

Definability in the embeddability ordering of finite directed graphs

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1 Introduction

In this section, we sketch the basic concepts used throughout the paper. We assume some knowledge of basic mathematical concepts. A binary relation $\rho \subseteq \mathcal{A} \times \mathcal{A}$ is said to be a *partial order on the set* $\mathcal{A}$ if it is reflexive, antisymmetric and transitive. Partial orders are often denoted by $\leq$. We call the pair $(\mathcal{A}, \leq)$ a *partially ordered set (poset)* if $\leq$ is a partial order on the set $\mathcal{A}$. An n -ary relation on a set $\mathcal{A}$ is just a subset of the Cartesian product

$$\mathcal{A}^n = \underbrace{\mathcal{A} \times \mathcal{A} \times \cdots \times \mathcal{A}}_{n \text{ times}},$$

so it consists of n -tuples whose components are from the set $\mathcal{A}$. Specifically, the 2-ary relations are the well-known binary relations and the 1-ary relations called *unary relations* are just subsets of the set $\mathcal{A}$. Let $(\mathcal{A}, \leq)$ be an arbitrary partially ordered set. An n -ary relation R is said to be definable in $(\mathcal{A}, \leq)$ if there exists a first-order formula $\Psi(x_1, x_2, \dots, x_n)$ with free variables $x_1, x_2, \dots, x_n$ in the language of partially ordered sets such that for any $a_1, a_2, \dots, a_n \in \mathcal{A}$, $\Psi(a_1, a_2, \dots, a_n)$ holds in $(\mathcal{A}, \leq)$ if and only if $(a_1, a_2, \dots, a_n) \in R$. A subset of $\mathcal{A}$ is definable if it is definable as a unary relation. An element $a \in \mathcal{A}$ is said to be definable if the set $\{a\}$ is definable.

Example 1.1. Let us consider the poset $(D_{60}, |)$, where D_{60} denotes the positive divisors of 60.

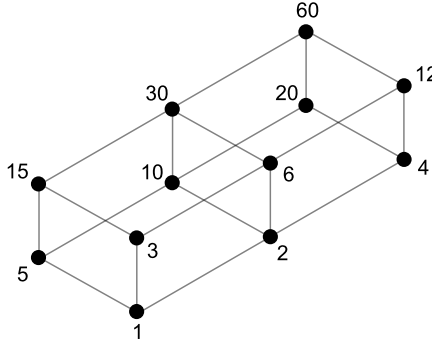


Figure 1: The Hasse diagram of $(D_{60}, |)$.

The elements 1 and 60 are definable in this poset, for

$$\{1\} = \{x : (\forall y)(x \leq y)\}, \quad \{60\} = \{x : (\forall y)(y \leq x)\}.$$

The set $\{2, 3, 5\}$ is definable as well, because

$$\{2, 3, 5\} = \{x : 1 \prec x\} = \{x : \neg(x \in \{1\}) \wedge ((y \leq x) \Rightarrow ((y = x) \vee (y \in \{1\})))\}.$$

We can distinguish between the element 2 and the elements 3 and 5 in $\{2, 3, 5\}$, for 2 has three covers, while 3 and 5 have only two. Thus we can define the sets $\{2\}$ and $\{3, 5\}$. The

question we may now ask is whether we can define the set $\{3\}$ or equivalently distinguish between 3 and 5 with first-order formulas. We might feel that we can not do this. How can we prove such a conjecture? Checking all the first-order formulas is obviously not an option. It is easy to see that if there exists an automorphism of the poset $(\mathcal{A}, \leq)$ that maps the element $a \in \mathcal{A}$ to $b \in \mathcal{A}$ then a and b are indistinguishable in $(\mathcal{A}, \leq)$ with first-order formulas. Now the map $1 \mapsto 1, 2 \mapsto 2, 4 \mapsto 4, 3 \mapsto 5, 5 \mapsto 3, 6 \mapsto 10, 10 \mapsto 6, 15 \mapsto 15, 30 \mapsto 30, 12 \mapsto 20, 20 \mapsto 12, 60 \mapsto 60$ is an automorphism of $(D_{60}, |)$, which verifies our conjecture.

Let us denote the isomorphism types of finite groups by $\mathcal{G}$. We can define a partial order $\leq$ on $\mathcal{G}$ with $G \leq G'$ for $G, G' \in \mathcal{G}$ if and only if G is isomorphic to some subgroup of G' . In this way $(\mathcal{G}, \leq)$ becomes a poset. More generally let $\mathcal{S}$ be the set of isomorphism types of all finite structures of a given type (for example lattices, posets or—as we have just seen—groups, etc.). We define the binary relation $\leq$ on $\mathcal{S}$: for all $S, S' \in \mathcal{S}$, $S \leq S'$ if and only if S is isomorphic to some substructure of S' . With this definition $(\mathcal{S}, \leq)$ turns out to be a poset, which we call *substructure ordering*. In 2009–2010 J. Ježek and R. McKenzie published a series of papers ([1], [2], [3], [4]) in which they have examined (among other things) the first-order definability in such substructure orderings of finite mathematical structures with a given type and determined the automorphism groups of these orderings. They considered finite semilattices [1], ordered sets [2], distributive lattices [3] and lattices [4].

In this paper we consider (the isomorphism types of) finite directed graphs. Let us consider a nonempty set V and a binary relation $E \subseteq V^2$. We call the pair $G = (V, E)$ a *directed graph* or just *digraph*. The elements of $V (= V(G))$ and $E (= E(G))$ are called the *vertices* and *edges* of G , respectively. The directed graph $G^T := (V, E^{-1})$ is called the *transpose* of G , where E^{-1} denotes the inverse of the relation E . A directed graph is *finite* if the number of its vertices is finite. We use an embeddability definition different from the substructure ordering explained above and used in the papers [1], [2], [3], [4]. We say that the directed graph G is *embeddable* into G' if there exists an injective map $\varphi : V(G) \rightarrow V(G')$ such that $(v_1, v_2) \in E(G)$ implies $(\varphi(v_1), \varphi(v_2)) \in E(G')$. We denote this fact by $G \leq G'$. Let $\mathcal{D}$ denote the set of isomorphism types of finite digraphs. For arbitrary digraphs $G, H \in \mathcal{D}$ let $G \prec H$ mean that $G \neq H$, $G \leq H$ and $G \leq X \leq H$ implies $X = G$ or $X = H$ for every digraph $X \in \mathcal{D}$.

Lemma 1.2. $(\mathcal{D}, \leq)$ is a poset and the map $G \mapsto G^T$ is a non-trivial automorphism of it.

Proof. Transitivity and reflexivity is obvious. Antisimmetry can be checked the following way. Let G and G' be finite digraphs embeddable into each other and let the maps showing the mutual embeddability be

$$\varphi_1 : V(G) \rightarrow V(G'), \quad \varphi_2 : V(G') \rightarrow V(G).$$

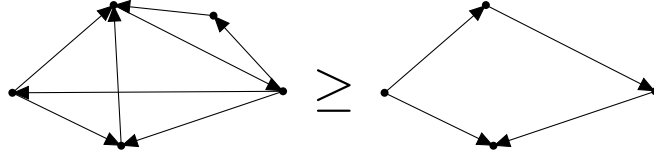


Figure 2: An example of our digraph embeddability concept.

