



BUDAPEST UNIVERSITY OF TECHNOLOGY AND ECONOMICS
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Analysis of the sensitivity of milling stability on the modal parameters

FINAL PROJECT

Author: Dávid Hajdu

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Supervisor: Tamás Insperger, associate professor
Department of Applied Mechanics

Project description

Student declaration

I, Dávid Hajdu, the undersigned, hereby declare that the present thesis work has been prepared by myself without any unauthorized help or assistance such that only the specified sources (references, tools, etc.) were used. All parts taken from other sources word by word or after rephrasing but with identical meaning were unambiguously identified with explicit reference to the sources utilized.

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Dávid Hajdu

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*To my parents and
grandparents*

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Introduction

One of the main optimization goals in manufacturing is to increase the material removal rate while maintaining the quality of the product [1]. High percent of the machine components are manufactured by cutting processes like turning and milling. If the revolution number is increased, the material removal rate can also be increased, and in case of high-speed cutting, very high cutting depths can be achieved and the required manufacturing time can be significantly decreased. Unfortunately, in many applications, undesired vibrations arise for lower revolution numbers also and it spoils the quality [2]. The amplitude of the vibration increases until the tool leaves the surface of the workpiece, flies over it and starts cutting again when returns. This self-excited vibration is called *chatter*. It is a challenging task to determine the limits of the stability and the corresponding machining parameters in advance, before instability appears.

The history of machine tool chatter goes back to almost a hundred years in the past when Taylor (1907) described the machine tool chatter as the 'most obscure and delicate problems facing the machinist' [3]. In the 1960s, two scientists (Tobias and Thusty) highlighted the importance of the so-called *regenerative effect*, which plays an important role in chip removal processes like drilling, milling or turning [2]. When the tool starts vibrating during the operation, its undesired motion is copied onto the surface of the workpiece. This phenomenon modifies the chip thickness, which is no longer constant. Since the workpiece or the tool rotates with given revolution, after one period, the wavy surface will have effect on the cutting force. The cutting force depends on the chip thickness, and therefore it varies with it. From the mathematical point of view, the cutting operation does not only depend on the actual state, but it also depends on the past state or states. The rough surface from the previous revolution excites the system. This phenomenon is described by delay-differential equations (DDEs), where the trivial solution can often become unstable [2].

Systems whose rate of change of state depends on states at deviating arguments are generally described by functional differential equations (FDEs). FDEs are equations involving the function $x(t)$ of one scalar argument t (called time) and its derivatives for several values of argument t . The classification of FDEs in case of time-delayed states, where the delay is generally denoted by τ , are as follows. If the rate of change of state depends on past states of the system, the the system is described by a retarded functional differential equation (RFDE). If the rate of change of state depends on its own past values as well, then the governing equation is a neutral functional differential equation (NFDE). If the rate of change of state depends on past values of higher derivatives of the state, then the system is described by an advanced functional differential equation (AFDE) [4].

Advanced type DDEs do not describe mechanical systems except for some special Newtonian examples, however retarded type and neutral type equations are frequently

used to model machine tool chatter, force or position control with feedback delay [5], population dynamics, wheel shimmy or ship stabilization just to mention a few [4].

For several decades, machine tool chatter research has had only a very limited influence on manufacturing industry, it often had an academic nature. The complexity of the involved model was too high and the parameter estimation and measurement was not enough accurate to efficiently handle the governing equations and determine the limits of stability. The improved experimental modal testing, more sophisticated mechanical models, results in nonlinear dynamics and the use of computer algebra made the latest research efforts more accessible for industrial applications during the last decade. This is true for turning and high-speed milling operations too [3].

Although the technology has improved a lot, the mathematical models in many cases require parameters, which are extracted from measured data and can be noisy and uncertain. The modal behavior of the machine is usually determined from impact or shaking tests [2]. The complexity of the model is always limited, for instance, only finite number of modes can be taken into account, and nonlinearities are hard to handle. It is also a question how different approximations of the measured frequency response function (FRF) describe the theoretically exact stability chart. In this work, parameter sensitivity of the stability chart is analyzed, when higher or non-dominant modes are neglected while noise and other uncertainties are taken into account.

Chapter 1

Modal analysis

In this chapter, a brief description is given on modal analysis and the characteristics of Multiple-Degrees-of-Freedom Systems (MDoF) is presented. First, proportionally damped systems are analyzed, then non-proportional damping is introduced. The parametric form of the frequency response function defined in the modal space is explained in details.

1.1 Proportional damping

The dynamical behavior of real structures in most of the cases can only be described by MDoF systems. A typical representation of a system like this can be seen in Figure 1.1. This model is widely used in practice where multiple modes are taken into account and modal parameters are estimated [1, 7]. Unless stated otherwise, MDoF systems are considered, for which the matrix differential equation of motion is given in the form

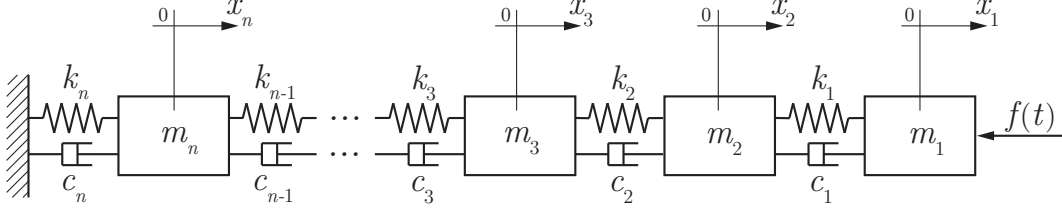
$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{0}, \quad (1.1)$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the general coordinate vector $\mathbf{M} \in \mathbb{R}^{n \times n}$ is the mass matrix, $\mathbf{C} \in \mathbb{R}^{n \times n}$ is the damping matrix, $\mathbf{K} \in \mathbb{R}^{n \times n}$ is the stiffness matrix and n is the number of degrees of freedom. According to the definition, a system is proportionally damped if the damping matrix can be written as

$$\mathbf{C} = \alpha_M \mathbf{M} + \alpha_K \mathbf{K}, \quad (1.2)$$

which means that it can be linearly constructed from the mass and stiffness matrices [6, 8]. Distribution of damping of the proportional type is sometimes, though not always, found to be plausible from a practical standpoint, i.e. the actual damping mechanisms are usually to be found in parallel with stiffness elements (for internal material or hysteresis damping) or with mass elements (for friction damping). Generally, a system is called proportionally damped if the following condition satisfies for the mass, damping and stiffness matrices [6]

$$(\mathbf{M}^{-1}\mathbf{K})(\mathbf{M}^{-1}\mathbf{C}) = (\mathbf{M}^{-1}\mathbf{C})(\mathbf{M}^{-1}\mathbf{K}). \quad (1.3)$$

Figure 1.1. Dynamical model for n -DoF case.

1.1.1 Analysis of undamped systems

Let us use the matrix differential equation of motion of the system defined in eq. (1.1). Using the exponential trial function $\mathbf{x}(t) = \mathbf{P}e^{\lambda t}$ and considering the undamped ($\mathbf{C} = \mathbf{0}$) and unforced system, the equation reads

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{0}. \quad (1.4)$$

After the substitution, and simplification with $e^{\lambda t} > 0$,

$$(\lambda^2 \mathbf{M} + \mathbf{K})\mathbf{P} = \mathbf{0} \quad (1.5)$$

is obtained, where $\mathbf{P} \in \mathbb{R}^n$ is the normal mode of the undamped system, $\lambda = i\omega$ (if there is no damping), and ω is the natural angular frequency of the undamped system. This equation states a free vibrations eigenproblem for a MDoF system. Moreover $\lambda^2 \mathbf{M} + \mathbf{K}$ is the so-called dynamic matrix, where

$$\det(\lambda^2 \mathbf{M} + \mathbf{K}) = 0 \quad (1.6)$$

is the characteristic equation for free undamped vibrations. This equation has $2n$ roots for an n -degrees-of-freedom case, where $\text{Im}(\lambda_i) = \pm\omega_{n,i}$. Here $\omega_{n,i}$ represents the i^{th} undamped natural angular frequency (subscript 'n' stands for natural). For undamped (and underdamped) case, the roots form complex conjugate pairs. The $\mathbf{P}_i$ mode shape vector corresponding to the i^{th} eigenvalue can be calculated according to

$$(\lambda_i^2 \mathbf{M} + \mathbf{K})\mathbf{P}_i = \mathbf{0}. \quad (1.7)$$

Since $\det(\lambda_i^2 \mathbf{M} + \mathbf{K}) = 0$ is singular by definition, one element of vector $\mathbf{P}_i$ can be chosen arbitrarily. Each mode shape vector can usually, though not always, be defined as

$$\mathbf{P}_i = \begin{pmatrix} 1 & P_{i,2} & P_{i,3} & \cdots & P_{i,n} \end{pmatrix}^T, \quad (1.8)$$

where $(\bullet)^T$ represents the transpose. If an undamped system is considered, since $\mathbf{M}$ and $\mathbf{K}$ are assumed to be symmetric matrices (non-symmetry can arise only in special cases [6]), the normal modes are not just real-valued, but enjoy the following orthogonality

properties with respect to the mass and stiffness matrices

$$\begin{aligned} \mathbf{P}_i^T \mathbf{M} \mathbf{P}_j &= 0 & \text{and} & \quad \mathbf{P}_i^T \mathbf{K} \mathbf{P}_j = 0 & \quad \text{if } i \neq j, \\ \mathbf{P}_i^T \mathbf{M} \mathbf{P}_i &= M_i > 0 & \text{and} & \quad \mathbf{P}_i^T \mathbf{K} \mathbf{P}_i = K_i > 0 & \quad \text{if } i = 1, 2 \dots n. \end{aligned} \quad (1.9)$$

M_i and K_i are also called modal mass and modal stiffness respectively. Furthermore, it can be shown that $\omega_{n,i} = \sqrt{K_i/M_i}$. Each $\mathbf{P}_i$ eigenvector can be normalized according to the following normalization criteria. Let us define $\boldsymbol{\phi}_i$ normalized eigenvector, such that

$$\boldsymbol{\phi}_i^T \mathbf{M} \boldsymbol{\phi}_i = 1 \quad \text{and} \quad \boldsymbol{\phi}_i^T \mathbf{K} \boldsymbol{\phi}_i = \omega_{n,i}^2 \quad \text{if } i = 1, 2 \dots n, \quad (1.10)$$

which are called mass-orthonormal eigenvectors. It can be obtained if the mode shape vector is normalized with respect to the mass matrix, i.e.

$$\boldsymbol{\phi}_i = \frac{\mathbf{P}_i}{\sqrt{\mathbf{P}_i^T \mathbf{M} \mathbf{P}_i}}. \quad (1.11)$$

The $\mathbf{q}(t)$ modal coordinates are introduced by the transformation $\mathbf{x}(t) = \boldsymbol{\Phi} \mathbf{q}(t)$, i.e.

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\phi}_1 & \boldsymbol{\phi}_2 & \cdots & \boldsymbol{\phi}_n \end{pmatrix} \begin{pmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_n(t) \end{pmatrix} = \underbrace{\begin{pmatrix} \phi_{11} & \phi_{21} & \cdots & \phi_{n1} \\ \phi_{12} & \phi_{22} & \cdots & \phi_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{1n} & \phi_{2n} & \cdots & \phi_{nn} \end{pmatrix}}_{\boldsymbol{\Phi}} \begin{pmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_n(t) \end{pmatrix}, \quad (1.12)$$

where $\boldsymbol{\Phi}$ is the modal transformation matrix. Based on eq. (1.10), the matrix equation of motion can be rewritten with the modal coordinates $\mathbf{q}(t)$ instead of the general coordinates $\mathbf{x}(t)$. Using the derived relation between the modal and general coordinates, substituting back into eq. (1.4) and multiplying both sides by $\boldsymbol{\Phi}^T$ from the left, it gives

$$\boldsymbol{\Phi}^T \mathbf{M} \boldsymbol{\Phi} \ddot{\mathbf{q}}(t) + \boldsymbol{\Phi}^T \mathbf{K} \boldsymbol{\Phi} \mathbf{q}(t) = \mathbf{0}. \quad (1.13)$$

Here, $\boldsymbol{\Phi}^T \mathbf{M} \boldsymbol{\Phi} = \mathbf{M}_m$ is the so-called modal mass matrix and $\boldsymbol{\Phi}^T \mathbf{K} \boldsymbol{\Phi} = \mathbf{K}_m$ is the modal stiffness matrix. Since the eigenvectors are mass-orthonormal, therefore $\boldsymbol{\Phi}^T \mathbf{M} \boldsymbol{\Phi} = \mathbf{M}_m = \mathbf{I}$ and $\boldsymbol{\Phi}^T \mathbf{K} \boldsymbol{\Phi} = \mathbf{K}_m = \text{diag}(\omega_{n,i}^2)$ hold. This method decouples the equations, since each coefficient matrix is diagonal.

1.1.2 Analysis of proportionally damped systems

The system is proportionally damped, if the damping matrix can be linearly constructed from the mass and from the stiffness matrices, as it was given in eq. (1.2). Consider now the damped equation of motion given by eq. (1.1) and using the modal decomposition, it yields

$$\boldsymbol{\Phi}^T \mathbf{M} \boldsymbol{\Phi} \ddot{\mathbf{q}}(t) + \boldsymbol{\Phi}^T \mathbf{C} \boldsymbol{\Phi} \dot{\mathbf{q}}(t) + \boldsymbol{\Phi}^T \mathbf{K} \boldsymbol{\Phi} \mathbf{q}(t) = \mathbf{0}, \quad (1.14)$$

where $\mathbf{\Phi}^T \mathbf{C} \mathbf{\Phi} = \mathbf{C}_m$ is the modal damping matrix. Application of the definition (1.2) of proportional damping gives

$$\mathbf{\Phi}^T \mathbf{C} \mathbf{\Phi} = \alpha_M \mathbf{\Phi}^T \mathbf{M} \mathbf{\Phi} + \alpha_K \mathbf{\Phi}^T \mathbf{K} \mathbf{\Phi} = \alpha_M \mathbf{I} + \alpha_K \text{diag}(\omega_{n,i}^2). \quad (1.15)$$

For Single-Degree-of-Freedom Systems (SDoF), the equation of motion can be written as

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0. \quad (1.16)$$

The analysis for damped cases is simpler, usually the following modal form is used after dividing by m , i.e.

$$\ddot{x}(t) + 2\xi\omega_n\dot{x}(t) + \omega_n^2x(t) = 0, \quad (1.17)$$

where ξ is the viscous damping factor and for underdamped cases $\xi < 1$. Giving the characteristic equation, and calculating the roots, one can obtain

$$\lambda^2 + 2\xi\omega_n\lambda + \omega_n^2 = 0 \quad \rightarrow \quad \lambda_{1,2} = -\sigma \pm i\gamma = -\xi\omega_n \pm i\omega_n\sqrt{1 - \xi^2}. \quad (1.18)$$

Note, that for undamped cases, the roots reduce to $\lambda_{1,2} = \pm i\omega_n$ as it was already defined previously. Otherwise, $\sigma = \xi\omega_n$ defines the decay rate of the amplitude of oscillation and $\gamma = \omega_n\sqrt{1 - \xi^2}$ is the natural angular frequency of the damped system. Based on the SDoF example, it can be shown that $\alpha_M + \alpha_K\omega_{n,i}^2 = 2\xi_i\omega_{n,i}$, which gives ξ_i as

$$\xi_i = \frac{1}{2} \left(\frac{\alpha_M}{\omega_{n,i}} + \alpha_K\omega_{n,i} \right). \quad (1.19)$$

This derivation points out that in case of proportional damping the equations can be decoupled since the modal transformation matrix $\mathbf{\Phi}$ makes $\mathbf{C}$ also diagonal. This also shows that the mode shapes (which are the right eigenvectors) are the same for undamped, and proportionally damped cases. Here, some remarks have to be made, which will be important for the following analysis. The properties of the proportional damping are the following [6]:

- Mode shapes are real valued, so-called normal modes.
- Mode shapes of the undamped and proportionally damped system are the same.
- Equations are decoupled in the modal space.
- The maximum and minimum points of the vibration are reached at the same time instant.
- The vibration can be represented by a standing wave.
- The coordinates of the mode shapes are totally in-phase or out-of-phase (phase delay is either 0° or 180°).
- The node points remain fixed in time.

Note, that for non-proportionally damped systems, the above derived expressions and properties do not hold. In those cases, the modal analysis is more complicated.

1.1.3 Solution in time domain

The $\mathbf{x}(t)$ solution can be given in terms of the eigenvalues and eigenvectors. To make the derivation more understandable, let us start with a SDoF example, where the equation of motion of the unforced system reads

$$\ddot{x}(t) + 2\xi\omega_n\dot{x}(t) + \omega_n^2x(t) = 0, \quad (1.20)$$

and the eigenvalues are $\lambda_{1,2} = -\sigma \pm i\gamma = -\xi\omega_n \pm i\omega_n\sqrt{1-\xi^2}$, if $\xi < 1$. The solution is

$$x(t) = b_1e^{\lambda_1 t} + b_2e^{\lambda_2 t} = b_1e^{-\xi\omega_n t + i\omega_n\sqrt{1-\xi^2}t} + b_2e^{-\xi\omega_n t - i\omega_n\sqrt{1-\xi^2}t}, \quad (1.21)$$

where b_1 and b_2 are complex valued coefficients depending on the initial conditions. Using the Euler formula, rewriting the equations, one can get

$$x(t) = \left(\underbrace{(b_1 + b_2)}_{B_1} \cos\left(\omega_n\sqrt{1-\xi^2}t\right) + \underbrace{(b_1 - b_2)i}_{B_2} \sin\left(\omega_n\sqrt{1-\xi^2}t\right) \right) e^{-\xi\omega_n t}, \quad (1.22)$$

where B_1 and B_2 must be real valued, since the $x(t)$ displacement function is real. This condition holds if $b_2 = \bar{b}_1$, and $(\bar{\bullet})$ indicates the complex conjugate, furthermore, if the system is underdamped ($\xi < 1$), then $\lambda_2 = \bar{\lambda}_1$. Therefore the solution can be written as $x(t) = b_1e^{\lambda_1 t} + \bar{b}_1e^{\bar{\lambda}_1 t}$. This property is used for MDoF systems too.

