{1, 2, \dots, r\}$, $\Psi_{N,2,i}^{(n)}$ is a continuous function. It follows that $\Psi_N^{(n)}$ is measurable, and this is what we had to show. $\square$

5.2. Lemma. (i) $\mathcal{C}(\mathbb{R}_+, (\mathbb{R}^p)^r)$ is closed in the topology generated by (2.1), therefore we have $\mathcal{C}(\mathbb{R}_+, (\mathbb{R}^p)^r) \in \mathcal{D}_\infty(\mathbb{R}_+, (\mathbb{R}^p)^r)$.

(ii) The linear mapping $(\mathbf{I} - \mathbf{\Pi}^{\oplus r} \text{circ}_r(\mathbf{m}_\xi)) : (\mathbb{R}^p)^r \rightarrow (\mathbb{R}^p)^r$ is measurable.

(iii) $\pi_0 : \mathcal{C}(\mathbb{R}_+, (\mathbb{R}^p)^r) \rightarrow (\mathbb{R}^p)^r$ is measurable.

Therefore, $C = \mathcal{C}(\mathbb{R}_+, (\mathbb{R}^p)^r) \cap \pi_0^{-1}\left((\mathbf{I} - \mathbf{\Pi}^{\oplus r} \text{circ}_r(\mathbf{m}_\xi))^{-1}(\{\mathbf{0}\})\right)$ is measurable.

Proof. Before we prove any of the assertions, we note that with the definition

$$\text{circ}_r(\mathbf{m}_\xi) := \begin{bmatrix} \mathbf{I} & \mathbf{m}_\xi & \dots & \mathbf{m}_\xi^{r-2} & \mathbf{m}_\xi^{r-1} \\ \mathbf{m}_\xi^{r-1} & \mathbf{I} & \dots & \mathbf{m}_\xi^{r-3} & \mathbf{m}_\xi^{r-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{m}_\xi^2 & \mathbf{m}_\xi^3 & \dots & \mathbf{I} & \mathbf{m}_\xi \\ \mathbf{m}_\xi & \mathbf{m}_\xi^2 & \dots & \mathbf{m}_\xi^{r-1} & \mathbf{I} \end{bmatrix},$$

C can be written in the form

$$C = \mathcal{C}(\mathbb{R}_+, (\mathbb{R}^p)^r) \cap \pi_0^{-1}\left((\mathbf{I} - \mathbf{\Pi}^{\oplus r} \text{circ}_r(\mathbf{m}_\xi))^{-1}(\{\mathbf{0}\})\right),$$

where $\mathbf{\Pi}^{\oplus r} = \mathbf{\Pi} \oplus \dots \oplus \mathbf{\Pi}$ and $\pi_0 : \mathcal{C}(\mathbb{R}_+, (\mathbb{R}^p)^r) \rightarrow (\mathbb{R}^p)^r$ denotes the projection $\pi_0(f) = f(0)$.

(i): To prove that $\mathcal{C}(\mathbb{R}_+, (\mathbb{R}^p)^r)$ is closed, it is enough to show that its complement is open. Let $\beta \in (\mathcal{C}(\mathbb{R}_+, (\mathbb{R}^p)^r))^c$. This means that there exists a $t_0 \in \mathbb{R}_+$ such that $\|\beta(t_0) - \beta(t_0-)\| > 0$. Suppose that $N \leq t_0 < N+1$ for some $N \in \mathbb{N}$. Then from the definition of (2.1), it is obvious that for any $\alpha \in \mathcal{C}(\mathbb{R}_+, (\mathbb{R}^p)^r)$ continuous function,

$$\delta_{N+1}(\alpha, \beta) \geq \frac{\|\beta(t_0) - \beta(t_0-)\|}{2} > 0$$

and because of this,

$$\delta(\alpha, \beta) \geq 2^{-(N+1)} \min\{1, 2^{-1}\|\beta(t_0) - \beta(t_0-)\|\} =: \varepsilon(\beta) > 0.$$

This means that no continuous function can be closer to β than $\varepsilon(\beta)$. With this, we have

$$(\mathcal{C}(\mathbb{R}_+, (\mathbb{R}^p)^r))^c = \bigcap_{\beta \in (\mathcal{C}(\mathbb{R}_+, (\mathbb{R}^p)^r))^c} \mathbf{B}(\beta, 2^{-1}\varepsilon(\beta)),$$

where $\mathbf{B}(\beta, \varepsilon)$ is the open ball centered at β with radius ε . Thus $(\mathcal{C}(\mathbb{R}_+, (\mathbb{R}^p)^r))^c$ is the union of open sets, therefore it is open. The proposition (ii) is trivial.

