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Introduction

The pioneering Black-Scholes formula had smashing impacts for option pricing. It has not only a result, but also a remarkable approach, which accelerated the development of mathematical finance in the 70's. Whereas as time wore on it became clear that the Black-Scholes framework is not proper to describe real-life events flawlessly. Namely it is unable to produce the heavy-tail effect, for this reason assumption that volatility parameter is deterministic is not valid anymore. As a consequence, we obtain a volatility smile or skew if we draw the graph of implied volatility in the Black-Scholes model.

Different approaches have evolved to cope with the problem like local volatility models, jump diffusions and stochastic volatility models. See Derman (2006) for further details.

Considering the volatility as a stochastic process, stochastic volatility models have two factors to deal with. Best-known stochastic volatility models are Hull-White (1987), Scott (1987), Wiggins (1987), Stein-Stein (1991) and Heston (1993).

The most popular of them is the Heston model due to it has a semi-closed solution for European options. Furthermore, it has the advantage that can produce skew and leptocurtic distributions with suitable parametrization, thereby gives more realistic estimation for the volatility surface far from the maturity.

Despite all its comfortable feature, it is not attainable to price American type options analytically in it. However, in the previous years, impressive articles have been written on both of the fields of numerical techniques and analytical researches. Aim of this study is to present and implement them comparing their main features.

In the first chapter, we will examine the Heston model and following Heston (1993) we will deduce the pricing formula of European options. In the second chapter, we will analyze American options. Widely spread Longstaff-Schwartz method will be presented after introducing some discretization schemes for simulations. Performance of simulation based pricing will be shown also. Finally, the tree-based approach of Nawalkha and Believa will be investigated, which transforms the basic equations of the Heston model in a form, when tree for variance and stock prices can be modelled independently to save massive computation time.

Chapter 1

Heston model

1.1 Dynamic of the model

In the model, stock prices follow a stochastic dynamic similar to geometric Brownian motion, however diffusion term is not constant unlike the Black-Scholes model. v_t , the variance, follows a CIR process, Cox, Ingersoll and Ross (1985):

$$\begin{aligned}dS_t &= \mu S_t dt + \sqrt{v_t} S_t dW_t^S \\ dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_t^V\end{aligned}$$

,where μ is the drift of the share, $\theta > 0$ is the mean of variance on long run, $\kappa > 0$ is the speed of reverse to the mean, $\sigma > 0$ is the volatility of the variance. Brownian motions, which drive the processes, are dependent: $E^\mathbb{P}[dW_t^S dW_t^V] = \rho dt$

Applying the Itô formula, it can be proved that $\sqrt{v}$ follows an Ornstein-Uhlenbeck process. Dynamic above is valid for the stock prices and for the variance under the physical measure, nonetheless we need it for the pricing under the risk-neutral measure. It can be attained using the Girsanov theorem:

$$\begin{aligned}dS_t &= rS_t dt + \sqrt{v_t} S_t d\widetilde{W}_t^S \\ dv_t &= [\kappa(\theta - v_t) - \lambda(S_t, v_t, t)]dt + \sigma\sqrt{v_t} d\widetilde{W}_t^V\end{aligned}$$

,where

$$\begin{aligned}\widetilde{W}_t^S &= \left(W_t^S + \frac{\mu - r}{\sqrt{v_t}} \right) \\ \widetilde{W}_t^V &= \left(W_t^V + \frac{\lambda(S_t, v_t, t)}{\sigma\sqrt{v_t}} t \right)\end{aligned}$$

In the Brownian of the variance under the risk-neutral measure, $\lambda(S_t, v_t, t)$ term appears, which is the market price of the volatility. Citing Breeden (1979), Heston (1993) assumes the market price of volatility to λv_t , where λ is constant. Thus dynamic of Heston model under the risk-neutral measure is the following:

$$\begin{aligned}dS_t &= rS_t dt + \sqrt{v_t} S_t d\widetilde{W}_t^S \\ dv_t &= [\kappa^*(\theta^* - v_t)]dt + \sigma\sqrt{v_t} d\widetilde{W}_t^V\end{aligned}$$

, where $\kappa^* = \kappa + \lambda$ and $\theta^* = \kappa\theta/(\kappa + \lambda)$ and $E^{\mathbb{Q}}[d\widetilde{W}_t^S d\widetilde{W}_t^V] = \rho dt$. For simplicity we omit the notation of the risk-neutral measure for parameters and the Wiener processes too.

1.2 Heston PDE

Following Black-Scholes (1973), it is straightforward to obtain the partial differential equation of the Heston model. This is a special case of the general PDE of stochastic volatility models, see Gatheral (2006) for further details.

Considering the fact that the Heston model is a two-factor model, hedge portfolio has to involve two instruments to hedge the option. Since usual delta-neutral hedging is non-sufficient due to vega-neutral hedging is required as well. Denote by $V(S, v, t)$ the derivative which is hedged, by δ the amount of underlying, by ϕ the amount of the other hedging instrument and by $U(S, v, t)$ the value of the other hedging instrument.

$$\Pi = V + \Delta S + \varphi U$$

assuming that the portfolio is self-financing

$$d\Pi = dV + \Delta dS + \varphi dU$$

applying the multidimensional Itô formula for determining dV and dU :

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial v}dv + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2}dt + \frac{1}{2}v\sigma^2\frac{\partial^2 V}{\partial v^2}dt + \rho\sigma vS\frac{\partial^2 V}{\partial S\partial v}dt.$$

Equation of dU corresponds with this equation writing U instead of V .

Substituting them into the equation of the self-financing portfolio and collecting terms, we get the expression:

$$\begin{aligned} d\Pi = & \left[\frac{\partial V}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2} + \rho\sigma vS\frac{\partial^2 V}{\partial S\partial v} + \frac{1}{2}v\sigma^2\frac{\partial^2 V}{\partial v^2} \right] dt \\ & + \varphi \left[\frac{\partial U}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \rho\sigma vS\frac{\partial^2 U}{\partial S\partial v} + \frac{1}{2}v\sigma^2\frac{\partial^2 U}{\partial v^2} \right] dt \\ & + \left[\frac{\partial V}{\partial S} + \varphi\frac{\partial U}{\partial S} + \Delta \right] dS + \left[\frac{\partial V}{\partial v} + \varphi\frac{\partial U}{\partial v} \right] dv. \end{aligned}$$

From where it arises that to construct the risk-neutral hedge strategy,

$$\varphi = -\frac{\partial V}{\partial v} \bigg/ \frac{\partial U}{\partial v}, \quad \Delta = -\varphi\frac{\partial U}{\partial S} - \frac{\partial V}{\partial S}$$

must be justified. Since the portfolio is risk-neutral, $d\Pi = r\Pi dt$ and it follows from that $d\Pi = r(V + \Delta S + \varphi U)dt$. Substituting φ and Δ , we receive that:

$$\begin{aligned}
& r \left(V + \left(\left(\frac{\partial V}{\partial v} \middle/ \frac{\partial U}{\partial v} \right) \frac{\partial U}{\partial S} - \frac{\partial V}{\partial S} \right) S - \left(\frac{\partial V}{\partial v} \middle/ \frac{\partial U}{\partial v} \right) U \right) dt = \\
& \quad \left[\frac{\partial V}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 V}{\partial S^2} + \rho \sigma v S \frac{\partial^2 V}{\partial S \partial v} + \frac{1}{2} v \sigma^2 \frac{\partial^2 V}{\partial v^2} \right] dt \\
& \quad + \varphi \left[\frac{\partial U}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 U}{\partial S^2} + \rho \sigma v S \frac{\partial^2 U}{\partial S \partial v} + \frac{1}{2} v \sigma^2 \frac{\partial^2 U}{\partial v^2} \right] dt
\end{aligned}$$

after simplifying:

$$\begin{aligned}
& \frac{\left[\frac{\partial V}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2} v \sigma^2 \frac{\partial^2 V}{\partial v^2} + \rho \sigma v S \frac{\partial^2 V}{\partial S \partial v} \right] - rV + rS \frac{\partial V}{\partial S}}{\frac{\partial V}{\partial v}} \\
& = \frac{\left[\frac{\partial U}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 U}{\partial S^2} + \frac{1}{2} v \sigma^2 \frac{\partial^2 U}{\partial v^2} + \rho \sigma v S \frac{\partial^2 U}{\partial S \partial v} \right] - rU + rS \frac{\partial U}{\partial S}}{\frac{\partial U}{\partial v}}
\end{aligned}$$

As both sides are function of V and U only, they can be considered as $f(S, v, t)$. Heston (1993) interprets this in the way:

$$f(S, v, t) = -\kappa(\theta - v) + \lambda v_t$$

where λv_t is the market price of risk mentioned above. Substituting this into the left side:

$$\begin{aligned}
& -\kappa(\theta - v) + \lambda v_t \\
& = \frac{\frac{\partial U}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 U}{\partial S^2} + \frac{1}{2} v \sigma^2 \frac{\partial^2 U}{\partial v^2} + \rho \sigma v S \frac{\partial^2 U}{\partial S \partial v} - rU + rS \frac{\partial U}{\partial S}}{\frac{\partial U}{\partial v}}.
\end{aligned}$$

Rearranging terms, we have the Heston PDE:

$$\frac{\partial U}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 U}{\partial S^2} + \frac{1}{2} v \sigma^2 \frac{\partial^2 U}{\partial v^2} + \rho \sigma v S \frac{\partial^2 U}{\partial S \partial v} - rU + rS \frac{\partial U}{\partial S} + [\kappa(\theta - v) - \lambda v] \frac{\partial U}{\partial v} = 0. \quad (1.1)$$