Since G, G' are finite and φ_1, φ_2 are injective, we have $|V(G)| = |V(G')|$ so φ_1 is bijective. It is easy to see that the maps

$$\phi_1 : E(G) \rightarrow E(G'), (v_1, v_2) \mapsto (\varphi_1(v_1), \varphi_1(v_2)),$$

$$\phi_2 : E(G') \rightarrow E(G), (v'_1, v'_2) \mapsto (\varphi_2(v'_1), \varphi_2(v'_2))$$

are injective too, so ϕ_1 is bijective and this implies that φ_1 is an isomorphism between the digraphs G and G' . The fact that $G \mapsto G^T$ is an automorphism of $(\mathcal{D}, \leq)$ is obvious, and it is non-trivial, for one can easily find a digraph not isomorphic to its transpose. $\square$

This lemma tells us that the strongest claim one can conjecture is that all sets of the form $\{G, G^T\}$ are definable with first-order formulas in the poset $(\mathcal{D}, \leq)$. This is the main result of this paper. We will prove—using this result—that there is no non-trivial automorphism of $(\mathcal{D}, \leq)$ different from $G \mapsto G^T$, which implies that the automorphism group of $(\mathcal{D}, \leq)$ is isomorphic to $\mathbb{Z}_2$.

We will use the notation $\tilde{G} := \{G, G^T\}$ in the rest of the paper. Our proof requires a lot of notations to be introduced, maybe too many to keep in mind, so Section 4 collects them to help the reader search their definitions in the paper.

2 The definability of sets $\{G, G^T\}$

Definition 2.1. Let us introduce the following digraphs:

$$E_1 : V(E_1) = \{v_1\}, E(E_1) = \emptyset,$$

$$L_1 : V(L_1) = \{v_1\}, E(L_1) = \{(v_1, v_1)\},$$

$$E_2 : V(E_2) = \{v_1, v_2\}, E(E_2) = \emptyset,$$

$$I_2 : V(I_2) = \{v_1, v_2\}, E(I_2) = \{(v_1, v_2)\}.$$

Lemma 2.2. *The digraphs E_1, L_1, E_2, I_2 are definable.*

Proof. E_1 is the unique digraph $X \in \mathcal{D}$ for which $X \leq G$ holds for every $G \in \mathcal{D}$. There are two elements covering E_1 , namely $L_1 (= L_1^T)$ and $E_2 (= E_2^T)$, hence the set $\{L_1, E_2\}$ is definable. In this set only L_1 has a unique cover so L_1 and E_2 are distinguishable, which

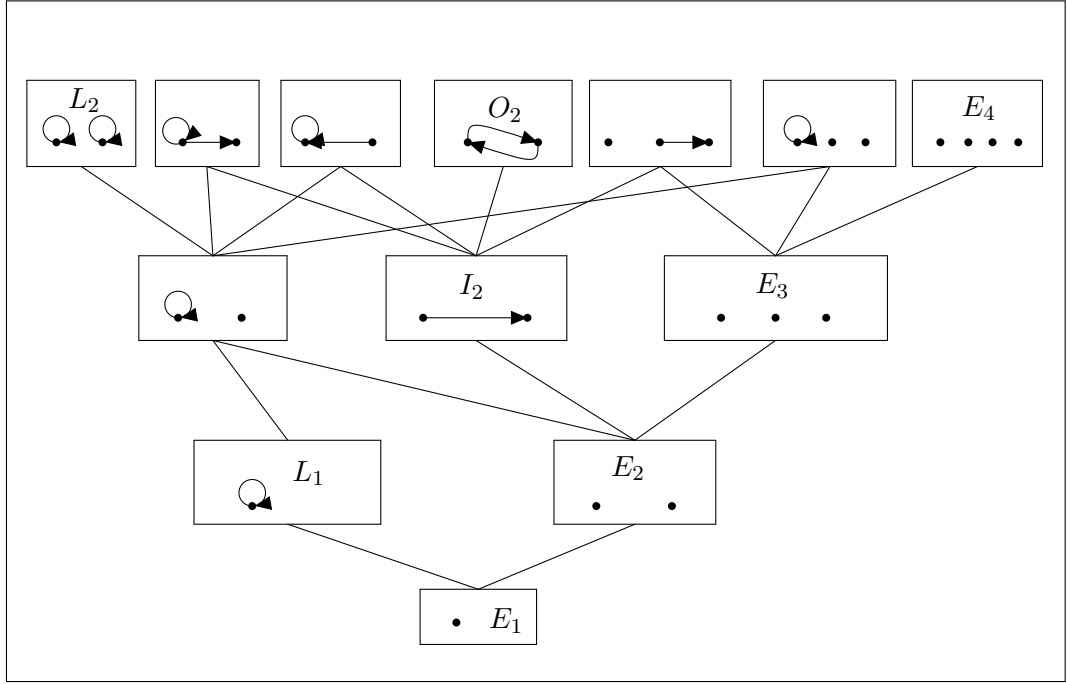


Figure 3: The “bottom” part of the Hasse diagram of $(\mathcal{D}, \leq)$.

implies that $\tilde{L}_1 = \{L_1\}$ is definable and so is $\tilde{E}_2 = \{E_2\}$. I_2 is the the unique element among the covers of E_2 which has 4 covers and does not cover L_1 (see Fig. 3.). $\square$

Definition 2.3. For a positive integer n , we say that $G \in \mathcal{D}$ is on level n if $|V(G)| + |E(G)| = n$. Let $\mathcal{S}_n$ denote the set of all digraphs on level n .

It is an easy observation that if $G \prec G'$ then G' is exactly one level above G , so our definition corresponds to the levels of our Hasse diagram (see Fig. 3.).

Lemma 2.4. *The set $\mathcal{S}_n$ is definable.*

Proof. $\mathcal{S}_1$ was already defined in Lemma 2.2. $\mathcal{S}_{n+1}$ can be defined recursively: $\mathcal{S}_{n+1}$ is the set of digraphs $X \in \mathcal{D}$ for which $Z \prec X$ for some $Z \in \mathcal{S}_n$. $\square$

Definition 2.5. Let E_n ($n = 1, 2, \dots$) be the “empty” digraph with n vertices: $V(E_n) = \{v_1, v_2, \dots, v_n\}$, $E(E_n) = \emptyset$. Let F_n ($n = 1, 2, \dots$) be the “full” digraph with n vertexes: $V(F_n) = \{v_1, v_2, \dots, v_n\}$, $E(F_n) = V(F_n)^2$. Let $\mathcal{T}_n = \{G \in \mathcal{D} : |V(G)| = n\}$ be the set of all digraphs having n vertices.

Lemma 2.6. *The digraphs E_n , F_n and the set $\mathcal{T}_n$ are definable for every positive integer n .*

Proof. The set $\mathcal{I} = \{E_n : n \in \{1, 2, \dots\}\}$ is definable, because its elements are those digraphs $X \in \mathcal{D}$ for which $L_1 \not\prec X$ and $I_2 \not\prec X$. In $\mathcal{I}$ we have $E_1 \prec E_2 \prec E_3 \prec \dots$, therefore $E_n (= E_n^T)$ is definable for every n . $\mathcal{T}_n$ contains exactly those digraphs $X \in \mathcal{D}$ for

which $E_n \leq X$ and $E_{n+1} \not\leq X$, so it is definable. F_n is the digraph $X \in \mathcal{D}$ which has n vertices and $X \prec Z$ implies that Z has $n + 1$ vertices. $\square$

Definition 2.7. Let us set notations for the following digraphs:

$$\begin{aligned} O_2 : V(O_2) &= \{v_1, v_2\}, E(O_2) = \{(v_1, v_2), (v_2, v_1)\}, \\ A : V(A) &= \{v_1, v_2, v_3\}, E(A) = \{(v_1, v_3), (v_2, v_3)\}, \\ I_3 : V(I_3) &= \{v_1, v_2, v_3\}, E(I_3) = \{(v_1, v_2), (v_2, v_3)\}, \\ L_2 : V(L_2) &= \{v_1, v_2\}, E(L_2) = \{(v_1, v_1), (v_2, v_2)\}, \\ L_{2+} : V(L_{2+}) &= \{v_1, v_2\}, E(L_{2+}) = \{(v_1, v_1), (v_2, v_2), (v_1, v_2)\}. \end{aligned}$$

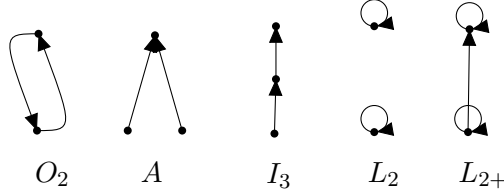


Figure 4: The digraphs O_2 , A , I_3 , L_2 , L_{2+} .