Let us now consider a general proportionally damped system, where the equation of motion is given in eq. (1.1). Then the eigenvalues are $\lambda_r = -\sigma_r + i\gamma_r$, and $\mathbf{P}_r$ is the corresponding real valued mode shape (eigenvector). Thus the solution is

$$\mathbf{x}(t) = \sum_{r=1}^n b_r \mathbf{P}_r e^{\lambda_r t} + \bar{b}_r \mathbf{P}_r e^{\bar{\lambda}_r t}. \quad (1.23)$$

Using Euler formula and the same steps, this equation can be rewritten as

$$\begin{aligned} \mathbf{x}(t) &= \sum_{r=1}^n \mathbf{P}_r \left(\underbrace{(b_r + \bar{b}_r)}_{B_{r1}} \cos(\gamma_r t) + \underbrace{(b_r - \bar{b}_r)i}_{B_{r2}} \sin(\gamma_r t) \right) e^{-\sigma_r t} \\ &= \sum_{r=1}^n \mathbf{P}_r (B_{r1} \cos(\gamma_r t) + B_{r2} \sin(\gamma_r t)) e^{-\sigma_r t}. \end{aligned} \quad (1.24)$$

Similarly to the SDoF example, B_{r1} and B_{r2} are real valued, and can be determined from the initial conditions. It shows that the modes shapes are described by standing waves.

1.1.4 FRF for proportionally damped systems

The dynamical behavior of a system can be determined from impact or shaking tests, from which the the frequency response function (FRF) is obtained [2]. Using some curve

fitting technique, the modal parameters can be approximated.

Let us now consider a proportionally damped and forced system as

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{f}(t), \quad (1.25)$$

where $\mathbf{f}(t) \in \mathbb{R}^n$ is the vector of excitation. For example, if the test is done with an impulse hammer, the excitation is theoretically a Dirac-impulse $f_0\delta(t)$ in a given direction and coordinate. The transfer function of a system is defined as a fraction of the Laplace transformation of the output and input. The Laplace transform of eq. (1.25) is

$$\mathbf{M}s^2\mathbf{X}(s) + \mathbf{C}s\mathbf{X}(s) + \mathbf{K}\mathbf{X}(s) = \mathbf{F}(s), \quad (1.26)$$

where $\mathbf{X}(s) = \mathcal{L}\{\mathbf{x}(t)\}(s)$, $\mathbf{F}(s) = \mathcal{L}\{\mathbf{f}(t)\}(s)$, $\mathcal{L}$ is the Laplace operator and s is the complex Laplace variable. Arranging the terms, one can get

$$\mathbf{X}(s) = (\mathbf{M}s^2 + \mathbf{C}s + \mathbf{K})^{-1} \mathbf{F}(s), \quad (1.27)$$

where $(\bullet)^{-1}$ denotes the inverse of a matrix. Since the input is the $\mathbf{f}(t)$ excitation, and the output is the $\mathbf{x}(t)$ displacement vector, the transfer function matrix is written as

$$\mathbf{W}(s) = \mathbf{X}(s)\mathbf{F}^{-1}(s) = (\mathbf{M}s^2 + \mathbf{C}s + \mathbf{K})^{-1}. \quad (1.28)$$

The frequency response function can be given, if $s = i\omega$ is formally replaced, thus

$$\mathbf{H}(\omega) = (-\mathbf{M}\omega^2 + \mathbf{C}i\omega + \mathbf{K})^{-1}. \quad (1.29)$$

Note, that although this derivation gives a matrix, later it will be shown that from the practical point of view it is enough to give only one element of the frequency response function matrix. In case of *tip-to-tip* machine tool measurements, when the tool is excited at the tool-tip and measured at the same point, the FRF for the stability analysis can be determined. For complete modal analysis, several measurements would be necessary at different points, but for the tool-tip measurement, only the displacement of one coordinate is necessary ($x_1(t)$).

Eq. (1.29) is not convenient for the curve fitting techniques. It is more appropriate to transform the equations into the modal space, therefore the initial equation is

$$\Phi^T \mathbf{M} \Phi \ddot{\mathbf{q}}(t) + \Phi^T \mathbf{C} \Phi \dot{\mathbf{q}}(t) + \Phi^T \mathbf{K} \Phi \mathbf{q}(t) = \Phi^T \mathbf{f}(t), \quad (1.30)$$

where $\Phi^T \mathbf{M} \Phi = \mathbf{I}$, $\Phi^T \mathbf{C} \Phi = \mathbf{C}_m$ and $\Phi^T \mathbf{K} \Phi = \mathbf{K}_m$. The Laplace transformation gives

$$(\mathbf{I}s^2 + \mathbf{C}_m s + \mathbf{K}_m) \mathbf{Q}(s) = \Phi^T \mathbf{F}(s), \quad (1.31)$$

where $\mathbf{Q}(s) = \mathcal{L}\{\mathbf{q}(t)\}(s)$. The transfer function can be expressed as

$$\mathbf{W}(s) = \mathbf{X}(s)\mathbf{F}^{-1}(s) = \Phi \mathbf{X}(s)\mathbf{F}^{-1}(s) = \Phi (\mathbf{I}s^2 + \mathbf{C}_m s + \mathbf{K}_m)^{-1} \Phi^T. \quad (1.32)$$

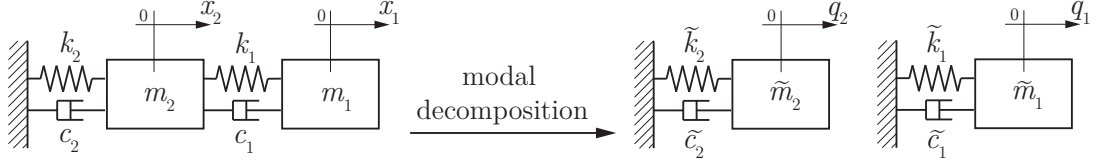


Figure 1.2. Modal transformation represented for a two-DoF system.

Since $(\mathbf{I}s^2 + \mathbf{C}_m s + \mathbf{K}_m)$ is diagonal, its inverse can be given with the reciprocates of the diagonal elements, i.e.

$$(\mathbf{I}s^2 + \mathbf{C}_m s + \mathbf{K}_m)^{-1} = \text{diag} (s^2 + 2\xi_i \omega_{n,i} s + \omega_{n,i}^2)^{-1}, \quad (1.33)$$

$$\mathbf{W}(s) = \mathbf{\Phi} \text{diag} \left(\frac{1}{s^2 + 2\xi_i \omega_{n,i} s + \omega_{n,i}^2} \right) \mathbf{\Phi}^T. \quad (1.34)$$

Here, one can simply obtain the following expression for the frequency response function

$$H_{ij}(\omega) = \sum_{r=1}^n \frac{\phi_{ri} \phi_{rj}}{-\omega^2 + 2\xi_r \omega_{n,r} \omega i + \omega_{n,r}^2}, \quad (1.35)$$

where n is the number of degrees of freedom, i is the row, j is the column of the FRF matrix, and ϕ_{ri} is the i^{th} element of the normalized eigenvector corresponding to the r^{th} eigenvector (see eq. (1.12)). The element $H_{11}(\omega)$, which is required for the stability analysis in case of a tip-to-tip impact test, can be given as

$$H_{11}(\omega) = \sum_{r=1}^n \frac{\phi_{r1}^2}{-\omega^2 + 2\xi_r \omega_{n,r} \omega i + \omega_{n,r}^2}, \quad (1.36)$$

where $\tilde{m}_r = 1/\phi_{r1}^2$ is the effective modal mass, $\tilde{k}_r = \tilde{m}_r \omega_{n,r}^2$ is the effective stiffness and $\tilde{c}_r = 2\xi_r \omega_{n,r} \tilde{m}_r$ is the effective damping. Now, it can be seen, what the biggest advantage of modal decomposition is. Instead of having one n -dimensional second-order coupled differential equation, the system falls apart to uncoupled, n second-order systems. This is illustrated in Figure 1.2. Furthermore the above equation can effectively be used for the curve fitting methods [2].

1.2 Non-proportional damping

The viscous damped systems can be categorized into proportional and non-proportional damping. In Section 1.1 the undamped and proportionally damped MDoF systems were analyzed in details. Real physical structures are comprised of many substructures and the connection between them can be very various, sometimes it can be characterized by non-proportional damping, or it can even be nonlinear [9].

1.2.1 Analysis of non-proportionally damped systems

Most of the assumptions and results derived for proportionally damped systems are not valid for the non-proportional cases. If $\mathbf{C} \neq \alpha_M \mathbf{M} + \alpha_K \mathbf{K}$, then the eigenvectors of the undamped system will not be identical to the damped system and the eq. (1.3) does not hold. The mode shapes are complex valued, and the modal decomposition cannot be used for eq. (1.1) as it was presented before, since the equations cannot be given in diagonal form.

Let us define eq. (1.1) in first-order form and introduce the state vector $\mathbf{v}(t)$ as

$$\mathbf{v}(t) = \begin{pmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{pmatrix}, \quad (1.37)$$

where $\mathbf{v}(t) \in \mathbb{R}^{2n \times 2n}$. If $\mathbf{M}^{-1}$ exists, the equations can be rearranged and the following form can be given

$$\begin{pmatrix} \dot{\mathbf{x}}(t) \\ \ddot{\mathbf{x}}(t) \end{pmatrix} = \underbrace{\begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{pmatrix}}_{\mathbf{Z}} \begin{pmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{pmatrix}, \quad (1.38)$$

where $\mathbf{Z} \in \mathbb{R}^{2n \times 2n}$. It is known, that the following equation can be given in the space of the eigenvectors using the eigenvalues and the transformation

$$\mathbf{T}^{-1}\mathbf{Z}\mathbf{T} = \mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \bar{\lambda}_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n & 0 \\ 0 & 0 & \cdots & 0 & \bar{\lambda}_n \end{pmatrix}, \quad (1.39)$$

where $\mathbf{T}$ is the transformation matrix. The eigenvalues independently from the different representation still form complex conjugate pairs (for underdamped system). From this equation the FRF can be formed, however the inverse of the transformation matrix has to be defined, which does not lead to the simplest representation. Instead of this, a different formulation will be used, which can be found in several references [6, 7, 8].

Let us use the same state vector $\mathbf{v}(t)$, but use the additional equation

$$\mathbf{M}\dot{\mathbf{x}}(t) - \mathbf{M}\dot{\mathbf{x}}(t) = \mathbf{0}. \quad (1.40)$$

Eq. (1.1) and eq. (1.40) can be formulated for unforced vibrations as

$$\underbrace{\begin{pmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{pmatrix}}_{\mathbf{A}} \begin{pmatrix} \dot{\mathbf{x}}(t) \\ \ddot{\mathbf{x}}(t) \end{pmatrix} + \underbrace{\begin{pmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{pmatrix}}_{\mathbf{B}} \begin{pmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \quad (1.41)$$

where $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{2n \times 2n}$ are symmetric matrices. Let us use the exponential trial function

similarly to the case of proportionally damped systems, $\mathbf{v}(t) = \mathbf{U}e^{\lambda t}$. Note, that $\mathbf{U}$ can be written as

$$\mathbf{U} = \begin{pmatrix} \mathbf{P} \\ \lambda \mathbf{P} \end{pmatrix}, \quad (1.42)$$

since $\mathbf{v}(t)$ is defined based on $\mathbf{x}(t) = \mathbf{P}e^{\lambda t}$ and its derivative $\dot{\mathbf{x}}(t) = \lambda \mathbf{P}e^{\lambda t}$, where $\mathbf{P}$ is the complex mode shape vector corresponding to the non-proportionally damped system. After the substitution, this derivation gives

$$(\lambda \mathbf{A} + \mathbf{B}) \mathbf{U} = \mathbf{0}, \quad (1.43)$$

which is analogous to eq. (1.6). This gives the characteristic equation as $\det(\lambda \mathbf{A} + \mathbf{B}) = 0$. The roots of the characteristic equation give the eigenvalues as complex conjugate pairs λ_i and $\bar{\lambda}_i$, and the corresponding eigenvectors which will similarly form conjugate pairs $\mathbf{U}_i$ and $\bar{\mathbf{U}}_i$. The eigenvectors can be normalized with respect to the following normalization criteria

$$\boldsymbol{\psi}_i = \frac{\mathbf{U}_i}{\sqrt{\mathbf{U}_i^T \mathbf{A} \mathbf{U}_i}}. \quad (1.44)$$

The corresponding transformation matrix reads

$$\boldsymbol{\Psi} = \begin{pmatrix} \boldsymbol{\psi}_1 & \bar{\boldsymbol{\psi}}_1 & \boldsymbol{\psi}_2 & \bar{\boldsymbol{\psi}}_2 & \cdots & \boldsymbol{\psi}_n & \bar{\boldsymbol{\psi}}_n \end{pmatrix},$$

and $\boldsymbol{\Psi} \in \mathbb{C}^{2n \times 2n}$ is a complex matrix. Using the properties of normalization, it can be shown, that $\boldsymbol{\Psi}^T \mathbf{A} \boldsymbol{\Psi} = \mathbf{I}$ and $\boldsymbol{\Psi}^T \mathbf{B} \boldsymbol{\Psi} = -\boldsymbol{\Lambda}$. The modal transformation matrix expressing the vectors give

$$\boldsymbol{\Psi} = \begin{pmatrix} \psi_{11} & \bar{\psi}_{11} & \psi_{21} & \bar{\psi}_{21} & \cdots & \psi_{n1} & \bar{\psi}_{n1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \psi_{1n} & \bar{\psi}_{1n} & \psi_{2n} & \bar{\psi}_{2n} & \cdots & \psi_{nn} & \bar{\psi}_{nn} \\ \lambda_1 \psi_{11} & \bar{\lambda}_1 \bar{\psi}_{11} & \lambda_2 \psi_{21} & \bar{\lambda}_2 \bar{\psi}_{21} & \cdots & \lambda_n \psi_{n1} & \bar{\lambda}_n \bar{\psi}_{n1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \lambda_1 \psi_{1n} & \bar{\lambda}_1 \bar{\psi}_{1n} & \lambda_2 \psi_{2n} & \bar{\lambda}_2 \bar{\psi}_{2n} & \cdots & \lambda_n \psi_{nn} & \bar{\lambda}_n \bar{\psi}_{nn} \end{pmatrix}. \quad (1.45)$$

1.2.2 Solution in time domain

It is interesting to see how the solution in time domain changes when non-proportional damping is assumed. Let us use eq. (1.1), which is valid for both cases. In case of non-proportional damping, the mode shapes are complex and the solution can be given as

$$\mathbf{x}(t) = \sum_{r=1}^n b_r \mathbf{P}_r e^{\lambda_r t} + \bar{b}_r \bar{\mathbf{P}}_r e^{\bar{\lambda}_r t}. \quad (1.46)$$

$\mathbf{P}_r = \bar{\mathbf{P}}_r$ only if $\mathbf{P}$ is real, but it would mean, that the damping is proportional. Using the Euler formula and expressing the exponential terms by trigonometric functions (furthermore $\lambda_r = -\sigma_r + \gamma_r i$ and $\bar{\lambda}_r = -\sigma_r - \gamma_r i$), after arranging the terms, one gets

$$\begin{aligned} \mathbf{x}(t) &= \sum_{r=1}^n \left(\underbrace{(b_r \mathbf{P}_r + \bar{b}_r \bar{\mathbf{P}}_r)}_{\mathbf{B}_{r1}} \cos(\gamma_r t) + \underbrace{(b_r \mathbf{P}_r - \bar{b}_r \bar{\mathbf{P}}_r)}_{\mathbf{B}_{r2}} i \sin(\gamma_r t) \right) e^{-\sigma_r t} \\ &= \sum_{r=1}^n (\mathbf{B}_{r1} \cos(\gamma_r t) + \mathbf{B}_{r2} \sin(\gamma_r t)) e^{-\sigma_r t}. \end{aligned} \quad (1.47)$$

In this case, $\mathbf{B}_{r1}$ and $\mathbf{B}_{r2}$ are real valued n -dimensional vectors, therefore the motion is also real, furthermore b_r and $\bar{b}_r$ values are determined by the initial conditions. Eq. (1.47) does not reflect a very important property of non-proportionally damped systems, and the mode shapes are difficult to extract from $\mathbf{B}_{r1,2}$ vectors.