In the case of a European call option with the boundary conditions:

$$\begin{aligned}
U(S, v, T) &= \max(S - K, 0) \\
U(0, v, t) &= 0 \\
\frac{\partial U}{\partial S}(\infty, v, t) &= 1 \\
U(S, \infty, t) &= S.
\end{aligned}$$

In many cases it is more simple to work with the transformation of the equation. Denote $x = \ln S$ and we draw the PDE as function of this instead of S :

$$\frac{\partial U}{\partial t} + \frac{1}{2} v \frac{\partial^2 U}{\partial x^2} + \frac{1}{2} v \sigma^2 \frac{\partial^2 U}{\partial v^2} + \rho \sigma v \frac{\partial^2 U}{\partial x \partial v} - rU + \left(r - \frac{1}{2} \right) \frac{\partial U}{\partial x} + [\kappa(\theta - v) - \lambda v] \frac{\partial U}{\partial v} = 0.$$

1.3 Price of European options in the Heston model

When value of a call option is expressed in the form of $C(K) = e^x P_1 - K e^{-r\tau} P_2$, then P_1 and P_2 stand for probabilities under different measures. Furthermore, option justifies the Heston PDE, hence substituting $C(K)$ in place of U , we obtain:

$$\frac{\partial C}{\partial t} + \frac{1}{2}v \frac{\partial^2 C}{\partial x^2} + \frac{1}{2}v\sigma^2 \frac{\partial^2 C}{\partial v^2} + \rho\sigma v \frac{\partial^2 C}{\partial x \partial v} - rC + \left(r - \frac{1}{2}\right) \frac{\partial C}{\partial x} + [\kappa(\theta - v) - \lambda v] \frac{\partial C}{\partial v} = 0.$$

Expressing suitable derivatives and choosing parameters practically, we have the following PDE for P_1 after simplifying:

$$\frac{\partial P_1}{\partial t} + \left(r + \frac{1}{2}v\right) \frac{\partial P_1}{\partial x} + \frac{1}{2}v \frac{\partial^2 P_1}{\partial x^2} + \rho\sigma v \frac{\partial^2 P_1}{\partial x \partial v} + [\rho\sigma v + \kappa(\theta - v) - \lambda v] \frac{\partial P_1}{\partial v} + \frac{1}{2}\sigma^2 v \frac{\partial^2 P_1}{\partial v^2} = 0.$$

Alike for P_2

$$\frac{\partial P_2}{\partial t} + \left(r - \frac{1}{2}v\right) \frac{\partial P_2}{\partial x} + \frac{1}{2}v \frac{\partial^2 P_2}{\partial x^2} + \rho\sigma v \frac{\partial^2 P_2}{\partial x \partial v} + [\kappa(\theta - v) - \lambda v] \frac{\partial P_2}{\partial v} + \frac{1}{2}\sigma^2 v \frac{\partial^2 P_2}{\partial v^2} = 0.$$

To ease notations, the following form is used for the two equations:

$$\frac{\partial P_j}{\partial t} + \frac{1}{2}v \frac{\partial^2 P_j}{\partial x^2} + \rho\sigma v \frac{\partial^2 P_j}{\partial x \partial v} + \frac{1}{2}\sigma^2 v \frac{\partial^2 P_j}{\partial v^2} + (r + u_j v) \frac{\partial P_j}{\partial x} + (a - b_j v) \frac{\partial P_j}{\partial v} = 0 \quad (1.2)$$

$j = 1, 2$, where $u_1 = \frac{1}{2}$, $u_2 = -\frac{1}{2}$, $a = \kappa\theta$, $b_1 = \kappa + \lambda - \rho\sigma$, and $b_2 = \kappa + \lambda$.

If we know characteristic function of f_j , P_j exercise probabilities can be assessed according to:

$$P_j = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left[\frac{e^{-i\phi \ln K} f_j(\phi; x, v)}{i\phi} \right] d\phi.$$

Heston (1993) assumes that characteristic function has a log-linear form:

$$f_j(\phi; x_t, v_t) = \exp(C_j(\tau, \phi) + D_j(\tau, \phi)v_t + i\phi x_t)$$

, where $\tau = T - t$ is time to maturity, C_j, D_j are coefficients and i is the imaginary unit. In accordance with Feynman-Kac formula, if a $f(\mathbf{x}_t, t)$ function justifies $\partial f / \partial t - r f + \mathcal{A} f = 0$, where $\mathcal{A}$ is Heston generator, then solution is the conditional expected value: $f(\mathbf{x}_t, t) = E[f(\mathbf{x}_T, T) \mid \mathcal{F}_t]$. We transform further this using that $f(\mathbf{x}_t, t) = E[e^{i\phi \ln S_T} \mid x_t, v_t]$ and f_j also justifies 1.2 PDE.

$$-\frac{\partial f_j}{\partial \tau} + \frac{1}{2}v \frac{\partial^2 f_j}{\partial x^2} + \rho\sigma v \frac{\partial^2 f_j}{\partial x \partial v} + \frac{1}{2}\sigma^2 v \frac{\partial^2 f_j}{\partial v^2} + (r + u_j v) \frac{\partial f_j}{\partial x} + (a - b_j v) \frac{\partial f_j}{\partial v} = 0$$

The following derivatives are needed for transforming the PDE:

$$\begin{aligned} \frac{\partial f_j}{\partial \tau} &= \left(\frac{\partial C_j}{\partial \tau} + \frac{\partial D_j}{\partial \tau} v \right) f_j, & \frac{\partial f_j}{\partial x} &= i\phi f_j, & \frac{\partial f_j}{\partial v} &= D_j f_j, \\ \frac{\partial^2 f_j}{\partial x^2} &= -\phi^2 f_j, & \frac{\partial^2 f_j}{\partial v^2} &= D_j^2 f_j, & \frac{\partial^2 f_j}{\partial v \partial x} &= i\phi D_j f_j. \end{aligned}$$

Substituting these, we receive the equation:

$$-\left(\frac{\partial C_j}{\partial \tau} + v \frac{\partial D_j}{\partial \tau}\right) + \rho \sigma v i \phi D_j - \frac{1}{2} v \phi^2 + \frac{1}{2} v \sigma^2 D_j^2 + (r + u_j v) i \phi + (a - b_j v) D_j = 0.$$

Rearranging

$$v \left(-\frac{\partial D_j}{\partial \tau} + \rho \sigma i \phi D_j - \frac{1}{2} \phi^2 + \frac{1}{2} \sigma^2 D_j^2 + u_j i \phi - b_j D_j \right) - \frac{\partial C_j}{\partial \tau} + r i \phi + a D_j = 0.$$

It can be expressed as two separate differential equations:

$$\frac{\partial D_j}{\partial \tau} = \rho \sigma i \phi D_j - \frac{1}{2} \phi^2 + \frac{1}{2} \sigma^2 D_j^2 + u_j i \phi - b_j D_j \quad (1.3)$$

$$\frac{\partial C_j}{\partial \tau} = r i \phi + a D_j \quad (1.4)$$

Equation 1.3 is a Riccati differential equation, while the second equation is a simple derivation, which can be simply solved knowing D_j . Recall that it is that assumed characteristic function is in the form:

$$f_j(\phi; x_t, v_t) = E[e^{i\phi x_T}] = \exp(C_j(\tau, \phi) + D_j(\tau, \phi)v_t + i\phi x_t)$$

At maturity, $x_T = \ln S_T$ is known, as a consequence expected value disappears. We have boundary conditions from this:

$$C_j(0, \phi) = 0, \quad D_j(0, \phi) = 0$$

1.3.1 Solution of Riccati equation

Express equation 1.3 in the following form:

$$\frac{\partial D_j}{\partial \tau} = P_j - Q_j D_j + R D_j^2$$

,where

$$P_j = u_j i \phi - \frac{1}{2} \phi^2, \quad Q_j = b_j - \rho \sigma i \phi, \quad R = \frac{1}{2} \sigma^2.$$

Corresponding second order ODE is:

$$\omega'' + Q_j \omega' + P_j R \omega = 0$$

,where $D_j = -\frac{1}{R} \frac{\omega'}{\omega}$. Roots of the auxiliary equation:

$$\alpha_j = \frac{-Q_j + \sqrt{Q_j^2 - 4P_j R}}{2} = \frac{-Q_j + d_j}{2}$$

$$\beta_j = \frac{-Q_j - \sqrt{Q_j^2 - 4P_j R}}{2} = \frac{-Q_j - d_j}{2}$$

,where

$$\begin{aligned} d_j &= \alpha_j - \beta_j = \sqrt{Q_j^2 - 4P_jR} \\ &= \sqrt{(\rho\sigma i\phi - b_j)^2 - \sigma^2(2u_j i\phi - \phi^2)}. \end{aligned}$$

Finally, the solution of the equation:

$$D_j = \frac{Q_j + d_j}{2R} \left(\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}} \right)$$

,where

$$g_j = \frac{Q_j - d_j}{Q_j + d_j}.$$

Consequently,

$$D_j(\tau, \phi) = \frac{b_j - \rho\sigma i\phi + d_j}{\sigma^2} \left(\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}} \right).$$

To get $C_j(\tau, \phi)$, we have to solve 1.4 equation. In view of D_j , it cannot be an obstacle:

$$C_j = \int_0^\tau ri\phi dy + a \left(\frac{Q_j + d_j}{\sigma^2} \right) \int_0^\tau \left(\frac{1 - e^{d_j y}}{1 - g_j e^{d_j y}} \right) dy + K_1$$

,where K_1 is constant. First integral can be computed simply, while at the second $x = \exp(d_j y)$ substitution is applied. Integrating by parts and considering $C_j(0, \phi) = 0$ initial condition, solution is provided for C_j .

$$C_j(\tau, \phi) = ri\phi\tau + \frac{a}{\sigma^2} \left[(b_j - \rho\sigma i\phi + d_j)\tau - 2 \ln \left(\frac{1 - g_j e^{d_j\tau}}{1 - g_j} \right) \right]$$

,where $a = \kappa\theta$.

