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Ambiguity neutrality

Ambiguitássemlegesség

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Chapter 1

Introduction

Uncertainty is omnipresent in all branches of economics. Throughout science from physics to genetics, the “natural” form of uncertainty plays an important role, requiring well-defined mathematical foundations. In modern economics however, uncertainty takes another form as well, since economic phenomena are greatly affected by the complex interactions of decision makers. These interactions tend to yield different outcomes in seemingly similar situations, which is why we call them the “human” form of uncertainty. Probability theory and game theory deal with the respective forms of uncertainty, while decision theory combines the two. The main question of decision theory is: how do human decision makers (which, by definition, are subjective beings) respond to situations marginally influenced by the natural (objective) uncertainty?

Originally, one of the motivations behind decision theory was to define and characterize the different attitudes towards risk. The intuition was that a decision maker can be risk averse, risk neutral or risk seeker, or of course, neither of these.

One of the results of early decision theory was that decision makers’ preferences over the alternatives can be represented by their expected utility indices à la von Neumann and Morgenstern (1947). The convexity of the Bernoulli utility functions determined the decision makers’ attitude towards risk. Ellsberg (1961) however, raised a question about a type of behavior that is easily observable in the real world but absent, or even contradictory in the classical expected utility theory.

His well-known example is as follows: Suppose you have an urn containing 30 red balls and 60 other balls that are either black or yellow. You do not know how many black or how many yellow balls there are, but that the total number of black balls plus the total number of yellow equals 60. The balls are well mixed so that each individual ball is as likely to be drawn as any other. You are now given a choice between Gambles A and B, and Gambles C and D.

Utility theory models the choice by assuming that in choosing between these gambles,

Gamble A	Gamble B
100\$, if a red is drawn	100\$, if a black is drawn
Gamble C	Gamble D
100\$, if a red or yellow is drawn	100\$, if a black or yellow is drawn

Table 1.1: The Ellsberg paradox

people assume a probability that the non-red balls are yellow versus black, and then compute the expected utility of the two gambles.

Since the prizes are exactly the same, it follows that you will prefer Gamble A to Gamble B if and only if you believe that drawing a black ball is less likely than drawing a black. This in our case would mean that you believe there are less than 30 black balls in the urn (according to expected utility theory). Also, there would be no clear preference between the choices if you thought that there are exactly 30 black balls. Similarly, it follows that you will prefer Gamble C to Gamble D if and only if you believe that drawing a red or yellow ball is more likely than drawing a black or yellow ball which in this case means that you think that there are more than 60 red and yellow balls. If you were to prefer Gamble A to Gamble B and at the same time Gamble D to Gamble C, you believe that there are at least 31 black balls and at least 61 red and yellow ones which would mean that there would be at least 92 balls in the urn which is, of course, not the case, resulting in that these two choices cannot coexist in the bounds of expected utility theory. Still, when surveyed most people strictly prefer Gamble A to Gamble B and Gamble D to Gamble C. Therefore, some assumptions of the expected utility theory are violated.

This experiment shows that decision makers prefer betting on events whose probabilities are known to betting on ones whose probabilities are uncertain. Since this is a perfectly valid point on a psychological level, and one that is inexplicable in the bounds of expected utility theory, a change of paradigm was necessary in order to implement the possibility of this choice and to define this previously unknown attitude.

One of the extensions of expected utility was the Choquet expected utility first formulated by Schmeidler (1989). This approach identifies the decision makers with their utility functions, and their subjective probability distributions of the events which can be non-additive. In the Ellsberg paradox, non-additivity means that a decision maker could indeed strictly prefer Gamble A to Gamble B, therefore think that it is more likely to draw a red ball than to draw a black ball, while strictly prefer Gamble D to Gamble C, and therefore think that it is more likely to draw a black or yellow ball than to draw a red or yellow ball, because the probability of drawing a red ball plus the probability of

drawing a yellow ball will not be necessarily equal to the probability of drawing either a red or a yellow ball, and so on, meaning that the total number of balls would not exceed 90. It is important to note that non-additivity is not supposed to be a consequence of irrational thinking, rather than a fear of the unknown.

The more general model required more general theorems, classifying the decision makers into behavior types following the analogy of risk attitude. Schmeidler's first classification theorem said that a decision maker is 'uncertainty averse' if and only if their subjective probability is convex. This result is noteworthy for us because of two main reasons: firstly, it does not take the utility functions into account, meaning that the new behavior type is determined only by the subjective probabilities and secondly, it implies that some concepts of cooperative game theory, most notably the core, will be of use by introducing convexity, which is a sufficient condition for the core to be nonempty (among others).

One of the problems with uncertainty aversion is that, as Epstein points out, there are some decision makers with preferences that clearly follow the ones in Ellsberg's example, but are not considered uncertainty averse in Schmeidler's classification. Following Ellsberg's original naming, the decision makers with preferences shown in the example were christened 'ambiguity averse', and a number of definitions for ambiguity aversion were formulated.

Two comparative definitions, from Epstein (1999) and Ghirardato and Marinacci (2002), are relevant for us. Both construe a comparison of decision makers by saying that one is 'more ambiguity averse' than the other if some intuitive condition is satisfied. Epstein divided the set of events into ambiguous and unambiguous events and formalized the available information for the decision makers as beliefs on the set of events. These beliefs were the same as the additive probabilities of the known, unambiguous events, and served as a lower or upper bound for the probabilities of the unknown, ambiguous events. His definition was that if, for pair of decision makers A and B, each time A prefers an unambiguous alternative to an ambiguous one B does so too, then B is more ambiguity averse than A. The only restricting condition for these preferences was that each had to be consistent with the decision makers' beliefs.

Ghirardato and Marinacci went further down the road and established a benchmark subjective probability which they assumed to be exhibiting ambiguity neutrality, allowing them to obtain an 'absolute' definition of ambiguity aversion. As benchmark, they chose the set of subjective probabilities that satisfy the conditions of expected utility theory, mainly, additivity. If a decision maker was more ambiguity averse than the assigned benchmark(s) then it was classified as ambiguity averse. Their result in the Choquet representation framework is that a decision maker is ambiguity averse if and only if their

subjective probability has a nonempty core.

Although this definition is tempting, it has some weaknesses whenever a preference is compared to its benchmark. If there is more than one subjective probability that is considered ambiguity neutral (which is usually the case), we have to choose one to which a decision maker can be compared to. The objective of this paper is to define some properties that are strict enough to narrow down the number of possible ambiguity neutral subjective probabilities to one. These properties, however, have to be plausible so that we do not drift far from our initial goal, which is to grasp ambiguity neutrality. They cannot be too strict either, lest there might be situations where we are unable to use the comparative concept.

The body of this paper is divided into three main parts. Chapter 2 is devoted to the basics of cooperative game theory from the definition of TU games to the two solution concepts we require for our results, the core and the nucleolus. In the end, we characterize the nucleolus on the set of balanced, monotonic games, and the set of balanced, monotonic capacities. The third chapter contains the introduction of the basics of decision theory, starting from the von Neumann-Morgenstern representation theorem and ending with Schmeidler's Choquet representation theorem. In chapter 4, we introduce Epstein's uncertainty field, which is a generalization of the classical probability field. We show a few basic examples, one of them being Ellsberg's urn situation. Subsequently, we introduce the decision makers to this environment and try to classify them based on their subjective probabilities. We present a set of axioms which may help narrow down the subjective probabilities available to the decision makers we assume to be ambiguity neutral. At the end of this chapter we define three types of behavior: ambiguity neutrality, ambiguity aversion and ambiguity seeking.

Chapter 2

TU games

Notations: $|N|$ is the cardinality of set N , $\mathcal{P}(N)$ denotes the class of all subsets of N . $\complement A$ is for the complement of set A . $A \subset B$ means $A \subseteq B$ but $A \neq B$. This chapter is mainly based on Peleg and Sudhölter (2003) but the notations and structure follow Pintér (2011).

Let $N \neq \emptyset$, $|N| < \infty$, and $v : \mathcal{P}(N) \rightarrow \mathbb{R}$ be such a function that $v(\emptyset) = 0$. Then N , v are called *set of players*, and *transferable utility game* (henceforth game) respectively. The class of games with players set N is denoted by $\mathcal{G}^N$.

It is easy to verify that $\mathcal{G}^N$ is isomorphic with $\mathbb{R}^{2^{|N|}-1}$. Henceforth, we assume the following fixed isomorphism between the two spaces: we take an arbitrary complete ordering on N , therefore $N = \{1, \dots, |N|\}$, and for all $v \in \mathcal{G}^N$: let $v = (v(\{1\}), \dots, v(\{|N|\}), v(\{1, 2\}), \dots, v(\{|N| - 1, |N|\}), \dots, v(N)) \in \mathbb{R}^{2^{|N|}-1}$, and regard $\mathcal{G}^N$ and $\mathbb{R}^{2^{|N|}-1}$ as identical.