In order to highlight the meaning of complex modes, let us rewrite the eigenvectors according to the following rule

$$\mathbf{P}_k = \begin{pmatrix} 1 \\ P_{k2} \\ \vdots \\ P_{kn} \end{pmatrix} = \begin{pmatrix} 1 \\ P_{k2}^{\text{Re}} + P_{k2}^{\text{Im}} i \\ \vdots \\ P_{kn}^{\text{Re}} + P_{kn}^{\text{Im}} i \end{pmatrix} = \begin{pmatrix} 1 \\ |P_{k2}| e^{i\vartheta_{k2}} \\ \vdots \\ |P_{kn}| e^{i\vartheta_{kn}} \end{pmatrix} \quad (1.48)$$

where k refers to the k^{th} mode shapes vector, ‘Re’ represents the real part, ‘Im’ the imaginary part, ϑ is the phase of the complex number and $|\bullet|$ is its absolute value. If the damping is proportional, then the imaginary parts are zero, and the phase shift is either 0° or 180° . Substitution of eq. (1.48) into eq. (1.46) yields

$$\mathbf{x}(t) = \sum_{r=1}^n \left(b_r \begin{pmatrix} 1 \\ |P_{r2}| e^{i\vartheta_{r2}} \\ \vdots \\ |P_{rn}| e^{i\vartheta_{rn}} \end{pmatrix} e^{\gamma_r t} + \bar{b}_r \begin{pmatrix} 1 \\ |P_{r2}| e^{-i\vartheta_{r2}} \\ \vdots \\ |P_{rn}| e^{-i\vartheta_{rn}} \end{pmatrix} e^{-\gamma_r t} \right) e^{-\sigma_r t}, \quad (1.49)$$

which can be rewritten as

$$\mathbf{x}(t) = \sum_{r=1}^n \begin{pmatrix} 1 \\ |P_{r2}| \\ \vdots \\ |P_{rn}| \end{pmatrix} \circ \left(b_r \begin{pmatrix} 1 \\ e^{i\vartheta_{r2}} \\ \vdots \\ e^{i\vartheta_{rn}} \end{pmatrix} e^{\gamma_r t} + \bar{b}_r \begin{pmatrix} 1 \\ e^{-i\vartheta_{r2}} \\ \vdots \\ e^{-i\vartheta_{rn}} \end{pmatrix} e^{-\gamma_r t} \right) e^{-\sigma_r t}, \quad (1.50)$$

where $\circ$ represents the Hadamard product of two vectors (element-by-element product). Using the Euler formula again, after the simplification from row-to-row, the solution can

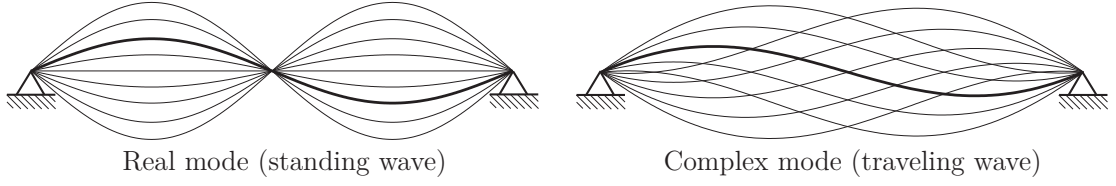


Figure 1.3. Difference between real and complex mode shapes [6].

be written as

$$\mathbf{x}(t) = \sum_{r=1}^n \begin{pmatrix} 1 \\ |P_{r2}| \\ \vdots \\ |P_{rn}| \end{pmatrix} \circ \left(B_{r1} \begin{pmatrix} \cos(\gamma_r t) \\ \cos(\gamma_r t + \vartheta_{r2}) \\ \vdots \\ \cos(\gamma_r t + \vartheta_{rn}) \end{pmatrix} + B_{r2} \begin{pmatrix} \sin(\gamma_r t) \\ \sin(\gamma_r t + \vartheta_{r2}) \\ \vdots \\ \sin(\gamma_r t + \vartheta_{rn}) \end{pmatrix} \right) e^{-\sigma_r t}, \quad (1.51)$$

and $B_{r1} = b_r + \bar{b}_r$, $B_{r2} = (b_r - \bar{b}_r)i$ are real valued, furthermore b_r and $\bar{b}_r$ are determined by the initial conditions. Eq. (1.47) and eq. (1.51) are equivalent, since the shifted trigonometric terms can be expressed by pure sines and cosines. However it clearly shows, that the mode shape can be described by a traveling wave, and the maximum and minimum points are reached at different time instants (shifted by a phase angle ϑ). In conclusion, the following properties hold for non-proportional damping [6]

- Mode shapes are complex valued, so-called complex modes.
- Mode shapes of the undamped and non-proportionally damped system are not the same.
- Equations cannot be decoupled in the conventional form, but it can be in the special first-order form.
- The maximum and minimum points of the vibration are not reached at the same time instant.
- The vibration can be represented by a traveling wave.
- The coordinates of the mode shapes are not in-phase or out-of-phase, there is a given phase shift between them.
- The node points do not remain fixed in time.

1.2.3 FRF for non-proportionally damped systems

It was presented that the conventional modal decomposition does not hold for non-proportionally damped systems, however in the first-order representation the transformed system matrices can have diagonal form. Let us use the equation of motion for forced vibrations

$$\mathbf{A}\dot{\mathbf{v}}(t) + \mathbf{B}\mathbf{v}(t) = \mathbf{f}_v(t), \quad (1.52)$$

where $\mathbf{f}_v(t) = (\mathbf{f}(t) \ \mathbf{0})^T$ is the general load vector. Similarly to proportional cases, the modal coordinates can be introduced as $\mathbf{v}(t) = \Psi \mathbf{q}_v(t)$. Using the Laplace transformation, and arranging the terms yields

$$(\mathbf{A}\Psi_s + \mathbf{B}\Psi) \mathbf{Q}_v(s) = \mathbf{F}_v(s), \quad (1.53)$$

where $\mathbf{V}(s) = \mathcal{L}\{\mathbf{v}(t)\}(s)$, $\mathbf{Q}_v(s) = \mathcal{L}\{\mathbf{q}_v(t)\}(s)$ and $\mathbf{F}_v(s) = \mathcal{L}\{\mathbf{f}_v(t)\}(s)$. Both sides of the above equation can be multiplied by Ψ^T , i.e.

$$(\Psi^T \mathbf{A} \Psi_s + \Psi^T \mathbf{B} \Psi) \mathbf{Q}_v(s) = \Psi^T \mathbf{F}_v(s), \quad (1.54)$$

and the transfer function matrix $\mathbf{W}(s)$ can be formulated as

$$\mathbf{W}(s) = \mathbf{V}(s) \mathbf{F}_v^{-1}(s) = \Psi \mathbf{Q}_v(s) \mathbf{F}_v^{-1}(s) = \Psi (\Psi^T \mathbf{A} \Psi_s + \Psi^T \mathbf{B} \Psi)^{-1} \Psi^T. \quad (1.55)$$

Since $\Psi^T \mathbf{A} \Psi = \mathbf{I}$ and $\Psi^T \mathbf{B} \Psi = -\Lambda$ are diagonal matrices, the inverse can be given with the reciprocates of the diagonal elements.

$$\mathbf{W}(s) = \Psi (\mathbf{I}s - \Lambda)^{-1} \Psi^T = \Psi \text{diag} \left(\begin{array}{cc} \frac{1}{s-\lambda_i} & 0 \\ 0 & \frac{1}{s-\bar{\lambda}_i} \end{array} \right) \Psi^T \Big|_{i=1 \dots n} \quad (1.56)$$

The frequency response function can simply be obtained

$$H_{ij}(\omega) = \sum_{r=1}^n \left(\frac{\psi_{ri} \psi_{rj}}{\omega i - \lambda_r} + \frac{\bar{\psi}_{ri} \bar{\psi}_{rj}}{\omega i - \bar{\lambda}_r} \right), \quad (1.57)$$

where ij represents the row and column of the frequency response function matrix. Assuming that the first coordinate is measured and excited, similarly to the proportional case, the corresponding $H_{11}(\omega)$ reads

$$H_{11}(\omega) = \sum_{r=1}^n \left(\frac{\psi_{r1}^2}{\omega i - \lambda_r} + \frac{\bar{\psi}_{r1}^2}{\omega i - \bar{\lambda}_r} \right). \quad (1.58)$$

Note, that if the system is proportionally damped, ψ_{ri}^2 is purely imaginary and the expressions of the non-proportional damping can be applied to proportional cases (see Appendix A.1).

1.3 Numerical examples

The introduced methods can be tested for MDoF systems if the FRF is obtained. In order to make the analysis more realistic and to take into account the effect of noise in the appropriate way, a few steps has to be made backwards. Instead of using the exact FRF, let us consider a fictitious measurement. In reality, in most of the cases, the modal parameters are determined from shaking or impact tests [2]. In this work, impact testing is considered (see Figure 1.4).

By measuring the input and output, the transfer function of the system can be determined according to eq. (1.28). As the formula defines, the displacement and the input force must be known. In practice, the acceleration and input force can be measured. Since the displacement can be calculated from the acceleration, the FRF can be theoretically measured. The accuracy of the approximation depends on the sampling rate f_s (given in Hz) and on the noise. For convenience, let us use the displacement as a ‘measured’ data and add artificial noise to it later. Since the length of the data is not infinite and the sampling is also present, the numerical examples will always have some inaccuracies.

1.3.1 Simulation

During the measurement, the tool-tip is hit by the impulse hammer and the resulting vibration is measured. In the numerical simulation, the excitation is not given, but an equivalent initial condition for the corresponding unit impulse can be calculated. In the following sections, the numerical examples are presented for a three-DoF system. The corresponding data can be seen in Table 1.1.

Type of damping	Mass [kg]			Damping $\left[\frac{Ns}{m}\right] \cdot 10^2$			Stiffness $\left[\frac{N}{m}\right] \cdot 10^6$		
	m_1	m_2	m_3	c_1	c_2	c_3	k_1	k_2	k_3
Proportional	4	5	100	5	2.5	25	5	2.5	25
Non-proportional	4	5	100	1	15	20	5	2.5	25

Table 1.1. Data for the three-DoF example with different damping models.

Before the impact the system is at rest, therefore the displacements and velocities are zero. At the moment of the impact, if the force is known, then the velocity of the first lumped mass (m_1) can be calculated from the impulse of the force

$$\Delta I_1 = m_1 v_1 + m_1 u_1 = - \int_{t_0}^{t_0+\tau} f(t) dt = -1, \quad (1.59)$$

where u_1 is the velocity before the impact, v_1 is the velocity after it, $f(t)$ is the force impulse (ideally only a Dirac-delta $\delta(t)$ impulse) and τ is the duration of the impact. Since c_1 is zero and the integral on the right hand side is -1 by definition, it yields

$$\begin{aligned} x_i(t_0) &= 0, \quad i = 1, 2, 3(\dots n), \\ \dot{x}_1(t_0) &= -\frac{1}{m_1} \quad \text{and} \quad \dot{x}_i(t_0) = 0, \quad i = 2, 3(\dots n). \end{aligned} \quad (1.60)$$

The solution for $x_1(t)$ can be obtained analytically by

$$\mathbf{v}(t) = e^{\mathbf{Z}(t-t_0)} \mathbf{v}(t_0), \quad (1.61)$$

where $\mathbf{v}(t)$ and $\mathbf{Z}$ are given in eq. (1.37) and eq. (1.38), while $\mathbf{v}(t_0)$ is the corresponding initial condition. After obtaining a sampled time-domain solution, using the Fast Fourier

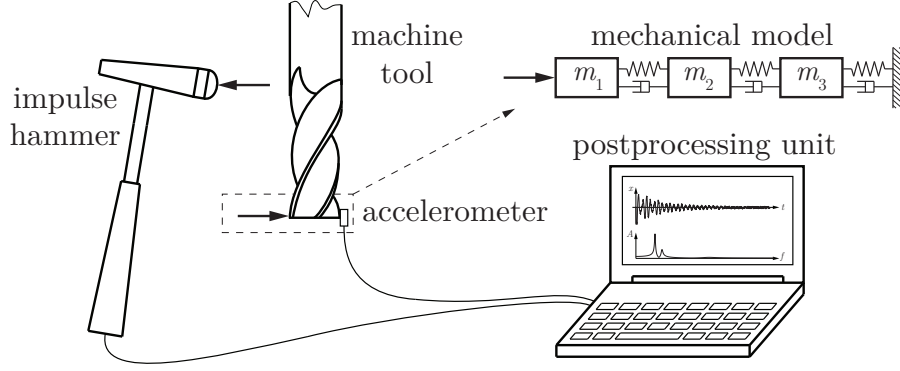


Figure 1.4. Experimental setup for impact testing

Transformation (FFT), the FRF can be obtained. For the given examples, the built-in functions of MATLAB were used (*fft* and *fftshift*). Figure 1.5 presents an example with $f_s = 4$ kHz.

1.3.2 Curve fitting

According to [10], some linear curve fitting techniques (Linear Least Squares) cannot handle the measured FRF with high precision, but nonlinear techniques, like Nonlinear Least Squares (NLLS) can give extremely good result in special cases. On the other hand, the frequently used *Rational Fraction Polynomial Method*, which leads to a solution of linear equations, has difficulties in practice [11]. During the fitting process, the stable roots must be selected, while the number of degrees of freedom is increased from step to step. This method is efficient if high number of degrees of freedom is considered [2, 8, 11].

In order to get accurate results for simple models, the built-in *lsqcurvefit* function of MATLAB was used, which utilizes an NLLS method. The function, which has to be minimized is

$$\min ||H_{\text{fitted}}(\omega) - H_{\text{measured}}(\omega)||_2^2, \quad (1.62)$$

where $H_{\bullet}(\omega)$ is the complex FRF. The algorithm fits the modal parameters for the real part and imaginary part of the frequency response function simultaneously. Although the selected function can handle complex data, it was found to be more suitable to separate the complex function into real and imaginary part, because the used trust-region-reflective algorithm operates with real data only. Therefore the correct minimization criterion is

$$\min \left\| \begin{pmatrix} H_{\text{fitted}}^{\text{Re}}(\omega) \\ H_{\text{fitted}}^{\text{Im}}(\omega) \end{pmatrix} - \begin{pmatrix} H_{\text{measured}}^{\text{Re}}(\omega) \\ H_{\text{measured}}^{\text{Im}}(\omega) \end{pmatrix} \right\|_2^2, \quad (1.63)$$

where ‘Re’ and ‘Im’ stands for the real and imaginary part. During the iteration, lower and upper boundaries for the unknown modal parameters can be given and unrealistic results can be avoided. The measured FRF can be approximated if the most dominant frequencies

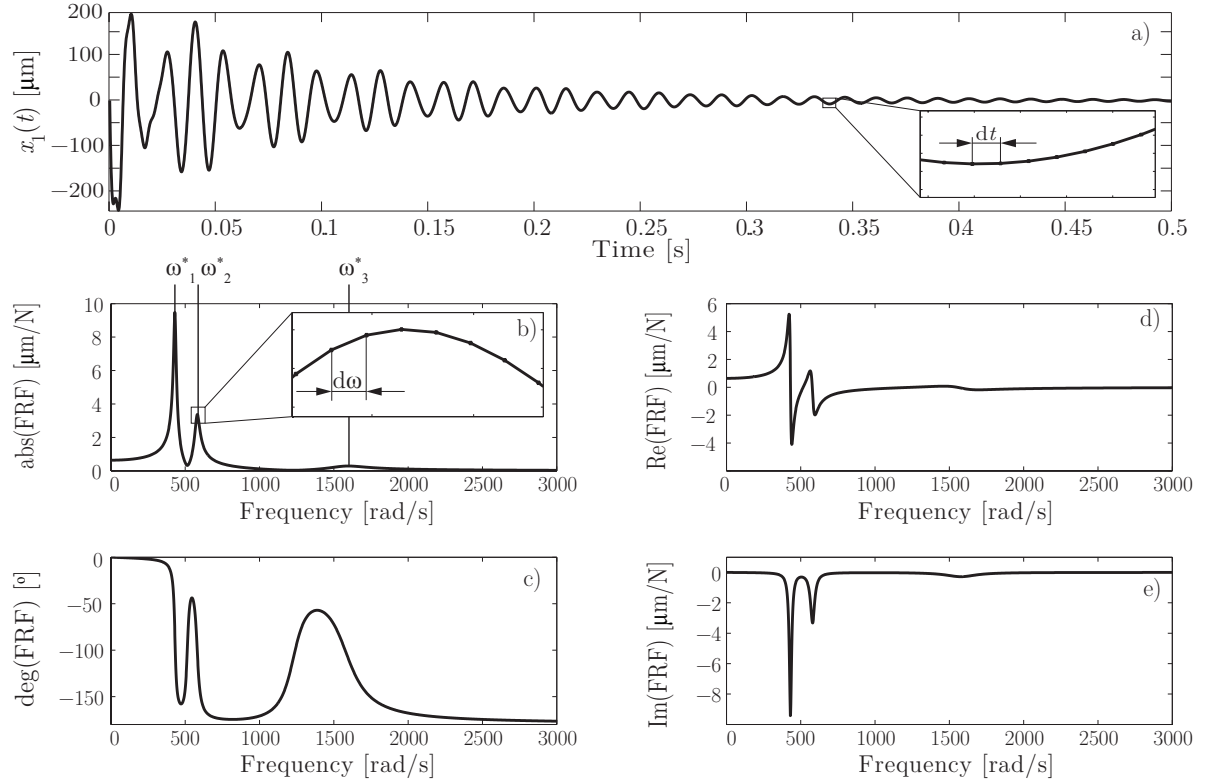


Figure 1.5. Displacement $x_1(t)$ and the different views of the corresponding frequency response function for $f_s = 4$ kHz.

are taken into account. The ω_i^* resonance frequency is close to the $\omega_{n,i}$ natural angular frequency for slightly damped cases. Therefore the initial values for the iterations of the *lsqcurvefit* function can be given assuming slight damping and measuring the resonance frequencies.

