

Thesis

Measurement and Allocation of Systemic Risk
A Game Theoretic Approach

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Chapter 1

Introduction

Financial firms will be required to follow the example of millions of families across the country that are saving more money as a precaution against bad times. They will be required to keep more capital and liquid assets on hand and, importantly, the biggest, most interconnected firms will be required to keep even bigger cushions. Now, higher capital requirements are an important step towards longer-term stability, but they are only the first step. (Treasury Secretary Timothy F. Geithner (2009))

During the turmoil of the financial crisis of 2007-2008, it turned out that the actual regulation framework for financial institutions to hold capital reserves was not sufficient and could not establish a stable system which is resilient enough to absorb shocks. Having run out of liquid funds, financial institutions could not meet liability to pay. Based on the deficiencies of Basel II regulations, the Basel Committee decided to build up a new regulation system in which, besides other important measures, new risk fields were incorporated as well. This new regulation framework became known as Basel III.

During the crises it turned out that the importance of systemic risk management is indisputable, so different approaches were built up to integrate systemic risk in different risk management practices. Acharya et al (2010) present a model in which they show how systemic risk can be measured and priced using systemic expected shortfall to measure institutions' risk contributions. Jo (2012) considers systemic risk from the perspective of a financial network under macroeconomic distress. Gauthier et al (2010) use a network based structural model to measure systemic risk and they allocate it to individual financial institutions based on different allocation methods. Webber and Willison (2011) develop a system-wide risk management approach to calculate systemic credit risk

for the entire financial system. Game theoretic approach of systemic risk has also gained ground in risk management, for instance, Drehmann and Tarashev (2011) and Tarashev et al (2010) apply the Shapley value (Shapley, 1953) to determine systemic risk and financial institutions' systemic importance.

In the thesis, we examine systemic risk deriving from liquidity problems and give a possible game theoretic approach to allocate risk among players who represent financial institutions. Our model is in accordance with the current Basel III regulation. In our context, we consider the members of a financial system and determine the risk of their retail mortgage portfolios. According to our intuition, systemic risk originates from the fact that when banks calculate their necessary capital reserves to cover their risks, they do not take into account what would happen if, due to any unfavorable event, the entire financial system acted in the same way.

In the thesis, we examine different scenarios in which banks have to liquidate certain parts of their real estate collaterals and they ignore others' behavior, that is, what would happen if all the financial institutions wanted to sell their real estates at the same time (hereafter banks' joint sale). In such cases, on illiquid markets, especially on the real estate market, joint sale would cause significant price decrease which would have further effects on banks' expected incomes. This is the reason why banks' capital reserves defined by Basel II regulations would not be sufficient to cover risk and banks would need additional capital to preserve their stability. Primarily, our aim is not to release figures about a financial system but to draw attention to a relevant source of systemic risk with a game theoretic methodology which can appropriately allocate capital requirements among the members of the financial system. We strongly believe that the previously described intuition is a relevant question for risk management and our model can serve as a guideline for the regulator to extend the current regulation framework and develop a methodology which can treat the previously described systemic risk factor properly.

In the thesis, we discuss transferable utility games and their basic properties with detailed examples, which makes the context easier to understand. We describe three solution concepts which include the core defined by Gillies (1959), the Shapley value introduced by Shapley (1953) and last but not least the nucleolus which was determined by Schmeidler (1969). Following Artzner et al (1999), we introduce coherent measures of risk which are going to be used in risk measurement. After defining the notion of liquidity and the liquidity requirement

(Acerbi and Scandolo, 2010), we consider risk allocation games introduced by Denault (2001). We define our model by modifying Csóka and Herings (2013)'s risk allocation games under liquidity considerations. To make the model easier to understand we build up our framework step by step with detailed examples for each part.

Moving on to the application, based on Balog et al (2011)'s article, we represent seven risk capital allocation methods and their most important properties. In the empirical part of the thesis we examine a financial system's retail mortgage portfolio and the risk deriving from banks' joint sale on the real estate market. Besides the description of our empirical analysis, we consider certain parts of Basel regulation framework in details which are going to be used in the empirical part of the thesis.

In the empirical example, as it was mentioned above, we focus on banks' retail mortgage portfolios and we assume that due to any unfavorable events, banks have to liquidate certain parts of their real estate portfolios which leads banks' joint sale on the real estate market. We also suppose that banks, during their risk capital reserve calculations, do not take joint sale into consideration. As real estate market is an illiquid market, with banks' joint sale, real estate prices decrease significantly which results that banks' income will be less than the amount their risk management previously calculated.

Due to insufficient publicly available data, we could not estimate real estate supply function in an appropriate way. This is the reason why we asked expert's opinion to determine price decrease in the case of banks' joint sale. At this point we note that with sufficiently detailed dataset about the real estate market, the estimation of the real estate supply function can be properly performed using the instrumental variables' method (see for instance in Brooks (2008)). With this methodology, the price drop can be estimated for each bank and for each bank coalition as well.

In the empirical analysis, the effect of banks' joint sale was incorporated into our model by the probability of default and the loss given default whose values were raised by the price decrease. In the analyses, we got results which confirms the fact that in some cases Basel II capital requirements are not sufficient to cover the financial system's risk which originates from banks' joint real estate sale. This phenomenon underpins the need of additional capital reserves in several cases. Applying our model and extending the current methodology, the regulator would allocate systemic risk which would correspond to the concept of Basel III

framework.

The structure of the thesis is the following: first, we introduce transferable utility games with their properties. Secondly, we move on to coherent measures of risk (Artzner et al, 1999), risk allocation games (Denault, 2001) and risk allocation games with liquidity requirement (Csóka and Herings, 2013). In Chapter 4, following Balog et al (2011), we represent seven risk capital allocation methods and finally we continue with the empirical analysis in Chapter 5. After the description of our empirical example, we examine the Basel capital function (BIS, 2005). In the last chapter of the thesis we summarize our results.

Chapter 2

Transferable Utility Games

Cooperative game theory deals with situations in which players can enter into coalitions and make enforceable binding agreements about payoffs. Within the class of cooperative games we can distinguish, transferable utility games, non-transferable utility games and games which are in-between the previous two types. In the thesis, we focus on transferable utility games (hereafter *TU* games) especially on risk allocation games with liquidity requirement.

Before moving on to the detailed examination of risk allocation games, in this chapter we briefly describe the class of *TU* games and highlight their most important properties and basic solution concepts used in this paper.

Let N define the finite, non-empty set of players. A $T \subseteq N$ set of players is called coalition and N refers to the grand coalition while $\emptyset$ is the empty coalition.

Definition 2.1. *Let N the finite, non-empty set of players and $v : \mathcal{P}(N) \rightarrow \mathbb{R}$ such a function that $v(\emptyset) = 0$, where $\mathcal{P}(N)$ refers to the class of all subsets of N . Then v is a *TU* game (transferable utility game). Let $\mathcal{G}^N$ denote the class of *TU* games with the player set N .*

In most of the cases, coalitions represent groups of people and payoffs are measured by cash (or by any arbitrary measures), which can be allocated arbitrarily among players entering into coalitions. In Definition 2.1 function v associates a real number $v(S)$ (payoff) with each $S \subseteq N$ coalition that they can gain together.

Definition 2.2. *Game $v \in \mathcal{G}^N$ is 0-normalized if $v(\{i\}) = 0$ for all $i \in N$.*

Definition 2.3. *A game $v \in \mathcal{G}^N$ is monotone if for all coalitions $S, T \subseteq N$, where $S \subseteq T$, we have $v(S) \leq v(T)$.*

Example 2.4. Let $N = \{1, 2, 3\}$ represent the set of players and the game v be given as $v(\{1\}) = 0$, $v(\{2\}) = 0$, $v(\{3\}) = 0$, $v(\{1, 2\}) = 0$, $v(\{1, 3\}) = 5$, $v(\{2, 3\}) = 3$ and $v(\{1, 2, 3\}) = 5$. The $\{i, j\}$ indicates the coalition of player i and j . It can be easily verified that this game is 0-normalized and monotone because the values of the single coalitions are not bigger than the values of the two-player or grand coalitions. At the same time, the two-player coalitions' values are less than or equal to the grand coalition's value. In this way, we proved that the game v is monotone.

Definition 2.5. A game $v \in \mathcal{G}^N$ is *superadditive* if for all coalitions $S, T \subseteq N$, where $S \cap T = \emptyset$, we have $v(S) + v(T) \leq v(S \cup T)$.

Example 2.6. Let $N = \{1, 2, 3\}$ represent the set of players and the game v be given as $v(\{1\}) = 0$, $v(\{2\}) = 0$, $v(\{3\}) = 0$, $v(\{1, 2\}) = 0$, $v(\{1, 3\}) = 3$, $v(\{2, 3\}) = 4$ and $v(\{1, 2, 3\}) = 4$. The assumption of $S \cap T = \emptyset$ implies that it is enough to consider the following coalition pairs: $\{1\} - \{2, 3\}$, $\{2\} - \{1, 3\}$, $\{3\} - \{1, 2\}$, $\{1\} - \{2\}$, $\{1\} - \{3\}$ and $\{2\} - \{3\}$ (it is clear that the proposition is met for the empty coalition). In the first three cases, the union of the coalitions gives the grand coalition and its value cannot be less than the sum of the single or two-player coalitions' values. In the other three cases, we get the two-player coalitions as unions whose values are not less than the sum of the single coalitions' values.

Definition 2.7. A game $v \in \mathcal{G}^N$ is *subadditive* if for all coalitions $S, T \subseteq N$, where $S \cap T = \emptyset$, we have $v(S) + v(T) \geq v(S \cup T)$.

Definition 2.8. A game $v \in \mathcal{G}^N$ is *additive* if for all coalitions $S, T \subseteq N$, where $S \cap T = \emptyset$, we have $v(S) + v(T) = v(S \cup T)$.

Definition 2.9. A game $v \in \mathcal{G}^N$ is *essential* if $v(N) > \sum_{i \in N} v(\{i\})$.

Definition 2.10. A game $v \in \mathcal{G}^N$ is *convex*, if for all coalitions $S, T \subseteq N$, $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$.

Example 2.11. Take the game v of Example 2.6. If we observe the single coalitions with two- and three-player coalitions we can find that they meet $S \cap T = \{i\}$, $i = 1, 2, 3$ or $v(S) + v(T) \leq v(S \cup T)$ because $S \cap T = \emptyset$. If we consider the intersection of two player coalitions, it gives a single coalition in every case for which $v(\{i\}) = 0$, $i = 1, 2, 3$ so that $v(S) + v(T) \leq v(S \cup T)$ is met. Finally, the intersection of a two-player coalition with the grand coalition always gives the two-player coalition so that the inequality in Definition 2.10 is satisfied with equality.

Definition 2.12. Let $v \in \mathcal{G}^N$, then a subgame of v is a game $v^T \in \mathcal{G}^T$ where $\emptyset \neq T \subseteq N$ and $v^T(S) = v(S)$ for all $S \subseteq T$.

Definition 2.13. The solution $\psi: A \rightrightarrows \mathbb{R}^N$ is a mapping on subclass of games $A \subseteq \mathcal{G}^N$, where $\rightrightarrows$ denotes a set-valued mapping.

In the thesis, we consider risk allocation games which are cost games. Cost games can be described as allocation situation in which the more the payoff is that a coalition gets, the worse the coalition's situation is. For example, players can represent people who want to subscribe to a certain service. In such a case, payoffs determine for each coalition the amount of money which has to be paid by the members of the coalitions and players with higher payoffs are in worse situations than those with lower values.

Let $\mathcal{G}_c^N$ denote the class of cost games with player set N .

In our model, coalitions represent the groups of financial institutions which minimize their risks deriving from their correlated behavior and payoffs are measured by capital requirement (or loss).

Definition 2.14. Let $v \in \mathcal{G}^N$, then the allocation $x \in \mathbb{R}^N$ is efficient if $\sum_{i \in N} x_i = v(N)$.

Definition 2.15. Let $v \in \mathcal{G}_c^N$, then the allocation $x \in \mathbb{R}^N$ is individually rational if $x_i \leq v(\{i\})$ for all $i \in N$.

Definition 2.16. Let $v \in \mathcal{G}_c^N$, then the pre-imputations of game v are defined as

$$I^*(v) = \left\{ x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N) \right\}.$$

Remark 2.17. The set of pre-imputation is not empty.

The set of pre-imputations consists of efficient allocations, that is, players allocate the value of the grand coalition.

Definition 2.18. Let $v \in \mathcal{G}_c^N$, then the imputations of game v are defined as

$$I(v) = \left\{ x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N), \text{ and } v(\{i\}) \geq x_i, i \in N \right\}.$$

In the case of cost games, allocations in which a player receives more than it could obtain on its own, without cooperating with anyone else, are unacceptable. This condition is known as individual rationality. The set of imputations consists of individually rational and efficient allocations.

Remark 2.19. The set of imputations can be empty.

2.1 Solution Concepts and Properties

2.1.1 The Shapley Value

The first solution concept considered in the thesis is the Shapley solution (Shapley, 1953). Shapley (1953) introduced a solution concept which can measure the value of each player in the game. This solution can be defined as follows:

Definition 2.20. [Shapley (1953)] Let $v \in \mathcal{G}^N$, then the Shapley value of player i in game v is given by

$$\varphi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N \setminus S| - 1)!}{|N|!} v'_i(S),$$

where $v'_i(S) = v(S \cup \{i\}) - v(S)$ defines player i 's marginal contribution to coalition S . Furthermore let φ denote the Shapley solution of game v .

Definition 2.20 can be interpreted as follows: let us suppose that players join an association in some order and all $n!$ ordering of players are equally likely. In this case, v'_i is the marginal contribution of player i as he/she joins the association. The payoff that player i receives, depends on the order in which the players joined the association. The Shapley value is the average payoff for each player if they join in a completely random order.

Definition 2.21. For a game $v \in \mathcal{G}^N$ players $i, j \in N$ are equivalent, $i \sim^v j$, if for all coalition $S \subseteq N$, such that $i, j \notin S$ satisfies $v'_i(S) = v'_j(S)$.