Lemma 2.8. *The digraphs O_2 , I_3 , L_2 , L_{2+} and the set $\tilde{A}$ are definable.*

Proof. $O_2 (= O_2^T)$ is the maximal digraph X that has 2 vertices and $L_1 \not\leq X$.

Let $G \in \mathcal{D}$ be the following digraph: $V(G) = \{v_1, v_2, v_3\}$, $E(G) = \{(v_1, v_2)\}$. Then $G (= G^T)$ is definable since it is the only element covering both I_2 and E_3 .

Now the set $\tilde{A} \cup \tilde{I}_3$ turns out to be definable, for it contains exactly those digraphs $X \in \mathcal{D}$ for which $G \prec X$, $O_2 \not\leq X$, $E_4 \not\leq X$ and $L_1 \not\leq X$ holds. Now $\tilde{I}_3 = \{I_3\}$ is the unique digraph $X \in \tilde{A} \cup \tilde{I}_3$ for which there exists $X \prec Z$ such that $W \prec Z$ implies $X = W$ (with the notation to be introduced in Definition 2.9, $Z = O_3$), meaning Z covers only X . From this we also get that $\tilde{A}$ is definable because $\tilde{A} = (\tilde{A} \cup \tilde{I}_3) \setminus \tilde{I}_3$.

The digraph L_2 is the maximal $X \in \mathcal{D}$ that has 2 vertices and for which $I_2 \not\leq X$. Finally, the digraph L_{2+} is the only $X \in \mathcal{D}$ having 2 vertices, being on level 5 for which $L_2 \leq X$. $\square$

Definition 2.9. Let I_n , O_n , L_n ($n = 2, 3, \dots$) denote the following digraphs:

$$\begin{aligned} V(I_n) &= V(O_n) = V(L_n) = \{v_1, v_2, \dots, v_n\}, \\ E(I_n) &= \{(v_1, v_2), (v_2, v_3), \dots, (v_{n-1}, v_n)\}, \\ E(O_n) &= \{(v_1, v_2), (v_2, v_3), \dots, (v_{n-1}, v_n), (v_n, v_1)\}, \\ E(L_n) &= \{(v_1, v_1), (v_2, v_2), \dots, (v_n, v_n)\}. \end{aligned}$$

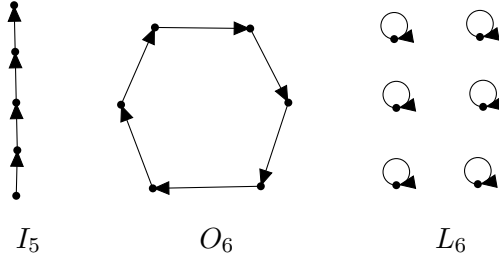


Figure 5: The digraphs I_5 , O_6 and I_6 .

Lemma 2.10. *The digraphs L_n , O_n , I_n ($n = 2, 3, \dots$) are definable.*

Proof. $L_n (= L_n^T)$ is the unique digraph X on level $2n$ that has n vertices and for which $I_2 \not\leq X$.

We define the digraphs $O_n (= O_n^T)$ and $I_n (= I_n^T)$ together, recursively. O_2 and I_2 were already defined in Lemma 2.8 and Lemma 2.2, respectively. Suppose that $O_2, I_2, \dots, O_n, I_n$ have already been defined. Then O_{n+1} is the unique digraph X that:

- has $n + 1$ vertices,
- $I_n \leq X$, $L_1 \not\leq X$, $O_2 \not\leq X$, $O_n \not\leq X$,
- is on level $2n + 2$, and
- there exists no $Z \in \tilde{A}$ for which $Z \leq X$.

Finally, the digraph I_{n+1} is the only element X for which $X \prec O_{n+1}$. □

Definition 2.11. Let G be an arbitrary finite directed graph with no loops. Let $L(G)$ denote the digraph that we get from G by adding loops to every vertex. For a set $\mathcal{G} \subseteq \mathcal{D}$ of finite digraphs with no loops let $\mathcal{L}(\mathcal{G}) := \{L(G) : G \in \mathcal{G}\}$.

Definition 2.12. We say that the digraph X is n level under Z if there exist $Z_1, \dots, Z_n \in \mathcal{D}$ such that $Z \succ Z_1 \succ \dots \succ Z_n = X$. Similarly, we say that the digraph X is n level above Y if there exist $Y_1, \dots, Y_n \in \mathcal{D}$ such that $Y \prec Y_1 \prec \dots \prec Y_n = X$.

Lemma 2.13. *Let $\mathcal{G} \subseteq \mathcal{D}$ be a finite definable set of finite digraphs with no loops. Then the set $\mathcal{L}(\mathcal{G})$ is definable.*

Proof. Let $N = \max\{|V(G)| : G \in \mathcal{G}\}$. Then $\mathcal{L}(\mathcal{G})$ consists of those $X \in \mathcal{D}$ for which:

- for every $1 \leq n \leq N$, if X has n vertices then $L_n \leq X$, and
- for every $1 \leq n \leq N$, if X has n vertices, $Z \leq X$ is n level under X and $L_1 \not\leq Z$, then $Z \in \mathcal{G}$. □

Definition 2.14. For an arbitrary $G \in \mathcal{D}$ let $M(G)$ denote the digraph we get by leaving all the loops out from G . For a set $\mathcal{G} \subseteq \mathcal{D}$ of finite digraphs put $\mathcal{M}(\mathcal{G}) := \{M(G) : G \in \mathcal{G}\}$.

Lemma 2.15. *Let $\mathcal{G}$ be an arbitrary finite subset of $\mathcal{D}$ that is definable. Then the set $\mathcal{M}(\mathcal{G})$ is definable.*

Proof. Let us consider $N = \max\{|V(G)| : G \in \mathcal{G}\}$ again. Then $\mathcal{M}(\mathcal{G})$ is the set of those digraphs $X \in \mathcal{D}$ for which either

- $L_1 \not\leq X$ and $X \in \mathcal{G}$

or

- $L_1 \not\leq X$ and there exists $Z \in \mathcal{G}$ such that for all $1 \leq n \leq N$, $L_n \leq Z$ and $L_{n+1} \not\leq Z$ together imply that X is n level under Z . $\square$

Definition 2.16. Let $O_{n,L}$ be the following digraph: $V(O_{n,L}) = \{v_1, v_2, \dots, v_n\}$, $E(O_{n,L}) = E(O_n) \cup \{(v_1, v_1)\}$, which means that

$$E(O_{n,L}) = \{(v_1, v_1), (v_1, v_2), (v_2, v_3), \dots, (v_{n-1}, v_n), (v_n, v_1)\}.$$

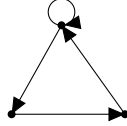


Figure 6: The digraph $O_{3,L}$.