1.3.3 Example for the estimation of proportional damping

As it was presented in Subsection 1.1.4, the parametric form of the FRF in the modal space in case of proportional damping for a model with n degrees of freedom reads

$$H(\omega) = \sum_{r=1}^n \frac{\phi_{r1}^2}{-\omega^2 + 2\xi_r \omega_{n,r} \omega i + \omega_{n,r}^2}. \quad (1.64)$$

The unknown parameters are ϕ_{r1} , ξ_r , $\omega_{n,r}$, furthermore $r = 1, 2 \dots n$ respectively and each of them are real valued numbers. This results $3n$ parameters.

Figure 1.6 presents an example for the curve fitting in case of a three-DoF model ($n = 3$). The displacement of the endpoint is measured with a sampling frequency of 4 kHz, while 5% noise was added to it (1% noise is a random distribution on the measured data with the amplitude of 1% of the maximum displacement). From the measured data, the FRF was determined by a FFT technique, and the fitting was performed. Black line

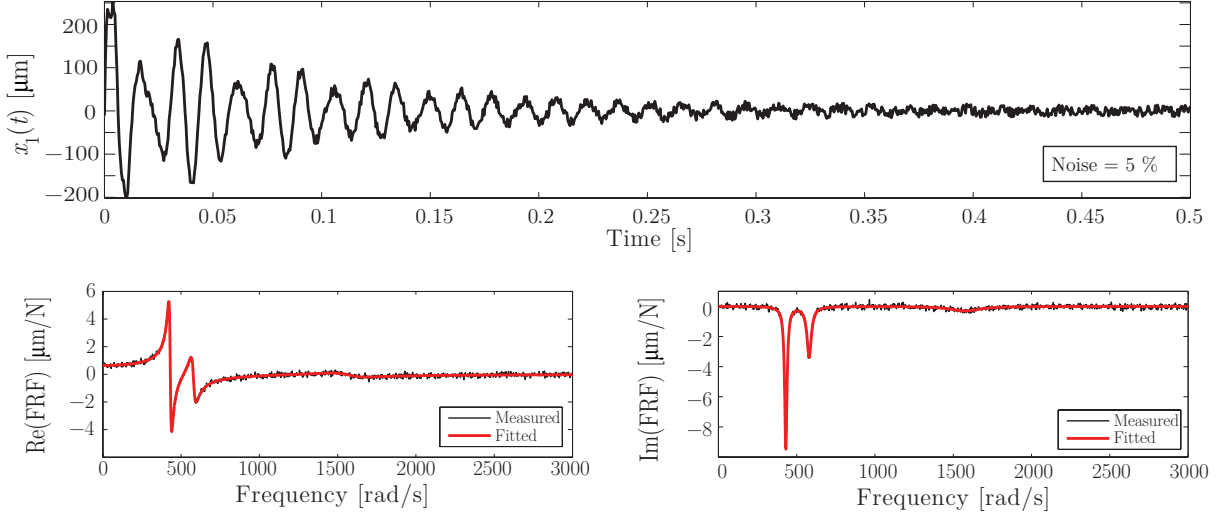


Figure 1.6. Fitted curve for the simulated noisy FRF assuming proportional damping ($f_s = 4$ kHz and $t_{\max} = 0.5$ s).

indicates the ‘measured data’, and red line indicates the fitted curve. Table 1.2 lists the results of the fitting for 0%, 2% and 5% noise respectively. When there is no noise, the fitting gives back the exact modal parameters extremely precisely although the sampling frequency is quite low. For higher sampling rate, the estimation converges to exact parameters. Even if the noise goes up to 2% or 5%, the estimation remains quite precise, the error of the estimated natural angular frequency is still under 1%. It can also be observed that the most dominant modes are estimated the most correctly, and the least significant modes can disappear in the noisy data. For instance, if the noise was larger, the identification of the third mode would be impossible, however, the first two frequencies would remain precisely estimable. Note that in this case, the used FRF is artificial, the fitting on real measured data can be more inaccurate.

Noise	ω_n [rad/s]			ξ [%]			ϕ [$\sqrt{\text{rad/kg}}$] $\cdot 10^2$		
Exact	430.88	579.85	1582.13	2.15	2.90	7.91	27.42	25.33	33.27
0%	430.79	579.85	1582.14	2.16	2.88	7.80	27.45	25.24	33.02
2%	430.79	579.94	1578.96	2.15	2.86	7.57	27.46	25.18	32.74
5%	430.81	579.90	1583.91	2.15	2.82	7.67	27.45	25.19	32.54

Table 1.2. Results of the parameter estimation for proportional damping ($f_s = 4$ kHz).

1.3.4 Example for the estimation of non-proportional damping

In several cases, the theory of non-proportional damping describes real phenomena in a more appropriate way, although it is still a simplification since nonlinearities and non-viscous damping are not taken into account. The steps of the fitting process are similar to the proportional case, but there are some differences. In this subsection, a non-

proportional case is studied where the parameters are listed in Table 1.1 ($n = 3$). This system is highly non-proportional, the phase of the complex mode shape vector varies between $1 \dots 50^\circ$ for the different modes.

As it was presented in Section 1.2.3, the FRF can be given in the form

$$H(\omega) = \sum_{r=1}^n \left(\frac{\psi_{r1}^2}{\omega i - \lambda_r} + \frac{\bar{\psi}_{r1}^2}{\omega i - \bar{\lambda}_r} \right), \quad (1.65)$$

where $\lambda_r = -\sigma_r + i\gamma_r$ and ψ_{r1}^2 is a complex number, which can be given as $\psi_{r1}^2 = U_{r1} + V_{r1}i$. Thus the FRF can be written in a more convenient form, which is suitable for the parameter fitting as

$$\begin{aligned} H(\omega) &= \sum_{r=1}^3 \left(\frac{U_r + V_r i}{\omega i - (-\sigma_r - \gamma_r i)} + \frac{U_r - V_r i}{\omega i - (-\sigma_r + \gamma_r i)} \right) \\ &= \sum_{r=1}^3 \left(\frac{U_r + V_r i}{\omega i + \sigma_r + \gamma_r i} + \frac{U_r - V_r i}{\omega i + \sigma_r - \gamma_r i} \right), \end{aligned} \quad (1.66)$$

furthermore $\sigma_r = \xi_r \omega_{n,r}$ and $\gamma_r = \sqrt{1 - \xi_r^2} \omega_{n,r}$. In this representation, the unknown modal parameters which must be determined are U_r , V_r , ξ_r , $\omega_{n,r}$ and $r = 1, 2 \dots n$, which results $4n$ parameters.

Similarly to the proportional case, a nonlinear curve fitting technique was utilized in order to achieve the best result. The initial values of the iteration were given by assuming slight damping and the resonance frequencies were measured for the estimation of the natural angular frequencies. The results are listed in Table 1.3 and can be seen in Figure 1.7. In this simple example, it can be seen that the estimation of the modal parameters are excellent, even for the noisy results.

Noise	ω_n [rad/s]			ξ [%]			ψ [$\sqrt{\text{rad} \cdot \text{s/kg}}$] $\cdot 10^3$		
Exact	436.11	575.93	1573.79	5.72	10.13	6.78	5.64-7.53i	7.09-4.13i	3.72-4.70i
0%	436.11	576.10	1574.41	5.71	10.13	6.82	5.64-7.53i	7.08-4.14i	3.71-4.73i
2%	436.09	576.53	1570.63	5.68	10.09	6.84	5.65-7.50i	7.03-4.13i	3.77-4.70i
5%	436.01	576.64	1566.13	5.67	10.32	7.46	5.63-7.50i	6.98-4.21i	3.90-4.69i

Table 1.3. Results of the parameter estimation for non-proportional damping ($f_s = 4$ kHz).

1.4 Toward higher order modal models

In reality, most of the structures cannot be characterized by only a purely one-dimensional model. When several modes play an important role in the system's dynamics, then the estimation of the modal parameters require a sophisticated modal analysis of the combined structure-tool-workpiece system [4]. In this section, a brief introduction is given how the FRF matrix can be formulated for more complicated models.

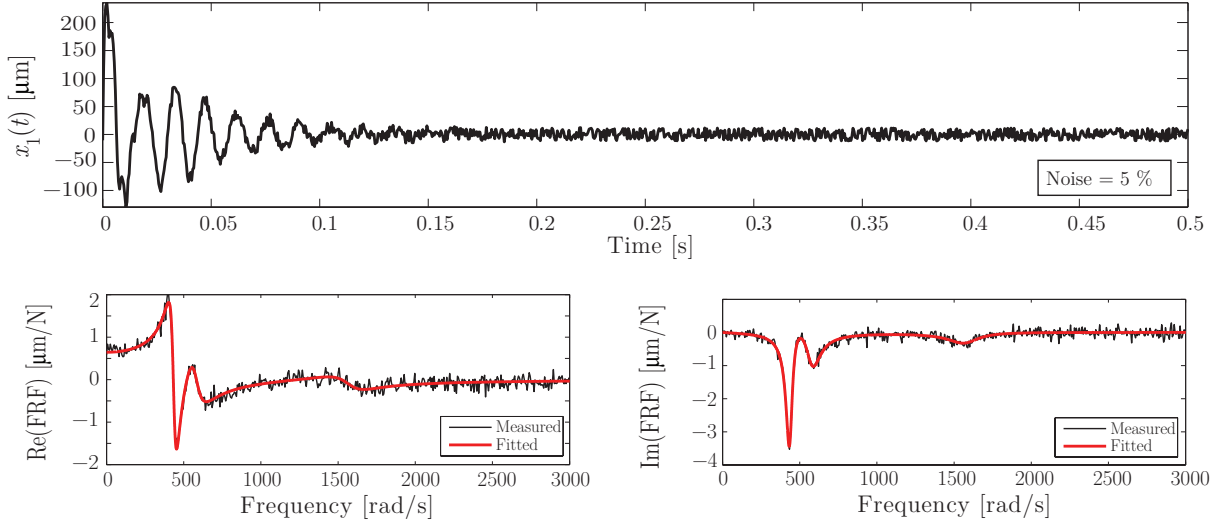


Figure 1.7. Fitted curve for the simulated noisy FRF assuming non-proportional damping ($f_s = 4$ kHz and $t_{\max} = 0.5$ s).

1.4.1 Coupled modes

The first step for the extension of the dynamical model is the consideration of a two-dimensional system, which is widely used to describe the modal behavior of real structures [2, 4, 8, 15]. A simple two-dimensional model is presented in Figure 1.8 a), which characterizes the dominant modes in direction x and y . The equation of motion in this case can be formulated as

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} \ddot{x}(t) \\ \ddot{y}(t) \end{pmatrix} + \begin{pmatrix} c_x & 0 \\ 0 & c_y \end{pmatrix} \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} + \begin{pmatrix} k_x & 0 \\ 0 & k_y \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} f_x(t) \\ f_y(t) \end{pmatrix}. \quad (1.67)$$

It can be seen that the system matrices are already diagonal. The excitation of the system in direction x does not induce vibration in direction y (and vice versa), therefore the natural frequencies of direction x does not appear in the frequency response function corresponding to direction y . The vibration modes can be decoupled in the main directions, and the measured FRFs can be fitted independently.

A different two-degrees-of-freedom model is presented in Figure 1.8 b). It can be shown that if the system is excited either in direction x or y , the mass vibrates in both directions. The two natural frequencies also appear in the frequency response function measured in both directions. In general, the equation of motion for a coupled case can be formulated as [18]

$$\begin{pmatrix} m_{xx} & m_{xy} \\ m_{yx} & m_{yy} \end{pmatrix} \begin{pmatrix} \ddot{x}(t) \\ \ddot{y}(t) \end{pmatrix} + \begin{pmatrix} c_{xx} & c_{xy} \\ c_{yx} & c_{yy} \end{pmatrix} \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} + \begin{pmatrix} k_{xx} & k_{xy} \\ k_{yx} & k_{yy} \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} f_x(t) \\ f_y(t) \end{pmatrix}, \quad (1.68)$$

where $(\bullet)_{xx}$ and $(\bullet)_{yy}$ are the main modal parameters, while $(\bullet)_{xy} = (\bullet)_{yx}$ are the symmetric cross terms expressing the coupling effect.

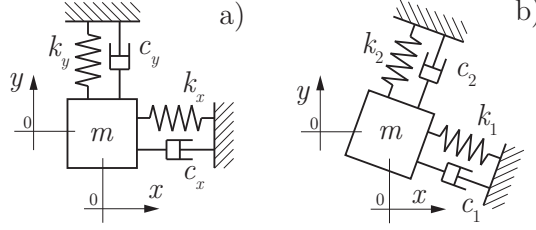


Figure 1.8. Two-dimensional model with planar mode shapes.

The frequency response function can be formulated similarly to the previous sections. Let us define the modal transformation matrix Φ of the proportional system as

$$\Phi = \begin{pmatrix} \phi_{xx} & \phi_{xy} \\ \phi_{yx} & \phi_{yy} \end{pmatrix} \quad (1.69)$$

and based on eq. (1.34) and eq. (1.35), the frequency response function matrix can be formulated as

$$\mathbf{H}(\omega) = \begin{pmatrix} H_{xx}(\omega) & H_{xy}(\omega) \\ H_{yx}(\omega) & H_{yy}(\omega) \end{pmatrix}. \quad (1.70)$$

The frequency response functions for the main and cross directions read

$$H_{xx}(\omega) = \frac{\phi_{xx}\phi_{xx}}{-\omega^2 + 2\xi_x i\omega_{n,x}\omega + \omega_{n,x}^2} + \frac{\phi_{xy}\phi_{xy}}{-\omega^2 + 2\xi_y i\omega_{n,y}\omega + \omega_{n,y}^2}, \quad (1.71)$$

$$H_{xy}(\omega) = \frac{\phi_{xx}\phi_{yx}}{-\omega^2 + 2\xi_x i\omega_{n,x}\omega + \omega_{n,x}^2} + \frac{\phi_{xy}\phi_{yy}}{-\omega^2 + 2\xi_y i\omega_{n,y}\omega + \omega_{n,y}^2}, \quad (1.72)$$

$$H_{yx}(\omega) = \frac{\phi_{yx}\phi_{xx}}{-\omega^2 + 2\xi_x i\omega_{n,x}\omega + \omega_{n,x}^2} + \frac{\phi_{yy}\phi_{xy}}{-\omega^2 + 2\xi_y i\omega_{n,y}\omega + \omega_{n,y}^2}, \quad (1.73)$$

$$H_{yy}(\omega) = \frac{\phi_{yx}\phi_{yx}}{-\omega^2 + 2\xi_x i\omega_{n,x}\omega + \omega_{n,x}^2} + \frac{\phi_{yy}\phi_{yy}}{-\omega^2 + 2\xi_y i\omega_{n,y}\omega + \omega_{n,y}^2}, \quad (1.74)$$

where $H_{xy}(\omega)$ indicates that the displacement is measured in direction x , but the system is excited in direction y . If the cross terms are assumed to be zero ($\phi_{xy} = \phi_{yx} = 0$), the non-diagonal elements of the FRF matrix are also zero, i.e. $H_{xy}(\omega) = H_{yx}(\omega) = 0$. Note, that $H_{xy}(\omega) = H_{yx}(\omega)$ holds if the system matrices $\mathbf{M}$, $\mathbf{K}$, $\mathbf{C}$ are symmetric, and as a result the theoretical frequency response function matrix is also symmetric ($\mathbf{H}(\omega) = \mathbf{H}^T(\omega)$) [2]. Also note, that the natural frequencies $\omega_{n,x}$ and $\omega_{n,y}$ appear in each direction, which means that those have a general vibration mode and does not correspond purely to direction x or y .

In practice, the modal behavior of the structure cannot be determined only from one measurement. In the simplest case, when a tip-to-tip measurement is performed, the frequency response functions must be calculated in the main directions (xx and yy) and for the cross directions (xy and yx) too, resulting four different complex functions. The above presented simple formulation therefore has to be extended for MDoF cases. Although the

spatial representation is convenient and can be visualized, the description in the modal space provides a simpler way to form the frequency response functions. There is no need to specify x_i and y_i coordinates and the corresponding spatial parameters, only the modal ones. If the elements of the modal transformation matrix is defined as ϕ_{ij} respectively, one can obtain the same parametric expression as before (see Section 1.1.4), i.e.

$$H_{ij}(\omega) = \sum_{r=1}^n \frac{\phi_{ir}\phi_{jr}}{-\omega^2 + 2\xi_r\omega_{n,r}\omega i + \omega_{n,r}^2}. \quad (1.75)$$

Note that the order of the subscripts in the numerator are changed compared to the one used in Section 1.1.4 in order to follow the standard notation of the literature [6]. In case of a two-degree-of-freedom system shown in Figure 1.8, the substitution $i = x, y$, $j = x, y$ and $r = x, y$ gives $H_{xx}(\omega)$, $H_{xy}(\omega)$, $H_{yx}(\omega)$, $H_{yy}(\omega)$ shown above. If an n -degrees-of-freedom model is assumed, ϕ_{ir} , ϕ_{jr} , $\omega_{n,r}$ and ξ_r must be fitted ($r = 1, 2 \dots n$). It means that in case of proportional damping, four parameters must be determined for each degree of freedom. Once these are obtained, the i^{th} and j^{th} row of the modal transformation matrix Φ is known. Note, that in case of tip-to-tip measurements only four FRFs are known, two for the main directions and two for the cross functions. The example on stability analysis of milling processes is presented in Section 2.3.

In case of non-proportional damping, the same theory holds. The rows of Ψ can be determined by fitting the frequency response function

$$H_{ij}(\omega) = \sum_{r=1}^n \left(\frac{\psi_{ir}\psi_{jr}}{\omega i - \lambda_r} + \frac{\bar{\psi}_{ri}\bar{\psi}_{rj}}{\omega i - \bar{\lambda}_r} \right) \quad (1.76)$$

on the main directions and cross directions too. Therefore ψ_{ir} , ψ_{jr} , $\omega_{n,r}$ and ξ_r must be approximated ($r = 1, 2 \dots n$), where $\psi_{\bullet r}$ is complex, so altogether six parameters must be determined for each degree of freedom.