According to Definition 2.21, two players are equivalent in game v , if their marginal contributions for each coalition $S \subseteq N$ not containing them, are the same.

Definition 2.22. Let ψ be a single-valued solution on the class of games $A \subseteq \mathcal{G}^N$. The solution ψ satisfies

- *Pareto optimality (PO)*, if for all game $v \in A$: $\sum_{i \in N} \psi_i(v) = v(N)$,
- *Equal Treatment Property (ETP)*, if for all game $v \in A$ and players $i, j \in N$: $i \sim^v j$ implies $\psi_i(v) = \psi_j(v)$,
- *Strong Monotonicity (SM)*, if for all games $v, w \in A$ and player $i \in N$: $v'_i \geq w'_i$ implies $\psi_i(v) \geq \psi_i(w)$.

According to Definition 2.22 solution ψ satisfies PO, if for all games, the sum of the payoffs associated with the players equals to the value of the grand coalition. The solution ψ satisfies ETP, if for all games equivalent players are associated with the same amount of payoffs. Finally, the solution ψ satisfies SM, if each player whose contribution is not less in a game v than in another game w , obtains at least as much payoff in game v as in game w .

Theorem 2.23 ((Young, 1985)). *Let ψ be a single-valued solution on the entire class of games. Then solution ψ satisfies Pareto optimality, Equal Treatment Property and Marginality if and only if ψ is the Shapley solution.*

Example 2.24. Let $N = \{1, 2, 3\}$ be the set of players and game v is defined as $v(\{1\}) = 0$, $v(\{2\}) = 0$, $v(\{3\}) = 0$, $v(\{1, 2\}) = 8$, $v(\{1, 3\}) = 9$, $v(\{2, 3\}) = 7$ and $v(\{1, 2, 3\}) = 12$. In this game, the Shapley value of player 1 can be calculated as follows

$$\begin{aligned}\varphi_1(v) &= \sum_{S \subseteq \{1, 2, 3\} \setminus \{1\}} \frac{|S|!(|\{1, 2, 3\} \setminus S| - 1)!}{|\{1, 2, 3\}|!} [v(S \cup \{1\}) - v(S)] \\ &= \frac{|\emptyset|!(|\{1, 2, 3\} \setminus \emptyset| - 1)!}{|\{1, 2, 3\}|!} [v(\emptyset \cup \{1\}) - v(\emptyset)] \\ &\quad + \frac{|\{2\}|!(|\{1, 2, 3\} \setminus \{2\}| - 1)!}{|\{1, 2, 3\}|!} [v(\{2\} \cup \{1\}) - v(\{2\})] \\ &\quad + \frac{|\{3\}|!(|\{1, 2, 3\} \setminus \{3\}| - 1)!}{|\{1, 2, 3\}|!} [v(\{3\} \cup \{1\}) - v(\{3\})] \\ &\quad + \frac{|\{2, 3\}|!(|\{1, 2, 3\} \setminus \{2, 3\}| - 1)!}{|\{1, 2, 3\}|!} [v(\{2, 3\} \cup \{1\}) - v(\{2, 3\})] \\ &= \frac{0!(3-1)!}{3!} [0 - 0] + \frac{1!(2-1)!}{3!} [8 - 0] + \frac{1!(2-1)!}{3!} [9 - 0] + \frac{2!(1-1)!}{3!} [12 - 7] = 4, 5\end{aligned}$$

The other players' Shapley values can be similarly calculated, and for the Shapley solution of game v we get

$$\varphi(v) = \begin{bmatrix} \varphi_1(v) \\ \varphi_2(v) \\ \varphi_3(v) \end{bmatrix} = \begin{bmatrix} 4, 5 \\ 3, 5 \\ 4 \end{bmatrix}.$$

2.1.2 The Core

In cooperative game theory besides the Shapley value (Shapley, 1953) and the nucleolus (Schmeidler, 1969), one of the well known solution concepts is the core

introduced by Gillies (1959). The core allocation associates payoffs with players in a way which is accepted by all the coalitions so that no coalition can improve on by deviating from the allocation.

Definition 2.25 (Core, (Gillies, 1959)). *Let $v \in \mathcal{G}_c^N$, then the core of game v is defined as*

$$C(v) = \left\{ x \in \mathbb{R} : \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \leq v(S), \ S \subseteq N \right\}.$$

Example 2.26. Let $N = \{1, 2, 3\}$ be the set of players and the cost game v is defined as $v(\{1\}) = 4$, $v(\{2\}) = 4$, $v(\{3\}) = 6$, $v(\{1, 2\}) = 8$, $v(\{1, 3\}) = 10$, $v(\{2, 3\}) = 8$ and $v(\{1, 2, 3\}) = 12$. Vector x is in the core of the game v if it satisfies the following conditions:

$$\begin{aligned} 4 - x_1 &\geq 0 & 8 - (x_1 + x_2) &\geq 0 \\ 4 - x_2 &\geq 0 & 10 - (x_1 + x_3) &\geq 0 \\ 6 - x_3 &\geq 0 & 8 - (x_2 + x_3) &\geq 0 \\ 12 - (x_1 + x_2 + x_3) &= 0 \end{aligned}$$

The conditions can be rearranged as

$$\begin{aligned} x_1 &\leq 4 & x_1 + x_2 &\leq 8 \\ x_2 &\leq 4 & x_1 + x_3 &\leq 10 \\ x_3 &\leq 6 & x_2 + x_3 &\leq 8 \\ x_1 + x_2 + x_3 &= 12. \end{aligned}$$

If we consider $x_2 + x_3 \leq 8$ and $x_1 + x_2 + x_3 = 12$ we can notice that $x_1 \geq 4$ and the first condition implies that $x_1 \leq 4$ so this is the reason why we finally get that $x_1 = 4$. Therefore, if we continue calculating the core of the game, only the following constraints must be met:

$$x_2 \leq 4, \quad x_3 \leq 6, \quad x_2 + x_3 = 8$$

All x_2, x_3 which meet the previous constraints also satisfy that $x_2, x_3 \geq 0$ and $x_2 = 8 - x_3$. Thus, the core of the game v consists of the allocations $x = (x_1, x_2, x_3)$ which meet $x_1 = 4$, $x_2 = 8 - x_3$ and $x_2 \geq 2, x_3 \geq 4$.

Definition 2.27. *Game $v \in \mathcal{G}^N$ is balanced if*

$$\max_{\lambda \in \mathbb{R}_+^{\mathcal{P}(N)}, \sum_{C \in \mathcal{P}(N)} \lambda_C \chi_C = \chi_N} \sum_{C \in \mathcal{P}(N)} \lambda_C v(C) \leq v(N),$$

where χ_N is the characteristic function of set N .

In the case of finitely many players, Bondareva (1963) and Shapley (1967) proved that for TU games, the balancedness is a necessary and sufficient condition for the non emptiness of the core which is stated in the following theorem.

Theorem 2.28 (Bondareva (1963) and Shapley (1967)). *The game $v \in \mathcal{G}^N$ has a non-empty core if and only if it is balanced.*

Example 2.29. Consider the game given in Example 2.26. This game is balanced because its core is non-empty.

Definition 2.30. *Game $v \in \mathcal{G}^N$ is totally balanced if every subgame of v is balanced.*

Definition 2.31. *Game $v \in \mathcal{G}^N$ is exact if for every $C \in \mathcal{P}(N)$ coalition there exists a core allocation x such that $x(C) = v(C)$.*

Definition 2.32. *Game $v \in \mathcal{G}^N$ is totally exact if every subgame of v is exact.*

Remark 2.33. The class of totally exact games coincides with the class of convex games.

2.1.3 The Prenucleolus

As we have mentioned in the previous parts of the thesis, in the case of transferable utility games, players can allocate payoffs arbitrarily. The nucleolus of the game, introduced by Schmeidler (1969), is a solution concept which lexicographically minimizes players' largest dissatisfaction. Thus, the nucleolus of the game chooses the allocation which places the coalition in the worst case into the best possible situation for the coalition. Before defining the nucleolus of the game, we introduce the following notions:

Definition 2.34. *Let $v \in \mathcal{G}^N$ then the excess function $e(\cdot, \cdot) : \mathcal{G}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^{2^N}$ is defined as follows for all coalitions $S \subseteq N$ and $x \in \mathbb{R}^N$:*

$$(e(v, x))_S = v(S) - \sum_{i \in S} x_i.$$

Remark 2.35 (Core). Let $v \in \mathcal{G}_c^N$. The core of game v can be defined as

$$C(v) = \left\{ x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N) \text{ and } (e(v, x))_S \geq 0, S \subseteq N \right\}.$$

Definition 2.36. Let $\varrho : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be defined as follows: for all $x \in \mathbb{R}^d$ and $1 \leq i \leq d$:

$$(\varrho(x))_i = \min_{T \subseteq \{1,2,\dots,d\}, |T|=i} \max_{j \in T} x_j.$$

According to the definition of function ϱ , it sorts the coordinates of vector x in a non-decreasing order.

Example 2.37. Consider vector $x = (3, 2, 1, 4)$ and apply function ϱ on it. If we want to get the first coordinate of $(\varrho(x))$ we have to take all the singleton sets of the coordinates of vector x . The minimum of these elements gives $(\varrho(x))_1$. Thus, $(\varrho(x))_1 = 1$. The second coordinate of $(\varrho(x))$ is determined by the sets of two elements. In the example the sets of two elements are $\{3, 2\}, \{3, 1\}, \{3, 4\}, \{2, 1\}, \{2, 4\}, \{1, 4\}$. First, we consider the maximums of these two element-sets and then their minimum is the second coordinate and we get $(\varrho(x))_2 = 2$. To determine the third coordinate of $(\varrho(x))$, we consider the three element-sets of coordinates of vector x and we get $(\varrho(x))_3 = 4$. The fourth elements of $(\varrho(x))$ can be determined similarly and finally, we get that $(\varrho(x)) = (1, 2, 3, 4)$.

Example 2.38. Let $N = \{1, 2\}$ denote the set of players and the values of the coalitions be $v(\{1\}) = 2, v(\{2\}) = 4$, and $v(\{1, 2\}) = 5$ with the payoff vector $x = (3, 2)$. Then, the excesses of the coalitions are $(e(v, x))_{\{1\}} = 2 - 3 = -1$, $(e(v, x))_{\{2\}} = 4 - 2 = 2$ and $(e(v, x))_{\{1, 2\}} = 5 - (3 + 2) = 0$ and if we apply the ϱ function we get

$$\varrho \circ e(v, x) = \begin{bmatrix} (e(v, x))_{\{1\}} \\ (e(v, x))_{\{1, 2\}} \\ (e(v, x))_{\{2\}} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

Definition 2.39 (Prenucleolus, (Sobolev, 1975)). Let $v \in \mathcal{G}_c^N$, then the pre-nucleolus of game v is defined as

$$\mathcal{PN}(v) = \{x \in I^*(v) : \varrho \circ e(v, x) \geq_{lex} \varrho \circ e(v, z), z \in I^*(v)\},$$

where $\geq_{lex}$ is for lexicographic ordering.

Definition 2.40 (Nucleolus, (Schmeidler, 1969)). Let $v \in \mathcal{G}_c^N$, then the nucleolus of game v is defines as

$$\mathcal{N}(v) = \{x \in I(v) : \varrho \circ e(v, x) \geq_{lex} \varrho \circ e(v, z), z \in I(v)\}.$$

Remark 2.41. If the core of the game is non-empty then the nucleolus and the prenucleolus are the same, and both in the core.

Example 2.42 ((Solymosi et al, 2007)). Let $N=\{1, 2\}$ denote the set of players and game $v \in \mathcal{G}_c^N$ be essential. Then, the set of imputations of game v is

$$I(v) = \{(x_1, x_2) : x_1 + x_2 = v(\{1, 2\}), x_1 \geq v(\{1\}), x_2 \geq v(\{2\})\}$$

With these constraints we try to find the vector $x = (x_1, x_2)$ for which the minimal excess is maximal. Formally, because only the single-coalitions' excesses can change we try to find the vector x for which $\min\{v(\{1\}) - x_1, v(\{2\}) - x_2\}$ is maximal. It can be proved that $\min\{v(\{1\}) - x_1, v(\{2\}) - x_2\}$ is minimal if the excesses are equal and we get

$$x_i = \frac{v(\{1, 2\}) - v(\{1\}) - v(\{2\})}{2} + v(\{i\}) \quad \text{for } i = 1, 2.$$

Remark 2.43. In the case of two-player essential games, the nucleolus coincides with the Shapley value.

Chapter 3

Risk Allocation Games with Liquidity Requirements

In risk management it is not an easy question to answer how an appropriate measure of risk should be chosen. Risk typically derives from financial institutions' uncertain operational environment and this is the reason why we would like to measure the uncertainty of the environment. In this chapter, before describing our model, we define what risk and coherent measures of risk introduced by Artzner et al (1999) mean. We define risk allocation games following Denault (2001) and Csóka et al (2009), we determine liquidity risk and liquidity requirements (Acerbi and Scandolo, 2010) and finally we introduce our model by modifying Csóka and Herings (2013)'s risk allocation games with liquidity considerations.

3.1 Coherent Measures of Risk

Let us consider a financial unit with N subunits and collect the random vectors describing the portfolios of the subunits in matrix $X = [X_1, \dots, X_n]$. With the previous notations, row i of matrix X is called the realization vector of subunit i and $\mathbb{R}^S$ represents the set of realization vectors, where S refers to the finite, non-empty set of possible states of nature. State s occurs with probability $\pi_s > 0$ and $\sum_{s=1}^S \pi_s = 1$. Let δ denote the discount factor. Let us define $R_+ = [0, \infty)$ and $R_{++} = (0, \infty)$ respectively.

After reviewing the notations, let us move on to the examination of coherent measures of risk.

Definition 3.1. *The function $\rho : \mathbb{R}^S \rightarrow \mathbb{R}$ is a measure of risk.*

Function ρ determines the minimal amount of cash that needs to be added to the financial subunit's portfolio (or to invest in the reference instrument), such that the financial subunit's risk becomes acceptable to the regulator. We also assume that the reference instrument has the same payoff in each state of nature, which is 1 so its payoff vector is $1^S = (1, \dots, 1)^\top$. The reference instrument is riskless, which means that there is no uncertainty in its payoffs (i.e. zero coupon bond).