Chapter 2

Heston Model for American Options

At the end of the last deduction, we obtain the formula, which provides a semi-closed solution for European option in the Heston model. Nevertheless, in the case of American options, there are no closed formulas like in the Black-Scholes model. For this reason, we are constrained to apply numerical techniques or seek more complex formulas for the pricing.

In this chapter, we will show simulation based algorithm of Longstaff and Schwartz (2001), which allows us to generate stock paths with an arbitrary model. To provide input for LSM approach, we will discuss firstly a few discretization schemes for simulations especially for the Heston model. Finally, in the last part of the chapter we will present the two-dimensional tree method of Nawalkha and Believa (2010).

2.1 Simulations

Recall that share price and variance follow the processes:

$$\begin{aligned}dS_t &= \mu S_t dt + \sqrt{v_t} S_t dW_t^S \\ dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_t^V\end{aligned}$$

, where $E^{\mathbb{P}}[dW_t^S dW_t^V] = \rho dt$. These processes are continuous, however by the simulations, only discrete processes can be taken account of. Hence we have to find an efficient discretization method, which allows us to perform simulations quickly and accurately. There are such well-known discretization methods like Euler scheme and Milstein scheme, which are described by Gatheral (2006) in details, and some other methods exist also, which are suited especially for the Heston model. Van Haastrecht and Pelsser provide a thorough overview of them. We deal with the Euler scheme, Milstein scheme and Moment Matching scheme.

2.1.1 Discretization and Truncation Schemes

If we have a general stochastic process like X_t :

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t) dW_t$$

and know the value of X_t , then we can simply compute $X_{t+\Delta t}$ as:

$$X_{t+\Delta t} = X_t + \int_t^{t+\Delta t} \mu(X_s, s)ds + \int_t^{t+\Delta t} \sigma(X_s, s) dW_s$$

To start the iteration, we need the initial value of the X_t process. Furthermore, division of time period on which simulation is carried out, is almost always equally spaced, hence $\Delta t_1 = \Delta t_2 = \dots = \Delta t_n$.

It is important to note, that there are two stochastic processes in the Heston model, which are correlated. For this reason, we generate two standard normal random variables in a usual way, which have ρ correlation. In many discretization scheme, it can be occurred

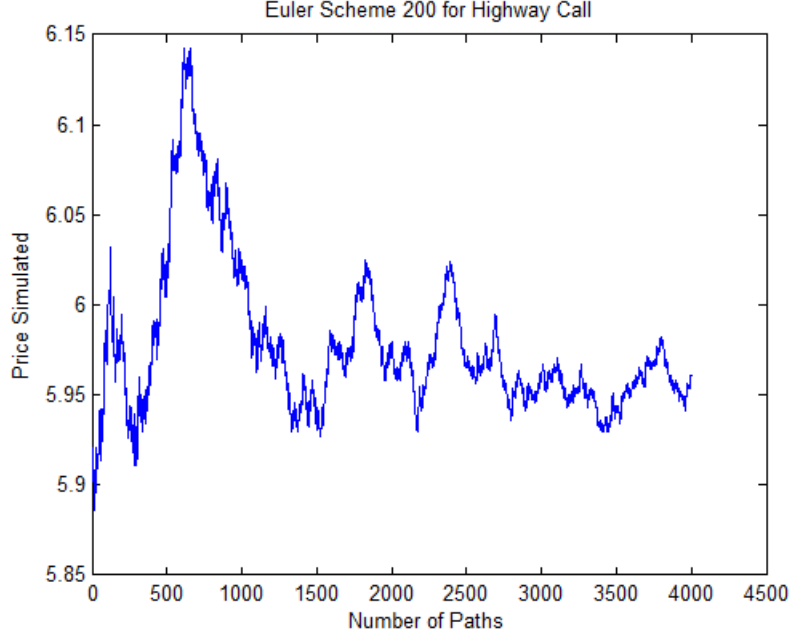


Figure 2.1: Convergence of Euler Scheme, the first 1000 paths are omitted

that the second equation generates negative variances, even if the parameters justify the Feller condition $2\kappa\theta > \sigma^2$. Therefore, some kind of truncation scheme must be implemented. The full truncation scheme sets the variance to zero, once that crosses over the zero line, hence v_t is replaced by $v_t^+ = \max(0, v_t)$. The other well-known truncation scheme is the reflection scheme, which reflects the v_t , when that is negative, so v_t is replaced by $|v_t|$ everywhere. We use the full truncation scheme in simulations. Having set up a discretization scheme, we will use it to price the Highway European options, which have the following parameters:

$$S = K = 100, \quad T = 1, \quad r = .03, \quad \kappa = 2, \quad \theta = .012, \quad v_0 = .014, \quad \sigma = 0.33, \quad \rho = -0.7$$

Heston price is 5.9044 for the call and 2.9490 for the put.

We will apply simulations to collect stock prices at maturity and evaluate the payoff function. After taking the mean of these, we receive the price of the option by discounting back the mean to time t .

$$C(S_t, t) = e^{-r(T-t)} \frac{1}{N} \sum_{i=1}^N \max(0, S_T^i - K)$$

$$P(S_t, t) = e^{-r(T-t)} \frac{1}{N} \sum_{i=1}^N \max(0, K - S_T^i)$$

2.1.2 Euler Scheme

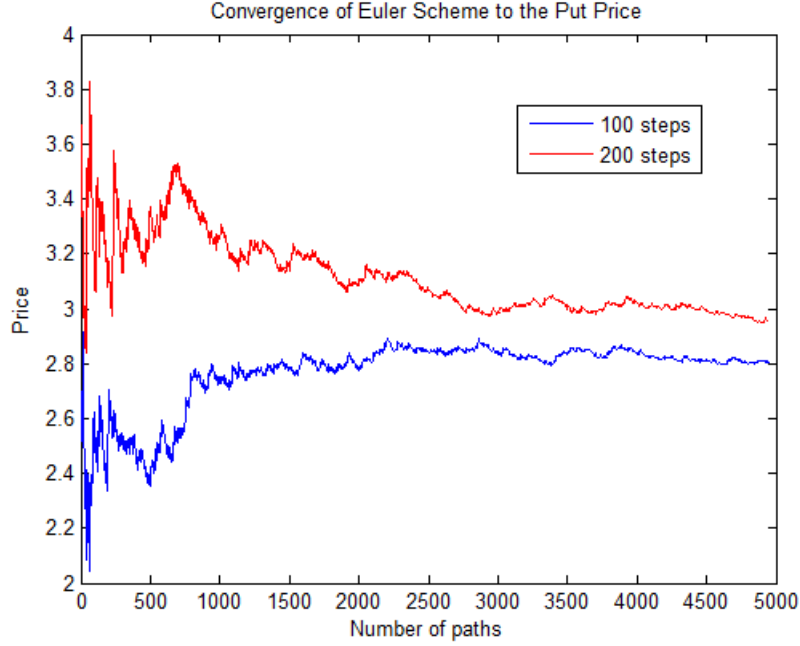


Figure 2.2: Convergence of Euler Scheme to the Put Option

Assuming that we know the values of $\mu(X_s, s)$ and $\sigma(X_s, s)$ and they will remain constant within the time period Δt , we can assess $X_{t+\Delta t}$ as:

$$X_{t+\Delta t} = X_t + \mu(X_t, t)\Delta t + \sigma(X_t, t) (W_{t+\Delta t} - W_t)$$

$$W_{t+\Delta t} - W_t \sim W_{\Delta t} \sim \sqrt{\Delta t}Z$$

$$X_{t+\Delta t} = X_t + \mu(X_t, t)\Delta t + \sigma(X_t, t) \sqrt{\Delta t}Z$$

,where $Z \sim N(0, 1)$. We can apply this for the share prices and for the variance:

$$S_{t+\Delta t} = S_t + rS_t\Delta t + \sqrt{v_t}S_t \sqrt{\Delta t}Z^S$$

$$v_{t+\Delta t} = v_t + \kappa(\theta - v_t)\Delta t + \sigma\sqrt{v_t} \sqrt{\Delta t}Z^V$$

,where $E^{\mathbb{Q}}[Z^SZ^V] = \rho$. It is common practice that we take the logarithm of the stock prices and simulate this rather than the stock prices. In this case $x_t = \ln S_t$ and its SDE:

$$dx_t = (r - \frac{1}{2}v_t)dt + \sqrt{v_t}dW_t^S.$$

Discretization of this is:

$$x_{t+\Delta t} = x_t + (r - \frac{1}{2}v_t)\Delta t + \sqrt{v_t}\sqrt{\Delta t}Z^S.$$

Finally, take the exponential to obtain S_t process:

$$S_{t+\Delta t} = S_t \exp\left((r - \frac{1}{2}v_t)\Delta t + \sqrt{v_t}\sqrt{\Delta t}Z^S\right).$$

Pricing with Euler Scheme

We apply the Euler scheme in two approaches. First, we set Δt to 1/100. Second we set it to 1/200. In both cases we generate 5000 paths.

In the case of the call, price amounts to 6.0802 with $\Delta t = .01$ and 5.9605 with $\Delta t = .005$. The figure 2.1.1 shows how the price converges to the analytical price. The first 1000 paths are omitted, as the price fluctuates wildly in that area and extends the scale of the graph. The computational times are 7.89 and 14.74 seconds.

In the case of a put option, the price is 2.8081 with 100 steps and 2.9580 with 200 steps. The computational times are similar to the call case.

