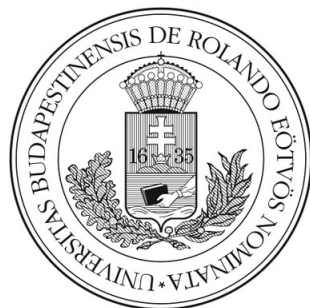


Eötvös Loránd Tudományegyetem



Real solutions of a problem in enumerative geometry

OTDK paper

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A Denjoy-Young-Saks több dimenziós változatai

Kivonat

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Szakdolgozatom témája a Denjoy-Young-Saks tétel különböző általánosításainak vizsgálata több dimenzióra. A tétel egy dimenzióban azt mondja ki, hogy tetszőleges $\mathbb{R} \rightarrow \mathbb{R}$ függvény Dini-deriváltjai majdnem mindenütt 3 különböző típusú konfigurációban léphetnek fel. A tétel általánosításaiban érzékeny pont a megfelelő derivált értelmezése és a deriváltak mérhetőségi tulajdonságai. A dolgozat célja az eddigi kiterjesztések megismerése, és továbbiak vizsgálata.

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Abstract

The subject of this paper is a variant of the following problem from enumerative geometry: Given 4 straight lines of general position in space, how many lines can one find, that intersect the given four? This problem can be generalised to higher dimensions: Given 4 subspaces of dimension $2k$ in general position, how many $2k$ -dimensional subspaces can one find, that intersects the given four in k dimensions? The answer depends on how many k -dimensional invariant subspaces a certain linear transformation has. So the answer in the complex case is always the same, but in the real case this depends on the aforementioned linear transformation. However, assigning signs to the solutions according to a certain rule, the signed sum must be constant, since this signed sum can be also given by an integral. In this paper we show how to determine the sign of a solution.

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1 Introduction

The types of problems introduced in the abstract, have already been studied in the 19th century. Schubert developed a method - called Schubert calculus - which gave an algorithm to calculate a wide range of enumerative problems. It took a long time to justify his methods by the effort of many mathematicians developing cohomology theory and intersection theory. However one important limitation of the method, is that it works only over the complex field. The question naturally arises over the reals as well. Recently finding real solutions has become a popular field of enumerative geometry, but no general theory has emerged yet.

In certain cases the cohomological methods can be adapted to the real situation but - in contrast with the complex case - the solutions come with a sign and we can calculate only the signed sum, giving only a lower bound instead of the exact answer. In some sense this is the best we can hope for, since the number of solutions depends on the configuration: the subvariety of non-generic configurations is one-codimensional, so in the complex case the space of generic configurations is connected, but in the real case we have "chambers". E.g. in the 4 lines problem the number of complex solutions is 2 for a generic configuration and we have 2 or 0 real solutions. Several calculations were done by Sottile et al. [3] and others but the general picture is not clear.

In this paper we study a class of problems, where the number of solutions can be determined by linear algebra (as explained to us by Tamás Terpai). So on one hand we know the signed sum from cohomology and on the other hand we know the unsigned sum by linear algebra.

In this paper we try to fill the gap, calculating the signs of the solutions. The main result is Theorem 2. Similar work has been done for the famous 27 lines on a cubic surface case by [2]. We hope that working out an infinite family of problems can help the emergence of a general theory of real enumerative geometry.

2 The number of unsigned solutions

The statement of the problem is the following:

Problem Given four $2k$ -dimensional subspaces of general position in $E := \mathbb{R}^{4k}$, how many $2k$ -dimensional subspaces are there, that intersect the given four in k dimensions?

The following approach is due to Tamás Terpai.

Denote the four subspaces $V_i \in \text{Gr}_{2k}(E)$, $i = 1, 2, 3, 4$. We shall denote an arbitrary solution by W . First, using that the subspaces are of general position, we have $E = V_1 \oplus V_2$. From this, it is clear that for each $v_1 \in V_1$, one has a uniquely determined $v_2 \in V_2$, such that $v_1 + v_2 \in V_3$. Thus V_3 and V_4 may both be regarded as graphs of linear transformations.