Consider two examples for measure of risk.

Definition 3.2. *If $\delta = 1$ the expected shortfall (ES) is defined as*

$$ES_\alpha = -\frac{1}{\alpha} \int_0^\alpha \overleftarrow{F}(p) \, dp, \quad (3.1)$$

where $\overleftarrow{F}(p) = \inf\{x | F(x) \geq p\}$ stands for the generalized inverse function..

Definition 3.3. *Let Value at Risk (VaR) be defined as*

$$VaR_\alpha = -\inf\{x : P(X \leq x) > \alpha\}. \quad (3.2)$$

At the significance level $1-\alpha$, expected shortfall determines the average loss in $\alpha\%$ of the worst cases, while value at risk is the best outcome among them.

Definition 3.4. *A function $\rho : \mathbb{R}^S \rightarrow \mathbb{R}$ is a coherent measure of risk (Artzner et al, 1999) if it satisfies the following axioms*

- *monotonicity: for all $X_1, X_2 \in \mathbb{R}^S$ such that $X_2 \geq X_1$, we have $\rho(X_2) \leq \rho(X_1)$,*
- *subadditivity: for all $X_1, X_2 \in \mathbb{R}^S$, we have $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$,*
- *positive homogeneity: for all $X \in \mathbb{R}^S$ and $h \in \mathbb{R}_{++}$, we have $\rho(hX) = h\rho(X)$,*
- *translation invariance: for all $X \in \mathbb{R}^S$ and $a \in \mathbb{R}$, we have $\rho(X + a1^S) = \rho(X) - \delta a$.*

Monotonicity implies that if the value of a portfolio in each state of nature is greater than or equal to the value of another one, then its risk should be less than or equal to the other's.

According to subadditivity, the merge of two portfolios cannot result in greater risk than the sum of the individual risks of the portfolios. If subadditivity is violated, there can be such situations in which it would be useful to divide the portfolio into sub-portfolios and thus the sum of the sub-portfolios' risks would result in smaller

value than the risk of the entire portfolio. Note that if subadditivity is not met, the portfolio diversification effect is violated.

Positive homogeneity requires that value of the portfolio should be directly proportional to the risk of the portfolio.

Translation invariance states that if we add a amount of the reference instrument to the portfolio, its risk should decrease by δa .

At this point we note that Expected Shortfall is a coherent measure of risk but Value at Risk fails to satisfy subadditivity.

3.2 Liquidity Risk and Coherent Measure of Risk

As far as the concept of liquidity is considered, we cannot find a universally accepted definition of liquidity. An asset is considered liquid if we can trade it in arbitrarily large amounts in a short period of time without affecting its market price. Thus, liquidity risk derives from the fact that illiquid assets cannot be immediately put into money and they cannot be traded in arbitrary large quantities without changing the market price. This is the reason why liquidity risk appears when an investor does not meet the liquidity requirement (Acerbi and Scandolo, 2010) and has to liquidate parts or the whole of his positions.

In the second part of the thesis, we consider liquidity risk in the case of financial institutions' retail mortgage portfolios. We focus on shocks which force banks to sell their real estate collaterals. In this way, the increasing real estate supply causes significant changes in the market price of real estates.

After reviewing the necessary notions, let us consider a financial unit with N subunits. Each subunit's portfolio consists of cash and investments in different illiquid assets belonging to the set J of assets. Let $\theta^i = (\theta_0^i, \theta_j^i)$ denote the portfolio of subunit i which shows the amount of cash and assets held by subunit i . Let $P = \mathbb{R} \times \mathbb{R}^J$ represent the space of portfolios held by the subunits. Let $p \oplus a$ denote the cash added to the portfolio which means that for a $p' \in P$, $p \oplus a$ results that $p'_0 = p_0 + a$ and $p'_j = p_j$, $j \in J$.

Definition 3.5. *The marginal supply-demand curve (MSDC) for asset $j \in J$ in state $s \in S$ is defined by the map $m_j^s : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ satisfying the followings*

- $m_j^s(x) \geq m_j^s(x')$ if $x < x'$;
- m_j^s is cadlag if $x < 0$ and ladcag if $x > 0$.

Remark 3.6. In Definition 3.5 *cadlag* (from the French term: *continue a droite, limite a gauche*) means right continuous with left limits and *ladcag* (*limite a droite, continue a gauche*) is left continuous with right limits.

Remark 3.7. In the case of marginal supply-demand curves, limit $m(0^-)$ and limit $m(0^+)$ are called best ask and best bid prices respectively.

Remark 3.8. The marginal supply-demand curve of θ_0 is the constant 1 function.

Definition 3.9. *The liquidation mark-to-market value of the portfolio $p \in P$ in state $s \in S$ is defined by*

$$l^s(p) = p_0 + \sum_{j \in J} \int_0^{p_j} m_j^s(x) \, dx . \quad (3.3)$$

In Definition 3.9, l is the income which comes from the liquidation of the portfolio. This value of the portfolio can be interpreted as an extremely prudent mark-to-market policy for portfolio p . We can also determine the completely opposite case as well:

Definition 3.10. *The uppermost mark-to-market value of a portfolio $p \in P$ in state $s \in S$ is given by*

$$u^s(p) = p_0 + \sum_{j \in J^+(p)} m_j^s(0^+) p_j + \sum_{j \in J^-(p)} m_j^s(0^-) p_j . \quad (3.4)$$

The uppermost mark-to-market value of the portfolio can be considered as the value of the portfolio in the long run. In the valuation of long positions we use the best bid prices while short positions are evaluated applying the best ask prices.

Definition 3.11. *The liquidation cost of a portfolio, $p \in P$ is defined as*

$$c^s(p) = u^s(p) - l^s(p) \in \mathbb{R}_+ . \quad (3.5)$$

Example 3.12. Consider a portfolio which consists of cash and a risky asset, $x = (3, 4)$. Let the marginal supply-demand curve be

$$m(x) = \begin{cases} 3, & \text{if } x \leq 0 \\ 2, & \text{if } 0 < x \leq 1 \\ 1, & \text{if } x > 1 \end{cases}$$

With these conditions the liquidation mark-to-market value of the portfolio is

$$l(x) = 3 + 2 \cdot 1 + 1 \cdot 3 = 8,$$

the uppermost-value is

$$u(x) = 3 + 2 \cdot 4 = 15,$$

and the liquidation cost equals to

$$c(x) = 15 - 8 = 7.$$

In a given market situation with the liquidation of parts or the whole of the portfolio, investors cannot attain an arbitrary portfolio. In other words, investors' initial portfolio determines the subset of portfolios which is available for them. This is said formally in the following definition.

Definition 3.13. *The set of portfolios attainable from some given portfolios $p \in P$ in state $s \in S$ by liquidating all or part of it is given by*

$$A^s(p) = \{q \in P : q_0 = l^s(p_0, p_J - q_J)\} . \quad (3.6)$$

Let $\mathcal{L}^s$ denote the liquidity requirement (Acerbi and Scandolo, 2010) in state $s \in S$. Liquidity requirement can be interpreted as regulations determined by the regulator to which the financial system as a whole and all the subunits must meet.

Assumption 3.14. *For every $s \in S$ it holds that*

- $\mathcal{L}^s$ is closed and convex,
- for all $p \in \mathcal{L}^s$ and $a > 0$ we have $p \oplus a \in \mathcal{L}^s$,
- for all $p \in \mathcal{L}^s$, we have $(p_0, 0) \in \mathcal{L}^s$,
- $A^s(\theta(N)) \cap \mathcal{L}^s \neq \emptyset$.

The first two points of Assumption 3.14 are inherited from Acerbi and Scandolo (2010). The second and third point of Assumption 3.14 state that if the investors add an arbitrary amount of cash to the portfolio or liquidate an arbitrary amount from the illiquid assets, the portfolio continues meeting the liquidity requirement.

3.3 Risk Allocation Games

In game theory, risk allocation problems were introduced by Denault (2001). In Denault (2001)'s article, risk allocation games are represented as cost games, and the games are supposed to be monotone and subadditive. In the definition of risk allocation games, we basically use the setup of Cs3ka et al (2009).

Applying the previously defined notations, for all coalitions $C \subseteq N$, risk allocation games can be defined as follows

$$X_s(C) = \sum_{i \in C} X_{s,i} ,$$

where $C \in \mathcal{P}(N)$.

Definition 3.15. *A risk environment is a tuple (N, S, π, ρ) , where N is the set of subunits, S refers to the number of states of nature, $\pi = (\pi_1, \dots, \pi_S)$ is the probability measure and ρ is a coherent measure of risk.*

Definition 3.16. *In a given risk environment (N, S, π, ρ) , a risk allocation game is a game $v \in \mathcal{G}_c^N$, where the value function $v : \mathcal{P}(N) \rightarrow \mathbb{R}$ is defined as*

$$v(C) = \rho(X(C)), \quad \text{for all } C \in \mathcal{P}(N) \setminus \{\emptyset\}$$

and $v(\emptyset) = 0$.

According to Denault (2001), the class of risk capital allocation games is balanced and as a subgame of risk allocation game is a risk allocation game as well, Csóka et al (2009) showed that the family of risk allocation games is totally balanced.

3.4 Risk Allocation Games with Liquidity Requirements

In the case of risk allocation games introduced by Denault (2001), no assumptions were made whether subunits' portfolios should meet supervisory requirements or not. The class of risk allocation games was extended with liquidity considerations by Csóka and Herings (2013). In the thesis we represent risk allocation games with liquidity requirement but we modify Csóka and Herings (2013)'s model.

Definition 3.17. *A risk environment with liquidity requirement is defined as a tuple $(N, S, \pi, m, \mathcal{L}, \rho)$, where N is the set of subunits, S refers to the number of states of nature, $\pi = (\pi_1, \dots, \pi_S)$ is the probability measure, $\mathcal{L}$ is the vector of liquidity requirements in different states of nature, ρ is a coherent measure of risk and m defines the matrix of marginal supply-demand curves.*

Definition 3.18. *Let $(N, S, \pi, m, \mathcal{L}, \rho)$ define a risk environment with liquidity requirement and $C \subset N$ refer to a coalition of subunits. The realization vector of*

coalition C is defined by

$$X^s(C) = u^s(q(C)) \rightarrow \sup \quad (3.7)$$

$$s.t. \quad q(C) \in A^s(\theta(C)) \cap \mathcal{L}^s, \quad s \in S.$$

For the grand coalition we can apply the previous definition as well.

Definition 3.19. *The realization vector $X(N) \in \mathbb{R}^S$ for the firm is defined*

$$X^s(N) = u^s(q(N)) \rightarrow \sup \quad (3.8)$$

$$s.t. \quad q(N) \in A^s(\theta(N)) \cap \mathcal{L}^s, \quad s \in S.$$

If for all $C \subseteq N$ the supremum can be reached, then it can be changed to maximum and let $q^s(C)$ denote one of the optimal solutions of (3.8) as

$$X^s(C) = u^s(q^s(C)) ; \quad (3.9)$$

and let $t^s(C)$ denote the corresponding trading vector of the coalition C satisfying

$$q^s(C) = \theta(C) + t^s(C). \quad (3.10)$$

The optimal trading vector shows how coalition C should modify its portfolio to meet the liquidity requirement.

Remark 3.20. We note that the calculation of the trading vector defined by Equation (3.10) is different from the trading vector applied by Csóka and Herings (2013). Csóka and Herings (2013) focus on how coalition C should modify its portfolio in such a way that the resulting portfolio of the main financial unit (grand coalition) is attainable and meets the liquidity requirement.

Definition 3.21. *Let $(N, S, \pi, m, \mathcal{L}, \rho)$ be a risk environment with liquidity requirement. A risk allocation game with liquidity requirement is a game $v \in \mathcal{G}_c^N$, where the value function $v : \mathcal{P}(N) \rightarrow \mathbb{R}$ is defined as*

$$v(C) = \rho(u(q(C))), \quad \text{for all } C \in \mathcal{P}(N) \setminus \{\emptyset\}$$

and $v(\emptyset) = 0$.

Example 3.22. Consider a two player game in which only one possible state of nature exists and $\mathcal{L} = \mathbb{R}$ represents the liquidity requirement. In this example, $\mathcal{L} = \mathbb{R}$ means that there are not liquidity regulations for banks. The matrix of realization vectors is defined in Table 3.1.

It is obvious that this game is additive.

State/ $X(C)$	$X(\{1\})$	$X(\{2\})$	$X(\{1, 2\})$
s_1	-100	-50	-150
$\rho(X)$	100	50	150
$v(C)$	100	50	150

Table 3.1: Additive game

Example 3.23. Consider a two player game where the set of states of nature is $S = \{s_1, s_2\}$ and $\mathcal{L} = \mathbb{R}$ represents the liquidity requirement. The matrix of realization vectors is defined in Table 3.2.

State/ $X(C)$	$X(\{1\})$	$X(\{2\})$	$X(\{1, 2\})$
s_1	-100	-20	-120
s_2	-5	-80	-85
$\rho(X)$	100	80	120
$v(C)$	100	80	120

Table 3.2: Subdditive game

Since $v(X(\{1\})) + v(X(\{2\})) > v(X(\{1, 2\}))$ is met, this game is subadditive.

Example 3.24. For the sake of simplicity, let us suppose, that on the market only two institutions operate and each of them has invested into asset $J = 1$ and cash. Their portfolios are $\theta^1 = (20, -11)$ and $\theta^2 = (26, -7)$ and both of them have short positions in the risky asset. Let $S = \{s_1, s_2\}$ define the set of states of nature with equal probability of occurrence in each state. The marginal supply-demand curves of the risky asset in the states are defined as

$$m_1^1(x) = \begin{cases} 5, & \text{if } x \leq 0 \\ 4, & \text{if } x > 0 \end{cases}$$

$$m_1^2(x) = \begin{cases} 10, & \text{if } x < -2 \\ 9, & \text{if } -2 \leq x < -1 \\ 8, & \text{if } -1 \leq x < 0 \\ 7, & \text{if } 0 \leq x < 1 \\ 6, & \text{if } x \geq 1 \end{cases}$$

Let $\mathcal{L}^{s_1} = \mathcal{L}^{s_2} = \{p \in P : p_1 \geq -10\}$ define the liquidity requirement which means that the amount of risky asset held by the institutions cannot exceed 10

positions either for an individual level or for the whole system (the grand coalition). Let the measure of risk be the maximum loss. With these conditions, it is obvious that institution 1 does not meet the liquidity requirement and the whole system fails as well because they have $11 + 7 = 18$ short positions from the risky asset.