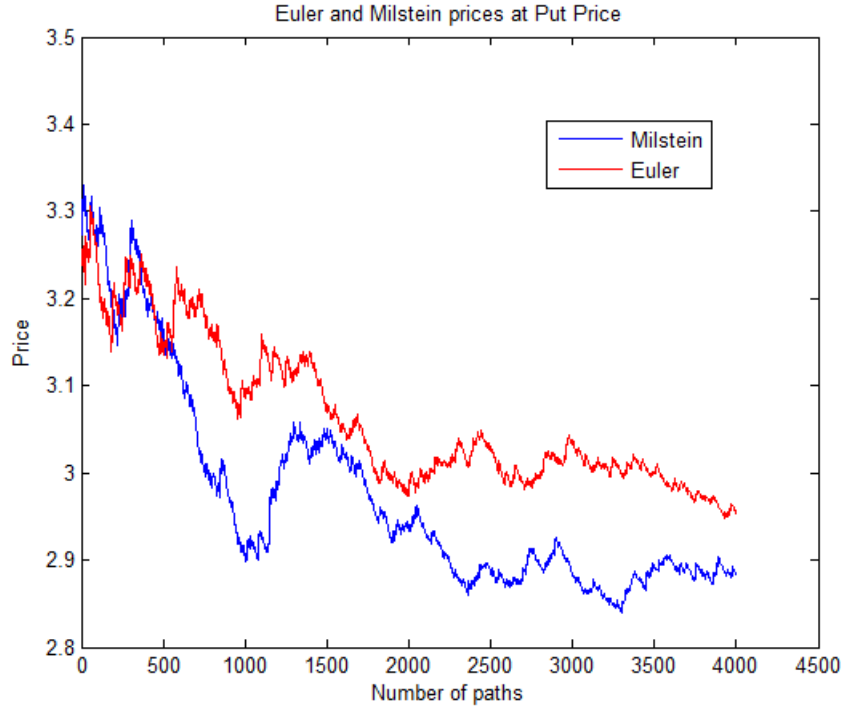


Figure 2.3: Convergence of Euler and Milstein Scheme to the Put Option

2.1.3 Milstein Scheme

This scheme is the enhanced version of the Euler scheme. Coefficients in the Heston processes are the function of the space variable, thereby we can expand them via Itô formula. So the discretization of the processes are:

$$S_{t+\Delta t} = S_t + rS_t\Delta t + \sqrt{v_t}S_t \sqrt{\Delta t}Z^S + \frac{1}{2}v_tS_t\Delta t((Z^S)^2 - 1)$$

$$v_{t+\Delta t} = v_t + \kappa(\theta - v_t)\Delta t + \sigma\sqrt{v_t} \sqrt{\Delta t}Z^V + \frac{1}{4}\sigma^2\Delta t((Z^V)^2 - 1).$$

We implement Milstein scheme in the same way as we have done in the Euler case. The call price is 5.8893 with 200 steps and 5.8879 with 100 steps. The computational times are 15.8 and 8.3 seconds. While put prices are 2.8822 and 2.8130 in the same order,

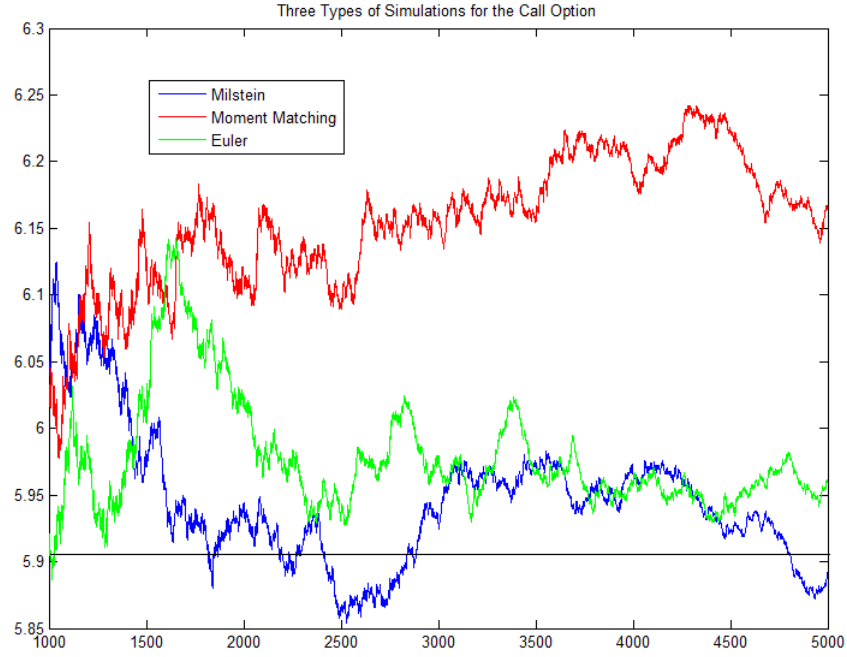


Figure 2.4: 3 types of discretization with 200 steps for the Highway call

times are similar. The figure 2.1.2 presents how the schemes with 200 steps converge to the put price.

2.1.4 Moment Matching Scheme

The scheme of Andersen and Brotherton-Ratcliffe (2005) produces positive variances only, since variance is under a lognormal distribution.

$$v_{t+\Delta t} = (\theta + (v_t - \theta)e^{-\kappa\Delta t}) \exp\left(-\frac{1}{2}\Gamma_t^2 + \Gamma_t Z^V\right)$$

,where

$$\Gamma_t = \ln\left(1 + \frac{\sigma^2 v_t (1 - e^{-2\kappa\Delta t})}{2\kappa(\theta + (v_t - \theta)e^{-\kappa\Delta t})^2}\right)$$

Results of this scheme are rather poor. Call prices are 6.1218 with 100 steps and 6.1601 with 200 steps, while put prices are 3.29 and 3.1502 in the same order.

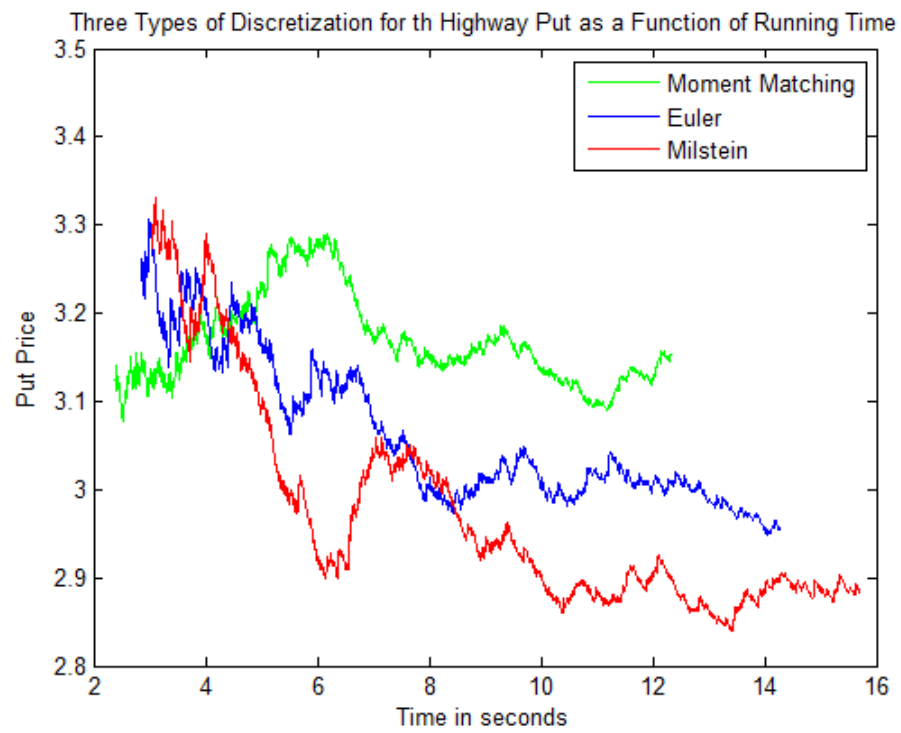


Figure 2.5: 3 types of discretization with 200 steps for the Highway put as a function of time

2.2 Least squares Monte Carlo

In this chapter we present the simulation-based algorithm of Longstaff and Schwartz (2001), which is applicable in every framework, where the stock prices can be simulated by stochastic process. It has remarkable advantages in multi-dimensional cases as well. We assume that there are K finite times to T maturity, when the holder can exercise the option. Let us denote with t_k , where $k = 1, \dots, K$ and $0 < t_1 < t_2 < \dots < t_K = T$.

If the option holders follow an optimal stopping strategy, they exercise the option at t_k , when the immediate exercise value is greater than the value of continuing to hold the option. The value $F(\omega, t_k)$ of continuing is simple the discounted expectation value of the remaining cash flows under the risk-neutral measure, thus we are highly interested in determining of these cash flows.

The amount of remaining cash flows is closely entwined with the exercise strategy, owing to if there will be an exercise prior to maturity at t_k , then the cash flow will be equal to the exercise value at t_k , else will be equal to the value of the payoff function at T . So to assess the conditional expected value at t_k , we should know whether the option will be exercised at any t_s , $s > t$ to which we should be able to compute $F(\omega, t_s)$ as well. Provided that we intended to use simulations for estimating the conditional expectation function, we would face with a bushy Monte Carlo problem. Avoiding this Longstaff and Schwartz determine the conditional expectation function backwards from T to 0 recursively. Let us see an example to present the LSM algorithm.

We would like to demonstrate the numerical example of Longstaff and Schwartz, but under the Heston model. They consider an American put option on a share of non-dividend-paying stock with $K = 1.1$ strike price. The option is exercisable at times 1, 2 and 3, furthermore 3 is the maturity time at the same time. The risk-free rate is six percent and the stock path are generated under the risk-neutral measure. We generated 1000 paths using the Milstein scheme in Heston framework divided the three years to 300 equal periods, but we list merely 10 randomly chosen paths in the matrix to remain transparent.