A further notation: let $v \in \mathcal{G}^N$ and $\beta \in \mathbb{R}^N$. Then $v \oplus \beta \in \mathcal{G}^N$ is a game such that for each $S \subseteq N$: $(v \oplus \beta)(S) = v(S) + \sum_{i \in S} \beta_i$, where β_i is component i of vector β .

Let $v \in \mathcal{G}^N$ and $i \in N$, and for each $S \subseteq N$: let $v'_i(S) = v(S \cup \{i\}) - v(S)$. v'_i is called player i 's marginal contribution function in game v . Put it differently, $v'_i(S)$ is player i 's marginal contribution to coalition S in game v .

Here, we consider the following classes of games: game $v \in \mathcal{G}^N$ is

- convex, if for each $S, T \subseteq N$: $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$,
- monotonic, if for each $S, T \subseteq N$, $S \subseteq T$: $v(S) \leq v(T)$,
- additive, if for each $S, T \subseteq N$, $S \cap T = \emptyset$: $v(S) + v(T) = v(S \cup T)$,
- concave, if for each $S, T \subseteq N$: $v(S) + v(T) \geq v(S \cup T) + v(S \cap T)$.

Let $v \in \mathcal{G}^N$ be a game, then players $i, j \in N$ are equivalent in game v , denoted by $i \sim^v j$, if for each $S \subseteq N$ such that $i, j \notin S$: $v'_i(S) = v'_j(S)$.

2.1 Solutions, axioms

One of the main objectives of cooperative game theory is to find a distribution (usually) of the value of the grand coalition $v(N)$ that satisfies some intuitive conditions about stability and fairness. The conditions of stability are meant to be such that the grand coalition does indeed form, requiring the “cooperation” of all the players, while the conditions of fairness are usually constructed such that more valuable players get higher payoffs, similar players get similar payoffs and less valuable players get lower payoffs.

Mapping $\psi : A \rightrightarrows \mathbb{R}^N$ is a *solution* on subclass of games $A \subseteq \mathcal{G}^N$.

In the next definition we list the axioms we use to characterize a single-valued solution.

Definition 2.1.1. *Single-valued solution ψ (for all $v \in A$, $|\psi(v)| = 1$) on $A \subseteq \cup_N \mathcal{G}^N$ is / satisfies*

- *Pareto optimal (PO), if for each $v \in A$: $\sum_{i \in N} \psi_i(v) = v(N)$,*
- *equal treatment property (ETP), if for each $v \in A$, $i, j \in N$: $(i \sim^v j) \Rightarrow (\psi_i(v) = \psi_j(v))$,*
- *covariant under strategic equivalence (COV), if for each $v \in A$, $\alpha > 0$, $\beta \in \mathbb{R}^N$ such that $\alpha v \oplus \beta \in A$: $\psi(\alpha v \oplus \beta) = \alpha \psi(v) + \beta$.*

Notice that COV can be interpreted not only for single-valued solutions, but also when ψ is a set function.

Brief interpretations of the above introduced axioms are the following:

Another commonly used name of axiom PO is efficiency. This axiom requires that the total value of the grand coalition must be distributed among the players. For example, consider a situation when the players are households of an apartment complex, and the goal is to distribute the total cost of water consumption among the apartments. In this case, PO means that the total price must be distributed, no more and no less.

ETP means that if two players have the same effects in the game, then their evaluations must be equal. Going back to our example, if two apartments are equivalent regarding water consumption, then their shares from the total cost must be equal.

Axiom COV states that the evaluation method be invariant under strategic equivalence. Going back to our example, if we change the format of the households' data on the prize of the consumption of water, for example from dollar to pound and assign some fixed amount of cost to every household (it can vary from household to household) then their shares must change in the same way.

Definition 2.1.2. (Maschler and Davis, 1965) Let $v \in \mathcal{G}^N$, $\emptyset \neq S \subseteq N$ and $x \in \mathbb{R}^N$. Then the reduced game $v_{(S,x)} \in \mathcal{G}^S$ of v with respect to S and x is defined as follows, for each $T \subseteq S$:

$$v_{(S,x)}(T) = \begin{cases} 0, & \text{if } T = \emptyset \\ v(N) - \sum_{i \in N \setminus S} x_i, & \text{if } T = S \\ \max_{Q \subseteq N \setminus S} (v(T \cup Q) - \sum_{i \in Q} x_i) & \text{otherwise} \end{cases}.$$

Definition 2.1.3. Solution φ on $A \subseteq \cup_N \mathcal{G}^N$ satisfies reduced game property (RGP), if for each $v \in A$, $\emptyset \neq S \subseteq N$ and $x \in \varphi(v)$ such that $v_{(S,x)} \in A$: $x_S \in \varphi(v_{(S,x)})$.

The reduced game as defined here is a way of omitting a coalition $N \setminus S$ from the set of players N . While the definition itself requires a solution vector $x \in \mathbb{R}^N$, only the payoffs of the deleted coalition, x_S is used. For our apartment complex example it says the following: suppose that some households decide to pay their own water consumption bills separately because, for example, their water usage habits are different. They agree with the rest of the households on their share of the total cost, then agree separately on a new distribution. However, they remain connected to the same network, meaning that their water consumption still affects the remaining households.

Peleg and Sudhölter describe RGP as a condition of self-consistency. Going back to our example, it means that the cost distribution of the remaining consumers must be the same as it was when all households paid together (if the agreed distribution, the share that the departing households must pay, does not change).

2.2 The core

This section is devoted to one of the most famous solution concepts, the core.

Definition 2.2.1. (Gillies, 1959) Let $v \in \mathcal{G}^N$, then the core of game v is defined as follows:

$$\text{Core}(v) = \left\{ x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S), S \subseteq N \right\}.$$

The core of a game is the set of all solutions that are acceptable for each coalition. The sum of the values assigned to the players in each subset of N can not be lower than the payoff they could achieve in the game by themselves. However, we cannot distribute more than what is available to the grand coalition and there is no reason to distribute less. This is a strong condition, meaning that the function $\text{Core}(\cdot)$ often yields the empty set as a result. Convexity is a sufficient condition for the core of a game to be nonempty.

Definition 2.2.2. Game $v \in \mathcal{G}^N$ is balanced if

$$\max_{\lambda \in \mathbb{R}_+^{\mathcal{P}(N)}, \sum_{A \in \mathcal{P}(N)} \lambda_A \chi_A = 1} \sum_{A \in \mathcal{P}(N)} \lambda_A v(A) \leq v(N) .$$

The Bondareva-Shapley theorem, formulated by Bondareva (1963) and Shapley (1967) states that a game is balanced if and only if it has a non-empty core. $\mathcal{G}_B^N$ denotes the set of monotonic, balanced games with player set N . $\mathcal{G}_b^N \subseteq \mathcal{G}_B^N$ denotes the set of capacities ($v(N) = 1$) in $\mathcal{G}_B^N$. Also we define $\Gamma_B = \cup_N \mathcal{G}_B^N$ and $\Gamma_b = \cup_N \mathcal{G}_b^N$.

2.3 The nucleolus

With the core we defined a stability concept for the solutions but we did not touch on their fairness yet. A number of solution concepts exist that deal with fairness all in different ways, perhaps the two most famous of these are the Shapley-value (Shapley, 1953) and the nucleolus (Schmeidler, 1969). We take a look at the latter one's definition and properties.

Definition 2.3.1. Let $v \in \mathcal{G}^N$. The imputations of game v are defined as follows:

$$I(v) = \left\{ x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N), \text{ and } x_i \geq v(\{i\}), i \in N \right\} .$$

The imputations of a game are the set of all distributions satisfying PO, that are also individually rational for the players. Each player gets at least as much as they could achieve alone.

Definition 2.3.2. The excess function $e(\cdot, \cdot) : \mathcal{G}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^{2^N}$ is defined as follows, for each $v \in \mathcal{G}^N$, $x \in \mathbb{R}^N$ and $S \subseteq N$:

$$(e(v, x))_S = v(S) - \sum_{i \in S} x_i .$$

The value of the excess function gives a natural estimation of how much each set of players are discontented with a given distribution. Negative values indicate that a coalition accepts the solution (they could not achieve more if they quit) while positive values show that the given coalition is unhappy with that distribution.

Remark 2.3.3. Let $v \in \mathcal{G}^N$. Then

$$\text{Core}(v) = \{x \in I(v) : e(v, x) \leq 0\}.$$

Definition 2.3.4. Let $\rho : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be defined as follows, for each $x \in \mathbb{R}^d$, $1 \leq i \leq d$:

$$(\rho(x))_i = \max_{T \subseteq \{1, 2, \dots, d\}, |T|=i} \min_{j \in T} x_j .$$

In other words, ρ puts a vectors components into nonincreasing order.

Definition 2.3.5. (*Schmeidler, 1969*) Let $v \in \mathcal{G}^N$. Then the nucleolus of game v is as follows:

$$\mathcal{N}(v) = \{x \in I(v) : \rho \circ e(v, x) \leq_{lex} \rho \circ e(v, y), y \in I(v)\},$$

where $\leq_{lex}$ is for the lexicographic ordering.