Lemma 2.17. *The digraphs $O_{n,L}$ ($n = 2, 3, \dots$) are definable.*

Proof. $O_{n,L}(= O_{n,L}^T)$ is the unique digraph X on level $2n + 1$ for which $O_n \leq X$ and $L_1 \leq X$. $\square$

Definition 2.18. For arbitrary finite directed graphs $G_1, G_2, \dots, G_n$ let us denote their disjoint union by $G_1 \dot{\cup} G_2 \dot{\cup} \dots \dot{\cup} G_n = \dot{\bigcup}_{i=1}^n G_i$, as usual.

Definition 2.19. Let $n_1, n_2, \dots, n_k$ be pairwise distinct integers greater than 1. Let $O(n_1, n_2, \dots, n_k) = \dot{\bigcup}_{i=1}^k O_{n_i}$ and $O_L(n_1, n_2, \dots, n_k) = \dot{\bigcup}_{i=1}^k O_{n_i, L}$.

Lemma 2.20. *The digraphs $O(n_1, n_2, \dots, n_k)$ and $O_L(n_1, n_2, \dots, n_k)$ are definable.*

Proof. $O(n_1, n_2, \dots, n_k)(= (O(n_1, n_2, \dots, n_k))^T)$ is the unique digraph X having $n_1 + n_2 + \dots + n_k$ vertices, being on level $2(n_1 + n_2 + \dots + n_k)$ for which $O_{n_i} \leq X$ ($i = 1, 2, \dots, k$) and there exists no $Z \in \tilde{A}$ such that $Z \leq X$.

$O_L(n_1, n_2, \dots, n_k)(= (O_L(n_1, n_2, \dots, n_k))^T)$ is the unique digraph X being on level $2(n_1 + n_2 + \dots + n_k) + k$ for which $O(n_1, n_2, \dots, n_k) \leq X$, $L_k \leq X$ and $O_{n_i, L} \leq X$ ($i = 1, 2, \dots, k$). $\square$

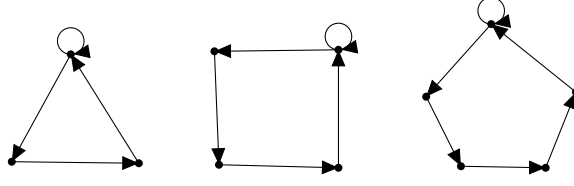


Figure 7: The digraph $O_L(3, 4, 5)$.

Definition 2.21. Let us define the digraph $O_{i,j,L1}$ the following way (see Fig. 8). Let $V(O_i) = \{v_1, \dots, v_i\}$ and $V(O_j) = \{v'_1, \dots, v'_j\}$. Now let $V(O_{i,j,L1}) = V(O_i \dot{\cup} O_j)$,

$$E(O_{i,j,L1}) = E(O_i \dot{\cup} O_j) \cup \{(v_1, v_1), (v'_1, v'_1), (v_1, v'_1)\}.$$

Let $O_{i,j,L2}$: $V(O_{i,j,L2}) = V(O_{i,j,L1})$, $E(O_{i,j,L2}) = E(O_{i,j,L1}) \cup \{(v'_1, v_1)\}$ and finally let $O_{i,j,1} = M(O_{i,j,L1})$, $O_{i,j,2} = M(O_{i,j,L2})$.

In words, $O_{i,j,L1}$ consists of two circles which are connected by an L_{2+} (see Fig. 8).

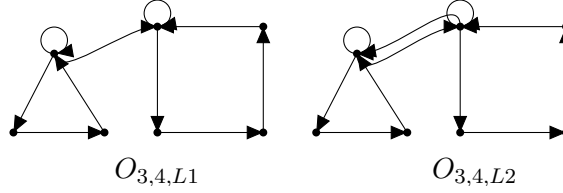


Figure 8: $O_{3,4,L1}$, $O_{3,4,L2}$

Lemma 2.22. Let i and j be different positive integers both bigger than 2. Then the sets $\tilde{O}_{i,j,L1}$, $\tilde{O}_{i,j,L2}$, $\tilde{O}_{i,j,1}$ and $\tilde{O}_{i,j,2}$ are definable.

Proof. $\tilde{O}_{i,j,L1}$ is the set of those digraphs X that:

- have $i + j$ vertices,
- are on level $2(i + j) + 3$, and
- $O_L(i, j) \leq X$, $L_{2+} \leq X$, $L_3 \not\leq X$.

$\tilde{O}_{i,j,L2}$ consists of those digraphs X that:

- are on level $2(i + j) + 4$,
- $F_2 \leq X$, and
- there exists $Z \in \tilde{O}_{i,j,L1}$ such that $Z \leq X$.

Finally, the definability of the digraphs $\tilde{O}_{i,j,1}$, $\tilde{O}_{i,j,2}$ follows from Lemma 2.15. $\square$

Definition 2.23. Let $\mathcal{O}_{i,j,1}^+$ denote the set of those digraphs $Z \prec X$ which we can get by adding an edge to Z that is not a loop, where $Z \in \tilde{\mathcal{O}}_{i,j,1}$. Similarly let $\mathcal{O}_{i,j,2}^+$ denote the set of those digraphs $Z \prec X$ which we can get by adding an edge to Z that is not a loop, where $Z \in \tilde{\mathcal{O}}_{i,j,2}$. (see Fig. 9).

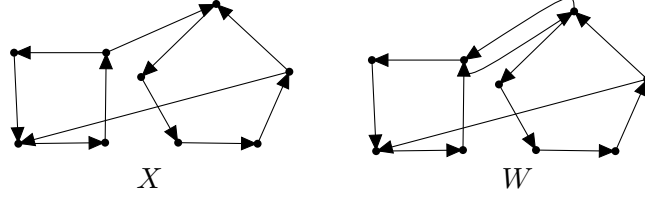


Figure 9: A digraph $X \in \mathcal{O}_{4,5,1}^+$ and a digraph $W \in \mathcal{O}_{4,5,2}^+$.

Lemma 2.24. The sets $\mathcal{O}_{i,j,1}^+$, $\mathcal{O}_{i,j,2}^+$ are definable for every distinct positive integers i, j both bigger than two.

Proof. $\mathcal{O}_{i,j,1}^+$ consists of those digraphs X for which $Z \prec X$ for some $Z \in \tilde{\mathcal{O}}_{i,j,1}$, $L_1 \not\subseteq X$ and $E_{i+j+1} \not\subseteq X$. Similarly, $\mathcal{O}_{i,j,2}^+$ is the set of those digraphs X for which $Z \prec X$ for some $Z \in \tilde{\mathcal{O}}_{i,j,2}$, $L_1 \not\subseteq X$ and $E_{i+j+1} \not\subseteq X$. $\square$

Definition 2.25. For a finite directed graph $G \in \mathcal{D}$ let

$$\begin{aligned} \mathcal{A}(G) &= \{X \in \mathcal{D} : V(X) = V(G), (\forall x, y \in V(X)) (\underbrace{|\{(x, y) \in E(X)\} \cup \{(y, x) \in E(X)\}|}_{\text{the number of edges between } x \text{ and } y \text{ in } X}) \\ &= \underbrace{|\{(x, y) \in E(G)\} \cup \{(y, x) \in E(G)\}|}_{\text{the number of edges between } x \text{ and } y \text{ in } G} \}. \end{aligned}$$

In words, $\mathcal{A}(G)$ is the set of those digraphs that can be obtained from G by reversing some edges.