Paths	t=0	t=1	t=2	t=3
1	1	0.84	1.04	1.10
2	1	1.03	1.06	1.15
3	1	0.9	0.946	0.65
4	1	1.10	1.2	1.41
5	1	0.99	0.79	0.90
6	1	1.06	1.17	1.56
7	1	0.88	1.09	1.04
8	1	1.15	1.29	1.50
9	1	0.75	0.79	0.82
10	1	1.21	1.37	1.44

In case of there are no exercise before the maturity, we know that the final value of the option is equal to the payoff function on every path, which is $(K - S(\omega, 3))^+$. It is obvious that this is the cash flow table as well if the option is not exercised prior time 3, therefore it is equal to the cash flow of the same European option.

Paths	Payoff
1	0
2	0
3	0.45
4	0
5	0.20
6	0
7	0.06
8	0
9	0.28
10	0

To reach a conditional expectation function to t_{K-1} we discount back one period the values of the payoff function at T . To illustrate the discounted payoff of the first 50 paths of simulation against its stock values at $t = 2$ can be seen on 2.2.

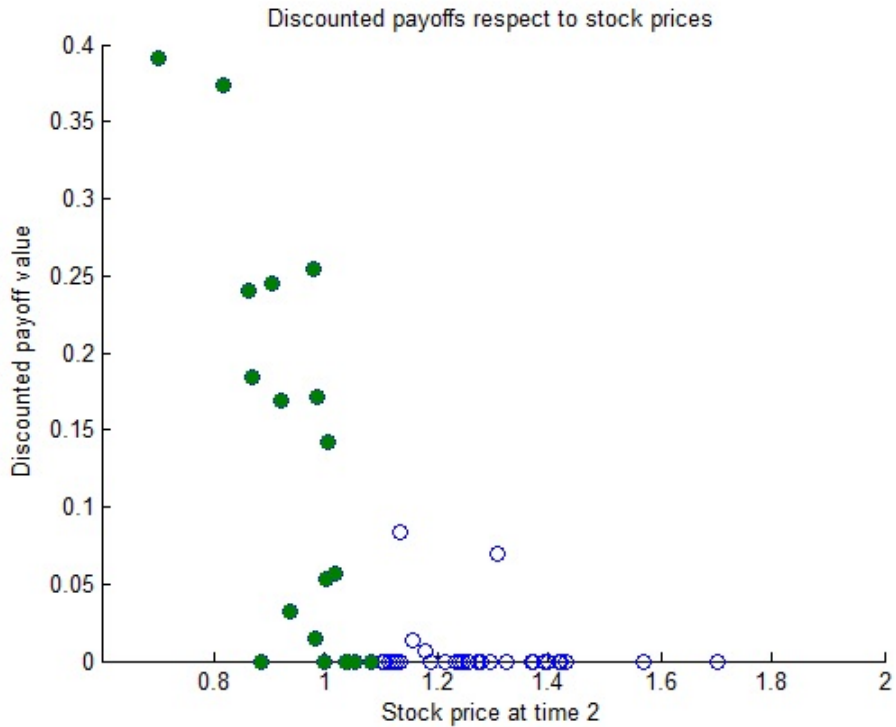


Figure 2.6: The filled circles are in-the-money paths at time 2

It is important to note that there are many paths where the stock price is under 1.1 at time 2, however they are out-of-the-money at maturity. On the contrary, a few circles, which have greater coordinate than 1.1 and are above the horizontal axis, indicate that

it can occur that the stock is out-of-the-money at time 2, yet it is in-the-money at the end. Nevertheless, Longstaff and Schwartz consider in the regression only in-the-money points at time 2 to accelerate the algorithm.

As Longstaff and Schwartz we assume that the sought function of $F(\omega, t_{K-1})$ can be represented as a linear combination of a countable set of $\mathcal{F}_{t_{K-1}}$ measurable basis function and use the Laguerre polynomials for this purpose:

$$F(\omega, t_{K-1}) = \sum_{j=0}^{\infty} a_j L_j(X),$$

The Laguerre polynomials are:

$$\begin{aligned} L_0(X) &= \exp(-X/2), \\ L_1(X) &= \exp(-X/2)(1 - X), \\ &\vdots \\ L_n(X) &= \exp(-X/2) \frac{e^X}{n!} \frac{d^n}{dX^n} (X^n e^{-X}) \end{aligned}$$

Every in-the-money path from 1000 are taken account of to the regression. The estimated conditional expectation function is shown on the 2.7.figure.

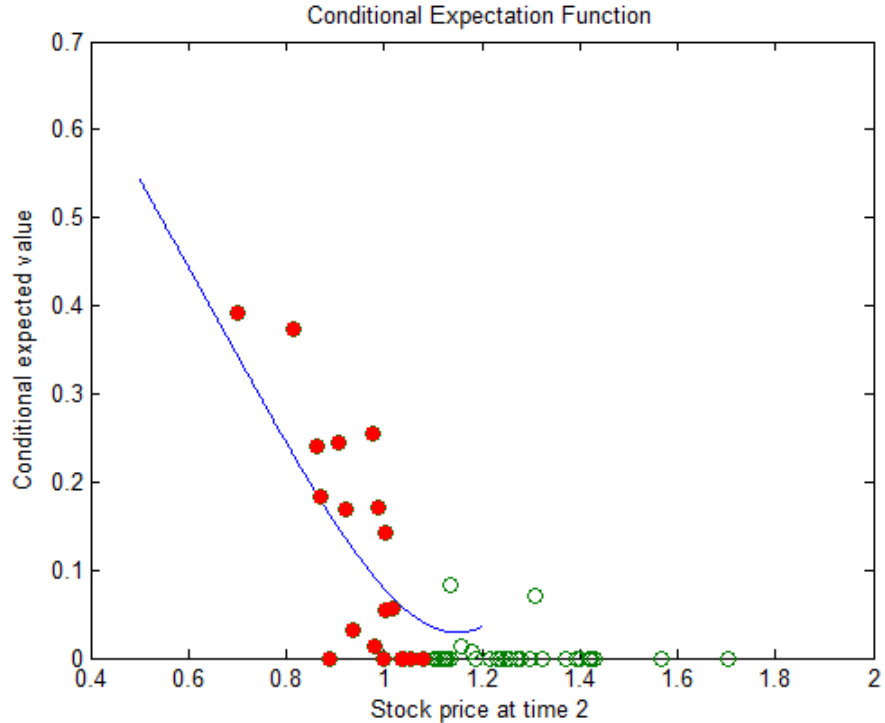


Figure 2.7: Conditional expectation function with the values of first 50 paths

Thus we can determine the continuation value at every in-the-money paths at time 2 by substituting in the conditional expectation function. Comparing the value of immediate

Paths	Stock price	Exercise value	Continuation value
1	1.04	0.061	0.058
2	1.06	0.043	0.050
3	0.94	0.159	0.120
4	(1.2)	-	-
5	0.79	0.313	0.257
6	(1.17)	-	-
7	1.09	0.011	0.038
8	(1.29)	-	-
9	0.79	0.308	0.252
10	(1.37)	-	-

exercise with the continuation value we assess on every in-the-money path whether the option is exercised. It is optimal to exercise the option at time 2 for the first, third, fifth and ninth paths. Whereby we are able to set the cash flow table at time 2.

Paths	t=1	t=2	t=3
1	-	0.061	-
2	-	-	0
3	-	0.159	-
4	-	-	0
5	-	0.313	-
6	-	-	0
7	-	-	0.06
8	-	-	0
9	-	0.308	-
10	-	-	0

Figure 2.8: Cash flow matrix at time 2

Since an American option can only be exercised once there are zero cash flows at time 3, where the option was exercised at time 2. Discounting this cash flow matrix back to time 1 the conditional expectation function can be estimated again and occurrent exercises are catchable. It is evident that at the beginning the holders do not exercise the option, because if there are no arbitrage, yet it made any sense, they would not buy the option at all. Therefore if the cash flow matrix has been assessed recursively to time 1 we can price the option discounting the cash flow on every path and taking the average of them. The price of the European option is 0.066 and the American price is 0.0953. The LSM method is simple to implement thanks to only a simple regression is involved. Computations took 3.5 second on a PC with 2.3 GHz processor and 1.75 GB RAM using MATLAB©.

2.2.1 LSM with Different Discretization Schemes

In this section, we intend to apply Least Squares Monte Carlo for pricing American options with discretization schemes presented above. Without creating new paths, we use extant paths generated for the European Highway options. Thus only LSM method takes up time

Option	Scheme	Euro		Amer	
		LSM	Heston	LSM	Control Var
Call	Euler	5.9605	5.9044	5.9628	5.9067
Call	Milstein	5.8893	5.9044	5.8853	5.9004
Call	Moment Matching	6.1601	5.9044	6.1531	5.8974
Put	Euler	2.958	2.949	3.1199	3.1109
Put	Milstein	2.8822	2.949	2.9952	3.062
Put	Moment Matching	3.1502	2.949	3.4607	3.2594

Figure 2.9: Table shows the prices attained from the LSM approach using control variate technique

to evaluate the price of the American Highway options. Results are congruent relative to call, however they are diverse for the put.

We would like to point it out that the use of Laguerre polynomials was problematic for the estimation of the conditional expected value function. Since the range of stock prices at the Highway option is much broader than it was at the example of Longstaff and Schwartz, more term from the infinite sequence of Laguerre polynomials is needed to approximate the conditional expected value function effectively. Figure 2.10 presents that adding extra terms from the sequence, estimation smooths down. Function presented by blue circles involve seven terms and it is a better approach than the red function, which contains three terms. Yet the simple first three powers of stock prices provide considerably superior choice. Furthermore, taking into consideration more term than four, leads to a matrix, which is almost singular in the regression. As a consequence, we implement the powers to the pricing. Running time of LSM method is about 1.7 second.