The basic idea behind the nucleolus is that from the solution vectors that satisfy PO and are individually rational, the ‘fairest’ solution is the one that maximizes the payoffs for the coalitions whose dissatisfaction factors (excess values) are the greatest. After that, we maximize the payoffs of the second most unhappy coalitions and so on.

2.3.1 Properties

$\mathcal{N}(v) \in \mathbb{R}^N$ denotes the nucleolus of v . The following lemmata are well-known results.

Lemma 2.3.6. Let $v \in \mathcal{G}^N$ be a game such that $I(v) \neq \emptyset$. Then $|\mathcal{N}(v)| = 1$.

Lemma 2.3.7. Let $v \in \mathcal{G}^N$ be a balanced game. Then $\mathcal{N}(v) \in \text{Core}(v)$.

Lemmata 2.3.6 and 2.3.7 may be verified by the reader.

Lemma 2.3.8. The Core is COV and it satisfies RGP on subclass of games Γ_B .

For proof of the RGP part of Lemma 2.3.8, see Peleg and Sudhölter (2003) page 30, Lemma 2.3.16. The reader may easily verify the COV part.

We now prove that Sobolev’s characterization of the prenucleolus, on the set of all games is (almost) the same as it is of the nucleolus on the set of monotonic games with nonempty core. For readers unfamiliar with the prenucleolus, it is a solution concept very closely related to the nucleolus which we did not introduce because it coincides with the nucleolus on the set of balanced games.

Proposition 2.3.9. Singleton valued solution φ is/satisfies ETP, COV and RGP for subclass of games $A = \Gamma_B$, if and only if $\varphi(v) = \mathcal{N}(v)$.

Proof. If: a well known result, see for example Snijders (1995). Note that $\mathcal{N}$ also satisfies PO.

Only if: Following the proof in Sobolev (1975) for the characterization of the prenucleolus on the class of TU games we use mathematical induction.

Step 1: First, we prove the characterization for $|N| = 1$. We can infer that for $v_0 = 0$, $v_0 \in \mathcal{G}_B^N$, $|N| = 1$. Then, for any $\alpha > 0$

$$\varphi(v_0) = \varphi(\alpha v_0) = \alpha \varphi(v_0)$$

due to the COV property of φ . Thus: $\varphi(v_0) = 0$.

Then, for each $v \in \mathcal{G}_B^N$, $|N| = 1$

$$v = v_0 + v(N) = \varphi(v_0) + v(N) = \varphi(v_0 \oplus v(N)) = \varphi(v)$$

again, due to the COV property of φ . With this, we have proven $\varphi(v) = \mathcal{N}(v)$ in the case where $|N| = 1$.

Step 2: Now we need $|N| = 2$. Without loss of generality, we assume $N = \{1, 2\}$, and $v(\{1\}) = a$, $v(\{2\}) = a + b$, $v(\{1, 2\}) = 2a + b + c$, where $a, b, c \geq 0$. We can see that all two player balanced games take this form without difficulty. Because of the RGP of φ ,

$$\varphi_2(v) = v_{(\{2\}, \varphi(v))}(\{2\}) = v(N) - \varphi_1(v)$$

so that $v(N) = \varphi_1(v) + \varphi_2(v)$, therefore PO also holds for φ when $|N| = 2$.

For $\beta = (b, 0)^\top$, consider the game $v \oplus \beta \in \mathcal{G}_B^N$. We can see that $(v \oplus \beta)(\{1\}) = (v \oplus \beta)(\{2\}) = a + b$ and $(v \oplus \beta)(N) = 2a + 2b + c$. Here, $\{1\} \sim^{v \oplus \beta} \{2\}$, so ETP and PO imply

$$\varphi_1(v \oplus \beta) = \varphi_2(v \oplus \beta) = a + b + \frac{c}{2},$$

meaning that

$$\varphi(v \oplus \beta) = \mathcal{N}(v \oplus \beta) \stackrel{COV}{\Rightarrow} \varphi(v) = \mathcal{N}(v).$$

Step 3: We now assume that φ is the nucleolus for $|N| = n - 1$. For $|N| = n$, we again, without loss of generality, suppose that $N = \{1, 2, \dots, n\}$. See that $v_{(N \setminus \{1\}, \varphi(v))} \in \mathcal{G}_B^{N \setminus \{1\}}$, because of Lemma 2.3.8¹. For $n - 1$ person games, we have

$$\varphi(v_{(N \setminus \{1\}, \varphi(v))}) = \mathcal{N}(v_{(N \setminus \{1\}, \varphi(v))}).$$

The value of the grand coalition in the reduced game is the following:

$$v_{(N \setminus \{1\}, \varphi(v))}(N \setminus \{1\}) = v(N) - \varphi_1(v) = \varphi_2(v_{(N \setminus \{1\}, \varphi(v))}) + \dots + \varphi_n(v_{(N \setminus \{1\}, \varphi(v))})$$

and because of RGP, $\varphi_i(v_{(N \setminus \{1\}, \varphi)}) = \varphi_i(v)$ for $i \in \{2, \dots, n\}$, meaning that

$$v(N) - \varphi_1(v) = \varphi_2(v) + \dots + \varphi_n(v).$$

¹Here we also need that $\varphi(v) \in \text{Core}(v)$, which is due to RGP and the induction condition.

This means that v has exactly one solution that is/satisfies ETP, COV and RGP. Since we already know that the nucleolus satisfies these conditions and there cannot be any other that do, we have proven that $\varphi(v) = \mathcal{N}(v)$. $\square$

It is worth noticing that the proof of Proposition 2.3.9 does not use the entirety of RGP, only the part when the player set leaving the grand coalition, $N \setminus S$ contains only one player. We formalize this by introducing a weaker form of RGP.

Definition 2.3.10. *Solution φ on $A \subseteq \cup_N \mathcal{G}^N$ satisfies weak reduced game property (WRGP), if for each $v \in A$, $i \in N$ and $x \in \varphi(v)$ such that $v_{(N \setminus \{i\}, x)} \in A$: $x_{N \setminus \{i\}} \in \varphi(v_{(N \setminus \{i\}, x)})$.*

Clearly, RGP implies WRGP. The converse is not true in general.

Corollary 2.3.11. *Singleton valued solution φ is/satisfies ETP, COV and WRGP on $A = \Gamma_B$ if and only if $\varphi(v) = \mathcal{N}(v)$.*

Proof. The same as the proof of Proposition 2.3.9. $\square$

Also, notice that for any $v_b \in \Gamma_b$ there exists $\alpha > 0$ such that $\alpha v_b = v_B$ where $v_B \in \Gamma_B$. Because of COV, we can apply the characterization of the nucleolus in the games in Γ_b as well. Of course, the application of WRGP is problematic in this case, because the reduction of a capacity does not result in a capacity. The concept of capacity reduction is meant to solve this problem.

Definition 2.3.12. *Let $v \in \mathcal{G}^N$ such that $v(N) = 1$, and $\emptyset \neq S \subseteq N$, and $x \in \mathbb{R}^N$, such that $v_{(S, x)}(S) \neq 0$. Then the capacity reduction of v with respect to S and x is the following for each $T \subseteq S$:*

$$v_{(S, x)}^c(T) = \frac{v_{(S, x)}(T)}{v_{(S, x)}(S)}.$$

Definition 2.3.13. *Solution φ on set of capacities A satisfies reduced capacity property (RCP), if for each $v \in A$, $\emptyset \neq S \subseteq N$ and $x \in \varphi(v)$ such that $v_{(S, x)}^c \in A$: $\frac{1}{v_{(S, x)}^c(S)} x_S \in \varphi(v_{(S, x)}^c)$.*

Definition 2.3.14. *Solution φ on set of capacities A satisfies weak reduced capacity property (WRCP), if for each $v \in A$, $i \in N$ and $x \in \varphi(v)$ such that $v_{(N \setminus \{i\}, x)}^c \in A$: $\frac{1}{v_{(N \setminus \{i\}, x)}^c(N \setminus \{i\})} x_S \in \varphi(v_{(N \setminus \{i\}, x)}^c)$.*

The reduced capacity is merely the normalization of the reduced game, meaning that the value of the grand coalition remains 1.

Similarly to RGP and WRGP, RCP implies WRCP.

Corollary 2.3.15. *Singleton valued solution φ is/satisfies ETP, COV and WRCP for subclass of games $A = \Gamma_b$, if and only if $\varphi(v) = \mathcal{N}(v)$.*

Proof. The proof follows an analogy with the proof of Proposition 2.3.9, because WRCP and COV together allow us to use everything we did with WRGP. The only modifications we need are for steps 1 and 2: Since there is only one game, where $v \in \mathcal{G}_b^N$, if $|N| = 1$, we cannot use COV the way we did earlier. Instead, from applying COV for $\alpha = \beta = \frac{1}{2}$, we have

$$\varphi(v) = \varphi\left(\frac{1}{2}v \oplus \frac{1}{2}\right) = \frac{1}{2}\varphi(v) + \frac{1}{2},$$

implying $\varphi(v) = 1 = \mathcal{N}(v)$.