Definition 2.26. Let u, v, w be 3 distinct vertices of a directed graph $G \in \mathcal{D}$. Let (u, v, w) denote the fact that in the digraph G exactly the following edges lie between the vertices u, v, w (and there are no loops on them): exactly one of (u, v) , (v, u) and exactly one of (v, w) , (w, v) . In case of (u, v, w) let $[u, v, w]_G$ denote the subgraph of G induced by the vertices u, v, w .

It is easy to see that either $[u, v, w]_G = I_2$ or $[u, v, w]_G \in \tilde{\mathcal{A}}$ holds. We say that the digraph $[u, v, w]_G$ is *determined* if it is known that which of the previous two relations hold. The following lemma states that for a weakly connected finite digraph $O_2 \not\subseteq G$ if $\mathcal{A}(G)$ is definable and $[u, v, w]_G$ is determined for all (u, v, w) , then the subset $\{G, G^T\} \subseteq \mathcal{A}(G)$ is also definable.

Lemma 2.27. *Let $G \in \mathcal{D}$ be a weakly connected digraph for which $O_2 \not\subseteq G$. Let $v_1, \dots, v_n$ be the vertices of G . Then*

$$\{X \in \mathcal{A}(G) : ([v_i, v_j, v_k]_X = I_2) \Leftrightarrow ([v_i, v_j, v_k]_G = I_2)\} = \{G, G^T\}. \quad (1)$$

Proof. We can suppose that G has an edge because otherwise G must have a single vertex which is a trivial case. We can also suppose that $(v_1, v_2) \in E(G)$, for it is only a matter of notation. Let H denote the left-hand side of (1). It is clear that for any $X \in H$, there is exactly 1 edge between the vertices v_1 and v_2 since $H \subseteq \mathcal{A}(G)$ and $O_2 \not\subseteq G$. Let us suppose first that for an $X \in H$, similarly to G , $(v_1, v_2) \in E(X)$ holds. We claim that $X = G$. Let us consider an arbitrary pair of vertices $v', v'' \in V(X)$ that are connected by an edge in X . Since G is weakly connected there exists a series

$$v_1 = w_1, v_2 = w_2, w_3, \dots, w_{k-2}, v' = w_{k-1}, v'' = w_k$$

of pairwise distinct vertices such that for all $j \in \{1, 2, \dots, k-1\}$ there is exactly one edge between the edges w_j, w_{j+1} . Let us consider the neighbouring vertices in this series. We know that $(w_1, w_2) \in E(G), E(X)$. Observe that the direction of the edge between the vertices w_2, w_3 in X is determined by the condition

$$([w_1, w_2, w_3]_X = I_2) \Leftrightarrow ([w_1, w_2, w_3]_G = I_2),$$

moreover it has the same direction in X as in G and so on, the direction of the edge between the vertices w_j, w_{j+1} is determined by the condition

$$([w_{j-1}, w_j, w_{j+1}]_X = I_2) \Leftrightarrow ([w_{j-1}, w_j, w_{j+1}]_G = I_2).$$

and it has the same direction in X and G . We proved that the direction of the edge connecting vertices v', v'' is the same in X as in G , therefore we proved $X = G$. If we suppose the converse: $(v_2, v_1) \in E(X)$ then $X^T = G$ by the previous case. $\square$

Definition 2.28. For pairwise distinct positive integers i, j, k greater than 2 we define the digraph $O_{i \rightarrow j \rightarrow k}$ the following way. Let $V(O_i) = \{v_1, v_2, \dots, v_i\}$, $V(O_j) = \{v'_1, v'_2, \dots, v'_j\}$ and $V(O_k) = \{v''_1, v''_2, \dots, v''_k\}$. Now let $V(O_{i \rightarrow j \rightarrow k}) = V(O_i \dot{\cup} O_j \dot{\cup} O_k)$ and

$$E(O_{i \rightarrow j \rightarrow k}) = E(O_i \dot{\cup} O_j \dot{\cup} O_k) \cup \{(v_1, v_1), (v'_1, v'_1), (v''_1, v''_1), (v_1, v'_1), (v'_1, v''_1)\}.$$

$O_{i \rightarrow j \leftarrow k}$ is defined similarly, by modifying the definition of $O_{i \rightarrow j \rightarrow k}$ naturally by replacing (v'_1, v''_1) with (v''_1, v'_1) .

In words, $O_{i \rightarrow j \rightarrow k}$ and $O_{i \rightarrow j \leftarrow k}$ consist of three disjoint circles O_i, O_j, O_k with one loop on each that are connected in the way $i \rightarrow j \rightarrow k$ and $i \rightarrow j \leftarrow k$ according to the sizes of the circles they are on (see Fig. 10).

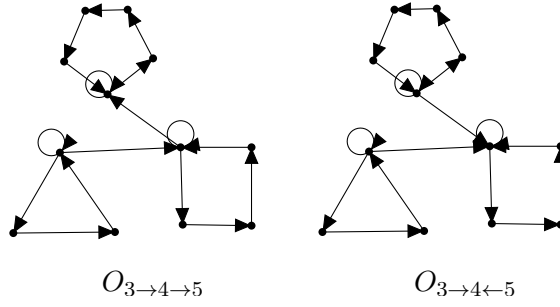


Figure 10: The digraphs $O_{3 \rightarrow 4 \rightarrow 5}$ and $O_{3 \rightarrow 4 \leftarrow 5}$.

Lemma 2.29. *Let i, j, k be pairwise distinct positive integers bigger than 2. Then the sets $\tilde{O}_{i \rightarrow j \rightarrow k}$ and $\tilde{O}_{i \rightarrow j \leftarrow k}$ are definable.*

Proof. $\tilde{O}_{i \rightarrow j \rightarrow k}$ is the set of those digraphs X that:

- have $i + j + k$ vertices,
- $O_L(i, j, k) \leq X$,
- are on level $2(i + j + k) + 5$,
- there exist $Z \in \tilde{O}_{i,j,L1}$ and $W \in \tilde{O}_{j,k,L1}$ such that $Z, W \leq X$, and
- $L(I_3) \leq X$.

We can define $\tilde{O}_{i \rightarrow j \leftarrow k}$ almost the same way as $\tilde{O}_{i \rightarrow j \rightarrow k}$, the difference is that we replace the condition $L(I_3) \leq X$ by: there exists $Z \in \mathcal{L}(\tilde{A})$ for which $Z \leq X$. $\square$

Definition 2.30. Let G be a finite directed graph with no loops and $\underline{v} = (v_1, v_2, \dots, v_n)$ be a vector containing all the vertices of G in some fixed order. Let us define the digraph $K(G, \underline{v})$ the following way. Let us consider the circles $\{O_i\}_{i=1}^n$ with $V(O_i) = \{v_{i,j} : 1 \leq j \leq n + i\}$. Now let

$$V(K(G, \underline{v})) = V\left(\bigcup_{1 \leq i \leq n} O_i\right),$$

$$E(K(G, \underline{v})) = E\left(\bigcup_{1 \leq i \leq n} O_i\right) \cup \{(v_{i,1}, v_{j,1}) : (v_i, v_j) \in E(G)\} \cup \{(v_{i,1}, v_{i,1}) : 1 \leq i \leq n\}.$$

In words we get $K(G, \underline{v})$ from G by adding loops and big, differently sized circles to all the vertices of G .

Example 2.31. Figure 11 shows $K(I_3, \underline{v})$ as an example (with $\underline{v} = (v_1, v_2, v_3)$).

This type of graph will be useful because in this graph—thanks to the big circles—we can distinguish the vertices of G which will allow us to define which pair of vertices are connected with how many edges, so we will be able to define the set $\mathcal{A}(G)$.