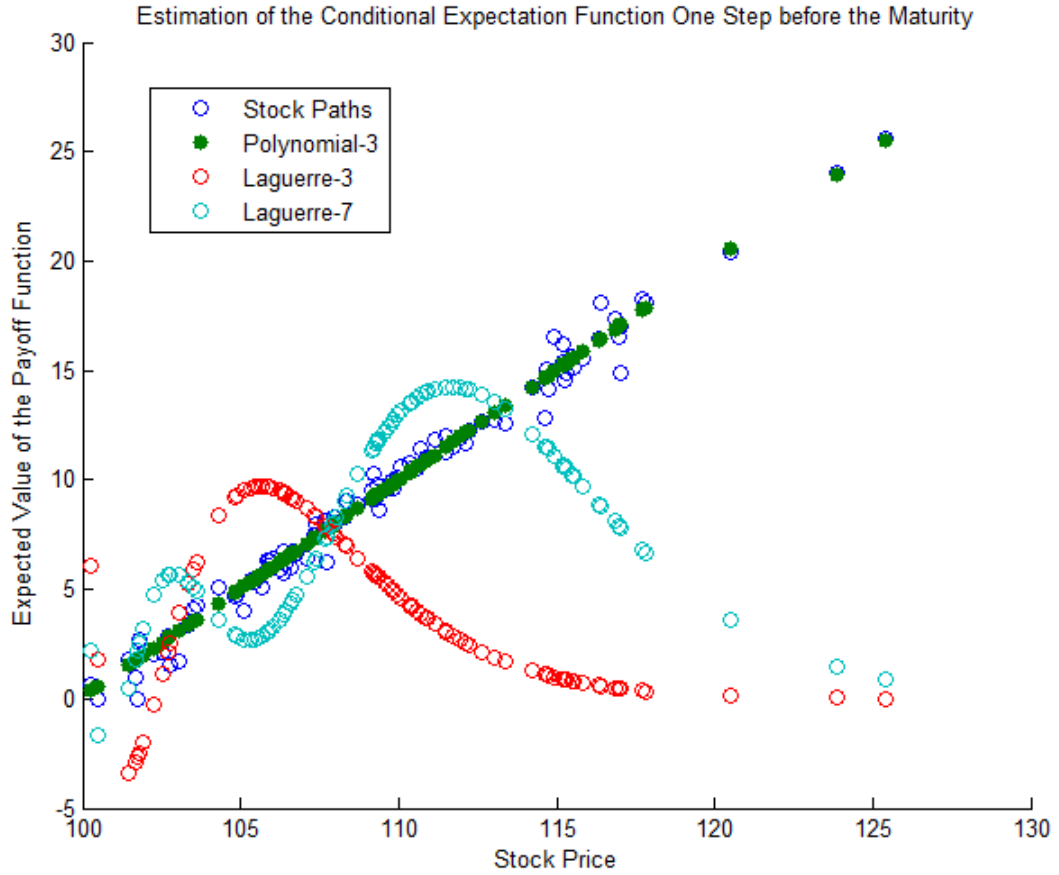


Figure 2.10: Conditional expectation function estimated by Laguerre polynomials and powers of stock price

2.3 Two-dimensional tree of Nawalkha and Believa

Nawalkha-Believa (2010) use a two-dimensional tree to price options in the Heston framework. In their approach, they can tackle problems arising from dependency of variance and stock by transforming the equation of the stock. In the transformed form the variance and the stock are independent, thus it is possible to model them with trinomial trees separately and merge them. The combination of the trees is straightforward, since the joint probabilities are the product of marginal probabilities. Having created the two-dimensional tree, the pricing of the option can be performed iteratively backward as usual.

2.3.1 Transformation of the stock price

The equations of the Heston model under the $\mathbb{Q}$ measure are the following:

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{v_t} S_t dW_t^S \\ dv_t &= \kappa(\theta - v_t) dt + \sigma \sqrt{v_t} dW_t^V \end{aligned}$$

,where $E^{\mathbb{Q}}[dW_t^S dW_t^V] = \rho dt$. The transformation of Nawalkha and Believa for the stock price process:

$$Y_t = \ln S_t - \frac{\rho}{\sigma} v_t - h_t \quad (2.1)$$

,where

$$h_t = \left(r - \frac{\rho \kappa \theta}{\sigma} \right) t$$

Applying the Itô-formula:

$$dY_t = \mu_Y(t)dt + \sigma_Y(t)d\widetilde{W}_t^S$$

,where

$$\mu_Y(t) = \left(\frac{\rho \kappa}{\sigma} - \frac{1}{2} \right) v_t, \quad \sigma_Y(t) = \sqrt{1 - \rho^2} \sqrt{v_t}. \quad (2.2)$$

and the transformed Brown motion:

$$d\widetilde{W}_t^S = \frac{dW_t^S - \rho dW_t^V}{\sqrt{1 - \rho^2}}.$$

It can be seen that $E^{\mathbb{Q}}[d\widetilde{W}_t^S dW_t^V] = 0$, so the transformed process and the variance process are independent and the joint distribution of (Y_t, v_t) process is equal to the product of the marginal distributions.

2.3.2 Tree for the Variance Process

As the variance tree is needed to construct the stock price tree, the first task to do is building the variance tree. It is known the variance follows CIR process and its volatility is $\sigma\sqrt{v_t}$. In order to be able to construct a recombining tree, it is practical to transform the variance process into a form, where the volatility is constant. The transformation of Nelson and Ramaswamy (1990) is suitable for this purpose.

$$x_t = \frac{2\sqrt{v_t}}{\sigma} \quad (2.3)$$

Using the Itô-formula and the relationship in equation 2.3, we can determine the stochastic process of x_t :

$$\begin{aligned} dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_t^V \\ dx_t &= \frac{1}{\sigma\sqrt{v_t}}dv_t + \frac{1}{4}\frac{-\sigma}{\sqrt{v_t}}dt \\ dx_t &= \frac{1}{x_t} \left(\frac{\kappa}{2} \left(\frac{4\theta}{\sigma^2} - x_t^2 \right) \frac{1}{2} \right) dt + dW_t^V \end{aligned}$$

The x_t process can be modelled with trinomial tree which has nodes equally spaced.

$$\begin{aligned} x_{t+\Delta t}^u &= x_t + b(J+1)\sqrt{\Delta t}, \\ x_{t+\Delta t}^m &= x_t + bJ\sqrt{\Delta t}, \\ x_{t+\Delta t}^d &= x_t + b(J-1)\sqrt{\Delta t} \end{aligned}$$

Where the b parameter is constant along the tree. As Nawalkha and Believa point out, b must be in the range $1 \leq b \leq \sqrt{2}$ in order to keep the probabilities non-negative. Nawalkha and Believa define b in the following way:

$$b = \begin{cases} b_c & \text{ha } |b_c - \sqrt{1,5}| < |b_e - \sqrt{1,5}| \\ b_e & \text{otherwise} \end{cases}$$

,where

$$b_e = \frac{x_0/\sqrt{\Delta t}}{[x_0/\sqrt{1,5\Delta t}]}, \quad b_c = \frac{x_0/\sqrt{\Delta t}}{[x_0/\sqrt{1,5\Delta t} + 1]}$$

,where $[.]$ is the nearest integer function. The J variable assures that the tree can follow the process when $\mu(x_t, t)dt$ is relatively large. In this case the middle node goes as close to the conditional expected value as possible. Furthermore, J can execute the truncation, see below.

$$J = \left\lceil \frac{\mu(x_t, t)\sqrt{\Delta t}}{b} + \frac{1}{b^2} \right\rceil.$$

The transition probabilities are determined in such a way that the first two moments of the discrete distribution matches to the conditional distribution of the continuous process.

$$\begin{aligned} p_v^u &= \frac{1}{2b^2} - \frac{J}{2} + \frac{1}{2b}\mu(x_t, t)\sqrt{\Delta t}, \\ p_v^m &= 1 - \frac{1}{b^2}, \\ p_v^d &= \frac{1}{2b^2} + \frac{J}{2} - \frac{1}{2b}\mu(x_t, t)\sqrt{\Delta t} \end{aligned}$$

Like the variance, the x_t process is greater than zero, for this reason the tree is truncated at zero. Furthermore, too high x_t -s and variance levels are unreal, therefore the tree is truncated from above as well. To ensure that the probabilities are non-negative, multiple jumps are allowed in the top and bottom area of the tree.

Having built the tree for x_t process, it is attainable to assess the tree for the variance, namely taking the inverse of the equation 2.3, we are able to construct the variance tree node by node. To illustrate, the figure 2.3.2 shows a tree when the initial value of variance is equal to θ , the figure 2.3.2 shows when the initial variance is over θ and finally figure 2.3.2 shows when initial variance is under θ .