For step 2, we again use COV a little differently. Here without loss of generality we assume $N = \{1, 2\}$, and $v(\{1\}) = a$, $v(\{2\}) = a+b$, $v(\{1, 2\}) = 2a+b+c$, where $a, b, c \geq 0$ and $2a + b + c = 1$. Let $\beta = (b, 0)^\top$ and $\alpha = \frac{1}{1+b}$. Then, $\alpha v \oplus \beta \in \Gamma_b$ and $\{1\} \sim^{\alpha v \oplus \beta} \{2\}$. We can prove that PO holds in a similar way as we did in the proof of Proposition 2.3.9. It then follows, because of PO and ETP that $\varphi(v) = \mathcal{N}(v)$.

For step 3, we need to apply WRCP instead of the weak part of RGP. □

Corollary 2.3.15 characterizes the nucleolus on Γ_b .

Chapter 3

Representing decisions

3.1 Decision under risk

Modeling ambiguity attitude generally follows the same steps as modeling risk attitude in the expected utility framework. In order to introduce the terminology used in the upcoming chapters and provide a jumping-off point for the analogy with risk we define risk aversion, neutrality and seeking based upon von Neumann and Morgenstern (1947). We retain some notations and most of the structure from Mas-Colell et al. (1995).

3.1.1 Preference relation

Consider X , a finite, nonempty set of prizes and $\Delta(X)$, the set of (additive) probability measures on X , called lotteries. $\succsim$ denotes a binary relation over $\Delta(X)$. Let $\succ$ and $\sim$ denote the strict and the symmetric part of $\succsim$ respectively. We call binary relation $\succsim$ preference relation if it is complete and transitive. For $\alpha_1, \alpha_2, \dots, \alpha_n \in [0, 1]$ such that $\sum_{i=1}^n \alpha_i = 1$, and lotteries $p_1, p_2, \dots, p_n \in \Delta(X)$ the addition $\alpha_1 p_1 + \alpha_2 p_2 + \dots + \alpha_n p_n \in \Delta(X)$ is defined pointwise.

Consider the following two axioms:

- **A1** (Continuity) For all $p, q, r \in \Delta(X)$ such that $p \succ q \succ r$ there are $\alpha, \beta \in (0, 1)$ such that $\alpha p + (1 - \alpha)r \succ q \succ \beta p + (1 - \beta)r$.
- **A2** (Independence) $p \succ q \Rightarrow \alpha p + (1 - \alpha)r \succ \alpha q + (1 - \alpha)r$ for any $\alpha \in (0, 1]$, $p, q, r \in \Delta(X)$.

The continuity axiom is generally accepted for preference relations. A number of different equivalent forms exist, the one presented here also appears in Schmeidler (1989). The independence axiom however, is debated. Mas-Colell et al. (1995) lists two situations, the Allais Paradox (Allais, 1953) and Machina's paradox (Machina, 1987) that object the independence axiom and one, the Dutch Book Principle (Green, 1987) that supports it.

The following theorem is a well known result.

Theorem 3.1.1. *(von Neumann and Morgenstern, 1947) A preference relation $\succsim$ on $\Delta(X)$ satisfies **A1** and **A2** if and only if there exists a function $U: X \rightarrow \mathbb{R}$ such that*

$$p \succsim q \Leftrightarrow \int U \, dp \geq \int U \, dq ,$$

furthermore, U is unique up to a positive affine transformation.

Functional $\int U: \Delta(X) \rightarrow \mathbb{R}$ is called the von Neumann-Morgenstern expected utility representation of preference relation $\succsim$, U is called a Bernoulli utility function.

3.1.2 Risk attitude

For $\alpha \in X$ let $\delta_\alpha \in \Delta(X)$ denote the Dirac-measure concentrated at α . Then, Bernoulli utility function U is defined pointwise over X as follows: $U(\alpha) = \int U \, d\delta_\alpha$. Suppose that X is a finite subset of $\mathbb{R}$.

Definition 3.1.2. $\succsim$ *exhibits risk aversion (seeking) if for each lottery in $p \in \Delta(X)$: $\delta_x \succsim (\preceq) p$, where $x = \int id_X \, dp$. Equivalently, if $\succsim$ has a von Neumann-Morgenstern expected utility representation, we can say that $\succsim$ is risk averse (seeker) if $U(x) \geq (\leq) \int U \, dp$. If $\succsim$ exhibits risk aversion and seeking then we say that $\succsim$ is risk neutral.*

Proposition 3.1.3. *Suppose that $\succsim$ is a preference relation and U is a representing Bernoulli utility function. Then $\succsim$ is risk averse (neutral, seeker) if and only if U is concave (linear, convex).*

Proposition 3.1.3 is one of the better-known results of expected utility theory. Our goal for this paper is to find a definition for ambiguity attitude following the analogy of Definition 3.1.2.

3.2 Choquet expected utility

In order to implement the choices in Ellsberg's paradox, we need two major modifications in our model. As the reader may recall, there were two kinds of prizes in each Gamble, 100\$ and 0\$, and they were assigned in some way to the three outcomes, red ball, yellow ball and black ball. Since the probabilities of the outcomes were sometimes unknown, we cannot assign lotteries to the prizes. A solution to this problem is the introduction of a set of outcomes Ω , independent from the set of prizes, and a set of acts A , whose elements assign the prizes to the outcomes. This way, we can define preference relation $\succsim$ on A with similar axioms as before.

The second modification is the change of axioms in order for them to allow the kind of preference observed in Ellsberg's example. This will lose us the von Neumann-Morgenstern representation, and give us the less intuitive, less comfortable Choquet expected utility representation.

This section is based on Schmeidler (1989). An intuitive description of the framework can be found in Anscombe and Aumann (1963) or, in greater detail, in Fishburn (1970).

3.2.1 The framework

Notice that the given definition for risk aversion does not use the particular set structure of $\Delta(X)$ only that it was convex. Therefore we can retain **A1** and **A2** for a preference relation $\succsim$ on any feasible set.

Ω denotes the finite set of outcomes and let the set of prizes X be a finite subset of $\mathbb{R}$. Then, $A = \{a: \Omega \rightarrow \Delta(X)\}$ is called set of acts. That way, similarly to $\Delta(X)$ itself, a convex structure can be defined for A as follows: for $\alpha_1, \alpha_2, \dots, \alpha_n \in [0, 1]$ and $a_1, a_2, \dots, a_n \in A$ such that $\sum_{i=1}^n \alpha_i = 1$, $\alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_n a_n = a$ is defined pointwise such that for each $\omega \in \Omega$, $\alpha_1 a_1(\omega) + \alpha_2 a_2(\omega) + \dots + \alpha_n a_n(\omega) = a(\omega)$ and $a \in A$.

Consider a preference relation $\succsim$ on A . For each lottery $p \in \Delta(X)$ there exists a constant act $c_p \in A$ such that for each outcome $\omega \in \Omega$, $c_p(\omega) = p$. This means that our preference relation can be extended for the lotteries too. Whenever we compare lotteries, we mean the comparison of their respective constant acts. For preference relation $\succsim$, we introduce three additional axioms.

- **A3** (Nondegeneracy) There exist $a, b \in A$ such that $a \succ b$,
- **A4** (Monotonicity) If for each $\omega \in \Omega$, $a(\omega) \succsim b(\omega)$ then $a \succsim b$.

Two acts, $a, b \in A$ are comonotonic if for all $\omega_1, \omega_2 \in \Omega$, $a(\omega_1) \succ a(\omega_2) \Rightarrow b(\omega_1) \succsim b(\omega_2)$, and $a(\omega_1) \prec a(\omega_2) \Rightarrow b(\omega_1) \precsim b(\omega_2)$. Our final axiom is for comonotonic acts.

- **A5** (Comonotonic independence) For acts $a, b, c \in A$, suppose that $a \succ b$ and a and c , and b and c are comonotonic. Then, for any $0 < \alpha \leq 1$: $\alpha a + (1 - \alpha)c \succ \alpha b + (1 - \alpha)c$.

The interpretation for the nondegeneracy axiom is relatively straightforward. If $\succsim$ is indifferent between any acts a and b we have neither the necessity nor the tools to characterize it, therefore only nondegenerate preference relations are considered.

Monotonicity shows the “double” use of preferences and how they are connected, the preference relation $\succsim$ on A must follow the one on $\Delta(X)$, meaning that if for two acts, one always yields a preferred lottery to the other's, then a decision maker must prefer that act.

Comonotonic independence is weaker than the debated independence axiom, addressing the issues raised by the Allais paradox.

These axioms are the necessary and sufficient conditions for the existence of the Choquet expected utility representation, which is analogous to the von Neumann-Morgenstern representation.

3.2.2 Choquet representation

First, we define the Choquet integral of step functions ($|R_f| < \infty$).