In State s_1 the institutions can buy risky asset at price 5 and sell that at price 4. According to the liquidity requirement, it is not allowed to have more than 10 short positions in the risky asset. This is the reason why 1 unit should be bought by institution 1 and 8 units by the system. In this case, in State s_1 , for institution 1, we determine the following results. For trading vector we get $t_1(\{1\}) = (-5, 1)$ and for portfolio $q(\{1\}) = (15, -10)$. The uppermost value of portfolio $q(\{1\})$ is $u^s(q(\{1\})) = 15 - 10 \cdot 5 = -35$ so the realization vector is $X(\{1\}) = -35$. It can be easily checked that we get the same result if institution 1 buy between 1 and 11 units of the risky asset. If it buys more than 11 units, that causes losses, because long positions are valued at price 4.

For institution 2, we get trading vector $t_1(\{2\}) = (0, 0)$ because it meets the liquidity requirement and the uppermost value of its portfolio $q(\{2\})$ is $u^s(q(\{2\})) = 26 - 7 \cdot 5 = -9$. For realization vector, we calculate $X(\{2\}) = -9$. In the case of the whole system, the following values are calculated: $t_1(\{1, 2\}) = (-40, 8)$, $q(\{1, 2\}) = (6, -10)$, $u^s(q(\{1, 2\})) = 6 - 10 \cdot 5 = -44$.

In State s_2 , institution 1 has to buy again 1 unit and the system 8 units similarly to the previous case. The difference can be found in the trading vectors: $t_2(\{1\}) = (-8, 1)$, $t_2(\{2\}) = (0, 0)$ and $t_2(\{1, 2\}) = (-77, 8)$ Finally, the results are the followings : $q(\{1\}) = (12, -10)$, $u^s(q(\{1\})) = 12 - 8 \cdot 10 = -68$, $X(\{1\}) = -68$, $q(\{2\}) = (26, -7)$, $u^s(q(\{2\})) = 26 - 8 \cdot 7 = -30$, $X(\{2\}) = -30$ and $t_2(\{1, 2\}) = (-77, 8)$, $q(\{1, 2\}) = (-31, -10)$, $u^s(q(\{1, 2\})) = -31 - 8 \cdot 10 = -111$, $X(\{1, 2\}) = -111$. In State 2. owing to the increasing prices, institution 1 and the system optimally buy 1 and 8 units from the risky asset. The summarized results can be found in Table 3.3:

State/ $X(C)$	$X(\{1\})$	$X(\{2\})$	$X(\{1, 2\})$
s_1	-35	-9	-44
s_2	-68	-30	-111
$\rho(X)$	68	30	111
$v(C)$	68	30	111

Table 3.3: Superdditive game

This game also illustrates that the core of risk allocation games with liquidity requirement can be empty. If we check the core constraints, we get

$$\begin{aligned} 68 - x_1 &\geq 0 \\ 30 - x_2 &\geq 0 \\ 111 - (x_1 + x_2) &= 0 \end{aligned}$$

and if we rearrange them, its result will be that $68 \geq x_1$, $30 \geq x_2$ and $111 = x_1 + x_2$ and there is no vector x which can satisfy these condition, so the core of the game is empty. Theorem 2.28 suggests that since the core is empty, it implies that the game is not balanced either.

Remark 3.25. At this point we note that in case of risk allocation games with liquidity requirement if we consider realization vectors X_1, X_2 then $\rho(X(\{1\}) + X(\{2\})) = \rho(X(\{1, 2\}))$ is not always met. For instance in Example 3.24 we have the results summarized in Table 3.4.

State/ $X(C)$	$X(\{1\})$	$X(\{2\})$	$X(\{1, 2\})$	$X(\{1\}) + X(\{2\})$
s_1	-35	-9	-44	-44
s_2	-68	-30	-111	-98
$\rho(X)$	68	30	111	98

Table 3.4: Superdditive game 2

In Table 3.4, we can see that the maximum loss risk measure is subadditive but the game is superadditive. This phenomenon is the result of the fact that in this model, coalitions' portfolios are not simply the sums of the institutions portfolios.

Example 3.26. Consider a game with the same marginal supply-demand curve and liquidity requirement as in Example 3.24 but with different portfolios: $\theta_1 = (20, -1)$, $\theta_2 = (1, -8)$. In this case, we summarize the results in Table 3.5.

State/ $X(C)$	$X(\{1\})$	$X(\{2\})$	$X(\{1, 2\})$
s_1	15	-39	-24
s_2	12	-63	-51
$\rho(X)$	-12	63	51
$v(C)$	-12	63	51

Table 3.5: Non-monotone game

The results show that this game is not monotone.

If we consider the previously described examples, we can state that our risk allocation games with liquidity considerations define a broad class of games. We showed additive, subadditive and superadditive games as well. At this point we note that in this context, superadditive games need further explanation.

Superadditive risk allocation games with liquidity requirement can be interpreted as situations in which banks only consider their individual risk and ignore the behavior of the rest of the financial system. This is a so-called market failure when banks miss to incorporate into their risk management a significant source of risk originating from the behavior of the others. If we consider Example 3.24 in details, we can note that financial institutions determine whether they individually meet the liquidity requirement and they ignore the system's risk. This remark proves to be important if we consider the development of the current financial regulations.

During the financial crises of 2008, it turned out that Basel II regulation was not sufficient to maintain the stability of the financial system and the Basel Committee introduced its extension known as Basel III in which systemic and liquidity risk appear and it confirms that our intuition is relevant.

Chapter 4

Risk Capital Allocation Methods

Risk management and risk capital allocation are essential for banks and companies exposed to risk and whose portfolio value is subject to uncertainty. In this chapter, our aim is to collect and represent the most important risk capital allocation methods. The description of the allocation methods is based on Balog et al (2011)'s article who have examined seven allocation methods. We present their analytical and simulation results as well. The allocation methods described in this chapter include Activity based method (Hamlen et al, 1977), Beta method (see for instance Homburg and Scherpereel (2008)), Incremental method (see for instance Jorion (2007)), Cost gap method (Driessen and Tijs, 1986), Euler method (Tasche, 2008), Shapley method (Shapley, 1953) and Nucleolus method (Schmeidler, 1969).

Definition 4.1. Let $X_N^\rho = \{N, \{X_i\}_{i \in N}, \rho\}$ denote the risk capital allocation situation (hereafter RCAS) and $RCAS_N$ the class of risk capital allocation situations with the player set N .

Definition 4.1 is closely related to the definition of the risk environment (see Definition 3.15) but we do not mention explicitly the states of nature S .

Definition 4.2. Function $\Psi : RCAS_N \rightarrow \mathbb{R}^N$ is called risk capital allocation method.

The application of risk capital allocation methods allows risk management to determine for each subunit, in each risk capital allocation situation the necessary amount of capital to hold for risk coverage. First, before describing the above-mentioned risk capital allocation methods, we examine what kind of properties should be satisfied to get a fair allocation.

Definition 4.3. The risk capital allocation method Ψ satisfies Equal Treatment Property (ETP), if for all risk capital allocation situations $X_N^\rho \in RCAS_N$ and

subunits $i, j \in N$ such that for any subunit coalition $S \subseteq N \setminus \{i, j\}$, $\rho(X_{S \cup \{i\}}) - \rho(X_S) = \rho(X_{S \cup \{j\}}) - \rho(X_S)$, we have $\Psi_i(X_N^\rho) = \Psi_j(X_N^\rho)$.

Equal Treatment Property ensures that subunits with the same risk characteristics have to be treated in the same way. This means that two subunits with the same risk contributions to each coalition not containing them, should be allocated the very same risk capital by the risk capital allocation method.

Definition 4.4. *The risk capital allocation method Ψ satisfies Strong Monotonicity (SM), if for all risk capital allocation situations $X_N^\rho, Y_N^\eta \in RCAS_N$ and subunit $i \in N$, $S \subseteq N$ such that $\rho(X_{S \cup \{i\}}) - \rho(X_S) \geq \eta(Y_{S \cup \{i\}}) - \eta(Y_S)$, we have $\Psi_i(X_N^\rho) \geq \Psi_i(Y_N^\eta)$.*

Strong Monotonicity requires that if a subunit's contribution of risk to any coalition is not decreasing, the risk allocated to this subunit should not decrease either. If a risk capital allocation method satisfies Strong Monotonicity, it is not worth for any coalition releasing higher risk parameters than the actual one in the hope of less capital requirement.

Definition 4.5. *The risk capital allocation method Ψ satisfies Core Compatibility (CC), if for all risk capital allocation situations $X_N^\rho \in RCAS_N$, $\rho(X_N) = \sum_{i \in N} \Psi_i(X_N^\rho)$ meets and $\rho(X_S) \geq \sum_{i \in N} \Psi_i(X_N^\rho)$ for all coalitions $S \subset N$.*

According to the property of Core Compatibility, the entire risk capital of the financial unit should be allocated among the financial subunits such that for each coalition, the sum of the members' risk capital determined by the risk capital allocation method is less than or equal to the risk of the particular coalition. If Core Compatibility is met, the risk capital allocation is accepted by all of the subunits and none of them want to object to it and leave it.

Csóka and Pintér (2010) proved that in case of general coherent measure of risk we cannot find any allocation method which is ETP, SM and CC. This is stated in the following theorem.

Theorem 4.6 (Csóka and Pintér (2010)). *Considering a general coherent measure of risk, there is no risk capital allocation method meeting the conditions of Core Compatibility, Strong Monotonicity and Equal Treatment Property.*

4.1 Allocation Methods

4.1.1 Activity Based Method

The Activity based method introduced by Hamlen et al (1977) allocates the financial unit's joint risk in proportion to the subunits individual and independent risk. For all risk capital allocation situations $X_N^\rho \in \text{RCAS}_N$ and subunit $i \in N$, the risk capital allocated by the Activity based method can be determined as follows

$$\Psi_i(X_N^\rho) = \frac{\rho(X_i)}{\sum_{j \in N} \rho(X_j)} \rho(X_N).$$

Remark 4.7. The major disadvantage of the Activity based method is that it ignores the dependence structure among the subunits and does not allocate smaller risk for those who negatively correlate with the other subunits.

4.1.2 Beta Method

Consider a $X_N^\rho \in \text{RCAS}_N$ risk capital allocation situation and let $\text{Cov}(X_i, X_N)$ denote the covariance of the realization vectors describing the financial position of subunit $i \in N$ and the main financial unit. β_i for subunit $i \in N$ is calculated as follows

$$\beta_i = \frac{\text{Cov}(X_i, X_N)}{\text{Cov}(X_N, X_N)}.$$

Beta method (see for instance Homburg and Scherpereel (2008)) allocates the main financial unit's risk among subunits in proportion to their β -s. For all risk capital allocation situations $X_N^\rho \in \text{RCAS}_N$ and subunit $i \in N$, the risk capital allocated by the Beta method can be determined as

$$\Psi_i(X_N^\rho) = \frac{\beta_i}{\sum_{j \in N} \beta_j} \rho(X_N).$$

Remark 4.8. Beta method cannot be applied in every case, if for all $i \in N$ subunits $\beta_i = 0$, this method does not work.

4.1.3 Incremental Method

The risk capital allocated by the Incremental method (see for instance Jorion (2007)) is determined in proportion to the subunits' individual risk contribution

(risk increment) to the main financial unit's risk. For all risk capital allocation situations $X_N^\rho \in \text{RCAS}_N$ and for subunit $i \in N$, the risk capital allocated by the Incremental method is

$$\Psi_i(X_N^\rho) = \frac{\rho(X_N) - \rho(X_{N \setminus \{i\}})}{\sum_{j \in N} \rho(X_N) - \rho(X_{N \setminus \{j\}})} \rho(X_N).$$

Remark 4.9. Incremental method cannot be applied in every case, for instance if for each subunit $i \in N$, $\rho(X_N) - \rho(X_{N \setminus \{i\}}) = 0$, the method does not work.

4.1.4 Cost Gap Method

To define Cost gap method (Driessen and Tijs, 1986), first, let γ_i denote the minimal cost gap of subunit $i \in N$ and be defined as

$$\gamma_i = \min_{\emptyset \neq S \subseteq N, i \in S} \left| \rho(X_S) - \sum_{j \in S} \rho(X_N) - \rho(X_{N \setminus \{j\}}) \right|.$$

Thus, for all risk capital allocation situations $X_N^\rho \in \text{RCAS}_N$ and subunit $i \in N$, the risk capital allocated by the Cost gap method is defined as follows

if $\sum_{j \in N} \gamma_j = 0$

$$\Psi_i(X_N^\rho) = \rho(X_N) - \rho(X_{N \setminus \{i\}}),$$

otherwise

$$\Psi_i(X_N^\rho) = \rho(X_N) - \rho(X_{N \setminus \{i\}}) + \frac{\gamma_i}{\sum_{j \in N} \gamma_j} \left(\rho(X_N) - \sum_{j \in S} (\rho(X_N) - \rho(X_{N \setminus \{j\}})) \right).$$

4.1.5 Euler Method

For all risk capital allocation situations $X_N^\rho \in \text{RCAS}_N$ and subunit $i \in N$, the risk capital allocated by the Euler method (Tasche, 2008) is defined as follows

$$\Psi_i(X_N^\rho) = \rho'(X_N, X_i),$$

where $\rho'(X_N, X_i)$ is the directional derivative of ρ at X_N along X_i .

4.1.6 Shapley Method

For all risk capital allocation situations $X_N^\rho \in \text{RCAS}_N$ and subunit $i \in N$, the risk capital allocated by the Shapley method (Shapley, 1953) equals to

$$\Psi_i(X_N^\rho) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N \setminus S| - 1)!}{|N|!} [\rho(X_{S \cup \{i\}}) - \rho(X_S)].$$

The interested reader can find further properties of Shapley value in Pintér (2011) and Pintér (2009) among others.

