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New bounds for 3-part Sperner families  
Új korlátok 3-részes Sperner családokra

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# NEW BOUNDS FOR 3-PART SPERNER FAMILIES

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ABSTRACT. In this paper we give a new upper and lower bound on the size of maximum 3-part Sperner families. We prove that  $1,05 < d_3 < 1,072$ . Further we disprove a conjecture of Aydinian, Czabarka, Erdős, Székely on the maximum size of  $k$ -part Sperner families for the case of equal parts of size  $2^\ell - 1$ .

## 1. INTRODUCTION

Let  $X$  be a finite set. Consider a partition of  $X$  into  $k$  subsets, so  $X = X_1 \cup X_2 \cup \dots \cup X_k$ , where  $X_i$  are pairwise disjoint. A family  $\mathcal{F} \subset \mathcal{P}(X)$  is called a  $k$ -part Sperner family on  $X$  with respect to the partition, if there are no two different elements  $F_1, F_2$  of  $\mathcal{F}$  such that  $F_1 \subset F_2$  and  $F_2 \setminus F_1 \subset X_j$  for some  $1 \leq j \leq k$ .

Let  $|X| = n$ ,  $|X_i| = n_i$ . We denote the maximum size of a  $k$ -part Sperner family on  $X$  by  $g(n_1, n_2, \dots, n_k)$ . Let  $f(n, k)$  be the maximum of  $g(n_1, n_2, \dots, n_k)$ . (It is generally believed that the maximum is attained at the balanced case, i. e. when all the  $n_i$  are almost equal.)

With this notation Sperner's theorem states that  $f(n, 1) = \binom{n}{\lfloor \frac{n}{2} \rfloor}$ . Katona[8] and Kleitman[9] proved the two-part Sperner theorem:  $g(n_1, n_2) = \binom{n_1+n_2}{\lfloor \frac{n_1+n_2}{2} \rfloor}$ . An important tool in their proof is the symmetric chain partition. If  $X$  is a finite set then a sequence of subsets  $(A_1, A_2, \dots, A_k)$  is a symmetric chain of length  $k$  if  $A_i \subset A_{i+1}$ ,  $|A_{i+1} \setminus A_i| = 1$  for all  $1 \leq i \leq k-1$  and  $|A_1| + |A_k| = |X|$ . A symmetric chain partition is a partition of  $\mathcal{P}(X)$  into symmetric chains. Such a partition always exists, see [2]. We can easily count the number of chains with a given length in a symmetric chain partition (which does not depend on the choice of partition). For our present purposes it suffices to consider the case  $|X| = 2n$ . In this case every symmetric chain has odd length. The number of chains with length at least  $2k+1$  is  $\binom{2n}{n+k}$ , because inclusion clearly gives a bijection between these chains and the  $n+k$  element subsets of  $X$ . So the number of chains with length  $2k+1$  is  $\binom{2n}{n+k} - \binom{2n}{n+k+1}$  for  $0 \leq k \leq n-1$ , and there is 1 chain with length  $2n+1$ .

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Symmetric chain partition also gives us a new proof of Sperner's theorem, because each chain can contain at most one element of a Sperner family. So the number of chains which is  $\binom{|X|}{\lfloor \frac{|X|}{2} \rfloor}$  is an upper bound to the size of a Sperner family. Our method to obtain an upper bound to 3-part Sperner families will be something similar in Section 2.

In the case of  $k$ -part Sperner families ( $k > 2$ ) there are several results by Füredi[6] and Griggs, Odlyzko, Shearer[7]. They proved that for fixed  $k$  the limit  $\lim_{n \rightarrow \infty} \frac{f(n,k)}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$  exists. We denote this limit by  $d_k$ . They also proved that  $d_k \sim \frac{\sqrt{k\pi}}{\sqrt{4 \log k}}$ . However, we don't know the exact value of any  $d_k$  except in the case of  $k = 1$  and 2, when  $d_1 = d_2 = 1$ . A good survey on this topic is [5] and [1].

In this paper we give new bounds for  $d_3$ . We prove that  $1.05 < d_3 < 1.072$  improving the bounds 1.036 and 1.131[5].

Griggs, Odlyzko, Shearer mentioned in their paper[7] that they managed to prove that  $\lim_{n \rightarrow \infty} \frac{g(n,n,n)}{\binom{3n}{\lfloor 1.5n \rfloor}} < 1.072$ , where the constant is the same as ours, and since this balanced case is believed to be the extremal they conjectured that this bound also holds for  $d_3$ . So we prove this conjecture. We will show stronger bounds if the ratio of  $n_i$ 's is fixed.

In Section 2 we give upper bounds for 3-part Sperner families using monochromatic chain partitions. However, to obtain the asymptotic value of these upper bounds we need some long and technical calculations which are contained in Section 3. The conclusions are made in Section 4.

In Section 5 we mention the result of Erdős, Katona[3] and Füredi, Griggs, Odlyzko, Shearer[7][5] about homogeneous families, we will see that finding the optimal  $k$ -part Sperner family is equivalent to finding the solution of an integer program. Solving this with computer in the case of  $k = 3, n_1 = n_2 = n_3 = 15$  we disprove the conjecture of Aydinian, Czaparka, Erdős, Székely[1], on the maximum size of  $k$ -part Sperner families for the case of equal parts of size  $2^\ell - 1$ . We also show some connection with monochromatic chain partitions.

In Section 6, to get a better lower bound we will basically use the same method as Füredi[6] and Griggs, Odlyzko, Shearer[7]. But where they made an explicit choice we find a better one using computer instead.

## 2. MONOCHROMATIC CHAIN PARTITIONS

**Definition 2.1.** Let  $X = X_1 \cup \dots \cup X_k$  be a set partitioned into  $k$  parts. A sequence  $(A_1, A_2, \dots, A_h)$  of subsets of  $X$  is called monochromatic chain of  $X$  if  $A_i \subset A_{i+1}$  for  $i = 1, 2, \dots, h-1$  and there exist  $j$  such that  $1 \leq j \leq k$  and  $A_{i+1} \setminus A_i \subset X_j$  for  $i = 1, 2, \dots, h-1$ , where  $h$  is the length of the chain,  $j$  is the color of the chain.

**Definition 2.2.** Let  $X = X_1 \cup \dots \cup X_k$  be a set partitioned into  $k$  parts. A set  $\mathcal{C}$  of monochromatic chains of  $X$  is called monochromatic chain partition, if every subset of  $X$  is contained in exactly one monochromatic chain in  $\mathcal{C}$ .

If  $\mathcal{F}$  is a Sperner family on  $X$ , then every monochromatic chain can contain at most one element of  $\mathcal{F}$ , so the lemma below clearly follows.

**Lemma 2.3.** *If  $\mathcal{F}$  is a Sperner family on  $X = X_1 \cup \dots \cup X_k$ ,  $\mathcal{C}$  is a monochromatic chain partition of  $X$ , then  $|\mathcal{F}| \leq |\mathcal{C}|$ .*

Let  $X = X_1 \cup \dots \cup X_k$  and  $Y = Y_1 \cup \dots \cup Y_m$  be disjoint sets with monochromatic chain partitions  $\mathcal{C}_1$  and  $\mathcal{C}_2$ . Our aim is to construct a monochromatic chain partition  $(\mathcal{C}_1 \square \mathcal{C}_2)$  of  $Z = X_1 \cup \dots \cup X_k \cup Y_1 \cup \dots \cup Y_m$ .

Let  $(A_1, A_2, \dots, A_h)$  and  $(B_1, B_2, \dots, B_g)$  be chains in  $\mathcal{C}_1$  and  $\mathcal{C}_2$ , respectively. We will show  $\min(h, g)$  monochromatic chains with length  $\max(h, g)$  such that they cover exactly those  $g \cdot h$  subsets of  $Z$  which have the form  $A_i \cup B_j$ .

If  $h \geq g$ , take the following monochromatic chains for  $1 \leq i \leq g$ :  $(A_1 \cup B_i, A_2 \cup B_i, \dots, A_h \cup B_i)$ .

If  $h < g$ , take the following monochromatic chains for  $1 \leq i \leq h$ :  $(A_i \cup B_1, A_i \cup B_2, \dots, A_i \cup B_g)$ .

By taking these monochromatic chains for all possible  $|\mathcal{C}_1| \cdot |\mathcal{C}_2|$  pairs of monochromatic chains, we get a monochromatic chain partition of  $Z$ . We denote it by  $\mathcal{C}_1 \square \mathcal{C}_2$ .

**Definition 2.4.** Let  $\mathcal{C}$  be a monochromatic chain partition. We define its profile vector as  $p(\mathcal{C}) = (k_1, k_2, k_3, \dots)$  where  $k_i$  is the number of chains in  $\mathcal{C}$  with length  $i$ . We will also use the following form:  $p(\mathcal{C}) = \sum_{i=1}^{\infty} k_i \mathbf{b}_i$ , where  $\mathbf{b}_i = (0, 0, \dots, 0, \overset{i}{1}, 0, \dots)$ .

**Definition 2.5.** Let  $A$  be an algebra over  $\mathbb{R}$  with linearly independent generators  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \dots$ . We define the product  $\times$  on the generators in the following way:  $\mathbf{b}_i \times \mathbf{b}_j = \min(i, j) \mathbf{b}_{\max(i, j)}$ . There is a unique way to extend this to the whole algebra to be distributive.

For  $u = \sum_{i=1}^{\infty} k_i \mathbf{b}_i$  we will sometimes use the notation  $(u)_i = k_i$ . It is easy to see that  $A$  will be a commutative, associative algebra. Moreover the following statement holds.

**Lemma 2.6.** *If  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are monochromatic chain partitions then  $p(\mathcal{C}_1 \square \mathcal{C}_2) = p(\mathcal{C}_1) \times p(\mathcal{C}_2)$ .*

**Definition 2.7.** We define the linear map  $L : A \rightarrow \mathbb{R}$  in the following way. For  $u = \sum_{i=1}^{\infty} k_i \mathbf{b}_i$ ,  $L(u) = \sum_{i=1}^{\infty} k_i$ .