If $H_{xx}(\omega)$, $H_{xy}(\omega)$, $H_{yx}(\omega)$, $H_{yy}(\omega)$ complex functions are measured, then the minimization criterion is

$$\min \left\| \begin{pmatrix} H_{xx,\text{fitted}}^{\text{Re}}(\omega) \\ H_{xx,\text{fitted}}^{\text{Im}}(\omega) \\ H_{xy,\text{fitted}}^{\text{Re}}(\omega) \\ H_{xy,\text{fitted}}^{\text{Im}}(\omega) \\ H_{yx,\text{fitted}}^{\text{Re}}(\omega) \\ H_{yx,\text{fitted}}^{\text{Im}}(\omega) \\ H_{yy,\text{fitted}}^{\text{Re}}(\omega) \\ H_{yy,\text{fitted}}^{\text{Im}}(\omega) \end{pmatrix} - \begin{pmatrix} H_{xx,\text{measured}}^{\text{Re}}(\omega) \\ H_{xx,\text{measured}}^{\text{Im}}(\omega) \\ H_{xy,\text{measured}}^{\text{Re}}(\omega) \\ H_{xy,\text{measured}}^{\text{Im}}(\omega) \\ H_{yx,\text{measured}}^{\text{Re}}(\omega) \\ H_{yx,\text{measured}}^{\text{Im}}(\omega) \\ H_{yy,\text{measured}}^{\text{Re}}(\omega) \\ H_{yy,\text{measured}}^{\text{Im}}(\omega) \end{pmatrix} \right\|_2^2. \quad (1.77)$$

Note, that overall six functions are fitted onto eight measured functions, since $H_{xy,\text{fitted}}(\omega) = H_{yx,\text{fitted}}(\omega)$ due to symmetry.

1.4.2 Non-symmetry in system matrices

All the cases considered until now have contained certain properties which make them less than the most general type of linear systems that can be encountered. There is another class of system which has to be mentioned. These system are affected by additional forces, for example gyroscopic forces, rotor-stator rub forces, electromagnetic forces, unsteady aerodynamic forces or time-varying fluid forces [6]. As a result, any of these phenomena can destroy the symmetry of the system matrices and none of the previously presented methods can be applied.

Fortunately, from the mathematical point of view, this problem can be treated in some cases, for instance in case of gyroscopic effect [6, 12]. The mode shapes are generally complex, and the non-proportional damping representation can be considered in the form

$$\mathbf{A}\dot{\mathbf{v}}(t) + \mathbf{B}\mathbf{v}(t) = \mathbf{f}_v(t), \quad \mathbf{v}(t) = \begin{pmatrix} \mathbf{x}(t) & \mathbf{y}(t) & \dot{\mathbf{x}}(t) & \dot{\mathbf{y}}(t) \end{pmatrix}^T, \quad (1.78)$$

where $\mathbf{A}$ and $\mathbf{B}$ can be both non-symmetric. The diagonal transformation can be performed if the left and right eigenvalues of system are calculated as [12]

$$(\mathbf{A}\lambda_i + \mathbf{B})\mathbf{U}_{R,i} = \mathbf{0}, \quad (1.79)$$

$$(\mathbf{A}^T\lambda_i + \mathbf{B}^T)\mathbf{U}_{L,i} = \mathbf{0}, \quad (1.80)$$

where $\mathbf{U}_{R,i}$ and $\mathbf{U}_{L,i}$ are the unnormalized right and left eigenvectors respectively. The physical meanings of the eigenvalues and right eigenvectors can be derived from the equation of motion of a system. However, the meaning of the left eigenvectors cannot be deduced in a straightforward manner [12]. According to [6], the right eigenvectors represent the mode shapes themselves while the left ones are associated with preferred excitation patterns.

The modal transformation matrices has to be normalized according to

$$\Psi_{R,i} = \frac{\mathbf{U}_{R,i}}{\sqrt{\mathbf{U}_{L,i}^T \mathbf{A} \mathbf{U}_{R,i}}} \quad \text{and} \quad \Psi_{L,i} = \frac{\mathbf{U}_{L,i}}{\sqrt{\mathbf{U}_{R,i}^T \mathbf{A}^T \mathbf{U}_{L,i}}}, \quad (1.81)$$

therefore the normalized eigenvectors satisfy the following properties

$$\Psi_L^T \mathbf{A} \Psi_R = \mathbf{I} \quad \text{and} \quad \Psi_L^T \mathbf{B} \Psi_R = -\Lambda. \quad (1.82)$$

The final structure of the modal matrices is

$$\Psi_{\bullet} = \begin{pmatrix} \psi_{11}^{\bullet} & \bar{\psi}_{11}^{\bullet} & \vdots & \psi_{1n}^{\bullet} & \bar{\psi}_{1n}^{\bullet} \\ \psi_{21}^{\bullet} & \bar{\psi}_{21}^{\bullet} & \vdots & \psi_{2n}^{\bullet} & \bar{\psi}_{2n}^{\bullet} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \psi_{n1}^{\bullet} & \bar{\psi}_{n1}^{\bullet} & \vdots & \psi_{nn}^{\bullet} & \bar{\psi}_{nn}^{\bullet} \end{pmatrix}, \quad (1.83)$$

where $\bullet$ denotes the R or L eigenvector. Based on Section 1.2, using the modal transformation $\mathbf{v}(t) = \mathbf{\Psi}_R \mathbf{q}_v(t)$ in the form

$$\mathbf{\Psi}_L^T \mathbf{A} \mathbf{\Psi}_R \dot{\mathbf{q}}_v(t) + \mathbf{\Psi}_L^T \mathbf{B} \mathbf{\Psi}_R \mathbf{q}_v(t) = \mathbf{\Psi}_L^T \mathbf{f}_v(t) \quad (1.84)$$

the frequency response function matrix can be formulated as

$$\mathbf{H}(\omega) = \mathbf{\Psi}_R \text{diag} \left(\begin{array}{cc} \frac{1}{i\omega - \lambda_i} & 0 \\ 0 & \frac{1}{i\omega - \bar{\lambda}_i} \end{array} \right) \mathbf{\Psi}_L^T \Big|_{i=1 \dots n}. \quad (1.85)$$

The parametric expression for each element reads

$$H_{ij}(\omega) = \sum_{r=1}^n \left(\frac{\psi_{ri}^R \psi_{rj}^L}{\omega i - \lambda_r} + \frac{\bar{\psi}_{ri}^R \bar{\psi}_{rj}^L}{\omega i - \bar{\lambda}_r} \right). \quad (1.86)$$

It can be shown, that $H_{ij} \neq H_{ji}$ meaning that $\mathbf{H}(\omega) \neq \mathbf{H}^T(\omega)$. Usually the measured cross frequency functions are close to each other, but sometimes significant differences can be observed, which can change the dynamical behavior.

Note, that during the fitting process, the measured non-symmetric cross frequency response functions have to be taken into account. The unknown parameters are $\omega_{n,r}$, ξ_r , ψ_{ri}^R , ψ_{ri}^L , ψ_{rj}^R , ψ_{rj}^L and $r = 1, 2 \dots n$. Since ψ is always complex, this results ten real parameters for each degree of freedom. The example on stability analysis of milling processes is presented in Section 2.3.

If $H_{xx}(\omega)$, $H_{xy}(\omega)$, $H_{yx}(\omega)$, $H_{yy}(\omega)$ complex functions are measured, then the minimization criterion is

$$\min \left\| \begin{pmatrix} H_{xx,\text{fitted}}^{\text{Re}}(\omega) \\ H_{xx,\text{fitted}}^{\text{Im}}(\omega) \\ H_{xy,\text{fitted}}^{\text{Re}}(\omega) \\ H_{xy,\text{fitted}}^{\text{Im}}(\omega) \\ H_{yx,\text{fitted}}^{\text{Re}}(\omega) \\ H_{yx,\text{fitted}}^{\text{Im}}(\omega) \\ H_{yy,\text{fitted}}^{\text{Re}}(\omega) \\ H_{yy,\text{fitted}}^{\text{Im}}(\omega) \end{pmatrix} - \begin{pmatrix} H_{xx,\text{measured}}^{\text{Re}}(\omega) \\ H_{xx,\text{measured}}^{\text{Im}}(\omega) \\ H_{xy,\text{measured}}^{\text{Re}}(\omega) \\ H_{xy,\text{measured}}^{\text{Im}}(\omega) \\ H_{yx,\text{measured}}^{\text{Re}}(\omega) \\ H_{yx,\text{measured}}^{\text{Im}}(\omega) \\ H_{yy,\text{measured}}^{\text{Re}}(\omega) \\ H_{yy,\text{measured}}^{\text{Im}}(\omega) \end{pmatrix} \right\|_2^2. \quad (1.87)$$

Here eight functions are fitted onto eight measured functions, since $H_{xy,\text{fitted}}(\omega) \neq H_{yx,\text{fitted}}(\omega)$ due to non-symmetry.

Chapter 2

Machine tool vibrations

Time delay takes an important role in several engineering application like wheel shimmy, car-following traffic models, control systems and machine tool chatter [4]. In this chapter, simply engineering models are considered for turning and milling processes and the sensitivity of the stability charts are investigated with respect to the variation of the frequency response function due to the change of modal parameters.

2.1 Turning

In the 1960s, after the extensive work of Tobias, Thusty and several other researchers, the so-called *regenerative effect* became the most commonly accepted explanation for machine tool chatter [4]. During the cutting process, the tool experiences bending vibration which modifies the chip thickness and leaves a wavy surface behind. Figure 2.1 represents the ideal case for a completely rigid and for a compliant tool considering the different vibration modes. In the latter case, the varying chip thickness induces time-dependent cutting force.

Stability properties of the machining processes are depicted by the so-called *stability lobe diagrams*, which plot the maximum stable depths of cut versus the spindle speed. These diagrams provide a guide to the machinist to select the optimal technological parameters in order to achieve maximum material removal rate without chatter [4].

A simple mechanical model can be seen in Figure 2.2, where the vibrations in direction y are neglected. In contrast, the vibrations in direction x , which is the direction of the feed are modeled by a MDoF model. Using this assumption, the governing equation reduces to simple retarded delay-differential equation with constant time delay.

Based on Figure 2.2, the equation of motion in case of MDoF systems reads

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{f}(t), \quad (2.1)$$

where $\mathbf{f}(t) = (F_x(t) \ 0 \dots 0)^T$, $\mathbf{f} \in \mathbb{R}^n$ and based on experimental results, the cutting-force is approximated as

$$F_x(t) = K_x w h^q(t). \quad (2.2)$$

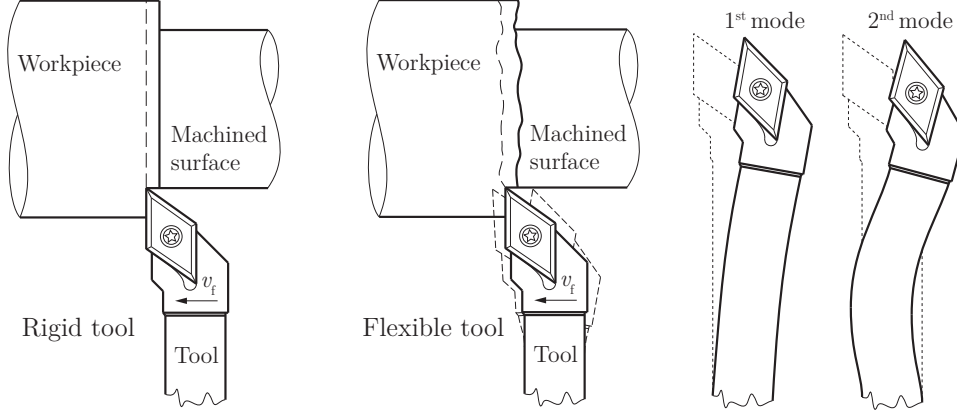


Figure 2.1. Chip removal process considering the flexibility of the tool.

K_x is the cutting-force coefficient in the tangential direction x , w is the depth of cut, $h(t)$ is the instantaneous chip thickness and q is the cutting-force exponent. If the tool was rigid, the chip thickness would be constant $h(t) = h_0$. However, since the tool experiences vibration, the wavy surface affects the cutting force after one revolution. The chip thickness is determined by the feed motion, the current tool position and the previous position of the tool one revolution before. For constant spindle speed, the time delay can be given explicitly as $\tau = 60/\Omega$, where Ω is the revolution number given in rpm (1/min). The instantaneous chip thickness can be calculated as

$$h(t) = v_f \tau + x_1(t - \tau) - x_1(t), \quad (2.3)$$

where v_f is the feed velocity. Therefore the governing equation can be given as

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \begin{pmatrix} K_x w (v_f \tau + x_1(t - \tau) - x_1(t))^q \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (2.4)$$

The stability of the system can be analyzed by considering only the linear part. The general solution can be given as $\mathbf{x}(t) = \mathbf{x}_p + \boldsymbol{\xi}(t)$, where $\mathbf{x}_p$ is related to the static deformation and $\boldsymbol{\xi}(t)$ is a small perturbation around the equilibrium. After the linearization of the cutting force, expanding it into Taylor series and neglecting the higher-order terms, the variational system is given by

$$\mathbf{M}\ddot{\boldsymbol{\xi}}(t) + \mathbf{C}\dot{\boldsymbol{\xi}}(t) + \mathbf{K}\boldsymbol{\xi}(t) = \boldsymbol{\kappa} (\boldsymbol{\xi}(t - \tau) - \boldsymbol{\xi}(t)), \quad (2.5)$$

and

$$\boldsymbol{\kappa} = \begin{pmatrix} \kappa & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad \kappa = K_x w q (v_f \tau)^{q-1} = K_x w q h_0^{q-1}, \quad (2.6)$$

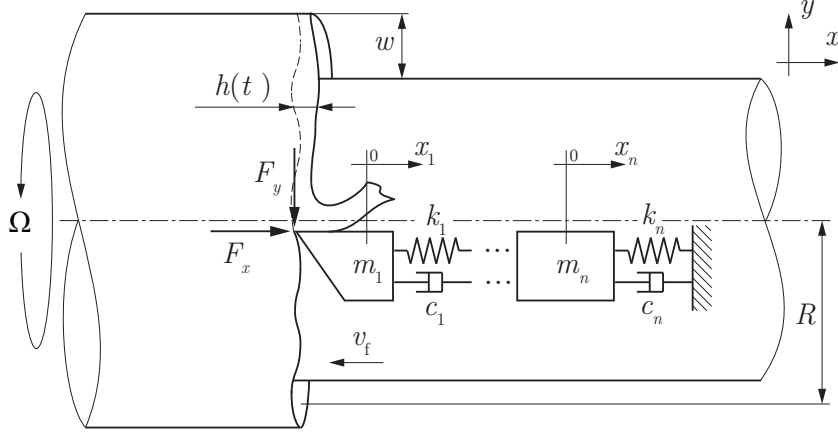


Figure 2.2. Surface regeneration in an orthogonal process.

where κ is the specific cutting-force coefficient and h_0 is the constant chip thickness. Note, that κ is linearly proportional to the depth of cut w which is an important machining parameter. The detailed derivations can be found in [4].

2.1.1 Frequency domain

Up to this point, the derivation was done in time domain. Due to the topic of the present work, frequency-domain stability analysis is presented using directly the FRF, but the stability boundaries can also be determined in time domain using the D-subdivision method if the parameters of the corresponding mechanical model are known [4].