4.1.7 Nucleolus Method

For all risk capital allocation situations $X_N^\rho \in \text{RCAS}_N$ and subunit $i \in N$, the risk capital allocated by the Nucleolus method (Schmeidler, 1969) equals to

$$\mathcal{N}_i(X_N^\rho) = \{x \in I(X_N^\rho) : \varrho \circ e(v, x) \geq_{lex} \varrho \circ e(v, z), z \in I(X_N^\rho)\},$$

where I is the set of imputations, $e(\cdot, \cdot) : \mathcal{G}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^{2^N}$ refers to the excess functions, $\geq_{lex}$ is for lexicographic ordering and ϱ defines the non-decreasing function (see Definition (2.36)).

4.2 Analytical and Simulation Results

In this part of the chapter we summarize the results of Balog et al (2011). First, their analytical results are represented in Table 4.1 which show that only two allocation methods meet two of the three fairness properties, the Shapley value and the Nucleolus. Table 4.1 also confirms Theorem 4.6 which states that none of the allocation methods satisfy all the fairness requirements.

As far as the simulation results of Balog et al (2011) are considered, they modeled the returns of a financial unit by generating return time series using random correlation matrices. During the simulation, they used normal and Student t -distributions. The aim of their research was to determine how many percent of the examined cases resulted in allocations which are not in the core of the game. In the case of three players, the performance of the Shapley value was about 60% while in the case of the four player-simulation the number of Core Compatible allocations decreased, it was about 45%. This means that in the case of the Shapley value, CC is often violated.

Based on Balog et al (2011) results, we chose the Shapley method to allocate risk capital in the empirical analysis.

Method/Property	Equal Treatment Property	Strong Monotonicity	Core Compatibility
Activity Based Method	✓	—	—
Beta Method	-	—	—
Incremental Method	✓	—	—
Cost Gap Method	✓	—	—
Euler Method	—	—	—
Shapley Method	✓	✓	—
Nucleolus Method	✓	—	✓

Table 4.1: Properties of allocation methods, (Balog et al, 2011)

Chapter 5

Application

After the theoretic review, we represent a methodology to measure and allocate systemic risk considering a financial system. In the thesis, we define systemic risk as a special type of risk which cannot be observed at banks' individual level; however in the case of the entire system it causes significant problems. Our analysis is based on the intuition which says that banks during their calculations of capital reserves do not take into account the systemic risk, which derives from the fact that in certain market situations banks do not act independently from each other (joint sale).

To capture the basic sense of the intuition, let us consider an example which is going to be used in the further parts of the thesis. Let us focus on bank's retail mortgage portfolio and consider a financial system with N institutions which have to liquidate certain parts of their real estate portfolio due to any unfavorable events. Furthermore, let us suppose that banks own real estates which can be described with average usual parameters and let y_i be the real estate portfolio of bank i . Let m refer to the marginal supply-demand curve of real estates and let m_1 be the best bid, m_2 the second-best bid price and so on where if $k < l$ then $m_k > m_l$ is met. Obviously every bank wants to sell its real estate portfolio at the best bid price so first, real estates are sold at the best observed bid prices.

If banks' real estates for sale are more than the amount available at the best bid price, the second-best bid price follows and it continuous like this. Thus, when bank i determines the expected revenue after selling y_i amount of real estate, it calculates the m_k prices at which such a transaction can be done. Formally, bank i 's income after the sale equals to *ceteris paribus*

$$\Pi_i = \int_0^{y_i} m(x) \, dx.$$

The problem occurs when more banks want to sell their real estate collaterals at the same time. As real estate market is illiquid, the more the number of orders is, the more the market price of real estates decreases. Thus, banks cannot obtain the expected incomes that they calculated in advance.

To sum up, we can conclude that in the case of banks' joint sale, real estate prices can significantly decrease which results that banks' revenues become insufficient to cover their risks. Based on this example, it can be seen that capital reserves determined according to Basel II regulation, are not sufficient in every case to ensure the necessary capital for systemic risk coverage (market failure). With our model it would become possible to treat systemic risk and determine additional capital requirements for banks to hold if it is necessary. This idea completely corresponds to the concept of Basel III regulation framework.

The convex optimization, applied in the theoretic parts of the thesis, appears in the model when banks determine the proportion of their real estate portfolio to liquidate owing to any unfavorable events.

All in all, we can say that the aim of this thesis is not to release exact calculation about banks' necessary capital requirements but to draw up a framework which proves that the previously introduced problem (banks' market failure) is a relevant economic issue. This model allows to incorporate systemic risk deriving from banks' joint sale into risk management which can serve as guideline to the regulator to integrate this type of systemic risk in the current regulation. Moreover, we also recommend a possible game theoretic solution to allocate additional capital reserves among the members of the financial system in proportion to their risk contributions. Our Intuition is in line with the Basel III regulation.

5.1 Basel Regulations

In this section we summarize certain parts of the Basel regulations which are going to be used in the empirical analyses. First, we start with Basel I Accord based on Hull (2012) and BIS (1988).

5.1.1 Basel I

In 1988 the Basel Committee published the first formal standard for capital requirements for internationally active banks. The aim of the regulation was to harmonize the regulatory and capital adequacy standards in the member states of

the Committee. Basel I Accord can be divided into four parts which include the followings:

- The constituents of capital
- Risk weighting
- The target standard ratio
- Transitional and implementing agreements

The key element of Basel I regulation was the uniform definition of capital. The accord determined not only the types of capital but also the necessary amount that banks should hold in each type. The risk weighting part of the accord introduced a comprehensive system for banks to weight the risk of each asset category in their balance sheet. In the accord, five asset categories were determined with risk weights of 0%, 20%, 50% or 100%. The target standard ratio aimed to define a universal standard for banks' capital requirements. The ratio was set at 8% of the risk weighted assets and it should be covered by Tier 1 and Tier 2 Capital reserves. The transitional and implementing agreements ensured national banks' appropriate surveillance in the countries who were involved in the accord. The main insufficiency of this framework is that it considers only credit risk. Having been widely criticized, the convention was extended in 1996 and market risk was incorporated into the regulation as well.

5.1.2 Basel II

During the 1990s, it turned out that capital requirements determined by the regulator and economically necessary capital to cover risks differ from each other. This phenomenon derived from the fact that Basel I applied ad hoc risk weights which did not correspond to financial institutions' risk management and portfolio quality. We can state that in Basel I Accord the definition of capital proved to be more important than the amount of capital to cover risk.

In response to the financial crises in the 1990s and to the insufficiency of Basel I Accord, the Basel Committee decided to supervise the current regulations and propose a new, comprehensive capital adequacy framework which is known as Basel II Accord. The new regulation maintains the rules laid in Basel I but extends

them, integrating operational risk and incorporating market-based surveillance and regulation.

Based on Mérő, K (2002), BIS (2001), BIS (2005) and BIS (2011) we describe the pillars of Basel II regulation which include the followings:

- Minimum capital requirements
- Surveillance and banking supervision
- Market discipline

Pillar 1

In the case of minimum capital requirements, Basel II regulation proposes several possible methodologies for financial institutions to determine the necessary capital reserves to cover credit, market and operational risk. According to the regulation, banks have the right to choose between several possible methodologies to measure their risk.

In the case of *credit risk*, banks can choose from three different methodologies to rate the risk of the bank's assets. First, we describe the *standardized approach (STA)* which extends the risk weighted methodology introduced in Basel I regulation. This methodology is mainly based on the risk weights assigned by external rating institutions (S&P, Moody's, Fitch). Banks can apply this methodology without any supervisory approval. The problem of this approach is that it ignores the institution specific risks. This methodology is recommended for financial institutions with less developed risk management practices.

Another possibility for financial institutions to rate their risks is the *internal rating based (IRB)* or *advanced internal rating based (IRB A)* approach. In the basic IRB approach, the risk weight function, the parameters of the risk distribution, except the probability of default (PD), are determined by the regulator. In the case of PD, banks can apply internal rating models in the estimation. Basel II framework allows banks with developed risk management practices to use the *advanced internal rating based* approach, in which only risk weight function is given by the regulator but probability of default (PD), exposure at default (EAD), loss given default (LGD) and maturity (M) are estimated with internal models. More parameters are estimated by the bank itself, more precisely risks can be determined and the effectiveness of risk measurements far exceeds the previous regulatory estimations.

The risk function, defined in the internal rating based approach (see for instance in Hull (2012) or BIS (2005)), is based on a one-factor Gaussian copula model in which the worst case default rate (WCDR, the rate which is 99.9% certain not to be exceeded) is computed as follows

$$WCDR = \Phi \left[\frac{1}{\sqrt{1-R}} \cdot \Phi^{-1}(PD) + \sqrt{\frac{R}{1-R}} \cdot \Phi^{-1}(0.999) \right],$$

where Φ is the distribution function of the Gaussian distribution and R refers to asset correlation. In this context, the value of risk weighted assets can be determined as follows

$$RWA = 1.06 \cdot 12.5 \cdot EAD \cdot LGD \cdot (WCDR - PD) \cdot MA, \quad (5.1)$$

where MA indicates the maturity adjustment. Based on Equation (5.1) in the internal rating based approach the capital requirement equals to

$$C = 0.08 \cdot RWA.$$

As it was mentioned above, in Basel II regulation framework *operational risk* is incorporated as well. For the banking system, three alternatives are proposed to choose from to determine reserves for operational risk. The first approach is the *basic indicator approach (BIA)* which determines to reserve the α percent of the gross income of the institution. In BIA α is equal to 15%.

The second alternative is the *standardized approach (STA)* which determines 8 business lines. For each business line a β risk weight is determined and the sum of the business line's capital requirements assigns the institutions total reserve for operational risk:

$$C_{STA} = \sum_{i=1}^8 \beta_i GI_i.$$

The eight business lines include corporate finance, trade and sales, retail banking, commercial banking, payment and settlement, agency services, asset management and retail brokerage.

The *alternative standardized approach (ASTA)* follows the logic of the standardized method in each business line except retail and commercial banking. The reason why this approach treats them differently is that for credit risk, reserves have already been determined in these fields. According to the ASTA, gross income for retail and commercial banking is determined as follows:

$$GI = mLA$$

where LA represents the amount of loans and advances and m is equal to 0.035.

The real operational risk management can be realized under the *advanced measurement approach (AMA)* which aims to map the institutions' operational risk using internal models. This approach is based on four key elements: internal and external data, scenario analysis and environmental factors. The models are estimated at the significance level of 99.9% for one year horizon and consider the extreme values at the edge of the distribution. In this approach, risk mitigation techniques can be applied for 20% of the total operational risk reserves. To introduce the AMA, institutions should meet strict regulatory requirements and acquire the regulatory approval in advance.

As it was mentioned above, *market risk* was incorporated into Basel I regulation in 1996. This modification of the framework allows financial institutions to cover their losses, deriving from market risk, from capital reserves. In the case of market risk, Basel II regulation did not bring notable changes, the approach is still based on the value at risk methodology with mark-to-market valuation (*standardized (STA)* or *internal model approach (IMA)*).

We summarize the structure of Pillar 1 on Figure 5.1.

Pillar 2

The supervision part of Basel II involves not only the compliance of the capital regulations but also the insurance of an appropriate risk control and management system. The role of the authorities is to control the financial institutions' appropriate capital allocation and to operate a supervisory system which can reveal the system's deficiencies and can intervene in an effective way if it is necessary. The supervision consists of two elements, the *Internal Capital Adequacy Assessment Process (ICAAP)* and *Supervisory Review and Evaluation Process (SREP)*. ICAAP aims to control the adequate capital to cover risk. This process is based on the institutions' own calculation in which banks determine the necessary capital to reserve, according to the risks they are exposed to. SREP includes the supervisory review of capital positions and the evaluation process of the institution's exposure to risks.

Pillar 3

The third pillar of Basel II regulation considers banks' disclosure requirements which include risk management objectives, strategies and policies for each type of risk. These disclosures allow market participants to collect the necessary information about risk exposures, risk assessment processes and capital allocation of the institutions. In this way clients can track the performance of the financial institutions more easily.

5.1.3 Basel III

Basel III regulation framework BIS (2011) can be considered as a response to the global economic crises of 2007-2008. The framework was officially accepted by the G20 countries in 2010 and it is strongly based on the previous Basel I and II regulations to strengthen global capital and liquidity standards with the aim of promoting a more resilient banking sector. The objective of Basel III regulation is to build up a framework which can improve the banking sector's ability to absorb shocks deriving from financial and economic stress. The Committee proposed micro-prudential and macro-prudential regulations as well which aim the risk management and the governments to strengthen banks' transparency and disclosure. The most important measures include the following fields:

- Capital requirements
- Counterparty credit risk
- Leverage ratio
- Global liquidity standards
- Systemically important financial institutions
- Systemic risk management

In this section we consider in details only certain parts of Basel III which are going to be important in our empirical analysis. With the new regulation, for the first time in Basel regulations' history the Committee aims to build up a strong, *uniform liquidity base* for the financial system. For this purpose, the Committee has introduced two minimum standards for funding liquidity (BIS, 2011):

- *Liquidity coverage ratio (LCR)* aims to improve short term liquidity resilience and compares the stock of high quality liquid assets held by a bank to its net cash outflows during a hypothetical 30-day stress scenario. The liquidity coverage ratio is set at a minimum of 100% and requires that high quality liquid assets completely cover the net cash outflows in the given scenario.
- *Net stable funding ratio (NSFR)* considers liquidity over a longer term horizon and motivate banks to finance their activities with stable and liquid sources of funding. The ratio compares the longer term stable fundings held a bank to the stable fundings that it requires over a one year period of extended stress. The minimum of NSFR is set at 100%.

The other major innovation in Basel III framework is the regulation of systemic risk. Higher capital requirements can improve the stability of the financial system but at the same time their interconnectedness increases the system's risk. This is the reason why the so-called *systemically important financial institutions (SIFIs)* could be subject to additional requirements reflecting the greater risks they expose the financial system to. This means that the failure of the systemically important financial institutions could cause further serious effects on the whole financial system's stability. The considered fields in the case of SIFIs include capital surcharges and the introduction of contingent capital or bail-in debt.