It is possible to choose basis vectors in both V_1 and V_2 , such that V_3 is the graph of the linear transformation $\alpha \in \text{Hom}(V_1, V_2)$ given by the identity matrix: $V_3 = \text{Graph}(\alpha)$. (Any choice of basis vectors determines a linear transformation, and by transforming the basis vectors one can obtain α as the identity.) In a similar fashion (with now fixed basis vectors), let us denote $\beta \in \text{Hom}(V_1, V_2)$, such that $V_4 = \text{Graph}(\beta)$.

Now we define a $\varphi \in \text{Aut}(V_1)$ linear automorphism of V_1 through α and β by $\varphi := \alpha^{-1} \circ \beta$. Denote by $W_i := W \cap V_i$, $i = 1, 2, 3, 4$. Note, that $\alpha(W_1) = W_2$ and $\beta(W_1) = W_2$, since $W = W_1 \oplus W_2$, thus for each $w \in W_i \subset W$ there exists (unique) $w_1 \in W_1, w_2 \in W_2$ such that $w = w_1 + w_2$, for both $i = 3, 4$.

This implies that $\varphi(W_1) = W_1$. This means that W_1 is a k -dimensional invariant subspace of φ . On the other hand, if one gives a k -dimensional invariant subspace W_0 of φ , then one obtains a W solution by taking $W := W_0 \oplus \alpha(W_0) = W_0 \oplus \beta(W_0)$. It follows that in the complex case one always has $\binom{2k}{k}$ solutions - each eigenvalue has a one-dimensional eigenspace associated to it, hence the $\binom{2k}{k}$ solutions. However in the real case φ might have complex conjugate eigenvalue pairs, and thus 2-dimensional invariant subspaces, hence less solutions, so the number of solutions can be stated as a function of φ . Notice, that in the case of multiple eigenvalues the number of solutions can be infinite, so we use the following definition:

Definition (Generic $2k$ -space) Four pairwise transversal $2k$ -spaces in E are called *generic* if the eigenvalues of the corresponding linear map α are all different.

3 The number of signed solutions

By cohomological methods the following result can be obtained¹

Theorem 1 *Fix $k = 2l$ ($l \in \mathbb{N}$). Given four generic $2k$ -dimensional subspaces in $E := \mathbb{R}^{4k}$, with real eigenvalues associated to them, the signed sum – the sign determined according to the rule detailed in section 3.4 – of the $2k$ -dimensional subspaces intersecting the given four in k dimensions is $\binom{2l}{l}$.*

We will reprove a this theorem by explicitly determining the signs of the solutions. Notice, that the Schubert calculus on the method of Tamás Terpai gives $\binom{2k}{k}$ complex solutions.

3.1 Determining the sign of a solution

Since all eigenvalues $\lambda_i \in \mathbb{R}$, $i = 1, \dots, k$ were supposed as distinct reals, all $\binom{2k}{k}$ solutions are realized. Thus the four given V_i subspaces may be written in the following form:

$$\begin{aligned} V_1 &= \langle x_1, x_2, \dots, x_{2k} \rangle \\ V_2 &= \langle x_{2k+1}, x_{2k+2}, \dots, x_{4k} \rangle \\ V_3 &= \langle x_1 + x_{2k+1}, x_2 + x_{2k+2}, \dots, x_{2k} + x_{4k} \rangle \\ V_4 &= \langle x_1 + \lambda_1 x_{2k+1}, x_2 + \lambda_2 x_{2k+2}, \dots, x_{2k} + \lambda_{2k} x_{4k} \rangle, \end{aligned}$$

In this way, the solutions can be given explicetly, by considering the (x_i, x_{2k+i}) as pairs ($i = 1, \dots, 2k$). The solutions W_S are now given by any k choice of these vector pairs; say in the case of the first k we have:

$$W_{S_0} = \langle x_1, x_{2k+1}, \dots, x_k, x_{3k} \rangle$$

S denotes a k -element subset of $\{1, 2, \dots, 2k\}$. It is easy to see that all the subspaces obtained in this manner satisfy the required conditions.