As it was already mentioned in Section 1.1.4, the frequency response function matrix of a system can be given by the Fourier transform of the output and the inverse of the input as

$$\mathbf{H}(\omega) = \mathbf{X}(\omega) \mathbf{F}^{-1}(\omega). \quad (2.7)$$

Therefore the excitation, which is the cutting force can be written as

$$\mathbf{f}(t) = \kappa (\mathbf{x}(t - \tau) - \mathbf{x}(t)) \xrightarrow{\mathcal{F}} \mathbf{F}(\omega) = \kappa (e^{-i\omega\tau} - 1) \mathbf{X}(\omega). \quad (2.8)$$

After arranging the terms in eq. (2.7) then substituting it back into eq. (2.8), the expression reads

$$\mathbf{F}(\omega) = \kappa (e^{-i\omega\tau} - 1) \mathbf{H}(\omega) \mathbf{F}(\omega). \quad (2.9)$$

The equation above can be written as

$$(\mathbf{I} - (e^{-i\omega\tau} - 1) \kappa \mathbf{H}(\omega)) \mathbf{F}(\omega) = \mathbf{0}. \quad (2.10)$$

Since $\mathbf{F}(\omega) \neq \mathbf{0}$, $\det(\mathbf{I} - (e^{-i\omega\tau} - 1) \kappa \mathbf{H}(\omega)) = 0$ must hold. This expression can be simplified drastically because only the first coordinate (the tool tip) is excited. After

the multiplications, only the first row of $\mathbf{H}(\omega)$ remains, but when the determinant is calculated, the non-diagonal elements will also fall out, i.e.

$$\det \left(\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} - (e^{-i\omega\tau} - 1) \begin{pmatrix} \kappa & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} H_{11}(\omega) & H_{12}(\omega) & \dots & H_{1n}(\omega) \\ H_{21}(\omega) & H_{22}(\omega) & \dots & H_{2n}(\omega) \\ \vdots & \vdots & \ddots & \vdots \\ H_{n1}(\omega) & H_{n2}(\omega) & \dots & H_{nn}(\omega) \end{pmatrix} \right) =$$

$$\det \begin{pmatrix} 1 - \kappa(e^{-i\omega\tau} - 1)H_{11}(\omega) & 1 - \kappa(e^{-i\omega\tau} - 1)H_{12}(\omega) & \dots & -\kappa(e^{-i\omega\tau} - 1)H_{1n}(\omega) \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}. \quad (2.11)$$

Finally it gives

$$\det \left(\mathbf{I} - (e^{-i\omega\tau} - 1) \kappa \mathbf{H}(\omega) \right) = 1 - \kappa(e^{-i\omega\tau} - 1)H_{11}(\omega) = 0, \quad (2.12)$$

and it yields the inverse of the frequency response function as

$$H_{11}^{-1}(\omega) = \kappa(e^{-i\omega\tau} - 1) = \Lambda_{\text{Re}} + i\Lambda_{\text{Im}}. \quad (2.13)$$

Using the identity $e^{-i\omega\tau} = \cos(\omega\tau) - i\sin(\omega\tau)$, the real and imaginary parts of eq. (2.13) can be written as

$$\Lambda_{\text{Re}} = \kappa(\cos(\omega\tau) - 1) \quad \text{and} \quad \Lambda_{\text{Im}} = -\kappa\sin(\omega\tau). \quad (2.14)$$

Taking the fraction of the two expressions, κ can be eliminated

$$\frac{\Lambda_{\text{Re}}}{\Lambda_{\text{Im}}} = \frac{1 - \cos(\omega\tau)}{\sin(\omega\tau)}. \quad (2.15)$$

This formulation leads to a trigonometric identity, using

$$\left. \begin{aligned} \sin^2\left(\frac{x}{2}\right) &= \frac{1 - \cos(x)}{2} \\ \cos^2\left(\frac{x}{2}\right) &= \frac{1 + \cos(x)}{2} \end{aligned} \right\} \text{tg}\left(\frac{x}{2}\right) = \frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} = \pm \sqrt{\frac{1 - \cos(x)}{1 + \cos(x)}} = \dots$$

$$\dots = \pm \sqrt{\frac{(1 - \cos(x))(1 - \cos(x))}{(1 + \cos(x))(1 - \cos(x))}} = \pm \sqrt{\frac{(1 - \cos(x))^2}{1 - \cos^2(x)}} = \frac{1 - \cos(x)}{\sin(x)} \quad (2.16)$$

finally leads to

$$\frac{\Lambda_{\text{Re}}}{\Lambda_{\text{Im}}} = \frac{1 - \cos(\omega\tau)}{\sin(\omega\tau)} = \text{tg}\left(\frac{\omega\tau}{2}\right). \quad (2.17)$$

After solving the equation for τ , it gives

$$\tau = \frac{2}{\omega} \left(\arctg \left(\frac{\Lambda_{\text{Re}}}{\Lambda_{\text{Im}}} \right) + j\pi \right), \quad j = 0, 1, 2 \dots \quad (2.18)$$

Since $\tau = 60/\Omega$, the revolution number can be expressed as

$$\Omega = \frac{30\omega}{\arctg \left(\frac{\Lambda_{\text{Re}}}{\Lambda_{\text{Im}}} \right) + j\pi}, \quad j = 0, 1, 2 \dots \quad (2.19)$$

The square of the specific cutting force coefficient is given as

$$\left. \begin{array}{l} \Lambda_{\text{Re}} + \kappa = \kappa \cos(\omega\tau) \\ \Lambda_{\text{Im}} = -\kappa \sin(\omega\tau) \end{array} \right\} \begin{array}{l} (\bullet)^2 \\ (\bullet)^2 \end{array} \rightarrow \Lambda_{\text{Re}}^2 + 2\Lambda_{\text{Re}}\kappa + \kappa^2 + \Lambda_{\text{Im}}^2 = \kappa^2 \quad (2.20)$$

It yields

$$\kappa = -\frac{\Lambda_{\text{Re}}^2 + \Lambda_{\text{Im}}^2}{2\Lambda_{\text{Re}}} \quad \text{and} \quad w = \frac{\kappa}{K_x q h_0^{q-1}}. \quad (2.21)$$

The stability boundaries can be determined numerically from eq. (2.19) and eq. (2.21) if the FRF is known and $\omega \in [0, \infty)$. Note, that the solution gives all of the bifurcation curves, and the stable domains must be identified in a following step. The advantage of this method is that it can be applied to measured data without fitting, although the precision highly depends on the accuracy of the measurement. Furthermore the required time for the computation of stability charts is lower and does not depend on the number of degrees of freedom.

Note, that the solution here was only presented for the receptance frequency response function (displacement-force). If the mobility (velocity-force) or accelerance (acceleration-force) is used, then the substitution $\mathbf{A}(s) = s\mathbf{V}(s) = s^2\mathbf{X}(s)$ must be considered in frequency domain, where $\mathbf{A}(s)$ is the Laplace transformed of the acceleration, $\mathbf{V}(s)$ is the transformed of the velocity, and $s = i\omega$ is the complex Laplace variable. It can be shown, that the different FRFs can be transformed into each other by multiplying or dividing the complex function $H_{ij}(\omega)$ by $i\omega$ or $(i\omega)^2 = -\omega^2$. The theoretical stability chart does not change. An example for a single-degree-of-freedom case can be seen in Figure 2.3.

2.1.2 Semi-discretization

The stability of the linearized equation (2.5) can also be analyzed by the semi-discretization method. The equations can be written in a first-order form

$$\begin{pmatrix} \dot{\xi}(t) \\ \dot{\xi}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}(\mathbf{K} + \boldsymbol{\kappa}) & -\mathbf{M}^{-1}\mathbf{C} \end{pmatrix} \begin{pmatrix} \xi(t) \\ \dot{\xi}(t) \end{pmatrix} + \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{M}^{-1}\boldsymbol{\kappa} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \xi(t-\tau) \\ \dot{\xi}(t-\tau) \end{pmatrix}. \quad (2.22)$$

In order to make the numerical calculation faster, let us rewrite the second matrix according to the followings. The mass matrix is diagonal, and $\boldsymbol{\kappa}$ contains only one non-zero

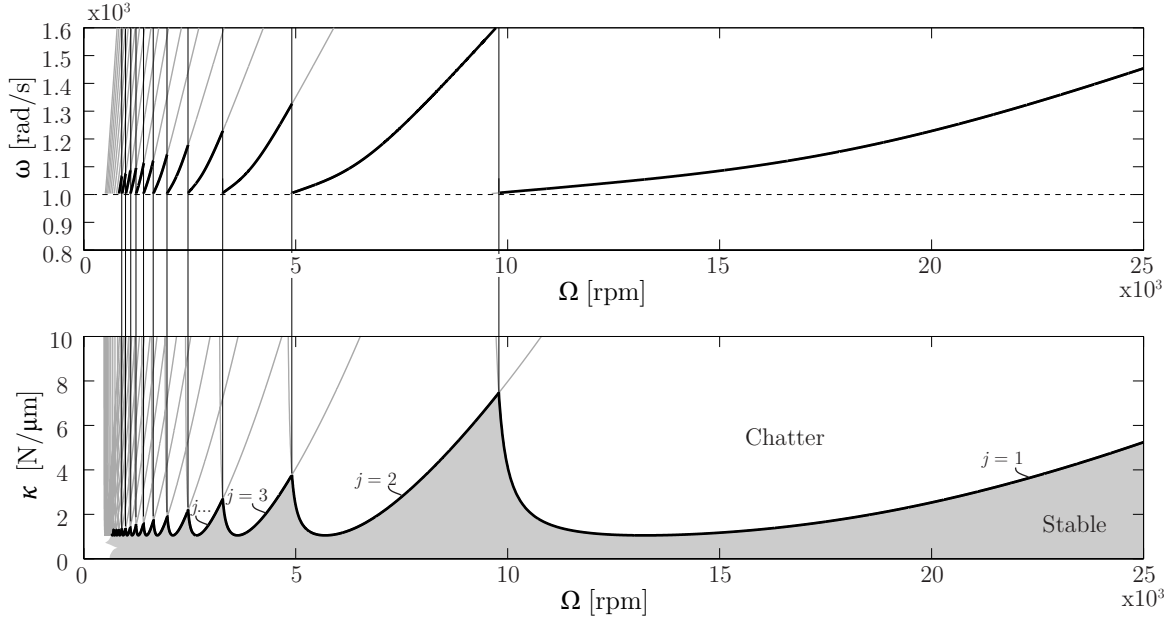


Figure 2.3. Stability lobe diagram and frequency diagram for a one-DoF system, where $\omega_n = 1000$ [rad/s], $\xi = 0.05$ [-], $\psi^2 = -5i \cdot 10^{-5}$ [rad · s/kg].

element. Instead of using a matrix with dimension $2n \times 2n$, the equations can be given by two $2n$ -dimensional vectors, i.e.

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{M}^{-1}\boldsymbol{\kappa} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \frac{\kappa}{m_1} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{\kappa}{m_1} \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}. \quad (2.23)$$

The state space representation can be given as

$$\begin{aligned} \dot{\mathbf{z}}(t) &= \mathbf{A}_s \mathbf{z}(t) + \mathbf{B}_s u(t - \tau), \\ u(t) &= \mathbf{C}_s \mathbf{z}(t), \end{aligned} \quad (2.24)$$

where

$$\begin{aligned} \mathbf{z}(t) &= \begin{pmatrix} \boldsymbol{\xi}(t) \\ \dot{\boldsymbol{\xi}}(t) \end{pmatrix}, \quad \mathbf{A}_s = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}(\mathbf{K} + \boldsymbol{\kappa}) & -\mathbf{M}^{-1}\mathbf{C} \end{pmatrix}, \\ \mathbf{B}_s &= \begin{pmatrix} 0 & \cdots & 0 & \frac{\kappa}{m_1} & \cdots & 0 \end{pmatrix}^T, \quad \mathbf{C}_s = \begin{pmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} \end{aligned} \quad (2.25)$$

and $\mathbf{z} \in \mathbb{R}^{2n}$ is the state vector, $u \in \mathbb{R}$, $\mathbf{A}_s \in \mathbb{R}^{2n \times 2n}$ is the system matrix, $\mathbf{B}_s \in \mathbb{R}^{2n}$ is the input matrix and $\mathbf{C}_s \in \mathbb{R}^{2n}$.

The main steps of the semi-discretization method are as follows [4]. The solution of

eq. (2.24) for the initial condition $\mathbf{z}(t_0) = \mathbf{z}_0$ reads (see Appendix A.2)

$$\mathbf{z}(t) = e^{\mathbf{A}_s(t-t_0)}\mathbf{z}(t_0) + \int_{t_0}^t e^{\mathbf{A}_s(t-s)}\mathbf{B}_s u(s-\tau)ds. \quad (2.26)$$

According to the semi-discretization method, assuming that the delay term is constant over the interval $t \in [t_i, t_{i+1})$, using $h = t_{i+1} - t_i$ numerical time step and $\tau = (r + 1/2)h$, where r is the resolution, the following discretized form can be obtained

$$\begin{aligned} \mathbf{z}_{i+1} &= e^{\mathbf{A}_s h} \mathbf{z}_i + \int_0^h e^{\mathbf{A}_s(h-s)} \mathbf{B}_s ds u_{i-r}, \\ u_i &= \mathbf{C}_s \mathbf{z}_i. \end{aligned} \quad (2.27)$$

This implies the discrete map $\mathbf{Z}_{i+1} = \mathbf{G}_i \mathbf{Z}_i$ as

$$\begin{pmatrix} \mathbf{z}_{i+1} \\ u_i \\ u_{i-1} \\ \vdots \\ u_{i-r+1} \end{pmatrix} = \begin{pmatrix} \mathbf{P} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{Q} \\ \mathbf{C}_s & 0 & \dots & 0 & 0 \\ \mathbf{0} & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & 0 & \dots & 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{z}_i \\ u_{i-1} \\ u_{i-2} \\ \vdots \\ u_{i-r} \end{pmatrix} \begin{matrix} 2n \\ 1 \\ 1 \\ \vdots \\ 1 \end{matrix}, \quad (2.28)$$

where

$$\mathbf{P} := e^{\mathbf{A}_s h}, \quad \mathbf{Q} := \int_0^h e^{\mathbf{A}_s(h-s)} \mathbf{B}_s ds. \quad (2.29)$$

Furthermore, if $\mathbf{A}_s^{-1}$ exists, then the integration gives

$$\mathbf{Q} = (e^{\mathbf{A}_s h} - \mathbf{I}) \mathbf{A}_s^{-1} \mathbf{B}_s. \quad (2.30)$$

The dimension of the matrices are $\mathbf{G}_i \in \mathbb{R}^{2n+r \times 2n+r}$, $\mathbf{P} \in \mathbb{R}^{2n \times 2n}$ and $\mathbf{Q} \in \mathbb{R}^{2n}$. Matrix $\mathbf{G}_i$ gives the monodromy matrix of the approximated system, which is stable if the largest eigenvalue of $\mathbf{G}_i$ is smaller than one. This condition has to be evaluated along a fixed interval on the stability chart. The general concept of the method and detailed derivation steps with explanations and mathematical proofs can be found in [4].

Semi-discretization for proportional damping

The stability of the equations can be analyzed in the modal space too. If proportional damping model is assumed, using the modal transformation $\Phi \mathbf{q}(t) = \xi(t)$, the governing equation (2.5) can be written as

$$\Phi^T \mathbf{M} \Phi \ddot{\mathbf{q}}(t) + \Phi^T \mathbf{C} \Phi \dot{\mathbf{q}}(t) + \Phi^T \mathbf{K} \Phi \mathbf{q}(t) = \Phi^T \kappa (\Phi \mathbf{q}(t - \tau) - \Phi \mathbf{q}(t)). \quad (2.31)$$

After simplifications, rewriting the equations in a first-order form, and using the identities for proportional damping presented in Section 1.1, the final form reads

$$\ddot{\mathbf{q}}(t) + \mathbf{C}_m \dot{\mathbf{q}}(t) + \mathbf{K}_m \mathbf{q}(t) = \boldsymbol{\kappa}_m (\mathbf{q}(t - \tau) - \mathbf{q}(t)), \quad (2.32)$$

where $\mathbf{C}_m = \text{diag}(2\xi_i \omega_{n,i})$, $\mathbf{K}_m = \text{diag}(\omega_{n,i}^2)$ and $\boldsymbol{\kappa}_m = \boldsymbol{\Phi}^T \boldsymbol{\kappa} \boldsymbol{\Phi}$. The modal parameters ξ_i , $\omega_{n,i}$ and $\phi_{i,1}$ can be obtained from the fitted FRF. Since $\boldsymbol{\kappa}$ has only one non-zero element, therefore

$$\begin{aligned} \boldsymbol{\kappa}_m &= \boldsymbol{\Phi}^T \boldsymbol{\kappa} \boldsymbol{\Phi} = \begin{pmatrix} \phi_{11} & . & \cdots & . \\ \phi_{21} & . & \cdots & . \\ \vdots & \vdots & & \vdots \\ \phi_{n1} & . & \cdots & . \end{pmatrix} \begin{pmatrix} \kappa & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \phi_{11} & \phi_{21} & \cdots & \phi_{n1} \\ . & . & \cdots & . \\ \vdots & \vdots & & \vdots \\ . & . & \cdots & . \end{pmatrix} \\ &= \kappa \begin{pmatrix} \phi_{11}\phi_{11} & \phi_{11}\phi_{21} & \cdots & \phi_{11}\phi_{n1} \\ \phi_{21}\phi_{11} & \phi_{21}\phi_{21} & \cdots & \phi_{21}\phi_{n1} \\ \vdots & \vdots & & \vdots \\ \phi_{n1}\phi_{11} & \phi_{n1}\phi_{21} & \cdots & \phi_{n1}\phi_{n1} \end{pmatrix} = \kappa \begin{pmatrix} \phi_{11} \\ \phi_{21} \\ \vdots \\ \phi_{n1} \end{pmatrix} (\phi_{11} \ \phi_{21} \ \cdots \ \phi_{n1}). \end{aligned} \quad (2.33)$$

The unknown elements of matrix $\boldsymbol{\Phi}$ (denoted by ‘.’) are unnecessary because those elements are multiplied by a zero element in $\boldsymbol{\kappa}$. The first-order representation is the same as it was given in eq. (2.24). It can be shown, that the matrices of the corresponding state space equations are changed to

$$\begin{aligned} \mathbf{z}(t) &= \begin{pmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{pmatrix}, \quad \mathbf{A}_s = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -(\mathbf{K}_m + \boldsymbol{\kappa}_m) & -\mathbf{C}_m \end{pmatrix}, \\ \mathbf{B}_s &= \kappa \begin{pmatrix} 0 & \cdots & 0 & \phi_{11} & \cdots & \phi_{n1} \end{pmatrix}^T, \quad \mathbf{C}_s = \begin{pmatrix} \phi_{11} & \cdots & \phi_{n1} & 0 & \cdots & 0 \end{pmatrix}. \end{aligned} \quad (2.34)$$

Using the same method, substituting back the modified $\mathbf{A}_s$, $\mathbf{B}_s$ and $\mathbf{C}_s$ matrices into eq. (2.29), then $\mathbf{P}$, $\mathbf{Q}$ into eq. (2.28), the stable domains can be calculated. The advantage of this method is that it does not require the exact $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ matrices, only the fitted modal parameters and the stable domains are directly identified.