Note that if  $u$  is a profile vector of a monochromatic chain partition then it gives us the number of chains.

Let  $X = X_1 \cup X_2 \cup X_3$  be a set partitioned into 3 parts. Let  $\mathcal{C}_i$  be the symmetric chain partition of  $\mathcal{P}(X_i)$ . (See Section 1.) Then  $L(p(\mathcal{C}_1) \times p(\mathcal{C}_2) \times p(\mathcal{C}_3))$  gives us an upper bound for the size of any 3-part Sperner family on  $X$  by Lemma 2.3. If  $|X_i| = n_i$ , we denote this upper bound by  $U(n_1, n_2, n_3)$ . First we try to determine  $U(2n_1, 2n_2, 2n_3)$ .

On a  $2n$  element set the symmetric chain partition has the following profile vector

$$\begin{aligned} p_{2n} &= \sum_{i=0}^{n-1} \left( \binom{2n}{n+i} - \binom{2n}{n+i+1} \right) \mathbf{b}_{2i+1} + \mathbf{b}_{2n+1} \\ &= \binom{2n}{n} \mathbf{b}_1 + \sum_{i=1}^n \left( \binom{2n}{n+i} \right) (\mathbf{b}_{2i+1} - \mathbf{b}_{2i-1}) = \sum_{i=0}^n \binom{2n}{n+i} \mathbf{d}_i, \end{aligned}$$

where  $\mathbf{d}_0 = \mathbf{b}_1$  and  $\mathbf{d}_i = \mathbf{b}_{2i+1} - \mathbf{b}_{2i-1}$ . Using the distributivity of  $\times$  and the linearity of  $L$

$$\begin{aligned} U(2n_1, 2n_2, 2n_3) &= L(p_{2n_1} \times p_{2n_2} \times p_{2n_3}) \\ &= \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \sum_{k=0}^{n_3} \binom{2n_1}{n_1+i} \binom{2n_2}{n_2+j} \binom{2n_3}{n_3+k} L(\mathbf{d}_i \times \mathbf{d}_j \times \mathbf{d}_k). \end{aligned}$$

Now we determine the value of  $L(\mathbf{d}_i \times \mathbf{d}_j \times \mathbf{d}_k)$ . Due to symmetry we can assume that  $i \geq j \geq k$ .

$$\begin{aligned} L(\mathbf{d}_0 \times \mathbf{d}_0 \times \mathbf{d}_0) &= 1, \\ L(\mathbf{d}_i \times \mathbf{d}_i \times \mathbf{d}_i) &= 6 - 4i \text{ if } i > 0, \\ L(\mathbf{d}_i \times \mathbf{d}_i \times \mathbf{d}_0) &= 2 \text{ if } i > 0, \\ L(\mathbf{d}_i \times \mathbf{d}_i \times \mathbf{d}_k) &= 4 \text{ if } i > k > 0, \\ L(\mathbf{d}_i \times \mathbf{d}_j \times \mathbf{d}_k) &= 0 \text{ if } i > j \geq k. \end{aligned}$$

We will use the following notations

$$\begin{aligned}
A(n_1, n_2, n_3) &= \sum_{i=0}^{\min(n_1, n_2)} \sum_{j=0}^{\min(n_3, i)} \binom{2n_1}{n_1+i} \binom{2n_2}{n_2+i} \binom{2n_3}{n_3+j}, \\
B(n_1, n_2, n_3) &= \sum_{i=0}^{\min(n_1, n_2, n_3)} \binom{2n_1}{n_1+i} \binom{2n_2}{n_2+i} \binom{2n_3}{n_3+i} i, \\
E(n_1, n_2, n_3) &= +11 \binom{2n_1}{n_1} \binom{2n_2}{n_2} \binom{2n_3}{n_3} \\
&\quad + 6 \sum_{i=1}^{\min(n_1, n_2, n_3)} \binom{2n_1}{n_1+i} \binom{2n_2}{n_2+i} \binom{2n_3}{n_3+i} \\
&\quad + 2 \sum_{i=1}^{\min(n_1, n_2)} \binom{2n_1}{n_1+i} \binom{2n_2}{n_2+i} \binom{2n_3}{n_3} \\
&\quad + 2 \sum_{i=1}^{\min(n_1, n_3)} \binom{2n_1}{n_1+i} \binom{2n_2}{n_2} \binom{2n_3}{n_3+i} \\
&\quad + 2 \sum_{i=1}^{\min(n_2, n_3)} \binom{2n_1}{n_1} \binom{2n_2}{n_2+i} \binom{2n_3}{n_3+i}.
\end{aligned}$$

With these notations we get

$$\begin{aligned}
U(2n_1, 2n_2, 2n_3) &= 4A(n_1, n_2, n_3) + 4A(n_2, n_3, n_1) + 4A(n_3, n_1, n_2) \\
&\quad - 4B(n_1, n_2, n_3) - E(n_1, n_2, n_3).
\end{aligned} \tag{1}$$

To obtain a common upper bound we prove the following theorem.

**Theorem 2.8.** *If  $3n = n_1 + n_2 + n_3$  then  $U(2n_1, 2n_2, 2n_3) \leq U(2n, 2n, 2n)$ .*

We need further lemmas to prove this theorem.

**Definition 2.9.** We define the linear map  $\nu : A \rightarrow A$  in the following way on the generators:  $\nu(\mathbf{b}_1) = \mathbf{b}_2$  and  $\nu(\mathbf{b}_i) = \mathbf{b}_{i-1} + \mathbf{b}_{i+1}$  for  $i > 1$ , and extend this linearly.

**Lemma 2.10.**  $\nu(\nu(p_{2n})) = p_{2n+2}$ .

*Proof.* It can be easily proved using the following identity:

$$\begin{aligned} & \left( \binom{n}{a} - \binom{n}{a+1} \right) + \left( \binom{n}{a+1} - \binom{n}{a+2} \right) \\ &= \left( \binom{n}{a} + \binom{n}{a+1} \right) - \left( \binom{n}{a+1} + \binom{n}{a+2} \right) \\ &= \binom{n+1}{a+1} - \binom{n+1}{a+2}. \end{aligned}$$

(Or we can recall how we constructed the symmetric chain partition of  $\mathcal{P}(X \cup \{t\})$  from the symmetric chain partition of  $\mathcal{P}(X)$ . See [8].)  $\square$

**Lemma 2.11.** *For  $u = \sum_{i=1}^{\infty} k_i \mathbf{b}_i$  and  $w = \sum_{i=1}^{\infty} \ell_i \mathbf{b}_i$  we have  $L(\nu(u) \times w) = 2L(u \times w) - \sum_{i=1}^{\infty} k_i \ell_i$ .*

*Proof.* Observe that  $L(\nu(\mathbf{b}_i) \times \mathbf{b}_i) = 2L(\mathbf{b}_i \times \mathbf{b}_i) - 1$  and  $L(\nu(\mathbf{b}_i) \times \mathbf{b}_j) = 2L(\mathbf{b}_i \times \mathbf{b}_j)$  if  $i \neq j$ . With distributivity and linearity we get the statement.  $\square$

To simplify the calculations we define  $\binom{m}{m+1}$  to be 0.

**Lemma 2.12.** *For fixed  $0 \leq j < k$  the following function is monotone decreasing in  $m$  where  $m \geq k$ :*

$$\frac{\binom{2m}{m+j} - \binom{2m}{m+k}}{\binom{2m}{m+k} - \binom{2m}{m+k+1}}.$$

*Proof.* For fixed  $a > 0$  the following function is monotone decreasing in  $m$  where  $m \geq a$ :

$$\frac{\binom{2m}{m+a-1} - \binom{2m}{m+a}}{\binom{2m}{m+a} - \binom{2m}{m+a+1}} = \frac{(2a-1)(m+a+1)}{(2a+1)(m-a+1)}.$$

(Note that the equality above also holds in the degenerate case of  $a = m$ .)

Multiplying functions like above we get that for fixed  $0 \leq j < k$  the following function is monotone decreasing in  $m$ , ( $m > j$ ):

$$\frac{\binom{2m}{m+j} - \binom{2m}{m+j+1}}{\binom{2m}{m+k} - \binom{2m}{m+k+1}}.$$

Summing this kind of functions we get the statement.  $\square$

**Lemma 2.13.** *If  $n_1 < n_2$  then  $U(2n_1 + 2, 2n_2, 2n_3) \geq U(2n_1, 2n_2 + 2, 2n_3)$ .*

*Proof.* We denote  $p_{2n_i}$  by  $q_i$ . By Lemma 2.10 we need to compare  $L(\nu(\nu(q_1)) \times q_2 \times q_3)$  and  $L(\nu(\nu(q_2)) \times q_1 \times q_3)$ . Further  $q_2 \times q_3$  has 0's in all its even coordinates, and  $\nu(q_1)$  has 0's in all its odd coordinates, so  $L(\nu(\nu(q_1)) \times q_2 \times q_3) = 2L(\nu(q_1) \times q_2 \times q_3)$ . Similarly,  $L(\nu(\nu(q_2)) \times q_3 \times q_3) = 2L(\nu(q_2) \times q_1 \times q_3)$ . So we have to compare  $L(\nu(q_1) \times q_2 \times q_3) = 2L(q_1 \times q_2 \times q_3) - \sum_{i=1}^{\infty} (q_1)_i (q_2 \times q_3)_i$  and  $L(\nu(q_1) \times q_2 \times q_3) = 2L(q_1 \times q_2 \times q_3) - \sum_{i=1}^{\infty} (q_2)_i (q_1 \times q_3)_i$ .