Definition 3.2.1. *A function $v: \mathcal{P}(\Omega) \rightarrow [0, 1]$ is called nonadditive or subjective probability if $v(\emptyset) = 0$, $v(\Omega) = 1$ and for $E \subseteq F \Rightarrow v(E) \leq v(F)$, $E, F \subseteq \Omega$.*

Definition 3.2.2. *(Choquet, 1953) For $f: \Omega \rightarrow \mathbb{R}$, such that $|R_f| < \infty$ and nonadditive probability v we define*

$$(C) \int f \, dv = \sum_{i=1}^n (f(\omega_i) - f(\omega_{i+1}))v(\{\omega_1, \omega_2, \dots, \omega_i\}) ,$$

where $\omega_1, \omega_2, \dots, \omega_n$ is an ordering of the outcomes in Ω such that $f(\omega_1) \geq f(\omega_2) \geq \dots \geq f(\omega_n)$, and $f(\omega_{n+1}) = 0$.

We list some simple properties of the Choquet integral:

- Monotonicity in the integrand: if $f \leq g$ then $(C) \int f \, dv \leq (C) \int g \, dv$,
- Monotonicity in the variable of the integration: for $f \geq 0$, if $v \leq w$ then $(C) \int f \, dv \leq (C) \int f \, dw$,
- Positive homogeneity: for all $\lambda \geq 0$, $(C) \int \lambda f \, dv = \lambda(C) \int f \, dv$,
- Comonotonic additivity: for comonotonic functions f and g it holds that $(C) \int f \, dv + (C) \int g \, dv = (C) \int (f + g) \, dv$.

Also notice that if v is additive, the Choquet integral is equivalent to the Lebesgue integral.

We have arrived at Schmeidler's Choquet expected utility representation theorem:

Theorem 3.2.3. *(Schmeidler, 1989) A binary relation $\succsim$ on A satisfies **A1**, **A3**, **A4** and **A5** if and only if there exist a unique nonadditive probability v on Ω and a utility function $u: \Delta(X) \rightarrow \mathbb{R}$ such that,*

$$a \succsim b \Leftrightarrow (C) \int u(a) \, dv \geq (C) \int u(b) \, dv ,$$

where composite function $u(a) = u \circ a$, furthermore, u is unique up to a positive affine transformation.

It is worth noting that this utility function u is different from the Bernoulli utility function, rather, it resembles the utility index of the lotteries, the representing functional $\int U$.

Schmeidler's approach to uncertainty goes by the introduction of another axiom:

- **A6** (Uncertainty aversion) $a \succsim b$ implies $\lambda a + (1 - \lambda)b \succsim b$ for each $\lambda \in [0, 1]$.

Corollary 3.2.4. *A binary relation $\succsim$ on A satisfies **A1**, **A3**, **A4**, **A5** and **A6** if and only if the unique subjective probability of $\succsim$ is convex, that is for each $E, F \subseteq \Omega$*

$$v(E) + v(F) \leq v(E \cup F) + v(E \cap F)$$

Chapter 4

Ambiguity neutrality

In this chapter we use a not-so-known special collection of sets called λ -system, d -system or Dynkin system. This is a system of subsets of a universal set Ω satisfying axioms weaker than the σ -algebra. It has two characterizations:

$D \subseteq \mathcal{P}(\Omega)$ is a λ -system if

1. $\Omega \in D$,
2. if $A, B \in D$ and $A \subseteq B$, then $B \setminus A \in D$,
3. if $A_1, A_2, \dots$ is a sequence of subsets in D and $A_n \subseteq A_{n+1}$ for all $n \geq 1$, then $\bigcup_{n=1}^{\infty} A_n \in D$.

Equivalently, D is a λ -system if

1. $\Omega \in D$,
2. if $A \in D$, then $\mathbb{C}A \in D$,
3. if $A_1, A_2, \dots$ is a sequence of subsets in D and $A_n \subseteq A_{n+1}$ such that $A_i \cap A_j = \emptyset$ for all $i \neq j$, then $\bigcup_{n=1}^{\infty} A_n \in D$.

4.1 Epstein's approach to ambiguity

While Schmeidler's axiom of ambiguity aversion does address the questions raised by the Ellsberg paradox and does follow an analogy with risk aversion, a number of technical and conceptual issues were pointed out about his approach, resulting in different definitions of ambiguity attitude. Epstein (1999) shows an example to demonstrate that convexity of the subjective probability is neither necessary nor sufficient condition for ambiguity averse behavior.

Epstein argued that, as in the original Ellsberg paradox, ambiguity is the result of Knightian uncertainty (Knight, 1921) of the outcomes in Ω . He constructed a model where the decision makers are presented with an imperfect information setting, prompting them to form their own subjective probabilities about the events which, in turn will ultimately show in their preferences. The first assumption made here is that there are two kinds of events: unambiguous ones, for which the exact probabilities are known to the decision makers, and ambiguous ones, for which only estimates, lower and upper bounds are known. For this new distinction between risk and ambiguity, the reader is referred to Epstein and Zhang (2001).

4.1.1 The uncertainty field

Epstein approaches ambiguity from as far as the basics of probability theory: the definition of the probability field. He divided the set of events into two disjoint, externally predetermined groups: ambiguous and unambiguous events. We believe that a “true” (additive) probability distribution exists on the given measurable space but only the unambiguous events’ probabilities are known. For all other events we only know the lower and upper bounds of the probabilities.

Naturally, if we know all the lower bounds, we can easily determine the upper bounds, hence we need not include both in our model. For a slightly easier compatibility with cooperative games the way we introduced them we use the lower bounds.

Definition 4.1.1. *Let $(\Omega, \mathcal{E})$ be a measurable space. $p: \mathcal{E} \rightarrow [0, 1]$ is called a lower bound function or belief function if*

1. $p(\emptyset) = 0, p(\Omega) = 1,$
2. *for each $A \subseteq B, p(A) \leq p(B), A, B \in \mathcal{E},$*
3. *there exists $p^* \in \Delta(\Omega, \mathcal{E}),$ such that $p \leq p^*.$*

The belief function formalizes the information available for the decision makers. Points 1 and 2 in Definition 4.1.1 formalize the trivial properties we expect from the lower bound of a probability measure. Point 3 states that such a probability measure must exist, otherwise the lower bound leads to internal contradiction.

Definition 4.1.2. *(Epstein, 1999) $F = (\Omega, \mathcal{M}, \mathcal{A}, p)$ tuple is called an uncertainty field, where*

1. Ω *is the finite set of outcomes,*
2. $\mathcal{M} \subseteq \mathcal{P}(\Omega)$ *is the set of unambiguous events, a λ -system on $\Omega,$*

3. $\mathcal{A}$ is such that $\mathcal{A} \cap \mathcal{M} = \emptyset$ and $\mathcal{A} \cup \mathcal{M} = \mathcal{P}(\Omega)$,
4. p is a lower bound function such that $p|_{\mathcal{M}}$ is additive, and for any $A \in \mathcal{A}$, $p|_{\{A\} \cup \mathcal{M}}$ is additive.

The set of uncertainty fields is denoted by $\mathcal{F}$.

Definition 4.1.3. For an uncertainty field $F = (\Omega, \mathcal{M}, \mathcal{A}, p)$, set of priors $\Delta(F)$ is the following: $p^* \in \Delta(F)$ if $p^* \in \Delta(\Omega, \mathcal{M} \cup \mathcal{A})$ and $p \leq p^*$.

The uncertainty field is merely a possible generalization of the well-known probability field. The events in the σ -algebra were separated into two parts, to unambiguous ($\mathcal{M}$) and ambiguous ($\mathcal{A}$) events. For $\mathcal{M}$ we demand two main things: The first concerns an event and its complement, if M is unambiguous with probability $p(M)$ than $\mathbb{C}M$ is also unambiguous with probability $1 - p(M)$. The second is about disjoint unions, for $M_1, M_2 \in \mathcal{M}$ and $M_1 \cap M_2 = \emptyset$ it must hold that $M_1 \cup M_2 \in \mathcal{M}$ and $p(M_1 \cup M_2) = p(M_1) + p(M_2)$. Thus, the set of all unambiguous events is a λ -system. That way, for any $p^* \in \Delta(F)$, and $M \in \mathcal{M}$, $p^*(M) = p(M)$ because the lower and upper bounds for the probability of M coincide.

In point 4 of Definition 4.1.2 it is stated that belief function must be additive over the union of $\mathcal{M}$ and any ambiguous event. To put it in a simple way, if $M \in \mathcal{M}$, $A \in \mathcal{A}$, such that $M \cap A = \emptyset$, $p(M \cup A) = p(M) + p(A)$ must hold because while we can only estimate the probabilities of ambiguous events A and $M \cup A$, for M we know the exact value, therefore the lower bound for $M \cup A$ must be exactly the same as for the event A plus $p(M)$.