Semi-discretization for non-proportional damping

In case of non-proportional damping, the linearized state space can be given in the form

$$\begin{pmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \dot{\boldsymbol{\xi}}(t) \\ \ddot{\boldsymbol{\xi}}(t) \end{pmatrix} + \begin{pmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{pmatrix} \begin{pmatrix} \boldsymbol{\xi}(t) \\ \dot{\boldsymbol{\xi}}(t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\kappa} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \left(\begin{pmatrix} \boldsymbol{\xi}(t - \tau) \\ \dot{\boldsymbol{\xi}}(t - \tau) \end{pmatrix} - \begin{pmatrix} \boldsymbol{\xi}(t) \\ \dot{\boldsymbol{\xi}}(t) \end{pmatrix} \right). \quad (2.35)$$

Let us write the equation above in the following form

$$\mathbf{A} \dot{\mathbf{v}}(t) + \mathbf{B} \mathbf{v}(t) = \tilde{\boldsymbol{\kappa}} (\mathbf{v}(t - \tau) - \mathbf{v}(t)), \quad (2.36)$$

which was presented in Section 1.2. Using the modal transformation $\Psi \mathbf{q}_v(t) = \mathbf{v}(t)$ and multiplying by Ψ^T from the left

$$\Psi^T \mathbf{A} \Psi \dot{\mathbf{q}}_v(t) + \Psi^T \mathbf{B} \Psi \mathbf{q}_v(t) = \Psi^T \tilde{\boldsymbol{\kappa}} \Psi (\mathbf{q}_v(t - \tau) - \mathbf{q}_v(t)) \quad (2.37)$$

can be obtained. According to the properties of the transformation presented in Section 1.2, and arranging the terms, the equation above can be written as

$$\dot{\mathbf{q}}_v(t) - \boldsymbol{\Lambda} \mathbf{q}_v(t) = \tilde{\boldsymbol{\kappa}}_m (\mathbf{q}_v(t - \tau) - \mathbf{q}_v(t)), \quad (2.38)$$

where $\boldsymbol{\Lambda} = -\Psi^T \mathbf{B} \Psi$ and $\tilde{\boldsymbol{\kappa}}_m = \Psi^T \tilde{\boldsymbol{\kappa}} \Psi$. Similarly to the previous cases, since only one element of $\tilde{\boldsymbol{\kappa}}$ is non-zero, therefore

$$\begin{aligned} \tilde{\boldsymbol{\kappa}}_m = \Psi^T \tilde{\boldsymbol{\kappa}} \Psi &= \begin{pmatrix} \psi_{11} & \cdot & \cdots & \cdot & \cdot \\ \bar{\psi}_{11} & \cdot & \cdots & \cdot & \cdot \\ \vdots & \vdots & & \vdots & \vdots \\ \psi_{n1} & \cdot & \cdots & \cdot & \cdot \\ \bar{\psi}_{n1} & \cdot & \cdots & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \kappa & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \psi_{11} & \bar{\psi}_{11} & \cdots & \psi_{n1} & \bar{\psi}_{n1} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \vdots & \vdots & & \vdots & \vdots \\ \cdot & \cdot & \cdots & \cdot & \cdot \end{pmatrix} \\ &= \kappa \begin{pmatrix} \psi_{11}\psi_{11} & \psi_{11}\bar{\psi}_{11} & \cdots & \psi_{11}\psi_{n1} & \psi_{11}\bar{\psi}_{n1} \\ \bar{\psi}_{11}\psi_{11} & \bar{\psi}_{11}\bar{\psi}_{11} & \cdots & \bar{\psi}_{11}\psi_{n1} & \bar{\psi}_{11}\bar{\psi}_{n1} \\ \vdots & \vdots & & \vdots & \vdots \\ \psi_{n1}\psi_{11} & \psi_{n1}\bar{\psi}_{11} & \cdots & \psi_{n1}\psi_{n1} & \psi_{n1}\bar{\psi}_{n1} \\ \bar{\psi}_{n1}\psi_{11} & \bar{\psi}_{n1}\bar{\psi}_{11} & \cdots & \bar{\psi}_{n1}\psi_{n1} & \bar{\psi}_{n1}\bar{\psi}_{n1} \end{pmatrix} = \kappa \begin{pmatrix} \psi_{11} \\ \bar{\psi}_{11} \\ \vdots \\ \psi_{n1} \\ \bar{\psi}_{n1} \end{pmatrix} \begin{pmatrix} \psi_{11} & \bar{\psi}_{11} & \cdots & \psi_{n1} & \bar{\psi}_{n1} \end{pmatrix}. \end{aligned} \quad (2.39)$$

Based on this derivation, the state space matrices can be given as

$$\begin{aligned} \mathbf{z}(t) &= \mathbf{q}_v(t), \quad \mathbf{A}_s = \boldsymbol{\Lambda} - \tilde{\boldsymbol{\kappa}}_m, \\ \mathbf{B}_s &= \kappa \begin{pmatrix} \psi_{11} & \bar{\psi}_{11} & \cdots & \psi_{n1} & \bar{\psi}_{n1} \end{pmatrix}^T, \quad \mathbf{C}_s = \begin{pmatrix} \psi_{11} & \bar{\psi}_{11} & \cdots & \psi_{n1} & \bar{\psi}_{n1} \end{pmatrix}. \end{aligned} \quad (2.40)$$

This formulation can be used in case on non-proportional damping by using only the fitted modal parameters. A two-degrees-of-freedom example is presented in Figure 2.4, where the stability boundaries are obtained by the semi-discretization method and directly from the FRF.

2.1.3 Sensitivity analysis of turning

The major aim of this work is the investigation of the sensitivity of machining processes. A two-degrees-of-freedom model is considered, which is approximated by a single-degree-of-freedom system taking into account some typical fitting mistakes. The variation of the stability chart of turning is analyzed due to the change of modal parameters.

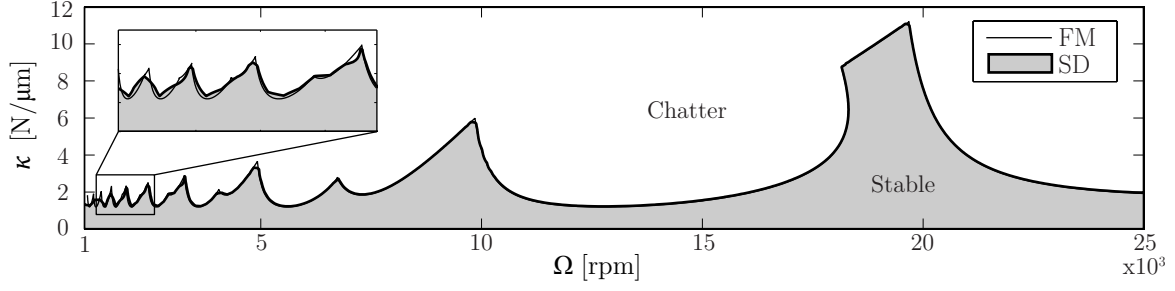


Figure 2.4. Comparison of the frequency method (FM) and the semi-discretization (SD, $r = 100, 200 \times 100$) method for a two-DoF system, where $\omega_{n,1} = 1000$ rad/s, $\omega_{n,2} = 2000$ rad/s, $\xi_1 = \xi_2 = 0.05$, $\psi_1^2 = \psi_2^2 = -5i \cdot 10^{-5}$ rad $\cdot$ s/kg.

Error in the estimation of modal parameters

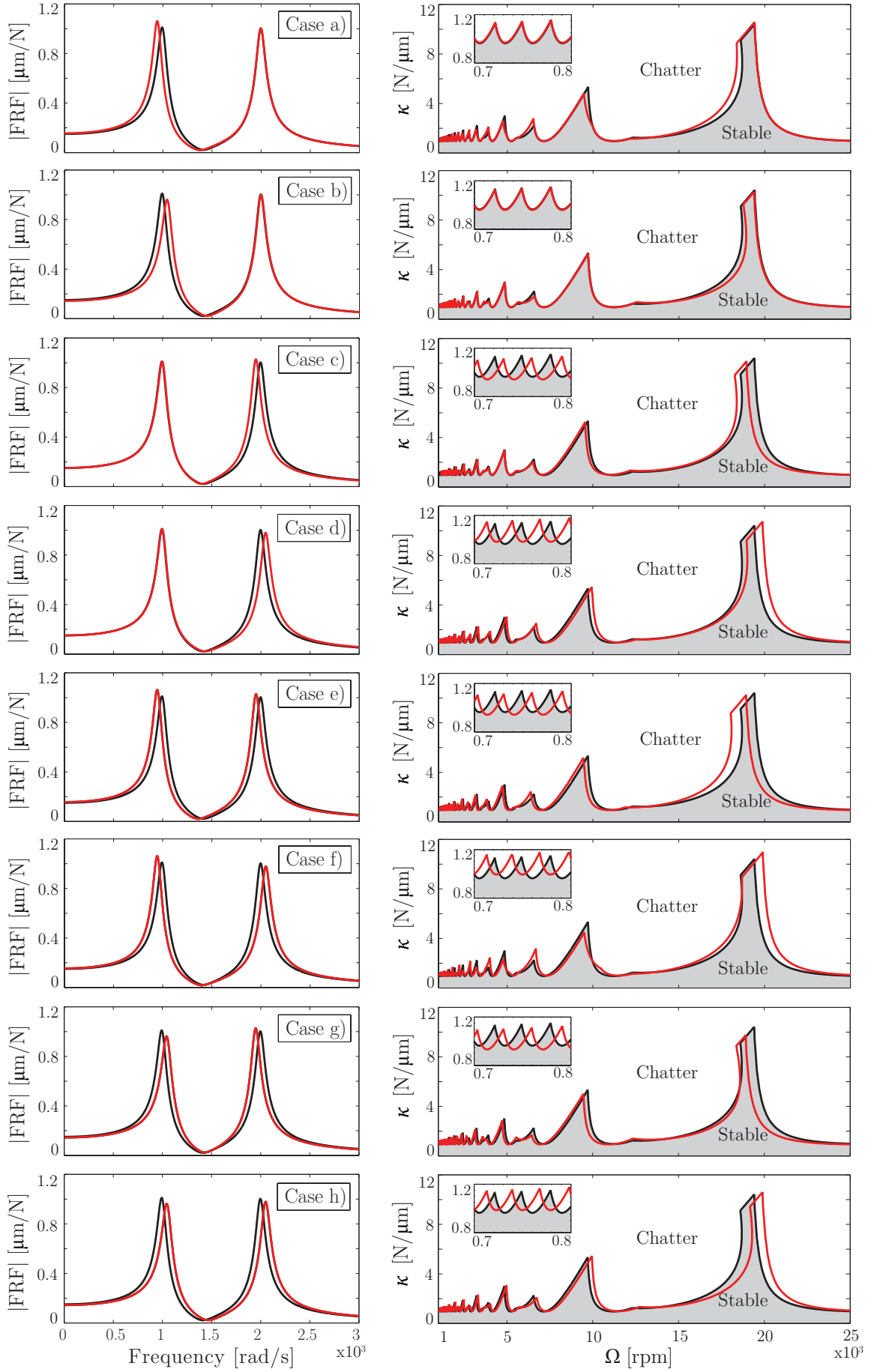
This is one of the simplest method, the modal parameters are shifted manually, and the difference between the original and the new stability chart is analyzed. In practice, when measured data is used, the fitting always contains errors. When a non-dominant mode is neglected or the data contains significant noise, usually the fitted mode cannot describe the original one perfectly. In Table 2.1, the modal parameters of a two-degrees-of-freedom system is given.

Mode	ω_n [rad/s]	ξ [%]	ψ^2 [rad $\cdot$ s/kg] $\cdot 10^{-5}$
1	1000.0	5.00	-5.00 <i>i</i>
2	2000.0	2.50	-5.00 <i>i</i>

Table 2.1. Modal parameters for the sensitivity analysis (two-degrees-of-freedom).

First, the sensitivity of the estimation of the natural angular frequencies is analyzed. The frequencies are shifted by ± 50 rad/s ($\pm 5\%$), while the other parameters are fixed. In Figure 2.5 all of the possible variations corresponding to the two-degrees-of-freedom system are presented, where grey domain indicates the originally stable solution and red curve indicates the shifted boundary due to the change of modal parameters. In this specific example the results are as follows. For Cases a) and b) where only the first mode is estimated incorrectly, the stability chart for lower spindle speeds almost does not change, and the shift in the boundaries for speeds is not significant. In case of the incorrect estimation of the second mode, Cases c) and d), the change of the stable domain is significant both for lower and higher spindle speeds. In this case, the second mode appears to be the dominant mode, the chart is more sensitive to its estimation. For Cases e–h), where both modes are incorrectly identified, the wandering of the boundaries are apparently the sum of the errors of the two modes.

The variation of the chart due to the change of the damping coefficient ξ ($\pm 10\%$) can be seen in Figure 2.6. For Cases a) and b) the change of the damping coefficient affects the

Figure 2.5. Sensitivity of the stability chart w.r.t. the estimation of ω_n (± 50 rad/sec).

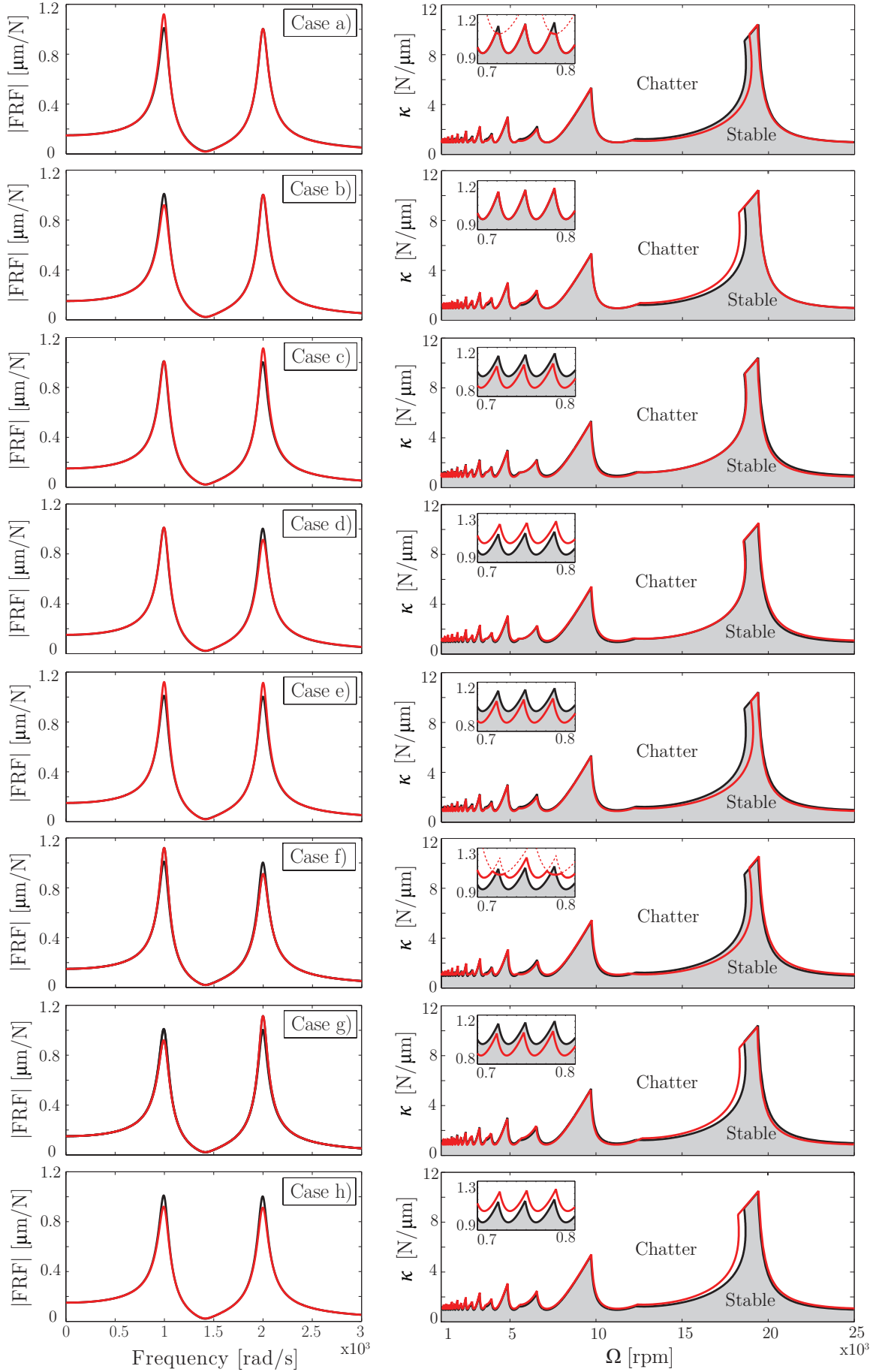


Figure 2.6. Sensitivity of the stability chart w.r.t. the estimation of ξ ($\pm 10\%$).