2.3.3 Tree for the Y_t Process

Nawalkha and Believa fit a recombining trinomial tree for the Y_t process. Basically the steps of the tree can be described with the following equations:

$$\begin{aligned} Y_{t+\Delta t}^u &= Y_t + \sigma_Y(0)\sqrt{\Delta t}, \\ Y_{t+\Delta t}^m &= Y_t, \\ Y_{t+\Delta t}^d &= Y_t - \sigma_Y(0)\sqrt{\Delta t} \end{aligned}$$

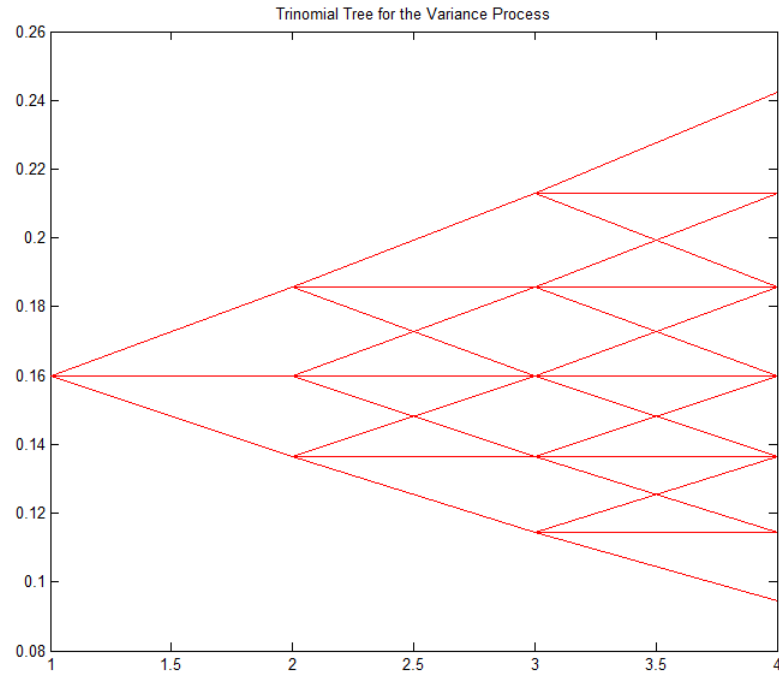


Figure 2.11: Recombining Variance Tree, Parameters: $v_0 = 0.16$, $\theta = 0.16$, $\kappa = 5$, $\sigma = 0.2$, $T = 0.25$

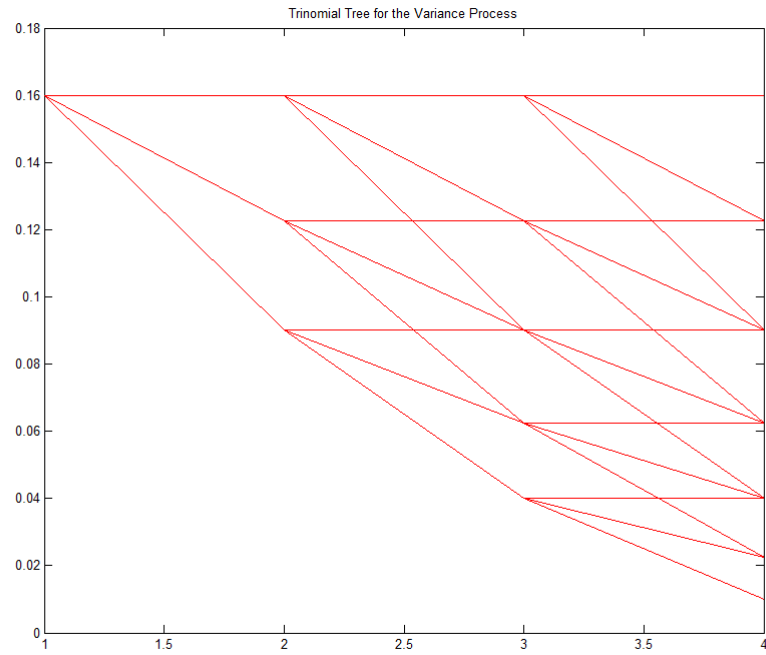


Figure 2.12: Recombining Variance Tree, Parameters: $v_0 = 0.16$, $\theta = 0.06$, $\kappa = 5$, $\sigma = 0.33$, $T = 0.25$

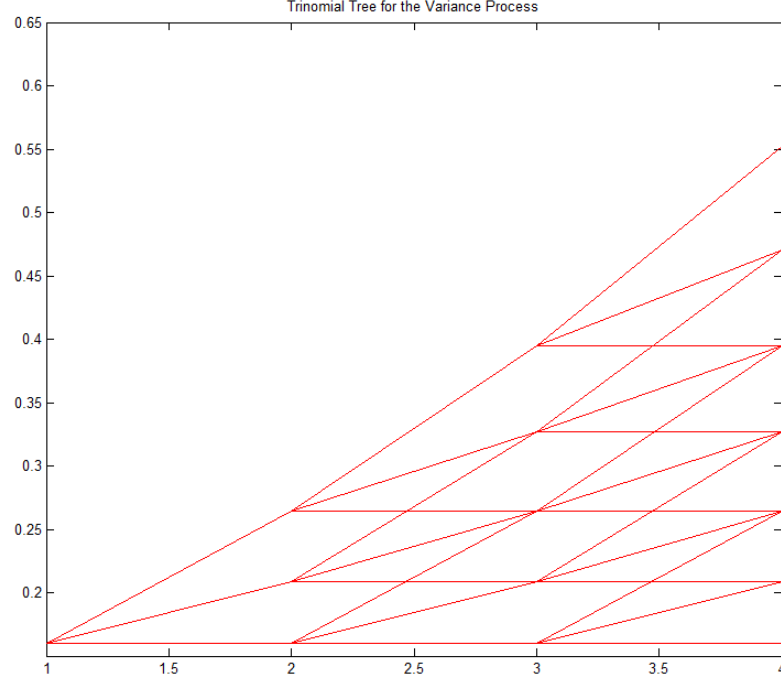


Figure 2.13: Recombining Variance Tree, Parameters: $v_0 = 0.16$, $\theta = 0.25$, $\kappa = 5$, $\sigma = 0.4$, $T = 0.25$

However, considering the fact that the variance of the Y_t process is not constant in the He-
 ston framework, some middle node probabilities are negative to match higher conditional
 variances than v_0 . To avoid this, multiple jumps are applied.

$$\begin{aligned} Y_{t+\Delta t}^u &= Y_t + k_t \sigma_Y(0) \sqrt{\Delta t}, \\ Y_{t+\Delta t}^m &= Y_t, \\ Y_{t+\Delta t}^d &= Y_t - k_t \sigma_Y(0) \sqrt{\Delta t} \end{aligned}$$

,where

$$k_t = \begin{cases} \lceil \sqrt{v_t/v_0} \rceil & \text{if } v_t > 0 \\ 1 & \text{otherwise.} \end{cases}$$

,where $\lceil \cdot \rceil$ is the ceiling function. This represents an efficient discretization of the Y_t
 process, if the number of steps is sufficiently large, hence Δt is small enough. Whereas,
 if there are few steps merely, it can be occur that the conditional expected value greater
 than $Y_{t+\Delta t}^u$ or typically less than $Y_{t+\Delta t}^d$, which leads negative transition probabilities as
 well. I_t is introduced to settle this problem.

$$I_t = \left\lceil \frac{\mu_Y(t) \sqrt{\Delta t}}{k_t \sigma_Y(0)} \right\rceil$$

Thereby, the middle node can be as close to the conditional expected value on the tree as possible.

$$\begin{aligned} Y_{t+\Delta t}^u &= Y_t + (I+1)k_t\sigma_Y(0)\sqrt{\Delta t}, \\ Y_{t+\Delta t}^m &= Y_t + I k_t\sigma_Y(0)\sqrt{\Delta t}, \\ Y_{t+\Delta t}^d &= Y_t + (I-1)k_t\sigma_Y(0)\sqrt{\Delta t} \end{aligned}$$

The starting value of the tree is $Y_0 = \ln S_0 - \frac{\rho}{\sigma} V_0$ and the S_t process can be obtained by inverting equation 2.1.

$$S_t = \exp\left(Y_t + \frac{\rho}{\sigma} v_t + h_t\right).$$

Having created the tree of the Y_t process, the final task is the assessment of transition probabilities. The marginal probabilities are solved by matching the conditional mean and variances and the sum of them is set at one.

$$\begin{aligned} p_Y^u &= \frac{1}{2} \frac{\sigma_Y^2(t)\Delta t + e_m e_d}{(k_t\sigma_Y(0))^2\Delta t}, \\ p_Y^m &= -\frac{\sigma_Y^2(t)\Delta t + e_u e_d}{(k_t\sigma_Y(0))^2\Delta t}, \\ p_Y^d &= \frac{1}{2} \frac{\sigma_Y^2(t)\Delta t + e_u e_u}{(k_t\sigma_Y(0))^2\Delta t} \end{aligned}$$

,where

$$\begin{aligned} e_u &= Y_{t+\Delta t}^u - Y_t - \mu_Y(t)\Delta t = (I+1)k_t\sigma_Y(0)\sqrt{\Delta t} - \mu_Y(t)\Delta t, \\ e_m &= Y_{t+\Delta t}^m - Y_t - \mu_Y(t)\Delta t = I k_t\sigma_Y(0)\sqrt{\Delta t} - \mu_Y(t)\Delta t, \\ e_d &= Y_{t+\Delta t}^d - Y_t - \mu_Y(t)\Delta t = (I-1)k_t\sigma_Y(0)\sqrt{\Delta t} - \mu_Y(t)\Delta t. \end{aligned}$$

2.3.4 Combination of Independent Trees

The configuration of the nodes in the two-dimensional tree is usual. The number of nodes at time t is the product of the numbers of nodes of the separated trees at time t . The tree of the variance has $2t+1$ nodes at time t prior to the truncation and constant number after that. The number of nodes of the stock-tree increases steadily, however the amount of growth depends on v_t through k_t indirectly. For instance, when $k(1) = 2$ at all nodes, then the tree of stock process has seven nodes at time 2, so the two-dimensional has 35 nodes a time 2. Since the trees are independent the transition probabilities are equal to the product of the marginal probabilities on every nodes:

$$p_{i,j} = p_Y^i \cdot p_v^j, \quad i = u, m, d; \quad j = u, m, d.$$

Once we designed the two-dimensional tree, then the pricing is conventional. The price can be obtained computing backward iteratively. For example, the value of an American put option at time t , when the stock price is S_t , is the maximum of the intrinsic value

and the value of continuing.

$$U(S_t, v_t) = \max \left\{ K - S_t, \right. \\ \left. e^{-r\Delta t} (p_{uu}U(S_{t+\Delta t}^u, v_{t+\Delta t}^u) + p_{um}U(S_{t+\Delta t}^u, v_{t+\Delta t}^m) + \cdots + p_{dd}U(S_{t+\Delta t}^d, v_{t+\Delta t}^d)) \right\}$$

2.3.5 Pricing with Nawalkha-Believa tree

In this section, we present an application of the Nawalkha-Believa tree. Firstly, we seek the prices of European options, which are comparable to the results obtained from the Heston model. Secondly, we proceed to price American options. As they do not have analytical prices, we compare the results of Nawalkha-Believa pricing to the outcomes of highly accurate computations of finite differences method.