It is sufficient to prove that  $(q_1)_i (q_2 \times q_3)_i \leq (q_2)_i (q_1 \times q_3)_i$  for every  $i$ .

$$\begin{aligned} (q_1)_i (q_2 \times q_3)_i &= i(q_1)_i (q_2)_i (q_3)_i + (q_1)_i (q_2)_i \sum_{j=1}^{i-1} j(q_3)_j + (q_1)_i (q_3)_i \sum_{j=1}^{i-1} j(q_2)_j, \\ (q_2)_i (q_1 \times q_3)_i &= i(q_1)_i (q_2)_i (q_3)_i + (q_1)_i (q_2)_i \sum_{j=1}^{i-1} j(q_3)_j + (q_2)_i (q_3)_i \sum_{j=1}^{i-1} j(q_1)_j. \end{aligned}$$

Some terms cancel out, we can divide by  $(q_3)_i$ , so we only have to prove the following:

$$(q_1)_i \sum_{j=1}^{i-1} j(q_2)_j \leq (q_2)_i \sum_{j=1}^{i-1} j(q_1)_j.$$

If  $i$  is even, both sides are zero, so assume that  $i = 2k + 1$ . If  $k > n_1$  then the left hand side is 0, because  $(q_1)_i = 0$ , the other side is nonnegative thus we are done. So we may assume that  $k \leq n_1 < n_2$ .

If we use that

$$\begin{aligned} \sum_{j=1}^{i-1} j(p_{2m+1})_j &= \sum_{j=0}^{k-1} (2j+1) \left( \binom{2m}{m+j} - \binom{2m}{m+j+1} \right) \\ &= \binom{2m}{m} + 2 \sum_{j=1}^{k-1} \binom{2m}{m+j} - (2k-1) \binom{2m}{m+k} \\ &= \left( \binom{2m}{m} - \binom{2m}{m+k} \right) + 2 \sum_{j=1}^{k-1} \left( \binom{2m}{m+j} - \binom{2m}{m+k} \right), \end{aligned}$$

then the inequality we have to prove is

$$\begin{aligned}
& \left( \binom{2n_1}{n_1+k} - \binom{2n_1}{n_1+k+1} \right) \\
& \left( \left( \binom{2n_2}{n_2} - \binom{2n_2}{n_2+k} \right) + 2 \sum_{j=1}^{k-1} \left( \binom{2n_2}{n_2+j} - \binom{2n_2}{n_2+k} \right) \right) \\
& \leq \left( \binom{2n_2}{n_2+k} - \binom{2n_2}{n_2+k+1} \right) \\
& \left( \left( \binom{2n_1}{n_1} - \binom{2n_1}{n_1+k} \right) + 2 \sum_{j=1}^{k-1} \left( \binom{2n_1}{n_1+j} - \binom{2n_1}{n_1+k} \right) \right).
\end{aligned}$$

So it is sufficient to show that for every  $0 \leq j \leq k-1$ ,

$$\begin{aligned}
& \left( \binom{2n_1}{n_1+k} - \binom{2n_1}{n_1+k+1} \right) \left( \binom{2n_2}{n_2+j} - \binom{2n_2}{n_2+k} \right) \\
& \leq \left( \binom{2n_2}{n_2+k} - \binom{2n_2}{n_2+k+1} \right) \left( \binom{2n_1}{n_1+j} - \binom{2n_1}{n_1+k} \right).
\end{aligned}$$

This follows from Lemma 2.12, if we rearrange it and use  $n_1 < n_2$ .  $\square$

*Proof.* (Theorem 2.8)

We can write Lemma 2.13 in the following form. If  $n_1 < n_2$  then  $U(2n_1+2, 2n_2-2, 2n_3) \geq U(2n_1, 2n_2, 2n_3)$ .

Suppose  $n_1 < n$  and  $n_2 > n$ . Then  $U(2n_1, 2n_2, 2n_3) \leq U(2n_1+2, 2n_2-2, 2n_3)$  and with this transformation we can reach  $U(2n, 2n, 2n)$  which proves the statement.  $\square$

### 3. ASYMPTOTIC CALCULATIONS

This section is mainly a quite long and technical calculation. We encourage the reader who is not interested in this to jump to Theorem 3.4.

We will determine the value of  $\lim_{n \rightarrow \infty} \frac{U(2p_n, 2q_n, 2r_n)}{\binom{2s_n}{s_n}}$  (and prove the existence of the limit), where  $p_n, q_n, r_n$  are sequences of positive integers,  $s_n = p_n + q_n + r_n$ ,  $\lim_{n \rightarrow \infty} s_n = \infty$ ,  $\lim_{n \rightarrow \infty} \frac{p_n}{s_n} = \alpha$ ,  $\lim_{n \rightarrow \infty} \frac{q_n}{s_n} = \beta$  and  $\lim_{n \rightarrow \infty} \frac{r_n}{s_n} = \gamma$  for fixed  $\alpha, \beta, \gamma > 0$  real numbers ( $\alpha + \beta + \gamma = 1$ ).

We will use the following local form of the de Moivre-Laplace theorem [4, VII. 3. Theorem 1.]:

**Lemma 3.1.** *If  $H_n$  is a sequence such that  $\lim_{n \rightarrow \infty} \frac{H_n^3}{n^2} = 0$ , then for all  $\varepsilon > 0$ , exists  $n_0$  such that for each  $n > n_0$  and  $|k| \leq H_n$*

$$1 - \varepsilon < \frac{\binom{2n}{n+k}}{\frac{1}{\sqrt{\pi n}} 2^{2n} \exp(-\frac{k^2}{n})} < 1 + \varepsilon$$

holds.

Let  $K_n$  be  $\lfloor s_n^{0.51} \rfloor$ . Here  $\frac{K_n^3}{(p_n)^2}, \frac{K_n^3}{(q_n)^2}, \frac{K_n^3}{(r_n)^2} \rightarrow 0$  when  $n \rightarrow \infty$  thus applying the lemma above yields.

**Lemma 3.2.** *For all  $\varepsilon > 0$  if  $n$  is sufficiently large and  $|k| \leq K_n$  the following inequalities hold*

$$\begin{aligned} 1 - \varepsilon &< \frac{\binom{2p_n}{p_n+k}}{\frac{1}{\sqrt{\pi p_n}} 2^{2p_n} \exp(-\frac{k^2}{p_n})} < 1 + \varepsilon, \\ 1 - \varepsilon &< \frac{\binom{2q_n}{q_n+k}}{\frac{1}{\sqrt{\pi q_n}} 2^{2q_n} \exp(-\frac{k^2}{q_n})} < 1 + \varepsilon, \\ 1 - \varepsilon &< \frac{\binom{2r_n}{r_n+k}}{\frac{1}{\sqrt{\pi r_n}} 2^{2r_n} \exp(-\frac{k^2}{r_n})} < 1 + \varepsilon. \end{aligned}$$

If  $n$  is large enough, then  $K_n < p_n, q_n, r_n$ , which allows us to define the following expression:

$$A'(p_n, q_n, r_n) = \sum_{i=0}^{K_n} \sum_{j=0}^i \binom{2p_n}{p_n+i} \binom{2q_n}{q_n+i} \binom{2r_n}{r_n+j}.$$

By Lemma 3.2 we have

$$A'(p_n, q_n, r_n) = (1 + o(1)) \frac{1}{\pi^{\frac{3}{2}} \sqrt{p_n q_n r_n}} 2^{2s_n} \sum_{i=0}^{K_n} \sum_{j=0}^i \exp(-\frac{i^2}{p_n} - \frac{i^2}{q_n} - \frac{j^2}{r_n}).$$

On the other hand

$$\begin{aligned} &\exp(-\frac{i^2}{p_n} - \frac{i^2}{q_n} - \frac{j^2}{r_n}) \\ &= s_n \int_{i \frac{1}{\sqrt{s_n}}}^{(i+1) \frac{1}{\sqrt{s_n}}} \int_{j \frac{1}{\sqrt{s_n}}}^{(j+1) \frac{1}{\sqrt{s_n}}} \exp(-\frac{(\frac{i}{\sqrt{s_n}})^2}{\frac{p_n}{s_n}} - \frac{(\frac{i}{\sqrt{s_n}})^2}{\frac{q_n}{s_n}} - \frac{(\frac{j}{\sqrt{s_n}})^2}{\frac{r_n}{s_n}}) dx dy. \end{aligned} \quad (2)$$

Define  $f_n$  in the following way,

$$f_n(x, y) = \begin{cases} \exp(-\frac{(\frac{\lfloor x\sqrt{s_n} \rfloor}{\sqrt{s_n}})^2}{\frac{p_n}{s_n}} - \frac{(\frac{\lfloor x\sqrt{s_n} \rfloor}{\sqrt{s_n}})^2}{\frac{q_n}{s_n}} - \frac{(\frac{\lfloor y\sqrt{s_n} \rfloor}{\sqrt{s_n}})^2}{\frac{r_n}{s_n}}) & \text{if } 0 \leq \lfloor x\sqrt{s_n} \rfloor \leq K_n, \\ & 0 \leq \lfloor y\sqrt{s_n} \rfloor \leq \lfloor x\sqrt{s_n} \rfloor, \\ 0 & \text{otherwise.} \end{cases}$$

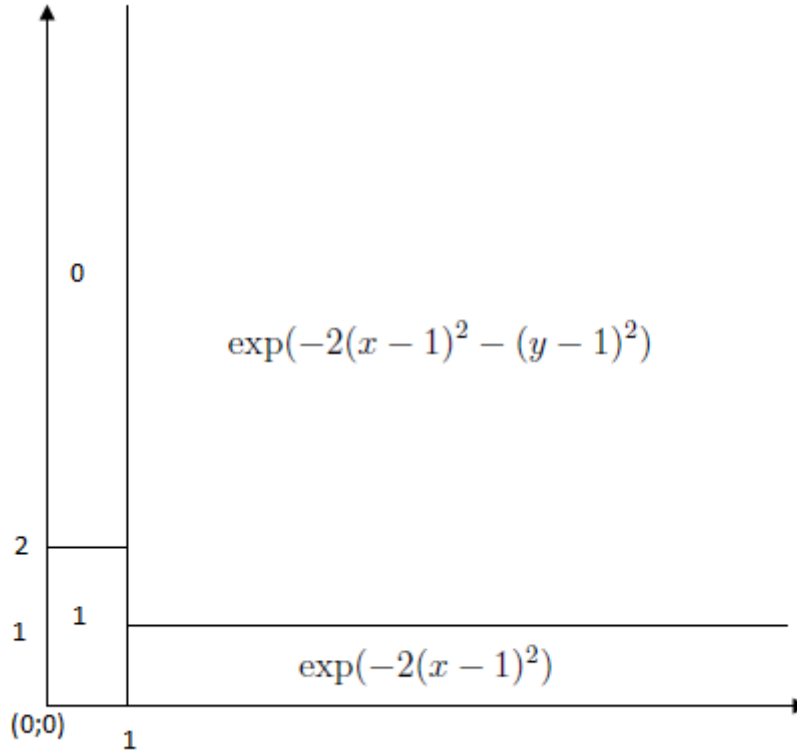
Now

$$\sum_{i=0}^{K_n} \sum_{j=0}^i \exp\left(-\frac{i^2}{p_n} - \frac{j^2}{q_n} - \frac{j^2}{r_n}\right) = s_n \int \int f_n(x, y) dx dy. \quad (3)$$