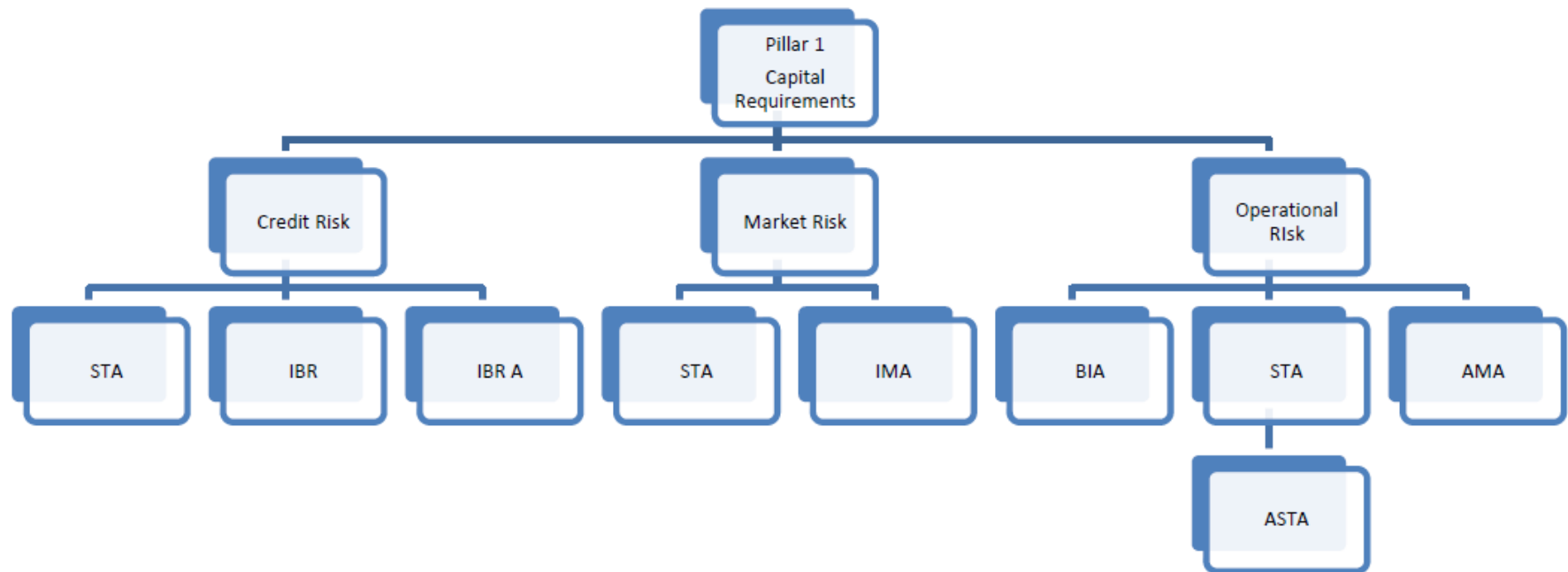


Figure 5.1: Pillar 1

5.2 Assumptions and The Data Used in The Analysis

In the determination of banks' characteristics, we worked with publicly available data of the Hungarian banking system's seven major banks. Banks' real estate portfolios and the distribution of real estates were calculated based on the Financial Stability Reports (MNB, 2013a), (MNB, 2013b) and Marsi (2013)'s article. Owing to the fact that we only had aggregated data about the entire financial system's loan portfolio, we allocated the aggregated amounts among banks according to their market shares. Furthermore, we determined the number of loans and real estates based on the amount of new real estate mortgages, supposing that every loan corresponds to only one real estate collateral. Maturity was assumed to be equal to 10 years in each case.

In our example we only considered four banks (denoted by $B1, B2, B3, B4$ respectively), which were treated completely anonymously as we wanted to emphasize that our aim is not to release exact figures, but to present the application of our model.

As we did not have sufficient available information about banks' risk management practices and their risk characteristics we determined the magnitude of their loss given default rates and probabilities of defaults based on BIS (2006) article, furthermore, we assumed that in risk management practices banks apply the same methodology. We summarize banks' characteristics in Table 5.1.

In the examination we suppose that banks' real estates can be described by average parameters.

Characteristics/Banks	$B1$	$B2$	$B3$	$B4$
EAD (HUF Billion)	361	357	821	261
PD	0.040	0.040	0.050	0.010
LGD	0.100	0.165	0.165	0.100
Real estate portfolio	104030	103000	236900	75190

Table 5.1: Properties of banks' retail mortgage portfolios

To determine the price decrease deriving from the liquidation of banks' real estate portfolios we asked expert's opinion. Due to the complexity of the problem, experts could not answer the question in a completely appropriate way but in the thesis we used their estimated values. Thus, we assume that in Hungary annual real estate turnover is about 90,000-100,000 while the quantity of real estates on the market is

about 200,000. We suppose that if the real estate supply doubles, market price of real estates decreases by 10%. Let the price drop be proportional to the previous value and in this way let 20,000 increase in the supply cause 1% price decrease in the market price of real estates.

In contrast, if banks have to immediately liquidate their real estate portfolios, they have to sell them in a very short period of time at a lower price which squeeze the competitors out of the market. In such a market situation, if the supply doubles, the value of the price drop can reach 20%. Similarly to the previous case, let the price decrease be proportional to the increase in the supply. Thus, 20,000 increase in the supply, cause 2% price decrease in the market price of real estates.

In Table 5.2 we summarize banks' and their coalitions' real estate portfolios and the values of price drops which occurs if banks liquidate their entire portfolio.

Bank	Real estate portfolio	Price decline
<i>B1</i>	104030	10.40 %
<i>B2</i>	103000	10.30 %
<i>B3</i>	236900	23.69 %
<i>B4</i>	75190	7.52 %
<i>B1 B2</i>	207030	20.70 %
<i>B1 B3</i>	340930	34.09 %
<i>B1 B4</i>	179220	17.92 %
<i>B2 B3</i>	339900	33.99 %
<i>B2 B4</i>	178190	17.82 %
<i>B3 B4</i>	312090	31.21 %
<i>B1 B2 B3</i>	443930	44.39 %
<i>B1 B2 B4</i>	282220	28.22 %
<i>B1 B3 B4</i>	416120	41.61 %
<i>B2 B3 B4</i>	415090	41.51 %
<i>B1 B2 B3 B4</i>	519120	51.91 %

Table 5.2: The price drop and coalitions' real estate sale

5.3 The Game

After the short review of the data and the assumptions, the game can be defined on the basis of our intuition. Let the members of the financial system be denoted by the

players and function v associate with every coalition the necessary amount of capital reserve to cover its risk. In our example, capital requirement to cover the risks of banks' real estate mortgage portfolio is determined by the Basel capital function (BIS, 2005).

On the basis of our intuition, we observe how coalitions' capital requirements changes if an unfavorable event occurs and banks have to liquidate certain parts of their real estate portfolio to cover their risks. For every coalition, the price decrease is calculated which occurs if the members of the coalition sell certain parts of their real estate portfolio. As in our example risk is measured by capital requirement determined by the Basel capital function, the extra risk deriving from price drop is incorporated into the function parameters. Thereby, for each coalition, the loss given default and the probability of default rates are increased by the value of the price decrease and with the new parameters in the case of sales, the new capital requirements are calculated for the coalitions of financial institutions. We emphasize that this incorporation of risk deriving from banks' joint sale, into the model is only one possible solution. The intuition can be integrated into any applied risk management methodology through different parameters and risk measurement methods. With this simple example we would like to draw attention to another relevant source of risk which risk management should take into consideration based on the current Basel regulations.

First, let us consider a market situation without assuming any unfavorable market event and calculate the value of the game. In the further parts, this case is going to be referred as benchmark. For each coalition the loss given default rates are calculated as the sum of the exposure weighted loss given default rates of the coalition members. Similarly, for each coalition, the probability of default is calculated as the sum of the exposure weighted portabilities of default. The results of the first calculation are seen in Table 5.3.

At this point we note that in order to avoid confusion, we do not incorporate diversification effect into the model, so the parameters of banks' coalitions are calculated only based on the members' probabilities of default, loss given default rates and exposures. With this example our aim is to focus on the systemic risk originated from banks' joint sale.

In the example systemic risk is allocated by the Shapley solution. Owing to Shapley solution's equal treatment property, banks with the same risk parameters are allocated the same capital requirements. Moreover, according to the efficiency,

Bank	LGD	PD	EAD (HUF Billion)	Capital (HUF Billion)	$v(C)$	Shapley value
<i>B1</i>	0.100	0.040	543	13.5	13.5	13.5
<i>B2</i>	0.165	0.040	538	22.0	22.0	21.9
<i>B3</i>	0.165	0.050	1237	57.0	57.0	56.6
<i>B4</i>	0.100	0.010	393	4.2	4.2	4.1
<i>B1 B2</i>	0.132	0.040	1081	35.5	35.5	
<i>B1 B3</i>	0.145	0.047	1780	69.8	69.8	
<i>B1 B4</i>	0.100	0.027	936	18.7	18.7	
<i>B2 B3</i>	0.165	0.047	1774	79.2	79.2	
<i>B2 B4</i>	0.138	0.027	930	25.6	25.6	
<i>B3 B4</i>	0.149	0.040	1629	60.7	60.7	
<i>B1 B2 B3</i>	0.150	0.045	2317	92.1	92.1	
<i>B1 B2 B4</i>	0.124	0.032	1473	39.9	39.9	
<i>B1 B3 B4</i>	0.137	0.040	2172	74.1	74.1	
<i>B2 B3 B4</i>	0.153	0.040	2167	82.7	82.7	
<i>B1 B2 B3 B4</i>	0.143	0.040	2710	96.1	96.1	

Table 5.3: The game without real estate sale

the grand coalition's (the financial system's) value is allocated in the game.

Based on these results, we can state that coalitions whose reserved capital is greater than or equal to their Shapley values, meet capital requirements to cover systemic risk and should not obtain extra capital. In every other case coalitions' individual capital reserves do not cover systemic risk and additional capital requirements are needed which are equal to the difference between coalitions' Shapley values and individual capital reserves.

Remark 5.1. As the game has empty core, the other "well performing" risk capital allocation method, the nucleolus, is not reasonable to consider.

At this point it is worth noting that without any sale assumptions it can occur that the coalitions' capital requirements are greater than the sum of the coalition members' individual capital reserves which means that the game is superadditive (market failure). As the Basel capital function is used in the example to measure risk, the explanation of the phenomenon is that the risk capital function is non linear in probability of default and in this way the union of banks bears systemic risk as well. Even though in practice the price decrease can be incorporated into

risk management frameworks in different ways, we want to examine how the value of the capital function depends on the choice of the initial parameters. We are going to return to this question in the next chapter.

Before moving on to the second scenario, let us describe how this game corresponds to the theoretic model. First, realization vectors represent banks' and their coalitions' capital requirements which they should hold to cover their risks. As we consider banks' retail mortgages, the risk measure applied in the analysis is the Basel capital function described in the previous section. According to our model, function v associates the capital reserve to hold for risk coverage with each coalition. When banks calculate their capital reserves, they take only their risk parameters into consideration and ignore others' behavior. The role of the liquidity requirement is indisputable in banks' optimization processes because they perform the optimization corresponding to the supervisory requirements. In the example, we do not consider either a detailed optimization or a liquidity requirement. Knowing the current financial situation with the applied regulations, if the researcher has sufficiently detailed data, the proper optimization can be performed. Now, let us move on to the second scenario.

In the second case, we observe how the financial system's capital requirement changes if we assume the occurrence of any unfavorable events which causes that banks have to liquidate certain parts of their real estate portfolio. As we do not have sufficiently detailed information about banks' real estate portfolios and risk management practices, we do not take optimization into consideration and we assume that in the first case, every bank wants to sell 25% of their real estate portfolio. Thus, we recalculate coalitions' capital requirements in a way that we increase the loss given default rates and probabilities of default in proportion to the value of the price decrease. Let us consider this calculation in the case of bank $B1$ and $B2$ so that the problem becomes easier to understand.

If bank $B1$ sells 25% of its real estate portfolio, it causes 2.6% of price decrease ceteris paribus. Then bank $B1$'s loss given default rate and probability of default are raised in proportion to the price drop and the capital requirement for bank $B1$ determined by the Basel capital function equals to HUF 14 billion. Similarly, bank $B2$'s real estate sale results in 2.58% price drop ceteris paribus and the capital requirement equals to HUF 22.9 billion. If we consider the coalition of bank $B1$ and $B2$ and suppose that the two banks want to sell 25% of their real estate portfolio that causes ceteris paribus 5.18% of price decrease which results in HUF 38.4 billion

capital reserves for the coalition. This value is grater than the sum of the individual capital reserves calculated for bank $B1$ and $B2$ separately.

Based on these calculations, we can state that if more bank liquidate certain parts of their real estate portfolio, banks' individual capital reserves do not cover risk deriving from banks' joint sale. The calculation results in details can be seen in Table 5.4.

Bank	LGD	PD	EAD (HUF Billion)	Capital (HUF Billion)	$v(C)$	Shapley value
$B1$	0.103	0.041	543	14.0	14.0	16.9
$B2$	0.169	0.041	538	22.9	22.9	26.2
$B3$	0.175	0.053	1237	62.2	62.2	66.8
$B4$	0.102	0.010	393	4.3	4.3	6.1
$B1\ B2$	0.139	0.042	1081	38.4	38.4	
$B1\ B3$	0.158	0.051	1780	79.1	79.1	
$B1\ B4$	0.104	0.029	936	20.1	20.1	
$B2\ B3$	0.179	0.051	1774	89.6	89.6	
$B2\ B4$	0.144	0.029	930	27.4	27.4	
$B3\ B4$	0.161	0.044	1629	68.1	68.1	
$B1\ B2\ B3$	0.166	0.050	2317	108.1	108.1	
$B1\ B2\ B4$	0.132	0.034	1473	44.4	44.4	
$B1\ B3\ B4$	0.151	0.044	2172	86.3	86.3	
$B2\ B3\ B4$	0.169	0.044	2167	96.2	96.2	
$B1\ B2\ B3\ B4$	0.161	0.045	2710	116.0	116.0	

Table 5.4: The game if banks liquidate 25% of their real estate portfolio

Thirdly, similarly to the previous two cases we examine how banks' necessary capital requirements change if we assume that they sell 50% of their real estate portfolio. The calculation results are summarized in Table 5.5.