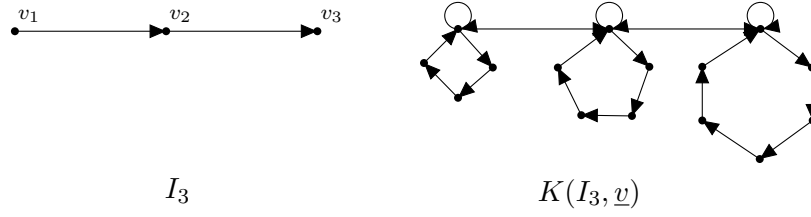


Figure 11: I_3 and a corresponding $K(I_3, \underline{v})$.

Lemma 2.32. *For an arbitrary weakly connected finite digraph $G \in \mathcal{D}$ with no loops the set $\tilde{G} = \{G, G^T\}$ is definable.*

Proof. First we consider all pairs of vertices that have edges in both directions between them and we clear one of the two edges for each such pair. We may get different digraphs clearing different edges but as for the proof it does not matter which one we take so let us fix one such digraph $O_2 \not\leq G'$ for the rest of the proof.

Let $\underline{v} = (v_1, v_2, \dots, v_n)$ be a fixed vector of the vertices of G . Let us define the set

$$\mathcal{H}_1 = \{K(G', \underline{v}), K((G')^T, \underline{v})\}.$$

This set contains exactly those digraphs X that:

- (1) have $(n+1) + (n+2) + \dots + (n+n) \left(= \frac{n(3n+1)}{2}\right)$ vertices,
- (2) $O_L(n+1, n+2, \dots, n+n) \leq X$,
- (3) $L_{n+1} \not\leq X$,
- (4) if there is no edge between the vertices v_i, v_j in G' then for all $Z \in \tilde{O}_{n+i, n+j, 1}$, $Z \not\leq X$ holds,
- (5) if there is an edge between the vertices v_i, v_j in G' then there is an $Z \in \tilde{O}_{n+i, n+j, L1}$ for which $Z \leq X$ but for all $W \in \mathcal{O}_{n+i, n+j, 1}^+$, $W \not\leq X$ holds,
- (6) if $[v_i, v_j, v_k]_{G'} = I_2$, there exists a $Z \in \tilde{O}_{(n+i) \rightarrow (n+j) \rightarrow (n+k)}$ for which $Z \leq X$, and
- (7) if $[v_i, v_j, v_k]_{G'} \in \tilde{A}$ there exists a $Z \in \tilde{O}_{(n+i) \rightarrow (n+j) \leftarrow (n+k)}$ for which $Z \leq X$.

The conditions (1)–(3) ensure the structure of “big” circles, determine the number of loops. The conditions (4)–(5) tell how many and what kind of edges are to be drawn between the “big circles”. The conditions (1)–(5) define the set

$$\{K(Z, \underline{v}) : Z \in \mathcal{A}(G')\}$$

while the conditions (6)–(7), by Lemma 2.27, choose the set $\mathcal{H}_1$ from it.

It is easy to see that the conditions (1)–(5) can similarly define the set

$$\mathcal{H}_2 := \{K(Z, \underline{v}) : Z \in \mathcal{A}(G)\}$$

with the condition

- (8) if there are edges between the vertices v_i, v_j in both directions then there exists a digraph $Z \in \tilde{\mathcal{O}}_{n+i, n+j, L2}$ for which $Z \leq X$ but for any $W \in \mathcal{O}_{n+i, n+j, 2}^+$, $W \not\leq X$ holds

added (and writing G instead of G' in every other condition, naturally). Now the set

$$\mathcal{H}_3 := \{K(G, \underline{v}), K((G)^T, \underline{v})\}$$

can be defined the following way. It consists of those digraphs X that:

- (9) $X \in \mathcal{H}_2$, and

- (10) there exists a digraph $Z \in \mathcal{H}_1$ for which $Z \leq X$.

Finally the set $\mathcal{H}_4 = \{L(G), L(G^T)\}$ consists of those X that:

- (11) have n vertices,

- (12) $L_n \leq X$,

- (13) $X \leq Z$ for some $Z \in \mathcal{H}_3$, and

- (14) X is maximal with the previous properties.

Finally, we prove the definability of $\tilde{G}$ using Lemma 2.15, for $\mathcal{M}(\mathcal{H}_4) = \{G, G^T\}$. $\square$

Let $G_1, G_2, \dots, G_n$ be the weakly connected components of an arbitrary $G \in \mathcal{D}$. Let $v_1 \in G_1, v_2 \in G_2, \dots, v_n \in G_n$ be arbitrary but fixed vertices. Let $N = |V(G)|$. Let us build the digraph H the following way. We add the vertices $v'_1, v'_2, \dots, v'_N$ to the digraph G so that they constitute an O_N circle. We make this digraph, having $n+1$ weakly connected components, weakly connected by adding the edges $(v'_1, v_1), (v'_2, v_2), \dots, (v'_n, v_n)$. It is clear that this construction depends only on the choice of the v_i 's.

Definition 2.33. Let us denote by $\mathcal{O}(G)$ the set of all digraphs H that can be created this way.

Example 2.34. In Figure 12 we can see a digraph G having 2 weakly connected components and an $H \in \mathcal{O}(G)$.

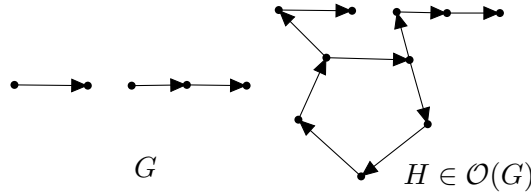


Figure 12: A digraph G and a corresponding $H \in \mathcal{O}(G)$.

Lemma 2.35. *For every finite directed graph $G \in \mathcal{D}$ with no loops the set $\{G, G^T\}$ is definable.*

Proof. We only need to deal with those $G \in \mathcal{D}$ that have more than one weakly independent components, for we have dealt with the other case before. Let us consider a digraph $H \in \mathcal{O}(G)$. Since H is weakly connected and does not have loops, we know that $\tilde{H}$ is definable by Lemma 2.32. Let H' be the digraph that we get from H by adding loops to all the vertices corresponding to G (that does not belong to the “big circle”).

Let $|V(G)| = m$. $\tilde{H}'$ is the set of those digraphs $X \in \mathcal{D}$ that

- there exists $Z \in \tilde{H}$ such that Z is m level under X , and
- $L_m \leq X$, $O_{m,L} \not\leq X$.

In H' there are loops exactly on those vertices that correspond to G which allows us to define $\mathcal{L}(\tilde{G})$ -t: it is the set of those digraphs X that:

- $X \leq Z$ for some $Z \in \tilde{H}'$,
- X has m vertices and $L_m \leq X$, and
- maximal with the previous properties.

We are done since $\tilde{G} = \mathcal{M}(\mathcal{L}(\tilde{G}))$ is definable by Lemma 2.15. □

Definition 2.36. Let σ_n denote the following digraph. Let $V(O_n) = \{v_1, \dots, v_n\}$ and let us define σ_n with $V(\sigma_n) = V(O_n) \cup \{v\}$ and $E(\sigma_n) = E(O_n) \cup \{(v_1, v)\}$. Let σ_n^L denote the digraph that is obtained from σ_n by the modification

$$E(\sigma_n^L) = E(\sigma_n) \cup \{(v, v)\}.$$

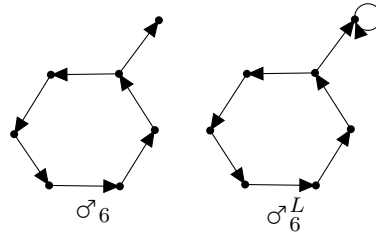


Figure 13: σ_6 and σ_6^L .