Our aim is to determine  $\lim_{n \rightarrow \infty} \int \int f_n$ . First we determine  $\lim_{n \rightarrow \infty} f_n(x, y)$  for every point  $(x, y)$ . In the case of  $0 \leq x$  and  $0 \leq y \leq x$ , if  $n$  is sufficiently large  $\lfloor x\sqrt{s_n} \rfloor \leq K_n$  and  $\lfloor y\sqrt{s_n} \rfloor \leq \lfloor x\sqrt{s_n} \rfloor$  is always true. Therefore, if  $n$  is sufficiently large  $f_n(x, y) = \exp\left(-\frac{(\frac{\lfloor x\sqrt{s_n} \rfloor}{\sqrt{s_n}})^2}{\frac{p_n}{s_n}} - \frac{(\frac{\lfloor x\sqrt{s_n} \rfloor}{\sqrt{s_n}})^2}{\frac{q_n}{s_n}} - \frac{(\frac{\lfloor y\sqrt{s_n} \rfloor}{\sqrt{s_n}})^2}{\frac{r_n}{s_n}}\right)$  and the limit of it is  $\exp\left(-\frac{x^2}{\alpha} - \frac{x^2}{\beta} - \frac{y^2}{\gamma}\right)$ . Otherwise  $f_n(x, y) = 0$  if  $n$  is large enough, so the limit is also 0.

Now we need a function  $g$  such that  $|f_n(x, y)| \leq g(x, y)$  for every  $(x, y)$  and  $\int \int g$  is finite. We will see that the function defined in Figure 1 satisfies the conditions.

FIGURE 1. The function  $g$



If  $x \geq 1$  and  $y \geq 1$ , then

$$\begin{aligned}
|f_n(x, y)| &\leq \exp\left(-\frac{\left(\frac{\lfloor x\sqrt{s_n} \rfloor}{\sqrt{s_n}}\right)^2}{\frac{p_n}{s_n}} - \frac{\left(\frac{\lfloor x\sqrt{s_n} \rfloor}{\sqrt{s_n}}\right)^2}{\frac{q_n}{s_n}} - \frac{\left(\frac{\lfloor y\sqrt{s_n} \rfloor}{\sqrt{s_n}}\right)^2}{\frac{r_n}{s_n}}\right) \\
&\leq \exp(-2(x-1)^2 - (y-1)^2).
\end{aligned}$$

If  $x \geq 1$  and  $1 \geq y \geq 0$ , then  $|f_n(x, y)| \leq \exp(-2(x-1)^2)$ . If  $0 \leq x \leq 1$  and  $0 \leq y \leq 2$ , then  $|f_n(x, y)| \leq 1$ . Finally  $|f_n(x, y)| \leq 0$  in any other case.

It is easy to see that the integral of  $g$  is finite, so using Lebesgue's dominated convergence theorem we obtain

$$\lim_{n \rightarrow \infty} \int \int f_n(x, y) dx dy = \int_D \exp\left(-\frac{\alpha+\beta}{\alpha\beta}x^2 - \frac{1}{\gamma}y^2\right),$$

where  $D = \{(x, y) | x \geq 0, x \geq y \geq 0\}$ . If  $\varphi(x, y) = (\frac{\sqrt{\alpha+\beta}}{\sqrt{\alpha\beta}}x, \frac{1}{\sqrt{\gamma}}y)$  and  $F(x, y) = \exp(-x^2 - y^2)$ , using integration with substitution we obtain

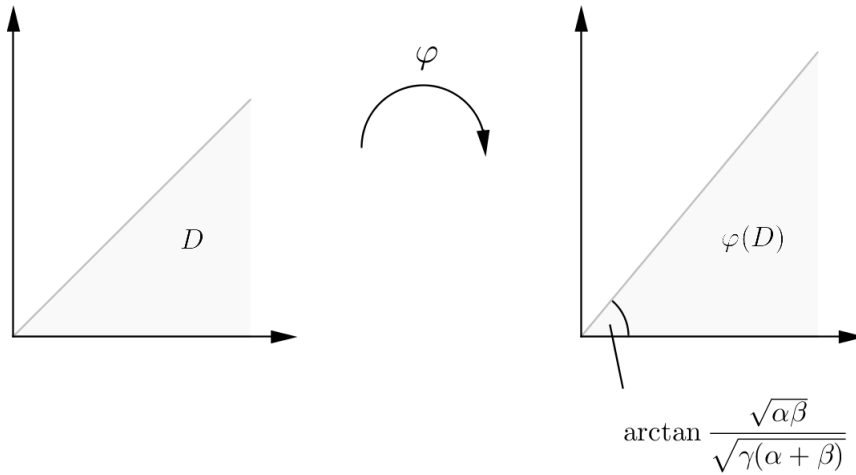
$$\int_{\varphi(D)} F = \int_D F \circ \varphi |det D \varphi|.$$

Therefore we get that

$$\int_D \exp\left(-\frac{\alpha+\beta}{\alpha\beta}x^2 - \frac{1}{\gamma}y^2\right) dx dy = \frac{\sqrt{\alpha\beta\gamma}}{\sqrt{\alpha+\beta}} \int_{\varphi(D)} \exp(-x^2 - y^2) dx dy.$$

The transformation of  $D$  by  $\varphi$  can be seen on Figure 2. The integral of  $\exp(-x^2 - y^2)$  on the whole plane is  $\pi$ , so the integral on  $\varphi(D)$  is  $\frac{1}{2} \arctan \frac{\sqrt{\alpha\beta}}{\sqrt{\gamma(\alpha+\beta)}}$ .

FIGURE 2. The transform of integration domain  $D$



By (2) and (3) we obtain:

$$\begin{aligned} A'(p_n, q_n, r_n) &= (1 + o(1)) \frac{s_n}{\pi^{\frac{3}{2}} \sqrt{p_n q_n r_n}} 2^{2s_n} \lim_{n \rightarrow \infty} \int \int f_n(x, y) dx dy \\ &= (1 + o(1)) \frac{s_n}{\pi^{\frac{3}{2}} \sqrt{p_n q_n r_n}} 2^{2s_n} \frac{\sqrt{\alpha \beta \gamma}}{2\sqrt{\alpha + \beta}} \arctan \frac{\sqrt{\alpha \beta}}{\sqrt{\gamma(\alpha + \beta)}}. \end{aligned} \quad (4)$$

We define

$$\begin{aligned} A''(p_n, q_n, r_n) &= \sum_{i=K_n+1}^{\min(p_n, q_n)} \sum_{j=0}^{\min(i, r_n)} \binom{2p_n}{p_n + i} \binom{2q_n}{q_n + i} \binom{2r_n}{r_n + j} \\ &= \sum_{i=K_n+1}^{\min(p_n, q_n)} \binom{2p_n}{p_n + i} \sum_{j=0}^{\min(i, r_n)} \binom{2q_n}{q_n + i} \binom{2r_n}{r_n + j} \\ &\leq \sum_{i=K_n+1}^{p_n} \binom{2p_n}{p_n + i} 2^{2(q_n + r_n)}. \end{aligned}$$

We will need the following inequality:

**Lemma 3.3** (Hoeffding inequality). *Consider  $n$  independent events, if each has probability  $p$ , then for all  $\varepsilon > 0$ , the probability of the event that at least  $n(p + \varepsilon)$  events occur is at most  $\exp(-2\varepsilon^2 n)$ .*

Using it in the special case of  $n = 2p_n$ ,  $p = \frac{1}{2}$  and  $\varepsilon = \frac{K_n+1}{2p_n}$ , we obtain that

$$\begin{aligned} A''(p_n, q_n, r_n) &\leq \sum_{i=K_n+1}^{p_n} \binom{2p_n}{p_n + i} 2^{2(q_n + r_n)} \\ &\leq 2^{2(p_n + q_n + r_n)} \exp\left(-\frac{(K_n + 1)^2}{2p_n}\right) \leq 2^{2s_n} \exp\left(-\frac{s_n^{0.02}}{2}\right). \end{aligned}$$

Using the fact that  $\lim_{n \rightarrow \infty} \frac{\binom{2s_n}{s_n}}{\sqrt{\pi s_n}} = 1$ , we obtain

$$0 \leq \lim_{n \rightarrow \infty} \frac{A''(p_n, q_n, r_n)}{\binom{2s_n}{s_n}} \leq \lim_{n \rightarrow \infty} \sqrt{\pi s_n} \exp\left(-\frac{s_n^{0.02}}{2}\right) = 0.$$