We can also consider the problem as being posed on the Grassmannian manifold $M := \text{Gr}_{2k}(E)$. For any given $V \in M$, define the singular submanifolds (subvarieties)

$$\sigma_V := \{W \in M : \dim(V \cap W) = k\}.$$

Using this definiton the solutions to our previously defined problem can be written as points of

$$\Sigma := \cap_{i=1}^4 \sigma_{V_i}.$$

The sign of the solutions are determined by calculating and comparing orientations of the tangent spaces of the subvarieties σ_{V_i} at the $W \in \Sigma$ solutions. First, fix an orientation of the tangent bundle TM (this is possible, since in our case $\text{Gr}_m(\mathbb{R}^n)$ $n = 2k$ is even, see later). Fix also an orientation of the $N_W \sigma_{V_i}$ normal spaces, for all $i = 1, 2, 3, 4$. Then

¹László Fehér, private communication

the sign of a W solution is determined by the orientation that the $(N_W \sigma_{V_i})_{i=1}^4$ (in this fixed order) give, as subspaces of $T_W M$ - this can be the same or the opposite as the fixed orientation of $T_W M$, giving a positive or a negative sign to the solution.

Remark 1 It is interesting to note, that the sign determined in this fashion is independent of the order of the $T_W \sigma_{V_i}$'s in the case that they are even dimensional. In other words we *don't even have to fix* an order of the σ_{V_i} 's. To see this, interchange two such tangent spaces - by considering the tangent vectors determining the orientation of the tangent space, it can be concluded that an even number of swaps have occurred, leaving the sign invariant.

3.2 Calculating the normal and tangent spaces

The following lemma is classical, see e.g. Kirillov [1] for a similar approach. To calculate the tangent spaces $T_W M$ of the Grassmannian, we use the following lemma about homogeneous spaces that we state without proof (see [1] Corollary 2.19, page 13):

Lemma 1 *Let G be a Lie group action on an M manifold ($G \curvearrowright M$), and denote the orbit of $m \in M$ by $O_m := G.m$. If O_m is an embedded manifold in M , then $T_m O_m = T_e G / T_e H$, where $H = \text{Stab}_G(m)$.*

By using the following lemma from linear algebra, we obtain an expression of both the tangent spaces $T_W M$ and the normal spaces $N_W \sigma_V$:

Lemma 2 *If $A = \text{Hom}(C, D)$, where C and D are finite-dimensional vector spaces, $C' \leq C$, $D' \leq D$ subspaces, then by noting $A' := \{\alpha \in A : \alpha(C') \subset D'\}$, we have*

$$A/A' \cong \text{Hom}(C', D/D').$$

Proof. Take the $\pi : D \rightarrow D/D'$ projection. By defining $\varphi : \text{Hom}(C, D) \rightarrow \text{Hom}(C', C/D')$ by $\varphi(\alpha) := \pi \circ \alpha|_{C'}$, we have $\ker(\varphi) = A'$. By applying the classical isomorphism theorem, this gives

$$A / \ker(\varphi) \cong \varphi(A) = \text{Hom}(C', D/D'), \quad (3.1)$$

which finishes the proof. $\square$

Corollary

$$T_W \text{Gr}(E) \cong \text{Hom}(W, E/W). \quad (3.2)$$

Proof. To apply Lemma 1 in order to calculate $T_W M$, we wish to have a group action G , such that $O_W = G.W = M$. Take the standard transitive group action $G := \text{GL}(E)$ on the Grassmannian, the stabilizer of W being

$$H := \{\alpha \in \text{GL}(E) : \alpha(W) \subset W\}.$$

By Lemma 1 we obtain

$$T_W \text{Gr}(E) = T_e G / T_e H$$

We use the fact that for a general vector space A , $T_e \text{GL}(A) = \text{Hom}(A, A)$, which gives $T_e G = \text{Hom}(E, E)$ and $T_e H = \{\alpha \in \text{Hom}(E, E) : \alpha(W) \subset W\}$.