Last but not least, for comparison let us represent the case when in an extreme scenario banks get during their optimization process that they do not have any other possible solution, just to liquidate their entire real estate portfolio. The results in such an extreme case can be seen in Table 5.6.

The previously described three cases show how banks' and the financial system's capital requirements change as a result of banks' real estate sale. The necessary amount of capital reserves increases on banks' individual level and on the entire

Bank	LGD	PD	EAD (HUF Billion)	Capital (HUF Billion)	$v(C)$	Shapley value
<i>B1</i>	0.105	0.042	543	14.6	14.6	20.5
<i>B2</i>	0.173	0.042	538	23.8	23.8	30.6
<i>B3</i>	0.185	0.056	1237	67.5	67.5	77.5
<i>B4</i>	0.104	0.010	393	4.4	4.4	8.2
<i>B1 B2</i>	0.146	0.044	1081	41.3	41.3	
<i>B1 B3</i>	0.170	0.055	1780	88.6	88.6	
<i>B1 B4</i>	0.109	0.030	936	21.5	21.5	
<i>B2 B3</i>	0.193	0.055	1774	100.4	100.4	
<i>B2 B4</i>	0.150	0.030	930	29.3	29.3	
<i>B3 B4</i>	0.173	0.047	1629	75.8	75.8	
<i>B1 B2 B3</i>	0.183	0.055	2317	124.8	124.8	
<i>B1 B2 B4</i>	0.141	0.037	1473	49.1	49.1	
<i>B1 B3 B4</i>	0.166	0.049	2172	99.0	99.0	
<i>B2 B3 B4</i>	0.185	0.049	2167	110.4	110.4	
<i>B1 B2 B3 B4</i>	0.180	0.051	2710	136.8	136.8	

Table 5.5: The game if banks liquidate 50% of their real estate portfolio

financial system's level if real estate sale is assumed. For example, in the case of the grand coalition, it should hold HUF 96.1 billion when no joint sale is assumed, while when all of the banks want to sell their real estates, it needs HUF 181.3 billion to reserve.

It becomes obvious that if banks are treated as one single financial unit, individual capital reserves determined during banks' internal risk calculation do not cover the financial system's systemic risk originating from banks' joint sale. The reason for this phenomenon is the fact that during their calculations, banks do not take into account what happens on the real estate market if, in a given market situation banks act in the same way. This phenomenon could cause significant changes in the market price of real estates.

In our example the financial system's capital requirement is allocated among banks by the Shapley solution. If banks' Shapley values are compared to their individual capital reserves we can determine whether additional capital reserves are needed to cover risk. The amount of additional capital can be calculated as the difference between players' Shapley values and their individual capital reserves.

Bank	LGD	PD	EAD (HUF Billion)	Capital (HUF Billion)	$v(C)$	Shapley value
<i>B1</i>	0.110	0.044	543	15.7	15.7	28.2
<i>B2</i>	0.182	0.044	538	25.6	25.6	40.2
<i>B3</i>	0.204	0.062	1237	78.5	78.5	100.2
<i>B4</i>	0.108	0.011	393	4.7	4.7	12.8
<i>B1 B2</i>	0.160	0.048	1081	47.4	47.4	
<i>B1 B3</i>	0.195	0.063	1780	108.6	108.6	
<i>B1 B4</i>	0.118	0.032	936	24.3	24.3	
<i>B2 B3</i>	0.221	0.063	1774	123.0	123.0	
<i>B2 B4</i>	0.162	0.032	930	33.1	33.1	
<i>B3 B4</i>	0.196	0.053	1629	91.8	91.8	
<i>B1 B2 B3</i>	0.216	0.065	2317	160.1	160.1	
<i>B1 B2 B4</i>	0.159	0.041	1473	58.8	58.8	
<i>B1 B3 B4</i>	0.194	0.057	2172	125.9	125.9	
<i>B2 B3 B4</i>	0.217	0.057	2167	140.3	140.3	
<i>B1 B2 B3 B4</i>	0.217	0.061	2710	181.3	181.3	

Table 5.6: The game if banks liquidate 100% of their real estate portfolio

With this example we have showed how capital requirement deriving from systemic risk can be allocated among the members of the financial system in a fair way.

Chapter 6

The Capital Function

The examination of the capital function (see for instance in Zsámboki (2007)) is a very new and complex problem with few available literature and, consequently in this way its analysis would require a more in-depth research. For this reason in the thesis the analysis of the capital function is only sketchy. In this chapter, we used Microsoft Excel and MATLAB to prepare the figures.

As we noted in the example of the previous chapter, we can easily get a superadditive game without assuming banks' joint sale when we consider the coalitions of banks. Thus, let us consider step by step whether the game is sub-, super- or additive if we change the initial parameters. This examination is relevant because in the case of superadditive games, the capital requirements do not cover banks' systemic risk (market failure).

The first three Figures show the case when capital requirement is calculated for only one bank as a function of the probability of default. The importance of the Figures is indisputable because when we incorporated the price decrease into our model, we increased the level of the initial loss given default rates and probabilities of default.

Figure 6.1 shows how the capital requirement changes as a function of the probability of default assuming different loss given default rates. In this case, exposure was supposed to be equal to 100.

Figure 6.2 shows how the capital requirement changes as a function of the probability of default assuming different exposures of default. In this case, loss given default was supposed to be equal to 10%.

Figure 6.3 shows how the capital requirement changes as a function of the probability of default assuming different exposures and loss given default rates.

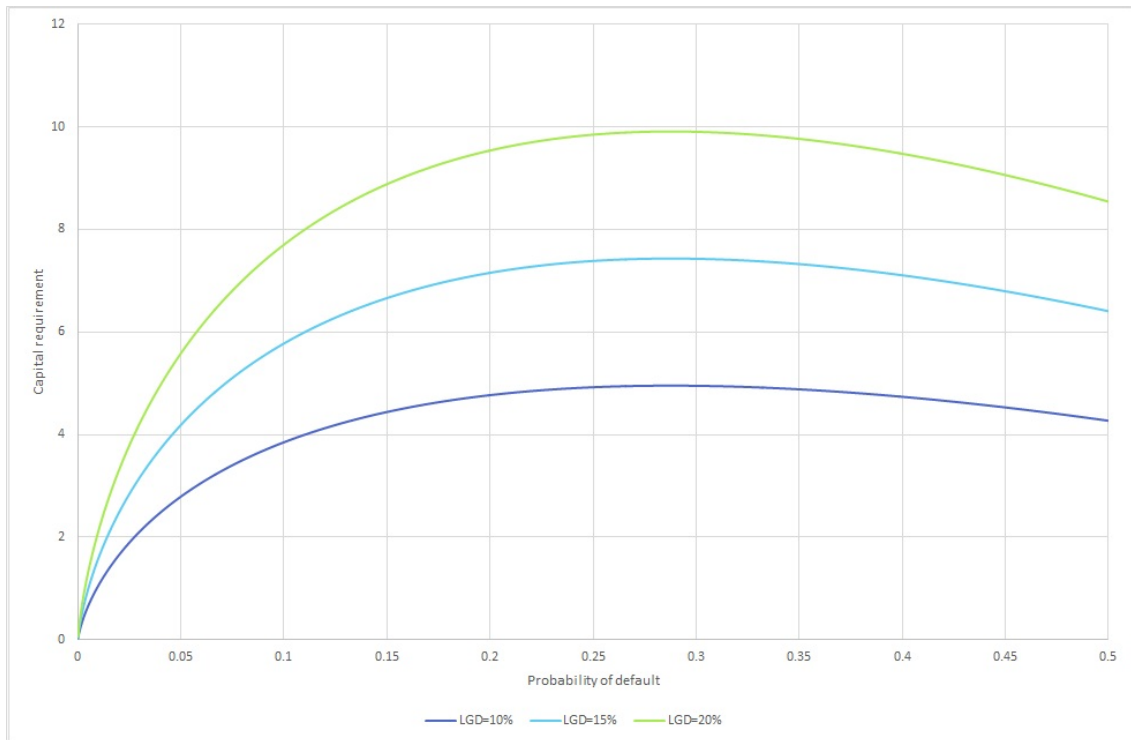


Figure 6.1: Different loss given default rates

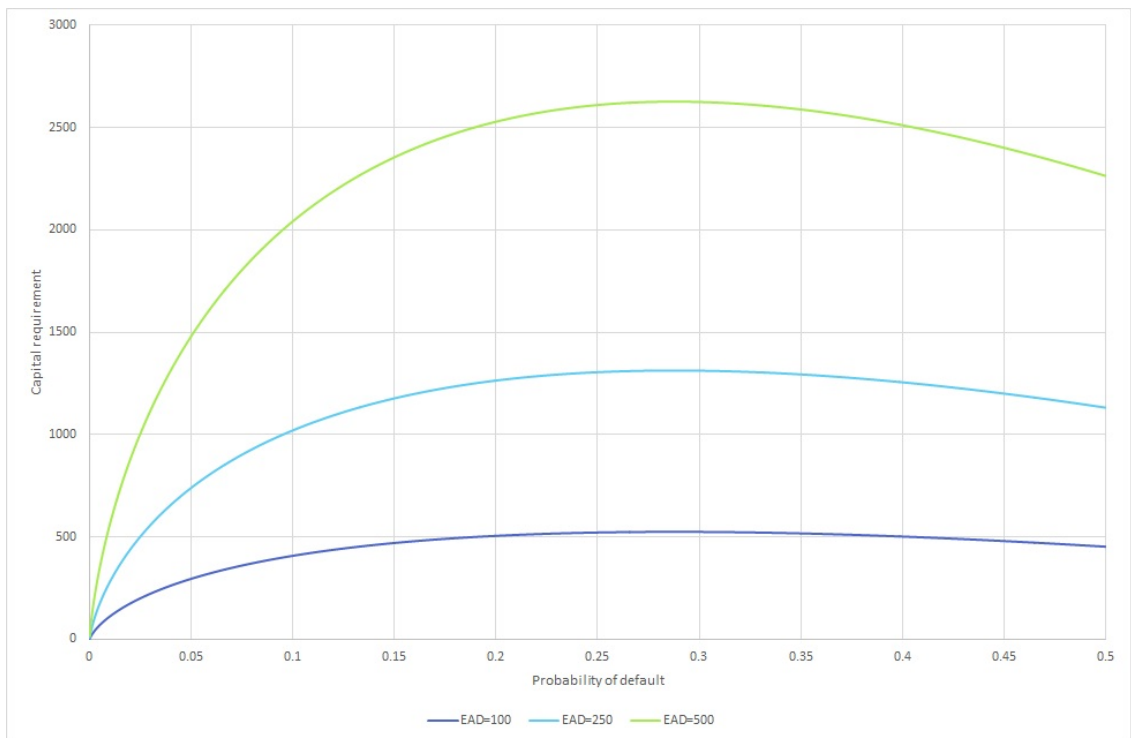


Figure 6.2: Different exposures of default

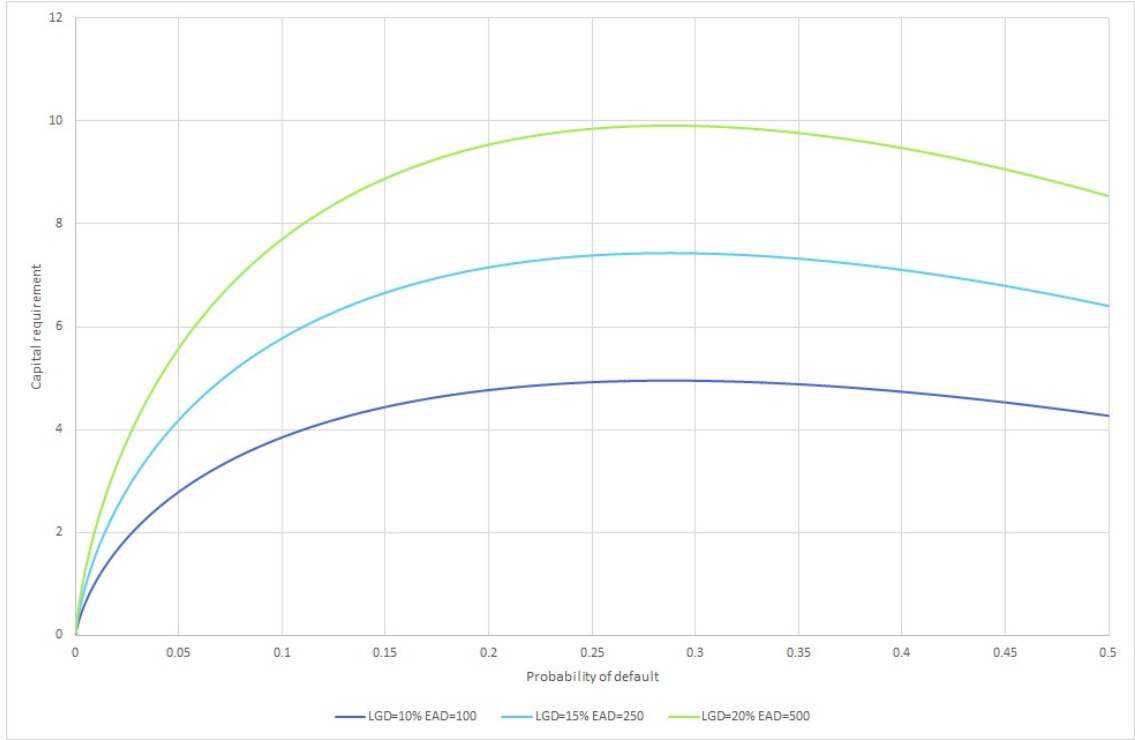


Figure 6.3: Different exposures at default and loss given default rates

In the next step, consider two banks denoted by $B1$ and $B2$ respectively. Let us suppose that the value of both banks residential mortgage portfolio is equal to 100, furthermore, for bank $B1$ loss given default is 10% and for $B2$ is 30%. Examine whether the game is subadditive, superadditive or additive as a function of the probability of default.