By (4) we have

$$\begin{aligned}
\lim_{n \rightarrow \infty} \frac{A'(p_n, q_n, r_n)}{\binom{2s_n}{s_n}} &= \lim_{n \rightarrow \infty} \frac{\frac{1}{\pi^{\frac{3}{2}} \sqrt{p_n q_n r_n}} 2^{2s_n} s_n \frac{\sqrt{\alpha\beta\gamma}}{2\sqrt{\alpha+\beta}} \arctan \frac{\sqrt{\alpha\beta}}{\sqrt{\gamma(\alpha+\beta)}}}{\frac{2^{2s_n}}{\sqrt{\pi s_n}}} \\
&= \lim_{n \rightarrow \infty} \frac{\frac{\sqrt{\alpha\beta\gamma}}{\sqrt{\alpha+\beta}}}{2\pi \sqrt{\frac{p_n}{s_n} \frac{q_n}{s_n} \frac{r_n}{s_n}}} \arctan \frac{\sqrt{\alpha\beta}}{\sqrt{\gamma(\alpha+\beta)}} \\
&= \frac{\frac{\sqrt{\alpha\beta\gamma}}{\sqrt{\alpha+\beta}}}{2\pi \sqrt{\alpha\beta\gamma}} \arctan \frac{\sqrt{\alpha\beta}}{\sqrt{\gamma(\alpha+\beta)}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{n \rightarrow \infty} \frac{A(p_n, q_n, r_n)}{\binom{2s_n}{s_n}} &= \lim_{n \rightarrow \infty} \frac{A'(p_n, q_n, r_n) + A''(p_n, q_n, r_n)}{\binom{2s_n}{s_n}} \\
&= \frac{1}{2\pi \sqrt{\alpha+\beta}} \arctan \frac{\sqrt{\alpha\beta}}{\sqrt{\gamma(\alpha+\beta)}}.
\end{aligned} \tag{5}$$

We will use similar methods to get the asymptotic value of  $B(p_n, q_n, r_n)$ . By Lemma 3.2 we have

$$\begin{aligned}
B'(p_n, q_n, r_n) &= \sum_{i=0}^{K_n} \binom{2p_n}{p_n+i} \binom{2q_n}{q_n+i} \binom{2r_n}{r_n+i} i \\
&= (1+o(1)) \frac{1}{\pi^{\frac{3}{2}} \sqrt{p_n q_n r_n}} 2^{2s_n} \sum_{i=0}^{K_n} \exp(-i^2 (\frac{1}{p_n} + \frac{1}{q_n} + \frac{1}{r_n})) i.
\end{aligned}$$

Here  $f_n(x)$  is defined in the following way:

$$f_n(x) = \begin{cases} \exp\left(-\left(\frac{\lfloor x\sqrt{s_n} \rfloor}{\sqrt{s_n}}\right)^2 \left(\frac{s_n}{p_n} + \frac{s_n}{q_n} + \frac{s_n}{r_n}\right)\right) \frac{\lfloor x\sqrt{s_n} \rfloor}{\sqrt{s_n}} & \text{if } 0 \leq \lfloor x\sqrt{s_n} \rfloor \leq K_n, \\ 0 & \text{otherwise.} \end{cases}$$

Hence we have

$$B'(p_n, q_n, r_n) = (1+o(1)) \frac{1}{\pi^{\frac{3}{2}} \sqrt{p_n q_n r_n}} 2^{2s_n} s_n \int_0^\infty f_n(x) dx.$$

For each  $x \geq 0$ ,  $\lim_{n \rightarrow \infty} f_n(x) = \exp(-(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma})x^2)x$ . We need a function  $g$  such that  $|f_n(x)| \leq g(x)$  if  $x \in [0, \infty)$ , and  $g$  has finite integral on  $[0, \infty)$ . We define  $g(x)$  such that  $g(x) = 1$  if  $0 \leq x \leq 1$ , and  $g(x) = \exp(-(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma})(x-1)^2)x$  otherwise, using Lebesgue's dominated convergence theorem:

$$\lim_{n \rightarrow \infty} \int_0^\infty f_n(x) dx = \int_0^\infty \exp\left(-\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right)x^2\right) x dx = \frac{\alpha\beta\gamma}{2(\alpha\beta + \beta\gamma + \gamma\alpha)}.$$

We define

$$B''(p_n, q_n, r_n) = \sum_{i=K_1+1}^{\min(p_n, q_n, r_n)} \binom{2p_n}{p_n+i} \binom{2q_n}{q_n+i} \binom{2r_n}{r_n+i} i.$$

With this notation  $\lim_{n \rightarrow \infty} \frac{B''(p_n, q_n, r_n)}{\binom{2s_n}{s_n}} = 0$  (proof like above), thus we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{B(p_n, q_n, r_n)}{\binom{2s_n}{s_n}} &= \lim_{n \rightarrow \infty} \frac{B'(p_n, q_n, r_n)}{\binom{2s_n}{s_n}} \\ &= \lim_{n \rightarrow \infty} \left( \frac{2^{2s_n}}{\sqrt{\pi s_n}} \right)^{-1} \frac{1}{\pi^{\frac{3}{2}} \sqrt{p_n q_n r_n}} 2^{2s_n} s_n \int_0^\infty f_n(x) dx \\ &= \frac{\sqrt{\alpha\beta\gamma}}{2\pi(\alpha\beta + \beta\gamma + \gamma\alpha)}. \end{aligned} \quad (6)$$

Using similar methods

$$\lim_{n \rightarrow \infty} \frac{E(p_n, q_n, r_n)}{\binom{2s_n}{s_n}} = 0. \quad (7)$$

Putting all the results together (equations (1), (5), (6) and (7)) we get the following theorem.

**Theorem 3.4.** *If  $p_n, q_n, r_n$  are sequences of positive integers,  $s_n = p_n + q_n + r_n$ ,  $\lim_{n \rightarrow \infty} s_n = \infty$ ,  $\lim_{n \rightarrow \infty} \frac{p_n}{s_n} = \alpha$ ,  $\lim_{n \rightarrow \infty} \frac{q_n}{s_n} = \beta$ ,  $\lim_{n \rightarrow \infty} \frac{r_n}{s_n} = \gamma$  for fixed  $\alpha, \beta, \gamma > 0$ , then  $\lim_{n \rightarrow \infty} \frac{U(2p_n, 2q_n, 2r_n)}{\binom{2s_n}{s_n}}$  exists, and has the value of  $u(\alpha, \beta, \gamma)$ , where*

$$\begin{aligned} &u(\alpha, \beta, \gamma) \\ &= \frac{2}{\pi} \left( \frac{\arctan \frac{\sqrt{\alpha\beta}}{\sqrt{\gamma(\alpha+\beta)}}}{\sqrt{\alpha+\beta}} + \frac{\arctan \frac{\sqrt{\alpha\gamma}}{\sqrt{\beta(\alpha+\gamma)}}}{\sqrt{\alpha+\gamma}} + \frac{\arctan \frac{\sqrt{\beta\gamma}}{\sqrt{\alpha(\beta+\gamma)}}}{\sqrt{\beta+\gamma}} - \frac{\sqrt{\alpha\beta\gamma}}{\alpha\beta + \beta\gamma + \gamma\alpha} \right). \end{aligned}$$

#### 4. FINAL STEPS OF THE PROOF OF THE UPPER BOUND

To deal with cases where not all parts are even, the following lemma will be useful.

**Lemma 4.1** (Griggs, Odlyzko, Shearer[7]).  $g(n_1, n_2, n_3) \leq 2g(n_1 - 1, n_2, n_3)$ .

*Proof.* Let  $X$  be  $X_1 \cup X_2 \cup X_3$ ,  $|X_i| = n_i$ . Let  $\mathcal{F}$  be a 3-part Sperner family on  $X$ , where  $g(n_1, n_2, n_3)$  is attained. Let  $t$  be an element of  $X_1$ .  $\mathcal{F}_1 = \{F \setminus \{t\} | F \in \mathcal{F}, t \in F\}$  and  $\mathcal{F}_2 = \{F | F \in \mathcal{F}, t \notin F\}$ . It is easy to see that  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are also 3-part Sperner families on the set  $X \setminus \{t\} = (X_1 \setminus \{t\}) \cup X_2 \cup X_3$  and the sum of their sizes is the size of  $\mathcal{F}$ .  $\square$

By applying this lemma repeatedly, the following lemma is obtained.

**Lemma 4.2.**  $g(2p + k_1, 2q + k_2, 2r + k_3) \leq 2^{k_1+k_2+k_3} g(2p, 2q, 2r)$ .  
( $k_1, k_2, k_3 \geq 0$ , integers.)

In the following proofs we will use similar arguments as Griggs, Odlyzko and Shearer[7] did in their paper.

**Lemma 4.3.** Let  $a_n, b_n, c_n$  be sequences of positive integers,  $m_n = a_n + b_n + c_n$ ,  $\lim_{n \rightarrow \infty} m_n = \infty$ ,  $\lim_{n \rightarrow \infty} \frac{a_n}{m_n} = \alpha$ ,  $\lim_{n \rightarrow \infty} \frac{b_n}{m_n} = \beta$ ,  $\lim_{n \rightarrow \infty} \frac{c_n}{m_n} = \gamma$  for fixed  $\alpha, \beta, \gamma > 0$ . Then  $\limsup_{n \rightarrow \infty} \frac{g(a_n, b_n, c_n)}{\binom{m_n}{\lfloor \frac{m_n}{2} \rfloor}} \leq u(\alpha, \beta, \gamma)$ .

*Proof.* Write  $a_n, b_n$  and  $c_n$  in the following form  $a_n = 2p_n + v_n$ ,  $b_n = 2q_n + w_n$  and  $c_n = 2r_n + t_n$ , where  $v_n, w_n, t_n$  are all 0 or 1,  $p_n, q_n, r_n$  are integers. Let  $s_n$  be  $p_n + q_n + r_n$ . Then  $\lim_{n \rightarrow \infty} s_n = \infty$ ,  $\lim_{n \rightarrow \infty} \frac{p_n}{s_n} = \alpha$ ,  $\lim_{n \rightarrow \infty} \frac{q_n}{s_n} = \beta$ ,  $\lim_{n \rightarrow \infty} \frac{r_n}{s_n} = \gamma$ . From Lemma 4.2 and Theorem 3.4:

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{g(a_n, b_n, c_n)}{\binom{m_n}{\lfloor \frac{m_n}{2} \rfloor}} &\leq \limsup_{n \rightarrow \infty} \frac{2^{v_n+w_n+t_n} g(2p_n, 2q_n, 2r_n)}{\binom{m_n}{\lfloor \frac{m_n}{2} \rfloor}} \\ &\leq \limsup_{n \rightarrow \infty} \frac{U(2p_n, 2q_n, 2r_n)}{\binom{2s_n}{s_n}} = u(\alpha, \beta, \gamma). \end{aligned}$$

$\square$

**Lemma 4.4.** Let  $a_n, b_n$  and  $c_n$  be sequences of nonnegative integers,  $m_n = a_n + b_n + c_n$ ,  $\lim_{n \rightarrow \infty} m_n = \infty$ ,  $\lim_{n \rightarrow \infty} \frac{a_n}{m_n} = 0$ . Then  $\limsup_{n \rightarrow \infty} \frac{g(a_n, b_n, c_n)}{\binom{m_n}{\lfloor \frac{m_n}{2} \rfloor}} \leq 1$ .