In the general case, when Ω is infinite, there are two main kinds of uncertainty situations where the environment above may be of use: The first is when need to extend an otherwise entirely known probability field and fill it with new, previously unknown events. In this case $\mathcal{M}$ is a σ -algebra itself and $\mathcal{A}$ contains the new events. The second case is when we are familiar with certain events' probabilities but we do not know the correlation between them, whether they are independent from each other, do they exclude or include each other, etc. In this case, the events in $\mathcal{M}$ may overlap many times and $\mathcal{A}$ contains their intersections.

The set of priors contains the probability measures that might be the “true” probability distributions. We assume that in some way, implicitly or explicitly this true distribution is the one that (certain) decision makers are trying to determine.

Example 4.1.4. Let us take a look at the Ellsberg paradox: An urn contains 90 balls, 30 of which are red. Each of the remaining 60 may be either black or yellow. The uncertainty field that best describes the situation where we have to guess a drawn ball's color is the following:

- $\Omega = \{\text{red}, \text{yellow}, \text{black}\},$
- $\mathcal{M} = \{\emptyset, \{\text{red}\}, \{\text{yellow}, \text{black}\}, \Omega\},$
- $\mathcal{A} = \{\{\text{black}\}, \{\text{yellow}\}, \{\text{red}, \text{yellow}\}, \{\text{red}, \text{black}\}\},$
- $p(\emptyset) = 0, p(\Omega) = 1, p(\{\text{red}\}) = \frac{1}{3}, p(\{\text{yellow}\}) = p(\{\text{black}\}) = 0, p(\{\text{red}, \text{yellow}\}) = p(\{\text{red}, \text{black}\}) = \frac{1}{3}, p(\{\text{yellow}, \text{black}\}) = \frac{2}{3}.$

A possible p^* prior would be: $p^*(\{\text{yellow}\}) = \frac{1}{4}, p^*(\{\text{black}\}) = \frac{5}{12}, p^*(\{\text{red}, \text{black}\}) = \frac{3}{4}, p^*(\{\text{red}, \text{yellow}\}) = \frac{7}{12},$ and $p^* = p$ in all other cases.

This is a simple example for the first of the situations mentioned above, as $\mathcal{M}$ is a (quite simple) σ -algebra, and we introduced the new, previously unknown event $\{\text{black}\}$ and $\{\text{yellow}\}.$

Example 4.1.5. Let us consider a cubic die. Nothing is known for certain about the numbers on this die only that they are whole numbers from 1 to 6. We know however, that the probability of rolling an even number is exactly $\frac{1}{2},$ and the probability of rolling a number divisible by three is $\frac{1}{3}.$ The uncertainty field that best describes the situation is the following:

- $\Omega = \{1, 2, 3, 4, 5, 6\},$
- $\mathcal{M} = \{\emptyset, \{2, 4, 6\}, \{3, 6\}, \{1, 2, 4, 5\}, \{1, 3, 5\}, \Omega\},$
- $\mathcal{A} = \mathcal{P}(\Omega) \setminus \mathcal{M},$
- $p(\emptyset) = 0, p(\Omega) = 1, p(\{2, 4, 6\}) = p(\{1, 3, 5\}) = \frac{1}{2}, p(\{3, 6\}) = \frac{1}{3}, p(\{1, 2, 4, 5\}) = \frac{2}{3},$
for all other cases $p(A) = \max_{M \in \mathcal{M}}(p(M) : M \subseteq A),$ meaning that we have no further information about the ambiguous events.

A possible p^* prior would be (listed only for the elements of Ω): $p^*(\{1\}) = \frac{1}{2}, p^*(\{2\}) = \frac{1}{6}, p^*(\{6\}) = \frac{1}{3}, p^*(\{3\}) = p^*(\{4\}) = p^*(\{5\}) = 0,$ so if the sides have an equal probability of being on top once rolled, we can have three 1s, one 2 and two 6s on the six sides.

This is an example for the second kind of situations mentioned above. The events in $\mathcal{M}$ intersect multiple times but the probabilities of the intersections are not known.

The belief function is also a TU game with set of players $\Omega.$ Then the three points in Definition 4.1.1 can be formalized for games as well. We now list the properties' and the classes of games they are affiliated with.

1. $p(\emptyset) = 0, p(\Omega) = 1 \Leftrightarrow p$ is a capacity,
2. for each $A \subseteq B, p(A) \leq p(B), A, B \in \mathcal{E} \Leftrightarrow p$ is monotonic,

3. there exists $p^* \in \Delta(\Omega, \mathcal{E})$, such that $p \leq p^* \Leftrightarrow \text{Core}(p) \neq \emptyset$.

The first two points are straightforward, the third however, is nontrivial so we elaborate on that further.

Proposition 4.1.6. *Let $F = (\Omega, \mathcal{M}, \mathcal{A}, p)$ be an uncertainty field and let $x \in \mathbb{R}^\Omega$. Then $x \in \text{Core}(p)$ if and only if the probability distribution generated by x is in $\Delta(F)$.*

Proof. $|\Omega| = n$.

Only if: Let $x = (x_1, x_2, \dots, x_n)^\top \in \text{Core}(p)$ arbitrarily fixed. Then for any $S \subseteq \Omega$:

$$\sum_{i \in S} x_i \geq p(S), \text{ and } \sum_{i \in \mathbb{C}S} x_i \geq p(\mathbb{C}S).$$

For any $S \in \mathcal{M}$, their complement $\mathbb{C}S \in \mathcal{M}$ also $p(S) + p(\mathbb{C}S) = 1$. Therefore

$$\sum_{i \in \Omega} x_i = 1 = p(S) + p(\mathbb{C}S),$$

meaning that for any $S \in \mathcal{M}$ it must hold that $\sum_{i \in S} x_i = p(S)$. Thus the distribution generated by x equals p for the events in $\mathcal{M}$. All other properties are trivial.

If: Let $p^* \in \Delta(\mathcal{F})$ and $S \subseteq \Omega$ arbitrarily fixed. Then:

$$p(S) \leq p^*(S) = \sum_{i \in S} p^*({i}),$$

which is the definition of the core. □

We conclude this section by formalizing an assumption about the decision makers' rationality. We believe that a consistency exists between the belief function and the subjective probabilities.

Assumption 4.1.7. *For any preference relation $\succsim$ on acts $A = \{a: \Omega \rightarrow \Delta(X)\}$ over uncertainty field $(\Omega, \mathcal{M}, \mathcal{A}, p)$, where $X \subset \mathbb{R}$ such that $|X| < \infty$, $\succsim$ has a Choquet expected utility representation with utility function u and subjective probability v , and $p \leq v$.*

Assumption 4.1.7 states that not only does a preference relation on A satisfy axioms **A1**, **A3**, **A4** and **A5**, it must also be such that the representing subjective probability is consistent with the lower bound function. This also means that for any unambiguous event $M \in \mathcal{M}$, $v(M) = p(M)$.

This might seem overly restrictive, after all, subjective probabilities in the Choquet expected utility framework generally have nothing to do with the actual likelihood of events, much less with their lower bounds. Our framework suggests that both the preference relation, and therefore the representing subjective probability have to be derived from a decision maker's knowledge or belief of the events' probabilities. This is an intuitive assumption and one that is necessary for progression.

4.2 Decision under ambiguity

The previous section described how we need to modify the probability field in order to implement the possibility of imperfect – ambiguous – information. Our question for the remainder of this section is, how do decision makers form their subjective probabilities based on the lower bound of the uncertainty field?

Through answering this question, we mean to differentiate the decision makers by their subjective probabilities. The goal in the end is to define attitudes towards ambiguity. Based on what we know about decision making under risk we believe that there exist at least three types of decision makers: those who are ambiguity averse, those who are ambiguity neutral and those who are ambiguity seekers. These stand for the people who assign lower subjective probabilities to ambiguous events, ambiguity has no effect on them, or assign higher probabilities to ambiguous events.

4.2.1 Axioms for ambiguity neutrality

We introduce a series of axioms based on our intuitions about ambiguity neutral decision makers. The first is due to Ghirardato and Marinacci (2002), and it determines the set of benchmarks we wish to narrow down with another three axioms.

- **ADD** (Additivity) For any uncertainty field F , the subjective probabilities of ambiguity neutral decision makers are additive and are in the prior set of F .

If we recall, risk neutral decision makers were the ones, where the expected value of the utilities in different outcomes was the same as the utility of the expected values, provided that the von Neumann-Morgenstern representation exists. In the case of ambiguity, we may say that if the Choquet expected utility representation theorem holds, and the Choquet integral can be replaced with a Lebesgue integral, then the decision maker is – in a way – unaffected by the presence of ambiguity. Conversely, if a decision maker’s subjective probabilities are not additive, we can say that their preference is biased through the simple presence of ambiguity in our model.

Since uncertainty field F contains all information available about the “true” distribution over Ω , only members of the prior set $\Delta(F)$ can be considered ambiguity neutral decision makers’ subjective probabilities.