(iii): From [8], Chapter VI, Theorem 1.14, we have that the Borel σ -field $\mathcal{D}_\infty(\mathbb{R}_+, (\mathbb{R}^p)^r)$ is the σ -field generated by projections. From this, (iii) immediately follows. $\square$

List of notations

- $\mathbb{N} := \{0, 1, 2, \dots\}$
- $\mathbb{R}_+ := [0, \infty)$
- $\mathbf{C}(\mathbb{R}_+, \mathbb{R}^p)$ is the space of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}^p$
- $\mathbf{D}(\mathbb{R}_+, \mathbb{R}^p)$ is the space of càdlàg functions from $\mathbb{R}_+$ to $\mathbb{R}^p$
- $\mathcal{D}_\infty(\mathbb{R}_+, \mathbb{R}^p)$ is the Borel σ -field of $\mathbf{D}(\mathbb{R}_+, \mathbb{R}^p)$
- $\xrightarrow{\mathcal{D}}$ and $\xrightarrow{lu}$ respectively denotes convergence in distribution and locally uniform convergence
- $(\mathbf{X}_n)_{n \in \mathbb{N}}$ is our critical, indecomposable p-type Galton-Watson process with immigration
- $\mathbf{m}_\xi$ is the offspring expectation matrix of $\mathbf{X}_n$ with r eigenvalues of maximum modulus
- $\mathbf{V}_{\xi_i}$ is the covariance matrix of the i -th type
- $\mathbf{x} \odot \mathbf{V}_\xi := \sum_{j=1}^p x_j \mathbf{V}_{\xi_j}$, where $\mathbf{x} = (x_1, x_2, \dots, x_p)^T$
- $\mathbf{u}, \mathbf{v}$ are the left and right Perron-vectors of $\mathbf{m}_\xi$
- $\mathbf{\Pi} := \mathbf{u}\mathbf{v}^T \in \mathbb{R}_+^{p \times p}$
- The $c \in \mathbb{R}_+$ and $\kappa \in (0, 1)$ constants are defined such that $\|\mathbf{m}_\xi^{nr} - \mathbf{\Pi}\| \leq c\kappa^n$
- The $c' \in \mathbb{R}_+$ constant is the upper bound of the $(\|\mathbf{m}_\xi^n\|)_{n \in \mathbb{N}}$ sequence
- $(\mathcal{X}_t^{(n)})_{t \in \mathbb{R}_+}$ is the n -th step process created from $(\mathbf{X}_n)_{n \in \mathbb{N}}$, see (3.8)
- $(\mathcal{X}_t)_{t \in \mathbb{R}_+}$ is the limit process of $(\mathcal{X}_t^{(n)})_{t \in \mathbb{R}_+}$, see Theorem 3.2
- $(\mathbf{M}_k)_{k \in \mathbb{N}}$ are the martingale differences created from $(\mathbf{X}_n)_{n \in \mathbb{N}}$, see (3.10)
- $(\mathcal{M}_t^{(n)})_{t \in \mathbb{R}_+}$ is the n -th step process created from $(\mathbf{M}_k)_{k \in \mathbb{N}}$, see (3.15)
- $(\mathcal{M}_t)_{t \in \mathbb{R}_+}$ is the limit process of $(\mathcal{M}_t^{(n)})_{t \in \mathbb{R}_+}$, see Theorem 3.4
- Ψ and $\Psi^{(n)}$ are the mappings which we use to reconstruct $(\mathcal{X}_t)_{t \in \mathbb{R}_+}$ and $(\mathcal{X}_t^{(n)})_{t \in \mathbb{R}_+}$ from $(\mathcal{M}_t)_{t \in \mathbb{R}_+}$ and $(\mathcal{M}_t^{(n)})_{t \in \mathbb{R}_+}$, see (3.36) and (3.35)

Acknowledgements

There are a lot of people whom I would like to thank, who contributed to this paper directly and indirectly. In fact, I have to thank everybody who have ever taught me mathematics, especially my teachers at the university. During my studies, I have received only kindness, patience and a lot of beautiful mathematics from them. Most importantly, I would like to give my utmost thanks to my advisor, Dr. Pap Gyula. He helped and guided me in mathematics from the beginning of my studies, and I am grateful for that.

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