*Proof.* By repeatedly applying Lemma 4.1 and using the two part Sperner theorem, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{g(a_n, b_n, c_n)}{\binom{m_n}{\lfloor \frac{m_n}{2} \rfloor}} &\leq \limsup_{n \rightarrow \infty} \frac{2^{a_n} g(0, b_n, c_n)}{\binom{m_n}{\lfloor \frac{m_n}{2} \rfloor}} \\ &= \limsup_{n \rightarrow \infty} \frac{2^{a_n} g(b_n, c_n)}{\binom{m_n}{\lfloor \frac{m_n}{2} \rfloor}} = \limsup_{n \rightarrow \infty} \frac{2^{a_n} \binom{b_n + c_n}{\lfloor \frac{b_n + c_n}{2} \rfloor}}{\binom{m_n}{\lfloor \frac{m_n}{2} \rfloor}} = \limsup_{n \rightarrow \infty} \frac{\sqrt{m_n}}{\sqrt{b_n + c_n}} = 1. \end{aligned}$$

□

Define  $u(\alpha, \beta, \gamma)$  to be 1 if  $\alpha, \beta$  or  $\gamma$  equals 0. From Lemma 4.3 and 4.4 follows

**Lemma 4.5.** *Let  $a_n, b_n, c_n$  be sequences of nonnegative integers,  $m_n = a_n + b_n + c_n$ ,  $\lim_{n \rightarrow \infty} m_n = \infty$ ,  $\lim_{n \rightarrow \infty} \frac{a_n}{m_n} = \alpha$ ,  $\lim_{n \rightarrow \infty} \frac{b_n}{m_n} = \beta$ ,  $\lim_{n \rightarrow \infty} \frac{c_n}{m_n} = \gamma$ . Then  $\limsup_{n \rightarrow \infty} \frac{g(a_n, b_n, c_n)}{\binom{m_n}{\lfloor \frac{m_n}{2} \rfloor}} \leq u(\alpha, \beta, \gamma)$ .*

Let  $u^*$  be the maximum of  $u(\alpha, \beta, \gamma)$  under the conditions:  $\alpha + \beta + \gamma = 1$ ,  $\alpha, \beta, \gamma \geq 0$ . By Theorem 2.8 we get that the maximum is attained at  $\alpha = \beta = \gamma = \frac{1}{3}$ .

**Theorem 4.6.**  $d_3 \leq u^*$ , that is  $d_3 \leq 1,072 \dots$

*Proof.* Consider the sequences  $a_n, b_n, c_n$  such that  $n = a_n + b_n + c_n$  and  $f(n, 3) = g(a_n, b_n, c_n)$ . Then  $d_3$  by definition is  $\lim_{n \rightarrow \infty} \frac{g(a_n, b_n, c_n)}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$ .

Let  $\lambda_n$  be  $(\frac{a_n}{n}, \frac{b_n}{n}, \frac{c_n}{n})$ .  $\lambda_n \in [0, 1]^3$ . Since  $[0, 1]^3$  is compact, there is a convergent subsequence of  $\lambda_n$ , that is there is an infinite sequence  $n_1 < n_2 < \dots$  such that  $\lim_{k \rightarrow \infty} (\frac{a_{n_k}}{n_k}, \frac{b_{n_k}}{n_k}, \frac{c_{n_k}}{n_k})$  exists. Applying Lemma 4.5 for  $a_{n_k}, b_{n_k}, c_{n_k}$  we get  $d_3 \leq u^*$ .

□

## 5. HOMOGENEOUS FAMILIES AND RELATED INTEGER PROGRAMS

For sake of simplicity all the theorems in this section will be stated for 3-part Spener-families, but one can easily generalize them for k-part Sperner families.

**Definition 5.1.** The set  $F$  has type  $(r_1, r_2, r_3)$  if  $|X_i \cap F| = r_i$ ,  $i = 1, 2, 3$ .

**Definition 5.2.** A family  $\mathcal{F} \subset \mathcal{P}(X)$  is homogeneous if  $F \in \mathcal{F}$  implies that all sets of the same type are in  $\mathcal{F}$ .

A homogeneous family  $\mathcal{F}$  will be a Sperner family, if there are no sets in  $\mathcal{F}$  with two different types  $(r_1, r_2, r_3)$  and  $(s_1, s_2, s_3)$  such that there are  $i \neq j \in \{1, 2, 3\}$ ,  $r_i = s_i$  and  $r_j = s_j$ .

**Lemma 5.3** (Füredi, Griggs, Odlyzko, Shearer[7][5] and Erdős, Katona[3]). *If  $\mathcal{F}_0$  is a maximum size Sperner family then there is a homogeneous Sperner family  $\mathcal{F}$  with the same size.*

Therefore it is enough to search for the maximum among the homogeneous families. Notice, that this is equivalent with solving the following integer program (Griggs, Odlyzko, Shearer[7]):  $x(i, j, k)$  are binary variables ( $0 \leq i \leq n_1, 0 \leq j \leq n_2, 0 \leq k \leq n_3$ ),  $x(i, j, k) = 1$  if the sets with type  $(i, j, k)$  are in  $\mathcal{F}$ , 0 otherwise. The constraints are the following:

$$\begin{aligned} \sum_{i=0}^{n_1} x(i, j, k) &\leq 1 \quad \forall 0 \leq j \leq n_2, 0 \leq k \leq n_3, \\ \sum_{j=0}^{n_2} x(i, j, k) &\leq 1 \quad \forall 0 \leq i \leq n_1, 0 \leq k \leq n_3, \\ \sum_{k=0}^{n_3} x(i, j, k) &\leq 1 \quad \forall 0 \leq i \leq n_1, 0 \leq j \leq n_2. \end{aligned}$$

Determine  $\max \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \sum_{k=0}^{n_3} \binom{n_1}{i} \binom{n_2}{j} \binom{n_3}{k} x(i, j, k).$

**Conjecture 5.4** (Aydinian, Czaparka, Erdős, Székely[1, Conjecture 7.1]). For every  $k \geq 1$  and every  $n_1 = n_2 = \dots = n_k = 2^\ell - 1$ ,  $\mathcal{F}$  is a maximum size homogeneous  $k$ -part Sperner family if and only if  $\mathcal{F} \in \mathfrak{F}_0^{(k)}$ .

(Here definition of  $\mathfrak{F}_0^{(k)}$  is omitted since it is a bit complicated. See [1, Conjecture 7.1] for details.)

In the case of  $k = 3$  and  $n_1 = n_2 = n_3 = 15$  the solution of the integer program shows that the conjecture is not true, since we can find a family of larger size than the size of the families in  $\mathfrak{F}_0^{(k)}$ . See [10] for details. (The authors of the conjecture verified it in the cases of  $k = 3$ ,  $\ell = 1, 2, 3$  with checking all the homogeneous families, but in the case of  $\ell = 4$  this is too slow, however the integer program can be solved in a few minutes.)

One of the reasons we can solve this integer program this fast is that the optimum value of the objective function of the integer program and the lp-relaxation happen to be the same. And the same happens in all the cases we have seen. Although we checked it in all the cases of  $n_1, n_2, n_3 \leq 15$  and they were equal, we believe that there must be a counter example for larger  $n_i$ 's.

We may formulate Lemma 2.3 in the following way:

$$\begin{aligned} & \max\{|\mathcal{F}| : \mathcal{F} \text{ is a 3-part Sperner family}\} \leq \\ & \min\{|\mathcal{C}| : \mathcal{C} \text{ is a monochromatic chain partition}\}. \end{aligned} \quad (8)$$

It is a natural question whether the inequality can be replaced by equality or not. We will show that if the lp and integer optimum are not the same, the inequality strictly holds.

Consider the lp-relaxation of the integer program above, its optimum value is at least the optimum value of the integer program. But the lp-optimum is equal to the optimum of the dual problem:

$$\begin{aligned} & y(*, j, k) + y(i, *, k) + y(i, j, *) \geq \binom{n_1}{i} \binom{n_2}{j} \binom{n_3}{k} x(i, j, k) \\ & \text{holds for all } 0 \leq i \leq n_1, \quad 0 \leq j \leq n_2, \quad 0 \leq k \leq n_3, \\ & y(*, j, k) \geq 0, \quad y(i, *, k) \geq 0, \quad y(i, j, *) \geq 0, \\ & \text{holds for all } 0 \leq i \leq n_1, \quad 0 \leq j \leq n_2, \quad 0 \leq k \leq n_3, \\ & \min \sum_{j=0}^{n_2} \sum_{k=0}^{n_3} y(*, j, k) + \sum_{i=0}^{n_1} \sum_{k=0}^{n_3} y(i, *, k) + \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} y(i, j, *). \end{aligned}$$

Therefore if we have a feasible solution to the dual problem we get an upper bound for the size of the three-part Sperner families. Now we show a method for getting a feasible solution to the dual problem from a monochromatic chain partition.

**Lemma 5.5.** *If  $\mathcal{C}$  is a monochromatic chain partition, there is a feasible solution to the dual problem with objective function value  $|\mathcal{C}|$ .*

*Proof.* A monochromatic chain  $C = (F_1, F_2, \dots, F_m)$  has type  $(*, j, k)$  if its color is 1,  $|F_1 \cap X_2| = j$  and  $|F_1 \cap X_3| = k$ . (Note that  $F_h \cap X_2$  are the same for  $h = 1, 2, \dots, m$ .)