Let v_F^N denote the subjective probability of an ambiguity neutral decision maker in uncertainty field F . Because of Proposition 4.1.6, any v_F^N can be identified with a solution on game p , $\varphi(p)$, where $\varphi: \Gamma_b^\Omega \rightarrow \mathbb{R}^\Omega$ that takes its values from the core. We now interpret the meanings of some axioms on φ for uncertainty fields. We retain the abbreviations from Chapter 2 but the names are changed to follow the new context.

- **ETP** (Equal evaluation of symmetric outcomes) If $\omega_1 \sim^p \omega_2$, then $\varphi(p)_{\omega_1} = \varphi(p)_{\omega_2}$, where $\omega_1, \omega_2 \in \Omega$.

The thought behind ETP is deep-rooted, a good example for that is the famous Laplace's Principle of Insufficient Reason. The introduction of Schmeidler (1989) supports it but points out its limitations. The limitations are, as Gilboa and Marinacci (2011) shows in an example is that there are numerous real-life situations where ETP is clearly not present. Their example is that, suppose that tomorrow the Third World War may erupt. We have two outcomes: it either erupts or not, and we have no way to differentiate between them. Under the rules of ADD and ETP, we must assign equal, $\frac{1}{2}-\frac{1}{2}$ probabilities to each of the two outcomes which is of course implausible. However, this example and the examples resembling it that oppose ETP tend to smuggle some concept of time into our simple, static model. In an uncertainty field, the outcomes are considered independent of time. The author would argue that there is a significant, albeit mainly philosophical difference between an urn situation and the above mentioned World War situation.

Example 4.2.1. Let us take a look at Example 4.1.4. The outcomes {black} and {yellow} are symmetric, but the prior does not satisfy ETP. In Example 4.1.5 outcomes {1} and {5} are symmetric and the given prior does not satisfy ETP either.

- **COV** (Additivity invariance) Consider two uncertainty fields, $F_1 = (\Omega, \mathcal{M}, \mathcal{A}, p)$ and $F_2 = (\Omega, \mathcal{M}, \mathcal{A}, \alpha p \oplus \beta)$, where $\beta \in \mathbb{R}^\Omega$, and $\alpha > 0$ such that $\alpha p \oplus \beta$ is a belief function. Then, for an ambiguity neutral decision maker's subjective probabilities in the two situations $\varphi(p)$ and $\varphi(\alpha p \oplus \beta)$ it must hold that $\varphi(\alpha p \oplus \beta) = \alpha \varphi(p) + \beta$.

COV forms a connection between ambiguity fields that differ from each other only in their belief functions. The basic thought behind it is that if we were to change an uncertainty field such that unambiguous events remain unambiguous and ambiguous events remain ambiguous by altering the ratios of already known events, then the ambiguity neutral subjective probabilities must change 'accordingly'.

Let p be a belief function and $b \in \Delta(\Omega)$, an (additive) probability measure. Then for each $\gamma \in [0, 1]$, convex combination $\gamma p + (1 - \gamma)b = \alpha p \oplus \beta$, where $\alpha = \gamma$, and $\beta \in \mathbb{R}^\Omega$ such that $\beta_\omega = (1 - \gamma)b(\{\omega\})$. Hence, if we mix p with a probability distribution b , the resulting belief function will have the same 'ambiguity structure' as p , meaning that an ambiguity neutral decision maker should, in a way, regard b as an absolute.

It is easy to see that $\alpha p \oplus \beta$ is a belief function if and only if α and β are such that $\beta_\omega \leq p(\{\omega\})$ and $\alpha = 1 - \sum_{\omega \in \Omega} \beta_\omega$.

Example 4.2.2. Again we demonstrate by using the uncertainty field from Example 4.1.4. Suppose that we added 10 black balls into the original urn of 90 balls. We now have 100 balls, and at least 10 of them are black so, for α and β the following must hold:

$$\beta = (0, \beta_{\{\text{black}\}}, 0)^\top ,$$

$$\alpha 0 + \beta_{\{\text{black}\}} = \frac{1}{10} ,$$

$$\alpha = 1 - \beta_{\{\text{black}\}} .$$

It is easy to see that $\beta = (0, \frac{1}{10}, 0)$ and $\alpha = \frac{9}{10}$. In this case COV means that the subjective probability of an ambiguity neutral decision maker decreases by 10% for each event, and increases by 0.1 for each event that contains $\{\text{black}\}$ in the new situation.

- **WRCP** (Weak reduced belief property) Consider an uncertainty field $(\Omega, \mathcal{M}, \mathcal{A}, p)$, and an outcome $\omega \in \Omega$. Then, for ambiguity neutral subjective probability $\varphi(p)$ it must hold that

$$\varphi_{\Omega \setminus \{\omega\}}(p) = \varphi(p_{(\Omega \setminus \{\omega\}, \varphi(p))}^c) p_{(\Omega \setminus \{\omega\}, \varphi(p))}(\Omega \setminus \{\omega\}) .$$

WRCP forms a connection between uncertainty fields that differ from each other in an outcome. It means that if a decision maker were to omit an outcome, their belief function would have to be updated following the reduced game in Definition 2.1.2.

This axiom allows us to apply the idea of conditional probabilities to the belief function. $\varphi(p_{(\Omega \setminus \{\omega\}, \varphi(p))}^c)$ stands for the decision maker's subjective probability distribution with the condition that ω is deleted from Ω . The axiom formalizes that this distribution must be exactly the same as the original distribution $\varphi(p)$ on $\Omega \setminus \{\omega\}$ normalized to one. Notice that this normalization constant, $p_{(\Omega \setminus \{\omega\}, \varphi(p))}(\Omega \setminus \{\omega\}) = p(\Omega) - \varphi_{\{\omega\}}(p) = 1 - \varphi_{\{\omega\}}(p)$, therefore an equivalent form of the axiom would be

$$\frac{\varphi_{\Omega \setminus \{\omega\}}(p)}{1 - \varphi_{\{\omega\}}(p)} = \varphi(p_{(\Omega \setminus \{\omega\}, \varphi(p))}^c) .$$

This form is slightly simpler and shows the analogy of conditional probabilities better.

Example 4.2.3. Once again, consider the uncertainty field of Example 4.1.4. We know that $\varphi(p) = (\varphi_{\{\text{red}\}}(p), \varphi_{\{\text{black}\}}(p), \varphi_{\{\text{yellow}\}}(p))^\top = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^\top$ from ADD and ETP. Suppose that the decision maker somehow learned that there are no yellow balls in the urn. Then, φ satisfies WRCP only if for the new field, it holds that $\varphi(p_{(\{\text{red}, \text{black}\}, \varphi(p))}) = (\frac{1}{2}, \frac{1}{2})^\top$.

Theorem 4.2.4. *Consider uncertainty field, $F = (\Omega, \mathcal{M}, \mathcal{A}, p)$, set of acts $A = \{a: \Omega \rightarrow \Delta(X)\}$, where $X \subset \mathbb{R}$ such that $|X| < \infty$, and an ambiguity neutral decision maker's preference relation $\succsim$ that has a Choquet expected utility representation, with representing utility function u and subjective probability v_N . If **ADD**, **ETP**, **COV** and **WRCP** hold, then $v_N(E) = \sum_{\omega \in E} \mathcal{N}_\omega(p)$, for all $E \subseteq \Omega$.*

Proof. Because of ADD, we may use Corollary 2.3.15, we only need to prove that the belief functions in the set of uncertainty fields $\mathcal{F}$ are indeed identifiable with the set of monotonic, balanced capacities, Γ_b . Let the set of such belief functions be denoted by P . Since each element of P is a monotonic, balanced capacity, we have no trouble seeing that $P \subseteq \Gamma_b$. The reverse is also true, since for each $v \in \mathcal{G}_b^N$ we can have a tuple $F = (N, \{\emptyset, N\}, \mathcal{P}(N), v)$ that satisfies all conditions required from an uncertainty field. Therefore $\Gamma_b \subseteq P$ also holds.

Since ETP, COV and WRCP characterize the nucleolus on Γ_b , the proof is complete. $\square$

4.2.2 Ambiguity aversion and seeking

Theorem 4.2.4 formalizes that a preference relation $\succsim$ is ambiguity neutral if and only if the representing subjective probability is the nucleolus of the belief function. We now use this subjective probability and the resulting preference relation as a benchmark to define ambiguity attitude. Our approach differs from Ghirardato and Marinacci (2002), as we compare each preference, each subjective probability to the same benchmark.

Definition 4.2.5. *Consider uncertainty field $F = (\Omega, \mathcal{M}, \mathcal{A}, p)$, set of acts $A = \{a: \Omega \rightarrow \Delta(X)\}$, where $X \subset \mathbb{R}$ such that $|X| < \infty$, and preference relation $\succsim$ on A that has a Choquet expected utility representation. The representing utility function and subjective probability are denoted by u and v respectively. Let us suppose that $u \geq 0$. If for all $a \in A$*

$$(C) \int u(a) \, dv \leq (=, \geq) (C) \int u(a) \, dv_N = \int u(a) \, dv_N ,$$

where $v_N(E) = \sum_{\omega \in E} \mathcal{N}_\omega(p)$, then $\succsim$ exhibits ambiguity aversion (neutrality, seeking).