Now use Lemma 2, with $C := E$, $D := E$, $C' := W$, and $D' := W$, $A = T_e G$, $A' = T_e H$, thus the lemma's statement:

$$T_W \text{Gr}(E) = T_e G / T_e H \cong \text{Hom}(W, E/W),$$

which finishes the proof. $\square$

Corollary

$$N_W \sigma_V \cong \text{Hom}(W \cap V, E/\langle W, V \rangle). \quad (3.3)$$

Proof. To apply Lemma 1 in order to calculate $T_W \sigma_{V_i}$, we wish to have a group action G' , such that $O'_W = G' \cdot W$ is the smooth part of σ_{V_i} . Take $G' = \{\alpha \in \text{GL}(E) : \alpha(V) = V\}$. This gives

$$T_e G' = \{\alpha \in \text{Hom}(E, E) : \alpha(V) \subset V\}.$$

Now use Lemma 2, with $C := W$, $D := E/W$, $C' := W \cap V$, and $D' := \pi(V)$, $A = T_W \text{Gr}(E)$. Using the (3.2) isomorphism, we obtain

$$T_W \sigma_V = \varphi(T_e G') = \{\beta \in \text{Hom}(W, E/W) : \beta(W \cap V) \subset \pi(V)\}, \quad (3.4)$$

which with Lemma 2's notation is $A' = T_W \sigma_V$, thus the lemma's statement:

$$N_W \sigma_V = T_W M / T_W \sigma_V \cong \text{Hom}(W \cap V, (E/W)/\pi(V)),$$

which finishes the proof. $\square$

3.3 Orienting the tangent spaces

For a moment consider the following general situation:

Lemma 3 *Let A and B both be even dimensional vector spaces with given $(a_i)_{i=1}^l \subset A$, and $(b_j)_{j=1}^m \subset B$ basis vectors and fix an arbitrary ζ ordering of the standard (a_i, b_j) basis of $\text{Hom}(A, B)$. Then this ζ ordering always defines the same orientation of $\text{Hom}(A, B)$, independently of the given bases.*

Proof. To see that this orientation is well-defined, take a different orientation on A , e.g. interchange two coordinates $a_i, a_j \in A$. On the standard basis of the $\text{Hom}(A, B)$ space this induces m swaps of coordinates $((a_i, b_k) \leftrightarrow (a_j, b_k))$, each swap corresponding to a reversal of the orientation. Since m is even, we obtain the same orientation. Similarly, if one interchanges two coordinates $b_i, b_j \in B$, an l number of swaps are introduced on the standard basis, and both dimensions were supposed to be even. $\square$

Returning to our case: by the $T_W \text{Gr}(E) \cong \text{Hom}(W, E/W)$ isomorphism, $\dim(W) = 2k$, $\dim(E/W) = 2k$ are both even. Using the $N_W \sigma_V \cong \text{Hom}(W \cap V, E/\langle W, V \rangle)$ isomorphism, we also have $\dim(W \cap V) = k$ and $\dim(E/\langle W, V \rangle) = k$ which are even by definition. Thus by fixing any ζ ordering of the Hom spaces we obtain a consistent orientation of the both the tangent and normal spaces.

3.4 Calculating the sign of a solution

In this section we explicitly give a formula that determines the sign of a solution.

To this end, we calculate the normal spaces $N_W\sigma_{V_i}$ explicitly, by using the previously obtained

$$N_W\sigma_V \cong \text{Hom}(W \cap V, E/\langle W, V \rangle).$$

We have seen that the W solutions are all spanned by pairs of $(x_i, x_{2k+i})_{i \in S}$, where $S \subset \{1, 2, \dots, 2k\}$ and $|S| = k$. Since choosing a W solution divides the $[(x_i, x_{2k+i})]_{i=1}^{2k}$ pairs into two groups (ones spanning the W solution, the others spanning its direct complement), denote the eigenvalues associated to the pairs spanning the solution by $(\alpha_i)_{i=1}^k$, the eigenvalues spanning its orthogonal complement by $(\beta_i)_{i=1}^k$. This also defines a permutation π_S of $(1, 2, \dots, 2k)$, by its first k elements being the elements of S in increasing order, and the last k elements belonging to S^c , also in increasing order. We shall use this later.