A monochromatic chain  $C = (F_1, F_2, \dots, F_m)$  has type  $(i, *, k)$  if its color is 2,  $|F_1 \cap X_1| = i$  and  $|F_1 \cap X_3| = k$ .

A monochromatic chain  $C = (F_1, F_2, \dots, F_m)$  has type  $(i, j, *)$  if its color is 3,  $|F_1 \cap X_1| = i$  and  $|F_1 \cap X_2| = j$ .

(If  $C = (F_1)$  then it has type  $(*, |F_1 \cap X_2|, |F_1 \cap X_3|)$ ).

Let  $y(*, j, k)$ ,  $y(i, *, k)$  and  $y(i, j, *)$  be the number of chains in  $\mathcal{C}$  with type  $(*, j, k)$ ,  $(i, *, k)$  and  $(i, j, *)$  respectively.

Consider an  $F$  which has type  $(i, j, k)$ . This  $F$  must be covered by a chain from  $\mathcal{C}$ , but it can only be covered with chains that have type  $(*, j, k)$ ,  $(i, *, k)$  or  $(i, j, *)$ . There are  $\binom{n_1}{i} \binom{n_2}{j} \binom{n_3}{k}$  sets with type  $(i, j, k)$ , so  $y(*, j, k) + y(i, *, k) + y(i, j, *) \geq \binom{n_1}{i} \binom{n_2}{j} \binom{n_3}{k}$ .  $\square$

*Remark 5.6.* This means that if the lp and integer optimum is not the same somewhere, then we have strict inequality in (8). Therefore we can not get the best bound with partitioning into monochromatic chains.

## 6. LOWER BOUNDS

Our method is a refinement of that of Füredi[6] and Griggs, Odlyzko, Shearer[7]. In this section we always assume that  $X = X_1 \cup X_2 \cup X_3$ ,  $|X_i| = n$  ( $i = 1, 2, 3$ ).

**Definition 6.1.** We call the following sets *cubes* with side  $t$ :

$$R_n(a_1, a_2, a_3, t) = \{F \in \mathcal{P}(X) \mid a_i \leq |X_i \cap F| < a_i + t, \quad (i = 1, 2, 3)\}.$$

**Definition 6.2.** The sets  $R_n(a_1, a_2, a_3, t)$  and  $R_n(b_1, b_2, b_3, u)$  are compatible if there exist  $i \neq j \in \{1, 2, 3\}$  such that  $[a_i, a_i + t) \cap [b_i, b_i + u) = \emptyset$  and  $[a_j, a_j + t) \cap [b_j, b_j + u) = \emptyset$ .

The proof of the following lemma is straightforward.

**Lemma 6.3.** *If  $R_n(a_1(i), a_2(i), a_3(i), t(i))$ ,  $i = 1, 2, \dots, N$  are pairwise compatible and  $\mathcal{F}_i \subset R_n(a_1(i), a_2(i), a_3(i), t(i))$  are three-part Sperner families then  $\mathcal{F} = \cup_{i=1}^N \mathcal{F}_i$  is also a three-part Sperner family.*  $\square$

Now we give a general method for finding large three-part Sperner families. Consider some Sperner families  $\mathcal{F}_i \subset R_n(a_1(i), a_2(i), a_3(i), t(i))$ , ( $i=1, 2, \dots, N$ ) and construct a graph  $G$  on the vertices  $\{1, 2, \dots, N\}$ . A pair of vertices  $(i, j)$  will be an edge of  $G$  if  $R_n(a_1(i), a_2(i), a_3(i), t(i))$  and  $R_n(a_1(j), a_2(j), a_3(j), t(j))$  are not compatible. A vertex  $i$  has the weight  $|\mathcal{F}_i|$ . Let  $H$  be an independent set in  $G$ . There will be a Sperner family with the size of the total weight of  $H$ . (Namely  $\cup_{i \in H} \mathcal{F}_i$  is a Sperner family with the size of  $\cup_{i \in H} |\mathcal{F}_i|$ .) So our aim is to find an independent set of weight near the maximum.

We will choose the cubes corresponding to the vertices of  $G$  in the following way. We will divide a big cube into  $(2k)^3$  smaller cubes with equal sides. Let the big cube be  $R_n(\lfloor \frac{n}{2} \rfloor - kt, \lfloor \frac{n}{2} \rfloor - kt, \lfloor \frac{n}{2} \rfloor - kt, 2kt)$ .

Then the small cubes will be  $C_n^t(a, b, c) = R_n(\lfloor \frac{n}{2} \rfloor + (a - k)t, \lfloor \frac{n}{2} \rfloor + (b - k)t, \lfloor \frac{n}{2} \rfloor + (c - k)t, t)$ , where  $0 \leq a, b, c \leq 2k - 1$ . We will use the triplet  $(a, b, c)$  to identify the corresponding vertex of  $G$ . Two different cubes  $C_n^t(a_1, b_1, c_1)$  and  $C_n^t(a_2, b_2, c_2)$  are not compatible if and only if at least two of the following equalities hold,  $a_1 = a_2$ ,  $b_1 = b_2$ ,  $c_1 = c_2$ . For a subset  $H$  of the vertices of  $G$  its characteristic vector is the vector  $(x(a, b, c))_{0 \leq a, b, c \leq 2k-1}$ , where  $x(a, b, c) = 1$  if  $(a, b, c) \in H$  and 0 otherwise.

It is easy to see that the binary solutions of the following integer program are exactly the characteristic vectors of the independent sets of  $G$ .

$$\begin{aligned} \sum_{a=0}^{2k-1} x(a, b, c) &\leq 1 (\forall 0 \leq b, c \leq 2k - 1), \\ \sum_{b=0}^{2k-1} x(a, b, c) &\leq 1 (\forall 0 \leq a, c \leq 2k - 1), \\ \sum_{c=0}^{2k-1} x(a, b, c) &\leq 1 (\forall 0 \leq a, b \leq 2k - 1), \end{aligned}$$

$$x(a, b, c) \geq 0 (\forall 0 \leq a, b, c \leq 2k - 1).$$

We denote this program by  $I_k$ .

**Lemma 6.4.** *Suppose  $\mathcal{F}_n(a, b, c) \subset C_n^t(a, b, c)$ . Let the weight of the vertex  $(a, b, c)$  be  $w_n^t(a, b, c) = |\mathcal{F}_n(a, b, c)|$ . Then the characteristic vector of the maximum weighted independent set can be determined as the optimum solution of the integer program  $I_k$  with the objective function  $\max \sum_{a=0}^{2k-1} \sum_{b=0}^{2k-1} \sum_{c=0}^{2k-1} w_n^t(a, b, c) \cdot x(a, b, c)$ . If the objective function value is  $M$ , we have a Sperner family with size  $M$ .  $\square$*

But how large  $w_n^t(a, b, c)$  can be chosen? The following simple lemma gives us a good bound.

**Lemma 6.5.** *For a given  $R_n(a_1, a_2, a_3, t)$  there exist a three-part Sperner family  $\mathcal{F}$  such that  $\mathcal{F} \subset R_n(a_1, a_2, a_3, t)$  and  $|\mathcal{F}| \geq \frac{1}{t} |R_n(a_1, a_2, a_3, t)|$ .*

*Proof.* We define  $\mathcal{F}_i = \{F \in R_n(a_1, a_2, a_3, t) \mid |F| \equiv i \pmod{t}\}$  for  $i = 1, 2, \dots, t$ .

Clearly  $\mathcal{F}_1, \mathcal{F}_2, \dots$  are pairwise disjoint, their union is  $R_n(a_1, a_2, a_3, t)$  and all of them are three-part Sperner families. So the largest of them will be proper.  $\square$

Therefore in Lemma 6.4 we can choose  $w_n^t(a, b, c)$  to be  $\frac{|C_n^t(a, b, c)|}{t}$ .

Now fix  $k$ . For a given  $n$  and various choices of  $t$  we can get large Spener families by applying Lemma 6.4 with  $w_n^t(a, b, c) = \frac{|C_n^t(a, b, c)|}{t}$ . But we want a bound for  $d_3$ , so we have to find some kind of limit of these integer programs. Our aim is to chose the parameter  $t = t(n)$  such that the following limits exist  $w(a, b, c) = \lim_{n \rightarrow \infty} \frac{w_n^{t(n)}(a, b, c)}{\binom{3n}{\lfloor 1.5n \rfloor}} =$

$$\lim_{n \rightarrow \infty} \frac{|C_n^{t(n)}(a, b, c)|}{t(n) \binom{3n}{\lfloor 1.5n \rfloor}}.$$

**Lemma 6.6.** *Consider the optimum solution of  $I_k$  respected to the objective function  $\max \sum_{a=0}^{2k-1} \sum_{b=0}^{2k-1} \sum_{c=0}^{2k-1} w(a, b, c) \cdot x(a, b, c)$ . This gives us an independent set  $H$  with total weight  $W$ .  $W$  is a lower bound for  $d_3$ .*

*Proof.* For each  $n$  we have a Sperner family with size  $\sum_{(a, b, c) \in H} w_n^{t(n)}(a, b, c) = \sum_{(a, b, c) \in H} (1 + o(1)) w(a, b, c) \binom{3n}{\lfloor 1.5n \rfloor} = (1 + o(1)) \binom{3n}{\lfloor 1.5n \rfloor} \sum_{(a, b, c) \in H} w(a, b, c) = (1 + o(1)) \binom{3n}{\lfloor 1.5n \rfloor} W$ , which means that  $W$  is a lower bound for  $d_3$ .  $\square$

The above mentioned limits will exist if we choose  $t(n)$  to be  $\lfloor \frac{s}{2} \sqrt{n} \rfloor$ , where  $s \in \mathbb{R}^+$  is constant as we will see in Lemma 6.8. We denote these limits by  $w(a, b, c, s, k)$ .