Figure 6.4 shows the superadditive case indicated by the brown surface. The green surface represents the sum of the individual capital requirements as a function of the probability of default. It can be seen that the two surfaces intersect each other along two lines where the game is additive. If we examine this figure, we can observe in which cases the game is additive, subadditive or superadditive.

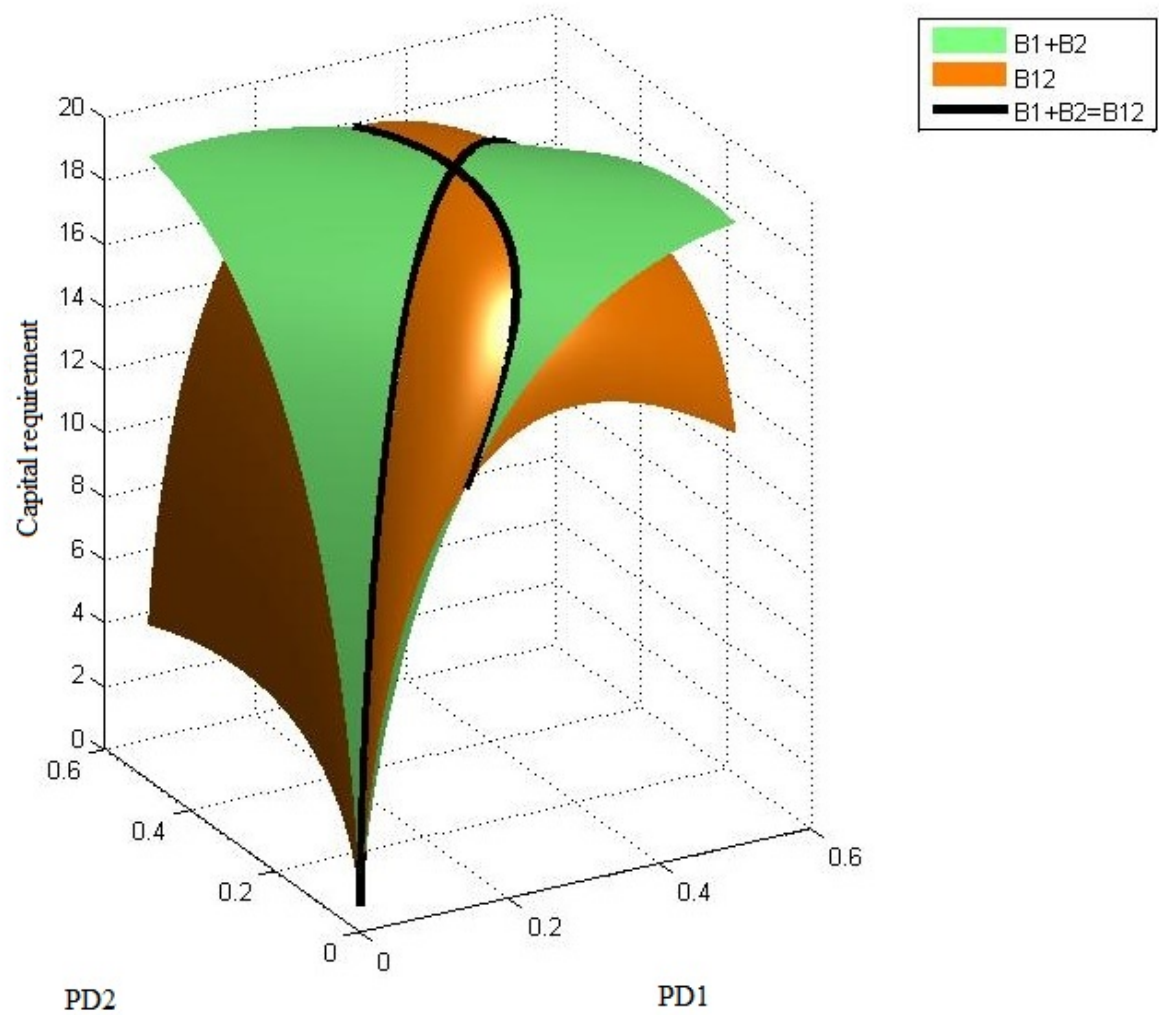


Figure 6.4: The case of two banks

Chapter 7

Conclusion

In the thesis we developed a game theoretic framework to measure and allocate systemic risk deriving from banks' correlated activities on illiquid markets. Our model is in line with the current Basel III regulation and can serve as a guideline for the regulator to extend the regulation. We determined how such systemic risk can be allocated among the members of a financial system. After building up the theoretic framework, we illustrated the application of the model with a detailed example as well.

In the empirical part of the thesis, we observed what happens if banks sell their retail real estate collaterals. We noticed that due to banks' joint sale, on the illiquid real estate market, prices significantly change and in this way banks need to hold additional capital reserves to cover their risks. In the model, capital requirement to cover systemic risk was allocated among the members of the financial system by the Shapley value known from game theory.

As we emphasized in the thesis, the described example is only an illustration of risk deriving from joint sale and it remains valid in the case of every illiquid asset in banks' portfolios. Obviously, risk, deriving from the fact that in some cases banks do not act independently from each other, can be incorporated into risk management models and practices in different ways by numerous parameters and estimations (not only by the parameters of the Basel capital function).

The aim of this thesis is to draw attention to a special source of risk which is not taken into account and currently not integrated into financial institutions' risk management practices. We think that the regulator with a uniform risk management methodology can build up a new extended framework into which this type of risk can be integrated. In this context, uniform risk management methodology means

that if the regulator was able to apply a common risk measurement methodology to determine financial institutions' capital requirements, the reserves to cover systemic risk could be allocated among banks. In this way, banks would modify their capital reserves determined by their individual estimations to cover systemic risk as well. We note that with such a new extended regulation in Hungary the currently applied quotas on banks' real estate sale would become unnecessary and avoidable.

To sum up, we can state that the problem described in the thesis is very complex and relevant in the current financial situation. The application of the model with appropriate data is a very complex and difficult task to complete, however, it can bring several important results for risk managers. The need for a comprehensive research in the topic seems unavoidable.

Appendix A

Notations

$A(p)$	set of attainable portfolios from portfolio p
$C(v)$	the core of game v
$c^s(p)$	liquidation cost of portfolio p in state s
$\varphi_i(v)$	the Shapley value of player i in game v
$\varphi(v)$	the Shapley solution of game v
$\Psi : \text{RCAS}_N \rightarrow \mathbb{R}^N$	risk capital allocation method
$\mathcal{G}^N$	the class of TU games with the players' set N
$\mathcal{G}_c^N$	the class of cost games with the players' set N
$I(v)$	the set of imputation of game v
$I^*(v)$	the set of pre-imputation of game v
J	set of assets
(N, S, π, X, ρ)	risk environment
$(N, S, \pi, X, m, \mathcal{L}\rho)$	risk environment with liquidity requirement
$\mathcal{N}(v)$	the nucleolus of game v
$\mathcal{L}$	liquidity requirement
$l^s(p)$	liquidation mark-to-market value of portfolio p in state s
m	MSDC curve
P	set of portfolios
$\mathcal{P}(N)$	the class of all subsets of N
$\mathcal{PN}(v)$	the prenucleolus of game v
π_i	the occurrence probability of state of nature i
ϱ	non-decreasing ordering

ρ	a coherent measure of risk
S	set of states of nature
$t(C)$	trading vector of coalition C
θ_i	the portfolio of subunit i
$u^s(p)$	liquidation uppermost value of portfolio p in state s
v	valuation function
v^T	the subgame of game v on coalition T
X	matrix of realization vectors
X_N^ρ	risk capital allocation situations

Appendix B

Figures

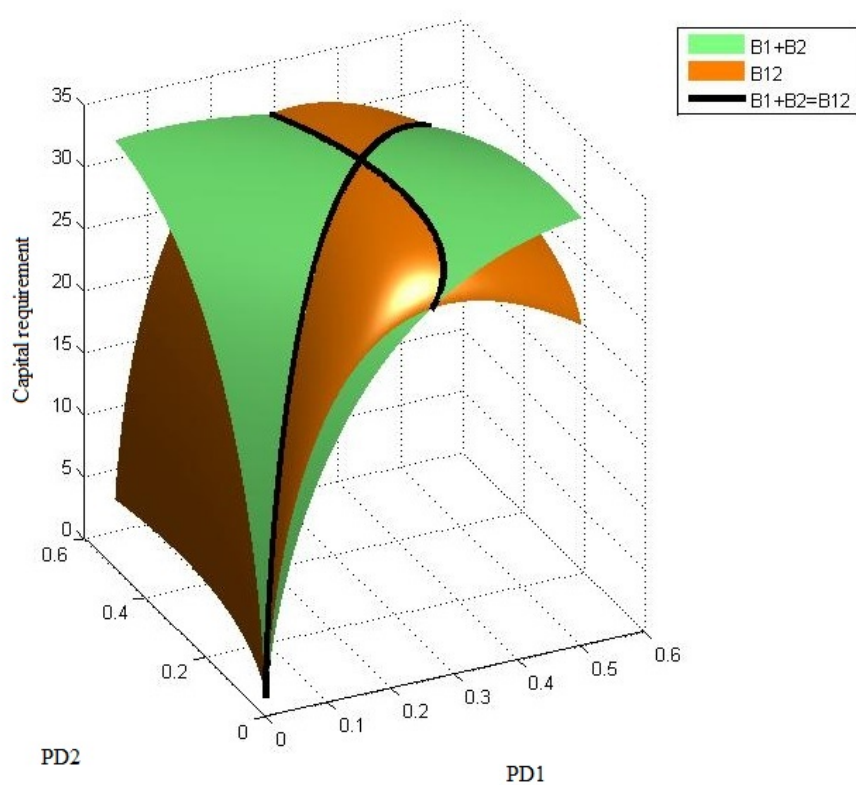


Figure B.1: $LGD_1=10\%$, $EAD_1=100$, $LGD_2=60\%$, $EAD_2=100$

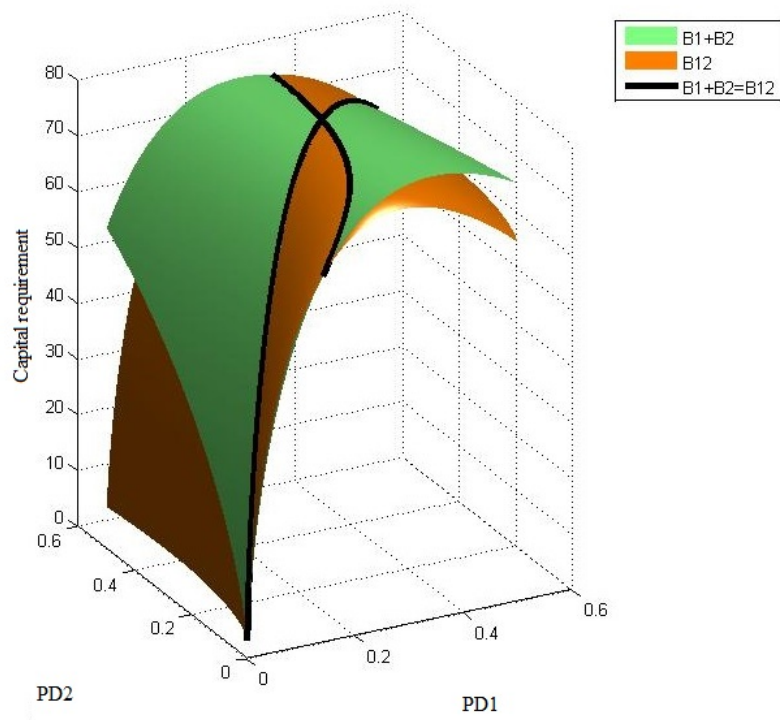


Figure B.2: $LGD_1=10\%$, $EAD_1=100$, $LGD_2=30\%$, $EAD_2=500$

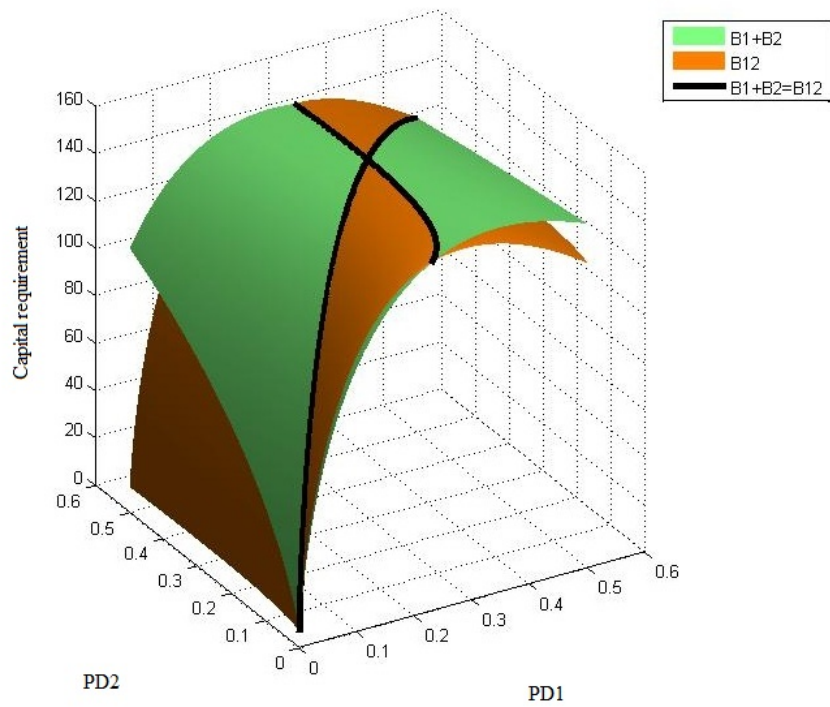


Figure B.3: $LGD_1=10\%$, $EAD_1=100$, $LGD_2=60\%$, $EAD_2=500$

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