3.4.1 Explicit form of the normal spaces

We specify the normal spaces as subspaces of the $4k^2$ -dimensional $T_W Gr = \text{Hom}(W, W^\perp)$ by introducing the $\langle \cdot, \cdot \rangle$ scalar product determined by the basis $(x_i)_{i=1}^{4k}$. The most convenient choice of basis seems to be writing the Hom spaces of W and $W^\perp$ by writing the $(x_i, x_{2k+i})_{i \in S}$ pairs in e.g. increasing order.

Consider the general case: A, B m -dimensional vector spaces, with given $(a_i)_{i=1}^m \subset A$, $(b_i)_{i=1}^m \subset B$ bases and take the $\langle \cdot, \cdot \rangle_A$ scalar product defined by A 's basis. Take $H \leq A$, $K \leq B$ l -dimensional subspaces; in our case we naturally obtain a $(h_i)_{i=1}^l \subset H$ and a $(k_j)_{j=1}^l \subset K$ basis. First, using the $\langle \cdot, \cdot \rangle_A$ scalar product, extend a general $\varphi \in \text{Hom}(H, K)$ transformation to $\tilde{\varphi} \in \text{Hom}(A, B)$ (with $\tilde{\varphi}|_{H^\perp} \equiv 0$.) We want to express the matrix of this $\tilde{\varphi}$ transformation in the given (a_i) és (b_j) bases. It is sufficient to express the $\tilde{e}_{ij}$ extensions of the $e_{ij} := (h_i \mapsto k_j)$'s in the standard basis $(a_i \mapsto b_j)$ of $\text{Hom}(A, B)$:

$$h_i = \sum_{j=1}^m \alpha_{ij} a_j$$

and

$$k_i = \sum_{j=1}^m \beta_{ij} b_j,$$

but in our case, since h_i and k_j are each the linear combination of only two a_k and b_l respectively, thus the general $\tilde{e}_{ij}$ transformation is also given by a 2×2 matrix (with 0's elsewhere). Since the general form of the h_i and k_j expressions are the same, we use the results of Appendix B and obtain the general form of a basis element.

Thus we obtain that $N_W \sigma_{V_i} = \text{span}(e_{11}^i, e_{12}^i, \dots, e_{kk}^i)$ for $i = 1, 2, 3, 4$ (we omit the W dependence), where

$$e_{rs}^i = e_{rs} \otimes b_{rs}^i,$$

where e_{rs} is the $k \times k$ matrix with a single 1 in the (r, s) position and 0's elsewhere, $\otimes$ denotes the Kronecker product and

$$b_{rs}^1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$b_{rs}^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$b_{rs}^3 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

$$b_{rs}^4 = \begin{pmatrix} \beta_r & \alpha_s \beta_r \\ -1 & -\alpha_s \end{pmatrix}$$

In other words, the basis consists of $2k \times 2k$ -sized matrices, divided into 2×2 blocks, with only one (r, s) block being b_{rs}^i . Notice, that only $N_W \sigma_{V_4}$ is dependent on the eigenvalues.

3.4.2 Calculating the orientation

In a general setting, if one has a V vector space with a fix $(f_i)_{i=1}^n$ basis, and an orientation determined by this basis, then the orientation given by a set of $(v_i)_{i=1}^n$ vectors can be determined by the sign of the determinant

$$\det \begin{pmatrix} v_{11} & v_{12} & \dots & v_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n1} & v_{n2} & \dots & v_{nn} \end{pmatrix}, \quad (3.5)$$

where $v_i = \sum_{j=1}^n v_{ij} f_i$.