Since σ_n is weakly connected and has no loops, $\tilde{\sigma}_n$ is definable by Lemma 2.32.

Lemma 2.37. *The digraphs $\tilde{\sigma}_n^L$ ($n = 2, 3, \dots$) are definable.*

Proof. $\tilde{\sigma}_n^L$ is the set of those digraphs X that:

- $Z \prec X$ for some $Z \in \tilde{\sigma}_n$, and

- $L_1 \leq X$ and $O_{n,L} \not\leq X$.

□

Lemma 2.38. *For an arbitrary weakly connected finite directed graph $G \in \mathcal{D}$ the set $\{G, G^T\}$ is definable.*

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of G . Let us consider the circles $\{O_{n+i}\}_{i=1}^n$ so that $V(O_{n+i}) = \{v_{n+1,j} : 1 \leq j \leq n+i\}$. We define the digraph G_{σ} with

$$V(G_{\sigma}) = V\left(G \dot{\cup} \left(\dot{\bigcup}_{i=1}^n O_{n+i}\right)\right)$$

$$E(G_{\sigma}) = E\left(G \dot{\cup} \left(\dot{\bigcup}_{i=1}^n O_{n+i}\right)\right) \cup \{(v_{n+i,1}, v_i) : 1 \leq i \leq n\}$$

In words, we add big circles to G and connect them to the vertices of G with edges pointing in G 's direction. (G_{σ} depends on the order of $v_1, v_2, \dots, v_n$, we do not emphasize this in the notation because we need this structure only once and here it is enough to think of any fixed order of the vertices.) An example of the construction is shown in Figure 14.

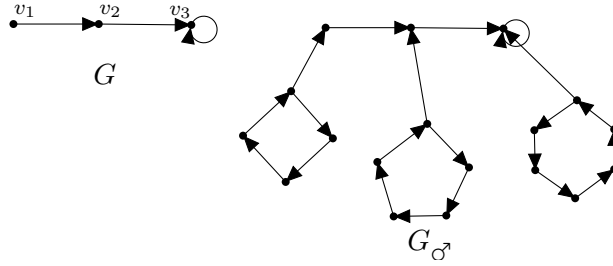


Figure 14: G_{σ} for a given G with a given order of vertices.

Since $M(G_{\sigma})$ is weakly connected and has no loops, the set $\mathcal{M}(\tilde{G}_{\sigma})$ is definable by Lemma 2.32. We can suppose that G has at least one loop because we dealt with the other case in Lemma 2.32. Let $v_{i_1}, v_{i_2}, \dots, v_{i_k}$ be the list vertices of G with loops. $\tilde{G}_{\sigma}$ is now the set of those digraphs X that:

- are k level above some $Z \in \mathcal{M}(\tilde{G}_{\sigma})$, and
- for every $1 \leq l \leq k$ there exists $Z \in \tilde{\sigma}_{n+i_l}^L$ for which $Z \leq X$.

Let $G'_{\sigma} = G \dot{\cup} \left(\dot{\bigcup}_{i=1}^n O_{n+i}\right)$. Then $\tilde{G}'_{\sigma}$ is the set of those X that:

- are n level under Z for some $Z \in \tilde{G}_{\sigma}$,
- $Z \in \tilde{\sigma}_{n+i}$ implies $Z \not\leq X$ for every $1 \leq i \leq n$, and
- $O_{n+i} \leq X$ for every $1 \leq i \leq n$.

Finally, $\tilde{G}$ consists of those X that:

- $X \leq Z$ for some $Z \in \tilde{G}'_{\sigma}$,

- have n vertices and $L_k \leq X$, and
- $Z \leq X$ for some $Z \in \mathcal{M}(\tilde{G})$. □

Lemma 2.39. *For a digraph $G \in \mathcal{D}$ that has at least one loop in every weakly connected components, the set $\{G, G^T\}$ is definable.*

Proof. We can still suppose that G have more than one weakly independent compotents, for we have dealt with the other case previously. Let us consider a digraph $H \in \mathcal{O}(G)$. Since H is weakly connected, the set $\tilde{H}$ is definable by Lemma 2.38. Let $n(> 1)$ be the number of weakly connected components of G and m be the number if its vertices. Let H' be the digraph that we get from H by leaving the edges out that connect the “big circle” to the components of G so we leave n edges out and get $H' = G \dot{\cup} O_m$. $\tilde{H}'$ contains exactly those digraphs X that

- are n level under Z for some $Z \in \tilde{H}$,
- there is no $Z \in \tilde{\sigma}_m$ for which $Z \leq X$, and
- $O_m \leq X$.

H' consists of $n+1$ weakly connected components from which exactly n has loops in them, exactly the components corresponding to G . Finally $\tilde{G}$ is the set of those X that:

- $X \leq Z$ for some $Z \in \tilde{H}'$,
- have m vertices and are on the same level as G ,
- have the same number of loops as G has, and
- $Z \leq X$ for some $Z \in \mathcal{M}(\tilde{G})$ ($\mathcal{M}(\tilde{G})$ is definable by Lemma 2.35). □

Theorem 2.40. *The set $\{G, G^T\}$ is definable for an arbitrary digraph $G \in \mathcal{D}$.*

Proof. We can still suppose G have more than one weakly independent compotents for the same reason as above. Let us consider a digraph $H \in \mathcal{O}(G)$ again. By Lemma 2.38 the set $\tilde{H}$ is definable. We can define $\tilde{H}'$ just as in the proof of Lemma 2.39. Let G_L denote the digraph that consists of those weakly connected components of G that contains loops. By Lemma 2.39 the set $\tilde{G}_L$ is definable. $\tilde{G}$ is the set of those X that:

- has the same number of vertices and are on the same level as G ,
- $X \leq Z$ for some $Z \in \tilde{H}'$,
- $Z \leq X$ for some $Z \in \mathcal{M}(\tilde{G})$, and
- $Z \leq X$ for some $Z \in \tilde{G}_L$. □

3 The automorphism group of $(\mathcal{D}, \leq)$

So far we know two automorphisms of $(\mathcal{D}, \leq)$, namely the trivial one and $G \mapsto G^T$. In this section we prove that there is no other, meaning that the automorphism group of $(\mathcal{D}, \leq)$ is isomorphic to $\mathbb{Z}_2$.

Lemma 3.1. $G^T \leq G \dot{\cup} O_n$ implies $G = G^T$ for every finite digraph G and integer $1 < n$.