Notice that the nonnegativity of u only means that all representing utility functions have a lower bound, because u is unique up to a positive affine transformation. This is satisfied whenever set of acts A is closed.

The thought behind Definition 4.2.5 is that as risk attitude was determined solely by the decision makers' utility functions, ambiguity attitude is determined by their subjective probabilities. If a decision maker's Choquet expected utility is not larger for any act then it would be, were they ambiguity neutral, then that decision maker exhibits ambiguity

aversion. Because of Assumption 4.1.7 $v(E)$ and $v_N(E)$ differ from each other only if E is ambiguous. The definition basically says that if the preference relation represented by u and v_N prefers an unambiguous act to an ambiguous one, $\succsim$ is ambiguity averse if it prefers it ‘even more’.

Proposition 4.2.6. *Consider uncertainty field $F = (\Omega, \mathcal{M}, \mathcal{A}, p)$, set of acts $A = \{a: \Omega \rightarrow \Delta(X)\}$, where $X \subset \mathbb{R}$ such that $|X| < \infty$, and preference relation $\succsim$ on A that has a Choquet expected utility representation. The representing utility function and subjective probability are denoted by u and v respectively. Let us suppose that $u \geq 0$. Then, $\succsim$ exhibits ambiguity aversion (neutrality, seeking) if and only if $v \leq (=, \geq) v_N$, where $v_N(E) = \sum_{\omega \in E} \mathcal{N}_\omega(p)$.*

Proof. If: It is the result of the Choquet integral’s monotonicity in the variable of integration property.

Only if: Suppose that for a set of outcomes $E \subset \Omega$, $v(E) > v_N(E)$. Let $p_0, p_1 \in \Delta(X)$ such that $u(p_0) = 0$ (this is nonrestrictive because u was unique up to a positive affine transformation), $p_0 \prec p_1$ and let act $a \in A$ be such that for all $\omega \in \Omega$

$$a(\omega) = \begin{cases} p_1, & \text{if } \omega \in E \\ p_0, & \text{if } \omega \notin E \end{cases}.$$

Naturally, we prove that $(C) \int u(a) dv > (C) \int u(a) dv_N$.

It is clear that $0 < u(p_1)$. Then, by the Choquet integral’s definition on step functions

$$(C) \int u(a) dv = (u(p_1) - 0) v(E) > (u(p_1) - 0) v_N(E) = (C) \int u(a) dv_N$$

The proof of the ambiguity seeker case follows the very same steps, with the inequalities reversed. $\square$

4.2.3 Making comparisons

In this subsection, we first compare Schmeidler’s **A6** (A9 in his original numbering) and our definition of ambiguity attitude and secondly, the results of Ghirardato and Marinacci (2002) and our own.

While both ambiguity aversion and uncertainty aversion are trying to address the issues raised by the Ellsberg paradox, the two approaches are so different that any connection between the two should seem incidental. Still, it is easy to see by ADD, that ambiguity neutral decision makers are uncertainty averse, because additive subjective probabilities are convex, moreover for any $E, F \subseteq \Omega$ it holds that

$$v_N(E) + v_N(F) = v_N(E \cup F) + v_N(E \cap F),$$

where v_N is a monotonic, additive capacity.

From this, we might get the sensation that ambiguity aversion could imply uncertainty aversion. This, however, is not true as Example 4.2.7 demonstrates.

Example 4.2.7. Consider a version of Ellsberg's urn. Again, we have an urn of 90 balls, but this time, we know nothing about them except that any of them might be red, yellow or black. The uncertainty field that describes the situation is:

- $\Omega = \{\text{red, yellow, black}\},$
- $\mathcal{M} = \{\emptyset, \Omega\},$
- $\mathcal{A} = \mathcal{P}(\Omega) \setminus \mathcal{M},$
- $p(\Omega) = 1, p(E) = 0 \text{ for any } E \subset \Omega.$

Now, consider that a decision maker has the following subjective probability: for $S \subseteq \Omega$

$$v(S) = \begin{cases} 0, & \text{if } S = \emptyset \\ \frac{8}{25}, & \text{if } |S| = 1 \\ \frac{15}{25}, & \text{if } |S| = 2 \\ 1, & \text{if } |S| = 3 \end{cases}.$$

It is easy to see that because of ADD and ETP, the ambiguity neutral subjective probability $v_N(S)$ of event $S \subseteq \Omega$ would be

$$v_N(S) = \begin{cases} 0, & \text{if } S = \emptyset \\ \frac{1}{3}, & \text{if } |S| = 1 \\ \frac{2}{3}, & \text{if } |S| = 2 \\ 1, & \text{if } |S| = 3 \end{cases}.$$

That way $v \leq v_N$, so because of Proposition 4.2.6 the decision maker is ambiguity averse while it is easy to see that v is not convex.

We have seen that ambiguity aversion is not a sufficient condition for uncertainty aversion. The reverse is also false, Example 4.2.8 shows that uncertainty aversion does not imply ambiguity aversion either.

Example 4.2.8. Consider the very same uncertainty field, with the following additive subjective probability: $v(\{\text{red}\}) = \frac{1}{3}$, $v(\{\text{yellow}\}) = 0$ and $v(\{\text{black}\}) = \frac{2}{3}$ and for all other events, v is defined so that additivity holds. This way, v is convex, but it does not represent ambiguity aversion.

Suppose that $0, 1 \in X$ and let $a_r, a_b, c_0, c_1 \in A$ denote the following acts:

- $a_r(\{\text{red}\}) = \delta_1, a_r(\{\text{yellow}\}) = \delta_0, a_r(\{\text{black}\}) = \delta_0,$

- $a_b(\{\text{red}\}) = \delta_0$, $a_b(\{\text{yellow}\}) = \delta_0$, $a_b(\{\text{black}\}) = \delta_1$,
- $c_0(\{\text{red}\}) = \delta_0$, $c_0(\{\text{yellow}\}) = \delta_0$, $c_0(\{\text{black}\}) = \delta_0$,
- $c_1(\{\text{red}\}) = \delta_1$, $c_1(\{\text{yellow}\}) = \delta_1$, $c_1(\{\text{black}\}) = \delta_1$.

The acts represent the following gambles:

- a_r : draw a ball, if it is red, you get 1\$, if not you get 0\$,
- a_b : draw a ball, if it is black, you get 1\$, if not you get 0\$,
- c_0 : you get 0\$,
- c_1 : you get 1\$.

Let $u_0, u_1 \in \mathbb{R}$ denote the following values: $u_0 = (C) \int u(c_0) dv$, $u_1 = (C) \int u(c_1) dv$. We suppose that getting one dollar is strictly preferred to getting zero dollars: $\delta_1 \succ \delta_0$. Because of the monotonicity axiom it follows that $c_1 \succ c_0$, and of course $u_1 > u_0$. Then,

$$(C) \int u(a_b) dv = \frac{2}{3}u_1 + \frac{1}{3}u_0 > \frac{1}{3}u_1 + \frac{2}{3}u_0 = (C) \int u(a_b) dv_N ,$$

meaning that $\succsim$ is not ambiguity averse.

To conclude, we compare our definition for ambiguity aversion to the result of Ghirardato and Marinacci. By using Epstein's comparative definition and using the additive subjective probabilities as benchmark, they proposed that v follows ambiguity aversion if and only if $\text{Core}(v) \neq \emptyset$.

The subjective probability in Example 4.2.8 had a nonempty core, meaning that balancedness does not imply ambiguity aversion in our sense. As for the reverse, however, we can see that in any situation, $v \leq v_N$ only if v is balanced, because v_N itself is an element of the core. Therefore, our definition is strictly stronger than Ghirardato and Marinacci's.

This is, of course, due to the fact that we had stronger conditions for ambiguity neutral decision makers. This paper argues that ambiguity neutrality is more than a bridge between ambiguity aversion and ambiguity seeking, it should also signify an unbiased evaluation of events. Going back to the original Ellsberg paradox, where we had 30 red and 60 black or yellow balls, if an individual thinks that there are exactly 15 black and 45 yellow balls, then they are not exhibiting ambiguity neutral behavior. Nor do they show ambiguity aversion or seeking either, this kind of behavior should be classified as 'additively biased' towards yellow balls.

On the grand scale, this paper argues that ambiguity neutrality should be characterized not only by concepts of stability (in this case, the nonemptiness of the core) but by

concepts of fairness as well. We presented a way of doing so and a way of defining ambiguity attitude. Of the three additional axioms, characterizing the nucleolus, we believe ETP is the most important and plausible one.

A possible question of further inquiry: do we even need COV and RGP? In this framework, we do, and fortunately they are intuitive and plausible, but so could be a number of other axioms of game theory solution concepts, that were not posed here. A deeper understanding of the model suggested by Epstein could lead to a modification of the framework, where a more general form of ETP, an axiom on symmetric events (not just outcomes) might alone characterize ambiguity neutrality.

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