Returning to our problem, we finally fix the orientation of the $\text{Hom}(W, W^\perp)$ space. First, we express the standard basis in the following way:

$$f_{rs}^i = e_{rs} \otimes c_i,$$

where the e_{rs} are as before and

$$c_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$c_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$c_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$c_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The orientation on $\text{Hom}(W, W^\perp)$ is the following: $((f_{rs}^1), (f_{rs}^2), (f_{rs}^3), (f_{rs}^4))$, where (f_{rs}^i) for fix i are ordered lexicographically.

Now by taking the just defined orientation, from the b_{rs}^i matrices, we express the determinant (3.5) for our problem:

$$N := \left(\begin{array}{c|c} I_{2k^2} & L \\ \hline 0_{2k^2} & M \end{array} \right),$$

whose determinant is equal to $\det(M)$, where

$$M := \left(\begin{array}{c|c} -I_{k^2} & -A \\ \hline I_{k^2} & B \end{array} \right),$$

where $A = I_k \otimes D_\alpha$, $B = D_\beta \otimes I_k$; D_α, D_β denote the diagonal matrices formed by $(\alpha_i)_{i=1}^k$ and $(\beta_i)_{i=1}^k$ respectively, and $\otimes$ denotes the Kronecker product.

The determinant of this M matrix is given by

$$d_W := \det(M) = \det(B - A) = \prod_{i=1}^k \prod_{j=1}^k (\beta_j - \alpha_i).$$

According to the previous discussion, it is the sign of this product, that determines the sign of a single W solution.

This establishes our main theorem:

Theorem 2 *The sign of a solution W is determined by the sign of the product*

$$\prod_{i=1}^k \prod_{j=1}^k (\beta_j - \alpha_i),$$

or the same, the sign of the π_S permutation using our previous notations.

Remark 2 Note also that the *genericity condition* can now be made more precise: according to the previous definition of genericity, all eigenvalues are distinct, thus the σ_{V_i} intersect transversally, since the $(N_W \sigma_{V_i})_{i=1}^4$ normal spaces span the $T_W \text{Gr}_{2k}(E)$ tangent space at each W solution.

Remark 3 Theorem 2 associates signs to the solutions also when k is odd, however the geometrical interpretation isn't clear, since the normal bundles are not orientable in this case.

Remark 4 It is interesting to note the resemblance of this determinant with the resultant of two polynomials. In our case $d_W = \text{Res}(\xi_W, \xi_{W^\perp})$, that is the resultant of the characteristic polynomials of the two linear transformations used in the first chapter, determined this time through W (for ξ_W) and its orthogonal complement (for $\xi_{W^\perp}$).

The following lemma is proved in Appendix A:

Lemma 4 *Take $k = 2l$. Consider permutations of $1, \dots, 2k$, divided into two consecutive groups of k and only those, in which elements of both groups are written in increasing order. Then the signed sum of the permutations (with the sign being that of the permutation) is $\binom{2l}{l}$. For k odd, the signed sum is 0.*

Using the previous lemma, the theorem implies the theorem stated in the introduction:

Corollary *The signed sum of the solutions is $\binom{2l}{l}$ (where $k = 2l$). This gives a lower bound on the number of real solutions.*

Note that in this proof we used the fact that the λ_i eigenvalues are real. The case with complex conjugate eigenvalues can be traced back to the case proved here, however due to time constraints the proof did not fit in the paper. However, it can be seen from the proof of the lemma found in the Appendix that the previous reasoning adapted to the complex conjugate case gives the same signed sum as has been determined.

A Proof of the signed sum of the solutions

We repeat Lemma 4 that is needed to establish the signed sum of the solutions:

Lemma Take $k = 2l$. Consider permutations of $1, \dots, 2k$, divided into two consecutive groups of k and only those, in which elements of both groups are written in increasing order. Then the signed sum of the permutations (with the sign being that of the permutation) is $\binom{2l}{l}$. For k odd, the signed sum is 0.