Proof. Our first easy observation is that $X \leq O_n$ implies $X = X^T$. Let us denote the set of weakly independent components of G by $\{G_a\}_{a \in A}$. Let $A = B \dot{\cup} C$ such that that $b \in B$ if and only if G_b is embeddable into O_n . Now let us suppose that $G^T \leq G \dot{\cup} O_n$. With the notation just introduced

$$\begin{aligned} \left(\dot{\bigcup}_{a \in A} G_a \right)^T &\leq \left(\dot{\bigcup}_{a \in A} G_a \right) \dot{\cup} O_n \\ \dot{\bigcup}_{a \in A} G_a^T &\leq \left(\dot{\bigcup}_{a \in A} G_a \right) \dot{\cup} O_n \\ \left(\dot{\bigcup}_{b \in B} G_b^T \right) \dot{\cup} \left(\dot{\bigcup}_{c \in C} G_c^T \right) &\leq \left(\dot{\bigcup}_{b \in B} G_b \right) \dot{\cup} \left(\dot{\bigcup}_{c \in C} G_c \right) \dot{\cup} O_n \end{aligned}$$

which obviously implies

$$\dot{\bigcup}_{c \in C} G_c^T \leq \left(\dot{\bigcup}_{c \in C} G_c \right) \dot{\cup} \underbrace{\left(\dot{\bigcup}_{b \in B} G_b \right)}_{:=X} \dot{\cup} O_n. \quad (2)$$

If there exists a $c \in C$ for which $G_c^T \leq X$, then $G_c^T \leq O_n$, for X consists of weakly connected components embeddable into O_n . Then, according to our first observation, $G_c^T = G_c$ which means $G_c \leq O_n$, a contradiction. This means there is no $c \in C$ for which $G_c^T \leq X$ so from (2) we deduce

$$\begin{aligned} \dot{\bigcup}_{c \in C} G_c^T &\leq \dot{\bigcup}_{c \in C} G_c \\ \left(\dot{\bigcup}_{c \in C} G_c \right)^T &\leq \dot{\bigcup}_{c \in C} G_c. \end{aligned}$$

By transposing both sides the embeddability stays true obviously, but we get the converse, implying

$$\left(\dot{\bigcup}_{c \in C} G_c \right)^T = \dot{\bigcup}_{c \in C} G_c.$$

Using our first observation once more, we obtain

$$\left(\dot{\bigcup}_{b \in B} G_b \right)^T = \dot{\bigcup}_{b \in B} G_b^T = \dot{\bigcup}_{b \in B} G_b.$$

Finally, putting together what we have we get

$$G = \dot{\bigcup}_{a \in A} G_a = \left(\dot{\bigcup}_{b \in B} G_b \right) \dot{\cup} \left(\dot{\bigcup}_{c \in C} G_c \right) = \left(\dot{\bigcup}_{b \in B} G_b \right)^T \dot{\cup} \left(\dot{\bigcup}_{c \in C} G_c \right)^T = G^T. \quad \square$$

Definition 3.2. Let us call X_n the digraph

$$V(X_n) = \{v, v_1, v_2, \dots, v_n\}, \quad E(X_n) = \{(v_1, v), (v_2, v), \dots, (v_n, v)\}.$$

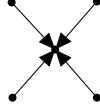


Figure 15: The digraph X_4 .

Theorem 3.3. *The poset $(\mathcal{D}, \leq)$ has exactly two automorphisms, namely the trivial and the one that maps every digraph to its transpose. Consequently, the automorphism group of $(\mathcal{D}, \leq)$ is isomorphic to $\mathbb{Z}_2$.*

Proof. It is easily seen that an automorphism can only move the elements of $\mathcal{D}$ inside definable sets, therefore from Theorem 2.40 it follows that it either does not move an element or maps it to its transpose. The method of the proof is the following. Let us consider an automorphism $\varphi : \mathcal{D} \mapsto \mathcal{D}$. If for every digraph $G \in \mathcal{D}$, $\varphi(G) = G$ holds then φ is trivial. If there is a digraph $G \in \mathcal{D}$ whose image is its transpose that is not isomorphic to it, then we show that $\varphi(A) = A^T$ (for the definition of A , see 2.7). Finally we show that if an automorphism $\varphi : \mathcal{D} \mapsto \mathcal{D}$ satisfies $\varphi(A) = A^T$ then for every $G \in \mathcal{D}$, $\varphi(G) = G^T$ holds. Briefly we will show the implications

$$(\exists G \neq G^T)(\varphi(G) = G^T) \Rightarrow \varphi(A) = A^T \quad (3)$$

$$\varphi(A) = A^T \Rightarrow (\forall G)(\varphi(G) = G^T). \quad (4)$$

These implications are of the form

$$\varphi(X) = X^T \Rightarrow \varphi(Y) = Y^T$$

and can be proved the following way. We show that by adding the digraph X to the language of partially ordered sets as a constant the element Y becomes definable.

Proof of (3). Let $G(\neq G^T)$ be an arbitrary digraph with $V(G) = \{v_1, v_2, \dots, v_n\}$. Let us add G to the language of posets as a constant. Lemma 3.1 lets us define $G \dot{\cup} O_{n+1}$ as the unique element from the definable set

$$\{G \dot{\cup} O_{n+1}, (G \dot{\cup} O_{n+1})^T = G^T \dot{\cup} O_{n+1}\}$$

that G is embeddable into. Let we use the notation

$$V(G \dot{\cup} O_{n+1}) = V(G) \cup \{v'_1, v'_2, \dots, v'_{n+1}\}.$$

We create a digraph G' by adding edges to $G \dot{\cup} O_{n+1}$ as follows:

$$E(G') = E(G \dot{\cup} O_{n+1}) \cup \{(v'_1, v_1), (v'_2, v_1), (v'_3, v_1), \dots, (v'_{n+1}, v_1)\}.$$

Now G' is definable as the unique element of the set $\{G', (G')^T\}$ that is $G \dot{\cup} O_{n+1}$ embeddable into. X_{n+1} is the unique element from the set $\{X_{n+1}, (X_{n+1})^T\}$ that is embeddable into G' so it is definable too. Finally, A is the unique element from the set $\{A, A^T\}$ that is embeddable into X_{n+1} .

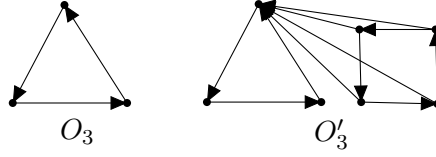


Figure 16: The digraph O_3 and a corresponding O'_3 .

Proof of (4). Now we add Q to the language of posets as a constant. Let G be an arbitrary finite digraph. If G is isomorphic to its transpose then its definability is provided by Theorem 2.40 so we can assume $G \neq G^T$. It is easy to check that we can do the same as in the proof of (3), backwards. The definability of G follows in the steps

$$A \Rightarrow X_{n+1} \Rightarrow G' \Rightarrow G \dot{\cup} L(O_{n+1}) \Rightarrow G.$$

□

4 Table of notations

In the following table we collect the notations used in the paper.

Notation	Number of definition	Page number
σ_n	2.36	15.
σ_n^L	2.36	15.
(u, v, w)	2.26	10.
$[u, v, w]_G$	2.26	10.
A	2.7	6.
$\mathcal{A}(\cdot)$	2.25	10.
E_n	2.5	5.
F_n	2.5	5.
$\dot{\cup}$	2.18	8.
I_n	2.9	6.
$K(G, \underline{v})$	2.30	12.
L_{2+}	2.7	6.
L_n	2.9	6.
$L(\cdot)$	2.11	7.
$\mathcal{L}(\cdot)$	2.11	7.
$M(\cdot)$	2.14	8.
$\mathcal{M}(\cdot)$	2.14	8.
O_n	2.9	6.
$O_{n,L}$	2.16	8.
$O(n_1, n_2, \dots, n_k)$	2.19	8.
$O_L(n_1, n_2, \dots, n_k)$	2.19	8.
$O_{i,j,1}$	2.21	9.
$O_{i,j,2}$	2.21	9.
$O_{i,j,L1}$	2.21	9.
$O_{i,j,L2}$	2.21	9.
$\mathcal{O}_{i,j,1}^+$	2.23	10.
$\mathcal{O}_{i,j,2}^+$	2.23	10.
$O_{i \rightarrow j \rightarrow k}$	2.28	11.
$O_{i \rightarrow j \leftarrow k}$	2.28	11.
$\mathcal{O}(\cdot)$	2.33	14.
$\mathcal{S}_n$	2.3	5.

$\mathcal{T}_n$	2.5	5.
X_n	3.2	19.

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