We denote the cumulative density function of standard normal distribution by  $\Phi(t)$ , that is  $\Phi(t) = \int_{-\infty}^t \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$ .

**Lemma 6.7.** *For any  $s \in \mathbb{R}^+$ ,  $x_1, x_2, x_3 \in \mathbb{R}$ , the limit*

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{\lfloor \frac{s}{2}\sqrt{n} \rfloor} |R_n(\lfloor \frac{n}{2} \rfloor + \lfloor \frac{x_1}{2}\sqrt{n} \rfloor, \lfloor \frac{n}{2} \rfloor + \lfloor \frac{x_2}{2}\sqrt{n} \rfloor, \lfloor \frac{n}{2} \rfloor + \lfloor \frac{x_3}{2}\sqrt{n} \rfloor, \lfloor \frac{s}{2}\sqrt{n} \rfloor)|}{\binom{3n}{\lfloor 1.5n \rfloor}}$$

*exists and its value is  $L_0(x_1, x_2, x_3, s) = \frac{2\sqrt{1.5\pi}}{s} \prod_{i=1}^3 (\Phi(x_i + s) - \Phi(x_i))$ .*

*Proof.* From the de Moivre-Laplace theorem

$$\sum_{j=\lfloor \frac{n}{2} \rfloor + \lfloor \frac{x_i}{2}\sqrt{n} \rfloor}^{\lfloor \frac{n}{2} \rfloor + \lfloor \frac{x_i}{2}\sqrt{n} \rfloor + \lfloor \frac{s}{2}\sqrt{n} \rfloor - 1} \binom{n}{j} = (1 + o(1))(\Phi(x_i + s) - \Phi(x_i))2^n.$$

From this and the fact that  $\lim_{m \rightarrow \infty} \frac{\binom{m}{\lfloor \frac{m}{2} \rfloor}}{2^m} = 1$  the statement follows.  $\square$

As a straightforward corollary we get the following lemma.

**Lemma 6.8.**  $w(a, b, c, s, k) = L_0((a - k)s, (b - k)s, (c - k)s, s)$ .  $\square$

From Lemma 6.6

**Lemma 6.9.** *For any  $s \in \mathbb{R}^+$ ,  $k \in \mathbb{Z}^+$ , the optimum value of the integer program  $I_k$  with objective function  $\max \sum_{a=0}^{2k-1} \sum_{b=0}^{2k-1} \sum_{c=0}^{2k-1} w(a, b, c, s, k) \cdot x(a, b, c)$  is a lower bound for  $d_3$ .*  $\square$

We could improve this bound if we could replace  $w(a, b, c, s, k)$  with something larger. So we need to improve the bound of Lemma 6.5 by finding a larger Sperner family which is a subset of  $R_n(a_1, a_2, a_3, t)$ .

We can get a better bound by dividing the cube into eight smaller cubes and choosing four of them which are compatible and then applying Lemmas 6.3 and 6.5. We can iterate this arbitrary many times. Now we will formalize it.

**Definition 6.10.** Let us define  $l_n(a, b, c, 2^u t_0, u)$  (where  $n, a, b, c, t_0, u$  are nonnegative integers,  $t_0 > 0$ ) in the following way:

For  $u = 0$ ,  $l_n(a, b, c, t_0, 0) = \frac{1}{t_0} |R_n(a, b, c, t_0)|$ .

For  $u > 0$ ,

$$\begin{aligned} l_n(a, b, c, 2^u t_0, u) \\ = \sum_{(p,q,r) \in S} l_n(a + p2^{u-1}t_0, b + q2^{u-1}t_0, c + r2^{u-1}t_0, 2^{u-1}t_0, u-1), \end{aligned}$$

where  $S = \{(p, q, r) \in \{0, 1\}^3 \mid p + q + r \equiv I_n(a) + I_n(b) + I_n(c) \pmod{2}\}$ , where  $I_n(x) = 1$  if  $x < \lfloor \frac{n}{2} \rfloor$  and 0 otherwise.

**Lemma 6.11.** *There exists a Sperner family  $\mathcal{F} \subset R_n(a, b, c, 2^u t_0)$  such that  $|\mathcal{F}| \geq l(a, b, c, 2^u t_0, u)$ .*

*Proof.* We will prove the statement by induction. For  $u = 0$ , we already proved it in Lemma 6.5. If  $u > 0$  by the induction hypothesis we know that for  $(p, q, r) \in S$  there are 3-part Sperner families  $\mathcal{F}(p, q, r) \subset R_n(a + p2^{u-1}t_0, b + q2^{u-1}t_0, c + r2^{u-1}t_0, 2^{u-1}t_0)$  such that  $|\mathcal{F}(p, q, r)| \geq l_n(a + p2^{u-1}t_0, b + q2^{u-1}t_0, c + r2^{u-1}t_0, 2^{u-1}t_0, u-1)$  for every  $(p, q, r) \in S$ . If we consider the sets  $R_n(a + p2^{u-1}t_0, b + q2^{u-1}t_0, c + r2^{u-1}t_0, 2^{u-1}t_0)$  for all  $(p, q, r) \in S$ , they are pairwise compatible, so  $\cup_{(p,q,r) \in S} \mathcal{F}(p, q, r) \subset R_n(a, b, c, 2^u t_0)$  is compatible due to Lemma 6.3 and has the desired size. (There is another possible choice of  $S$ , namely  $\{0, 1\}^3 \setminus S$ , but this gives a better bound.)  $\square$

For fixed  $u$ , let  $t$  be  $2^u t_0$ .

From Lemma 6.11 in Lemma 6.4 we can choose  $w_n^t(a, b, c)$  to be  $l_n(\lfloor \frac{n}{2} \rfloor + (a - k)t, \lfloor \frac{n}{2} \rfloor + (b - k)t, \lfloor \frac{n}{2} \rfloor + (c - k)t, 2^u t_0, u)$ . We would like to define a limit problem. We define  $t(n)$  to be  $2^u \lfloor \frac{s_0}{2} \sqrt{n} \rfloor$  for some constant  $s_0 \in \mathbb{R}^+$ . ( $s = 2^u s_0$ .) Then again  $\lim_{n \rightarrow \infty} \frac{w_n^{t(n)}(a, b, c)}{\binom{3n}{\lfloor 1.5n \rfloor}}$  exists as we will show it in the following lemmas.

**Definition 6.12.** Let us define  $L(x, y, z, 2^u s_0, u)$ , where  $x, y, z, s_0 \in \mathbb{R}, s_0 > 0, u$  is nonnegative integer, in the following way.

For  $u = 0$ ,  $L(x, y, z, s_0, 0) = L_0(x, y, z, s_0)$ .

For  $u > 0$ ,

$$\begin{aligned} L(x, y, z, 2^u s_0, u) \\ = \sum_{(p,q,r) \in S} L(x + p2^{u-1}s_0, y + q2^{u-1}s_0, z + r2^{u-1}s_0, 2^{u-1}s_0, u-1), \end{aligned}$$

where  $S = \{(p, q, r) \in \{0, 1\}^3 \mid p + q + r \equiv I(x) + I(y) + I(z) \pmod{2}\}$ , where  $I(x) = 1$  if  $x < 0$  and 0 otherwise.

**Lemma 6.13.** For  $x_1, x_2, x_3, s_0 \in \mathbb{R}, s_0 > 0$

$$\lim_{n \rightarrow \infty} \frac{l_n(\lfloor \frac{n}{2} \rfloor + \lfloor \frac{x_1}{2} \sqrt{n} \rfloor, \lfloor \frac{n}{2} \rfloor + \lfloor \frac{x_2}{2} \sqrt{n} \rfloor, \lfloor \frac{n}{2} \rfloor + \lfloor \frac{x_3}{2} \sqrt{n} \rfloor, 2^u \lfloor \frac{s_0}{2} \sqrt{n} \rfloor, u)}{\binom{3n}{\lfloor 1.5n \rfloor}} = L(x_1, x_2, x_3, 2^u s_0, u).$$

*Proof.* For  $u = 0$  we already proved it in Lemma 6.7.

For  $u > 0$  it can be easily proved by induction on  $u$ , because the recursive definition of  $L$  and  $l_n$  are basically the same.  $\square$

**Lemma 6.14.**  $\lim_{n \rightarrow \infty} \frac{w_n^{t(n)}(a,b,c)}{\binom{3n}{\lfloor 1.5n \rfloor}} = L((a-k)s, (b-k)s, (c-k)s, 2^u s_0, u).$

$\square$

**Lemma 6.15.** For any  $s_0 \in \mathbb{R}^+, k \in \mathbb{Z}^+, u \in \mathbb{Z}_0^+, s = 2^u s_0$ , the optimum value of the integer program  $I_k$  with objective function  $\max \sum_{a=0}^{2k-1} \sum_{b=0}^{2k-1} \sum_{c=0}^{2k-1} L((a-k)s, (b-k)s, (c-k)s, 2^u s_0, u) \cdot x(a, b, c)$  is a lower bound for  $d_3$ . We denote this bound by  $B(k, u, s_0)$ .  $\square$

We choose  $s_0 = 0.05$  ( $s = 0.2$ ),  $u = 2, k = 16$ , solving this with computer we get a feasible solution with objective function value 1.0506. We get that  $d_3 > 1.0506$ [10].

*Remark 6.16.* Of course the bound  $B(k, u, s_0)$  is smaller than  $B(2^u k, 0, s_0)$ , however the time complexity of our method depends on  $k$ , so  $k$  in practice cannot be chosen so large. We can use this method without linear optimization with  $k=1$ , as Griggs, Odlyzko and Shearer[7] did.

Their method was that they found a large Sperner family applying lemma 6.11 with carefully chosen parameters  $a, b, c, t, u$  and with lemma 6.13 they calculated the asymptotic value of the bounds. (This means that we take a large cube and divide it into smaller cubes, we choose some of them which are compatible, and use Lemma 6.5. Lemma 6.11 gives an explicit choice of cubes, our improvement is that we find a better choice using the above mentioned integer program.)

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