Proof. Consider first only those permutations, in which $(2j+1, 2j+2)$, $j = 0, \dots, k-1$ are in the same group (the case of all eigenvalues being complex conjugate). Since permuting pairs of consecutive numbers doesn't change the sign of the permutation, each solution has a positive sign, so the signed sum is $\binom{2l}{l}$.

The proof follows by induction, by dividing the pairs one-by-one. We have just proved the statement for $N = k$ pairs. Suppose the statement proved for N , and now prove the statement for $N - 1$ pairs. Denote the currently divided pair by (a, b) ($b = a + 1$). Comparing with the case of N pairs, there are an even number of new permutations, (maybe zero, in which case there is nothing to prove) and in these 'new' permutations a and b appear in different groups. (Otherwise, they were already present in the $N - 1$ case.) By swapping a and b the sign of the permutation will change, since b and a are consecutive. So to each new solution, by assigning the solution obtained by swapping a and b we have pairs of solutions of different signs, so the signed sum of the solutions doesn't change.

If k is odd, we obtain 0 for the signed sum, since at the first step of the induction, there are two solutions; one positive, and one negative (since we interchange two elements in an odd permutation), the rest of the proof goes unchanged. $\square$

This completes the proof of the Theorem 1.

Remark 5 Note also, that this approach also shows that the total number of solutions is congruent mod 4. (For $k > 1$ that is.) To see this, in the induction step take any already divided (c, d) pair. The new solutions (by considering the just divided (a, b) pair) are obtained by interchanging the elements of (a, b) with an already divided pair (in our case (c, d)). For a fix (c, d) pair, the new solutions are those, in which exactly one element is interchanged. It is easy to see, that there are 4 such new permutations ($\binom{4}{2} = 6$, 2 of which have been already considered).

B Calculating $N_W\sigma_{V_i}$'s for the 4-lines problem

To justify the calculations of the normal spaces, we calculate the normal and tangent spaces for the simplest case, of four lines in space.

The four V_i 's can be given as

$$\begin{aligned} V_1 &= \langle a, b \rangle \\ V_2 &= \langle c, d \rangle \\ V_3 &= \langle a + c, b + d \rangle \\ V_4 &= \langle a + \alpha c, b + \beta d \rangle, \end{aligned}$$

which give the following two solutions:

$$\begin{aligned} W_1 &= \langle a, c \rangle \\ W_2 &= \langle b, d \rangle. \end{aligned}$$

We take $W = W_1$ and calculate the tangent and normal spaces at this point:

$$T_W Gr = \text{Hom}(W, W^\perp)$$

which can be represented as 2×2 matrices ($M : \langle a, c \rangle \rightarrow \langle b, d \rangle$).

Using $N_W\sigma_V \cong \text{Hom}(W \cap V, E/\langle W, V \rangle)$:

since $W \cap V_1 = \langle a \rangle \subset E$ and $\langle W, V_1 \rangle^\perp = \langle d \rangle \subset E$ we have:

$$N_W\sigma_{V_1} \cong \left\langle \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\rangle$$

since $W \cap V_2 = \langle c \rangle \subset E$ and $\langle W, V_2 \rangle^\perp = \langle b \rangle \subset E$ we have:

$$N_W\sigma_{V_2} \cong \left\langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\rangle$$

since $W \cap V_3 = \langle a + c \rangle \subset E$ and $\langle W, V_3 \rangle^\perp = \langle b - d \rangle \subset E$ we have:

$$N_W\sigma_{V_3} \cong \left\langle \frac{1}{4} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \right\rangle$$

since $W \cap V_4 = \langle a + \alpha c \rangle \subset E$ and $\langle W, V_4 \rangle^\perp = \langle \beta b - d \rangle \subset E$ we have:

$$N_W\sigma_{V_4} \cong \left\langle \frac{1}{1 + \alpha^2} \cdot \frac{1}{1 + \beta^2} \begin{pmatrix} \beta & \alpha\beta \\ -1 & -\alpha \end{pmatrix} \right\rangle$$

where the normalising factors are obviously of no importance.

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