Steps	Nawalkha-Believa Prices	Relative Error
2	3.2172	-13.76%
5	3.3442	-10.35%
10	3.5448	-4.97%
16	3.6153	-3.08%
20	3.6385	-2.46%
40	3.6841	-1.24%
50	3.6933	-0.99%
100	3.7117	-0.5%

Figure 2.14: Nawalkha prices and errors in the pricing of Tyler call option

European Options

We aim to examine the price of the Tyler call and put options, which have the parameters:

$$S = K = 100, \quad T = 0.1, \quad r = 0.03, \quad \kappa = 2, \quad \theta = 0.1, \quad v_0 = 0.08, \quad \sigma = 0.4, \quad \rho = 0.1$$

Heston price of the call option is 3.7303. The figure 2.3.5 shows how the price of tree method converges to the Heston price when the number of steps are grown.

It is important to note that the prices converge from below monotonously and the convergence speed is remarkable, namely applying only several steps the results are close to the analytical price. When number of steps are sensible, the results are in the typical bid-ask spread range.

Let us see the put option. The Heston price of the put option is 3.4308. The figure 2.3.5 shows the estimated prices with some time steps. The properties of the convergence are very similar to the case of the call option. It can be seen that the relative errors follow some type of the inverse functions. Therefore, we draw three of the functions, which are identified with $f(x) = \frac{-1}{x^\alpha} + c$. The figure 2.3.5 shows them, from above $\alpha_1 = 1$, $\alpha_2 = 0.9$, $\alpha_3 = 0.8$. Since the tree method is asymptotically unbiased, the asymptote of the inverse function must be equal to the Heston price. Thus it seems that the best function to estimate the sequence of Nawalkha-Believa prices is

$$f(x) = \frac{-1}{x^{0.9}} + 3.4308.$$

Steps	Nawalkha-Believa Prices	Relative Error
2	3.0674	-10.59%
5	3.1045	-9.51%
10	3.2752	-4.53%
16	3.3345	-2.81%
20	3.3539	-2.24%
40	3.392	-1.13%
50	3.3997	-0.906%
100	3.4152	-0.45%

Figure 2.15: Nawalkha prices and errors in the pricing of Tyler put option

If we take the assumption that the sequence of Nawalkha-Believa prices follow the $-x^{-0.9} + AP$ function, where AP is the analytical price, we are able to assess the asymptote from the first few points fitting the function with a proper constant and save computational time.

American Options

Pricing the American form of the Tyler option, which has no dividend payments to the maturity, we received reasonable results, hence American call prices correspond with the European call prices and put prices are higher with about 2 cents than the European puts. Nevertheless, we cannot match the American prices of the Tyler option with exact control prices. For this reason, we examine the American put prices of Clarke and Parrott (1999) and Ikonen and Toivanen (2008). The parameters are:

$$K = 10, \quad r = 0.1, \quad T = 0.25, \quad \kappa = 5, \quad \theta = 0.16, \quad v_0 = 0.0625, \quad \sigma = 0.9, \quad \rho = 0.1$$

Using 50 steps for the tree, the results are presented in the figure 2.3.5. At in-the-money options, prices correspond with Ikonen-Toivanen prices, however as moving on options, which are out-of-the-money, errors are rising.

We would like to emphasize that according to our empirical findings, Nawalkha-Believa tree provides accurate prices in the case of call and put options as well, when μ_Y in equation 2.2 is small and performance of the tree is better with in-the-money options. For instance, at Tyler option $\mu_Y(t) = 0$ owing to $\rho\kappa/\sigma = 1/2$, while at Clarke and Perrott set-up $\mu_Y(t) = 0.056v_t$, which also leads to modest drift.

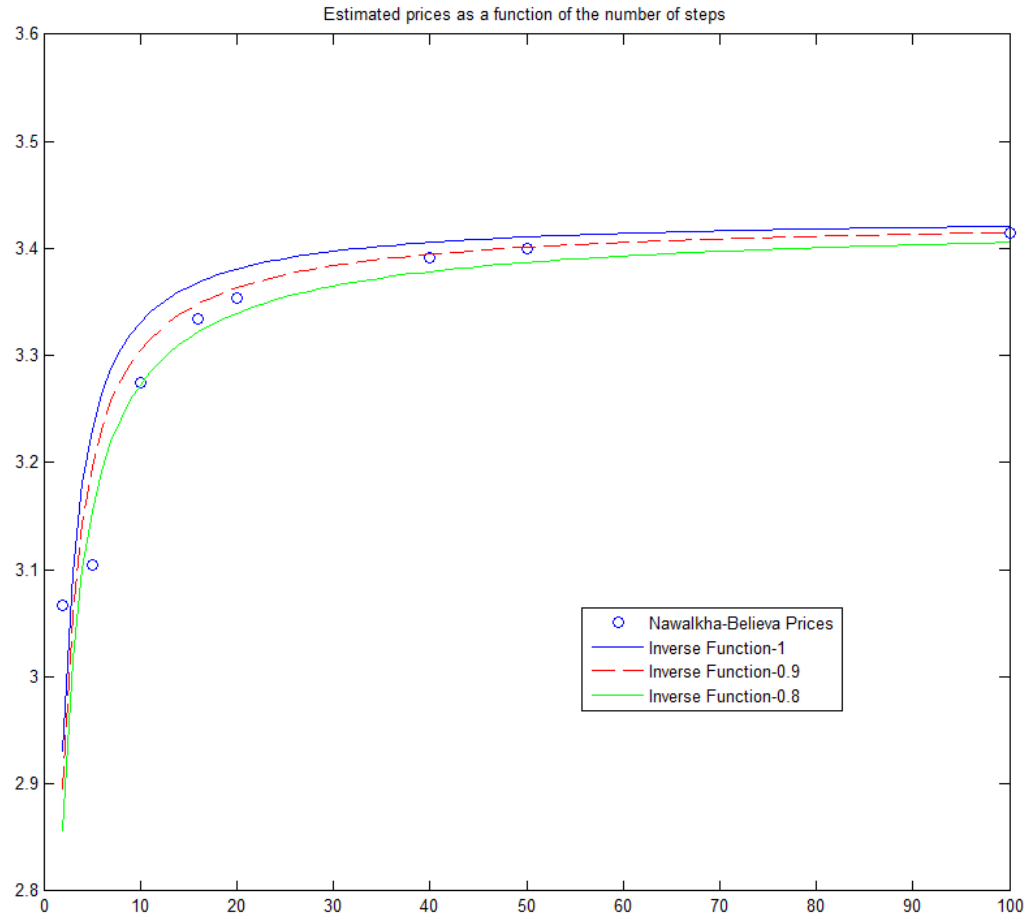


Figure 2.16: The Nawalkha-Believa prices for the put option at different number of steps estimated with inverse functions

Steps	Call			Put		
	European	American	Premium	European	American	Premium
2	3.2172	3.2172	0	3.0674	3.0674	0
5	3.3442	3.3442	0	3.1045	3.1234	0.018849
10	3.5448	3.5448	0	3.2752	3.294	0.018817
16	3.6153	3.6153	0	3.3345	3.3535	0.019052
20	3.6385	3.6385	0	3.3539	3.3731	0.019112
40	3.6841	3.6841	0	3.392	3.4113	0.019257
50	3.6933	3.6933	0	3.3997	3.419	0.019287
100	3.7117	3.7117	0	3.4152	3.4345	0.019347

Figure 2.17: Nawalkha-Believa prices for the Tyler option

S	Ikonen-Toivanen	Nawalkha-Believa	Relative Error
8	2	2	0
9	1.1076	1.1035	-0.37%
10	0.52003	0.51207	-1.53%
11	0.21367	0.20719	-3.03%
12	0.082036	0.078134	-4.76%

Figure 2.18: Ikonen-Toivanen and Nawalkha-Believa prices respectively

Conclusion

In the paper, we deduced the pricing formula for European options in the Heston model which was used as a control price in the case of simulations. Having received a check tool, we can compare different simulation techniques. In our empirical findings shows that Moment Matching method of Andersen performs poorly in the case of Highway options. Euler and Milstein schemes seemed very similar to each other.

Regarding these, we headed for pricing American options with the Least Squares Monte Carlo method. After executed LSM computations, we experienced that the majority of the computational time can be attributed to the production of paths in spite of pricing.

In addition, we found that use of Laguerre polynomials for the regression needs more accurate computations when nominal stock prices are greater.

Finally, we constructed a Nawalkha-Believa two-dimensional tree, which estimates very accurately the prices of Clarke-Perrott and Tyler options, however it becomes less accurate, when parameter set-up is not so favourable. Furthermore, it is important to emphasize that the computation time of this approach depends vastly on the coding technique as well. How to improve the speed of the tree could be a topic of further researches.

It was not discussed in this study, however field of enhanced numerical technique based on finite differences method is also attractive for pricing American options in the Heston model. Clarke and Perrott (1999), Ikonen and Toivanen (2008) provide more details.

Having looked at a wide-spread pricing model and an alternative numerical model, it seems that this field of option pricing is doubtless vivid and interesting breakthroughs are expectable.

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