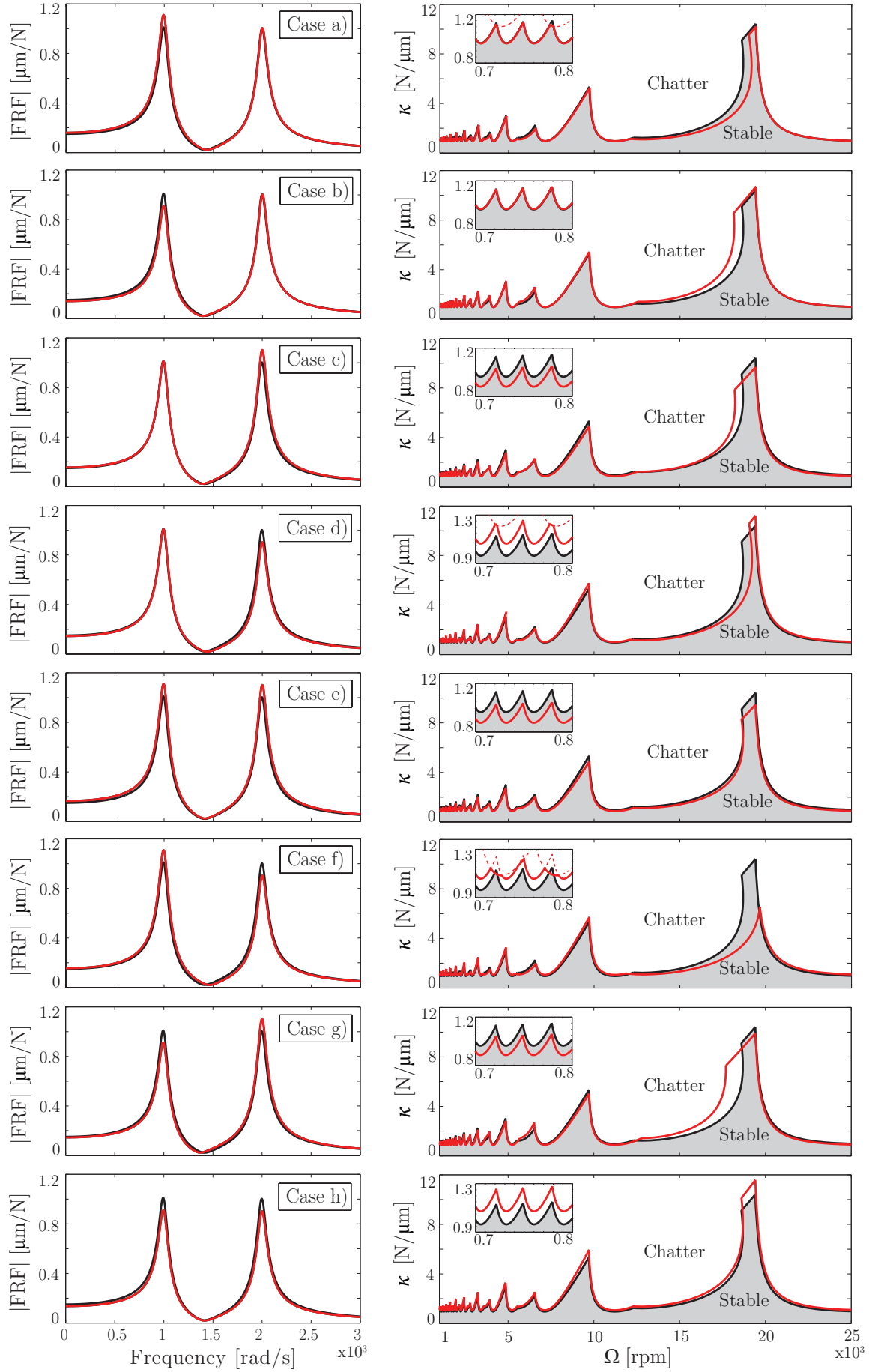
Figure 2.7. Sensitivity of the stability chart w.r.t. the estimation of ψ^2 ($\pm 10\%$).

chart for higher speeds and does not affect significantly for lower speeds, although it can happen that the bifurcation curves cross each other at new points and the peaks are cut down (Case a), dashed red line). For Cases c) and d), the chart is almost unaffected for high spindle speeds but the boundaries are significantly shifted for lower spindle speeds. In case of multiple errors (Cases e–h)), the wandering of the boundaries is approximately the sum of the two errors of the two modes.

The effect of the estimation of ψ^2 ($\pm 10\%$), which is related to the mode shape, can be seen in Figure 2.7. For Cases a) and b), the change is similar to Figure 2.6, although in this case the wandering of the boundaries are slightly larger. The chart is practically affected only for higher spindle speeds, but the shifted bifurcation curves can intersect and cut down the stable domains for lower speeds too. For Cases c) and d) the boundary is shifted for lower spindle speeds and the effect is significant even for higher speeds. For multiple errors (Cases e–h)) the wandering of the boundaries are apparently the sum of the errors.

Note, that this analysis is based on a simple two-degree-of-freedom example, and general conclusions cannot be drawn. Although it can be said, that the stability chart gets more sensitive for high speeds and more sensitive for the estimation of the modal parameters of modes at higher frequencies.

Neglected modes

The real measured structure theoretically can only be described by an infinite dimensional (continuum) system, but when curve fitting is applied, many modes are usually neglected and only finite number of modes are fitted. The analysis for this case is done in the following way. First, an artificial FRF is assumed and the stability charts are calculated. As a next step, a curve fitting technique is used based on the non-proportional damping model and the original function is approximated by a single-degree-of-freedom system. This results change in the modal parameters for the dominant modes too.

Case	Mode	ω_n [rad/s]	ξ [%]	ψ^2 [rad · s/kg] · 10 ⁻⁵	$\tilde{\omega}_n$ [rad/s]	$\tilde{\xi}$ [%]	$\tilde{\psi}^2$ [rad · s/kg] · 10 ⁻⁵
a)	1	1000.0	5.00	-5.00i	997.6	5.06	0.25-5.03i
	2	2000.0	10.00	-1.00i	×	×	×
b)	1	1000.0	5.00	-5.00i	992.5	5.24	0.78-5.09i
	2	2000.0	10.00	-3.00i	×	×	×
c)	1	500.0	20.00	-1.00i	×	×	×
	2	1000.0	5.00	-5.00i	1002.2	5.17	-0.24-5.15i
d)	1	500.0	20.00	-5.00i	×	×	×
	2	1000.0	5.00	-5.00i	1011.4	6.36	-1.36-5.98i

Table 2.2. Modal parameters for the sensitivity analysis of neglected modes, where $\tilde{\bullet}$ denotes the fitted results.

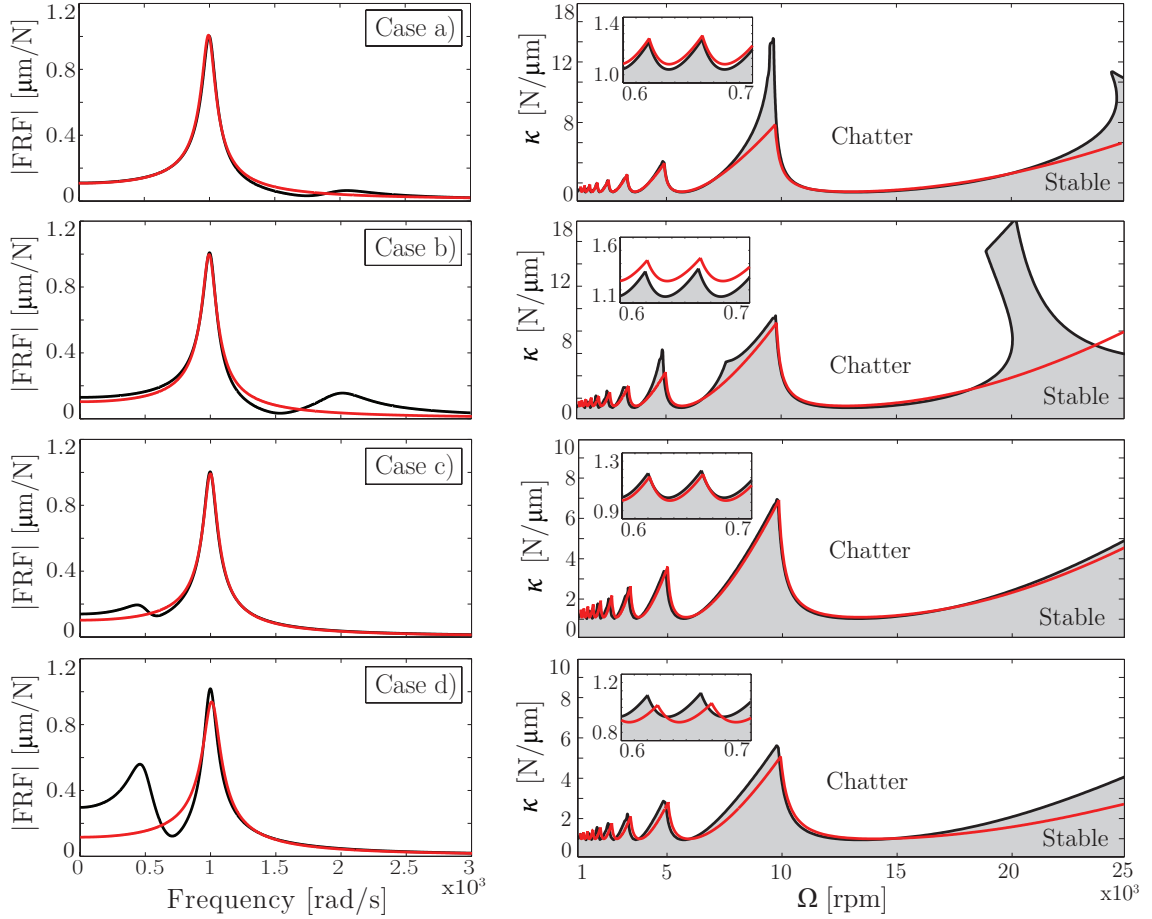


Figure 2.8. Results of the sensitivity analysis in case of neglected modes.

Table 2.2 lists the original and approximated modal parameters while Figure 2.8 presents the difference between the stability charts. In every real case there are modes which are neglected during the fitting. For Case a), where the second mode at higher frequency with smaller amplitude seems to be negligible, the estimated and exact boundaries show significant differences for higher spindle speeds. It can be seen that the difference can go up to 100% near to 10 000 rpm, which is remarkable since the machining time can be halved if the depth of cut is doubled. For lower speeds, since the dominant mode is approximately correct, the stability chart is also almost accurate. For Case b), the neglected mode is larger, the difference is more significant. It can also be observed that new bifurcation curves intersect the existing stable domain and reduce it.

Surprisingly, the chart is not that sensitive if the non-dominant mode is neglected at a lower frequency compared to the dominant mode. For Cases c) and d), even if the neglected mode is apparently significant, the stable domain is practically the same for Case c) and shows only a small change for Case d). The reason is the following, when the neglected mode is located at lower frequency compared to the dominant mode then the corresponding bifurcation curves are likely to be located at negative cutting-force coefficients. In other words, based on eq. (2.21), if $\Lambda_{Re} > 0$ then $\kappa < 0$ or $w < 0$ which is

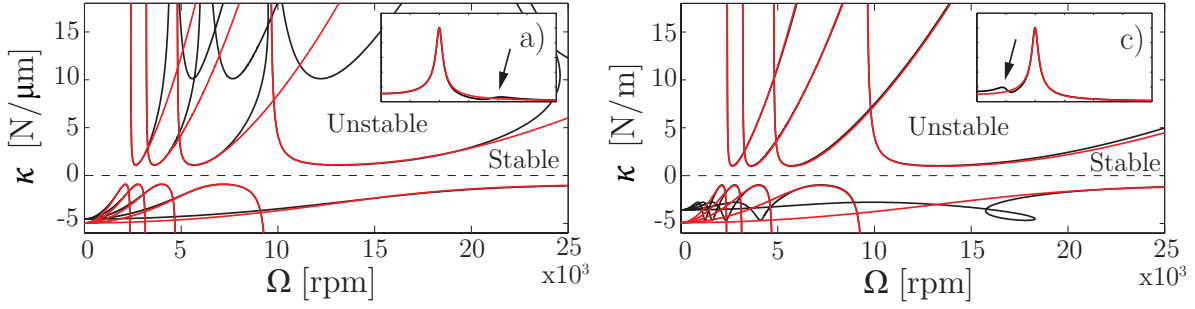


Figure 2.9. Change of the bifurcation curves in case of different neglected modes.

not relevant from the machining point of view. This phenomenon can be seen in Figure 2.9, whose subfigures correspond to Figure 2.8 Cases a) and c).

Without the loss of generality it can be said again that the chart is more sensitive if the modes which are neglected in the FRF appear at higher frequencies. In this case the stability chart can lose some important domains at high speeds, which have practical relevance in high speed machining.

Merged modes

In practice, the measured FRF contains many modes which are hard to approximate properly. In many cases, some modes are so close to each other that they are not even visible. In Figure 2.10 some typical cases are listed, when two adjacent modes are approximated by only one mode. The exact and fitted modal parameters are given in Table 2.3.

Case	Mode	ω_n [rad/s]	ξ [%]	ψ^2 [rad · s/kg] · 10 ⁻⁵	$\tilde{\omega}_n$ [rad/s]	$\tilde{\xi}$ [%]	$\tilde{\psi}^2$ [rad · s/kg] · 10 ⁻⁵
a)	1	850.0	10.00	-2.00i	1002.8	6.98	-1.00-6.81i
	2	1000.0	5.00	-5.00i	×	×	×
b)	1	1000.0	5.00	-4.00i	1045.0	7.46	-0.23-8.95i
	2	1080.0	4.50	-4.00i	×	×	×
c)	1	1000.0	5.00	-4.50i	1003.5	8.53	-1.71-7.78i
	2	1150.0	10.00	-4.00i	×	×	×

Table 2.3. Modal parameters for the sensitivity analysis of mode merging, where $\tilde{\bullet}$ denotes the fitted results.

For Case a), the neglected mode is located before the dominant mode. The fitted mode does not describe properly the dominant mode, however, the chart does not change significantly, even though the boundaries are affected at lower and higher spindle speeds too. The neglected mode appears at the negative half plane, similarly as it was presented in Figure 2.9, which is not relevant from the machining point of view. Note that the error is magnified on these figures, in practical examples the fitting is usually more accurate.

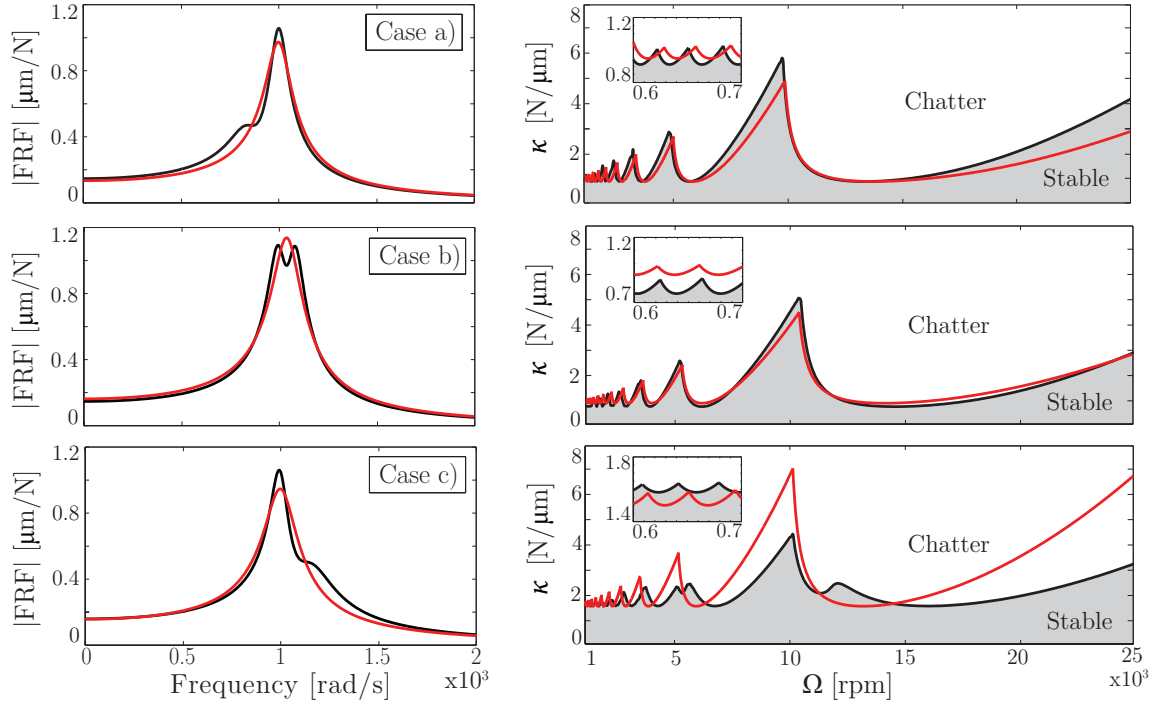


Figure 2.10. Results of the sensitivity analysis in case of mode merging.

When the two modes are very close to each other and none of them is more dominant than the other (Case b)), the fitted stable domain slightly differs from the exact. For high speeds, the wandering of the boundary is not significant, however a larger shift can be observed in the domains for lower spindle speeds.

Case c) is similar to Case a), but here the non-dominant mode follows the dominant one. The change of the stability chart is more significant than in Case a). As it is visible, the boundaries are shifted slightly at lower speeds but the difference is qualitatively larger for high speeds than in any other case before.

Based on the results of these simple cases, it can be said that the chart is more sensitive if the neglected modes are located at higher frequencies compared to the dominant modes.

2.1.4 Stable islands in turning

The required time of the manufacturing process depends on the material removal rate, which linearly depends on the cutting speed, the feed rate and the depth of cut. The manufacturing time can be reduced, if any of those parameters are increased. The maximum revolution is limited by the capabilities of the machine tool, on the other hand, the depth of cut cannot be increased arbitrarily. The machinist must choose the most suitable machining parameters to design an optimal process.

In some special cases, only for multiple-degrees-of-freedom systems, it can happen that a stable domain gets separated from a larger stable area. An example and its modal parameters are given in Table 2.4 and the stability charts can be seen in Figure 2.11 (the

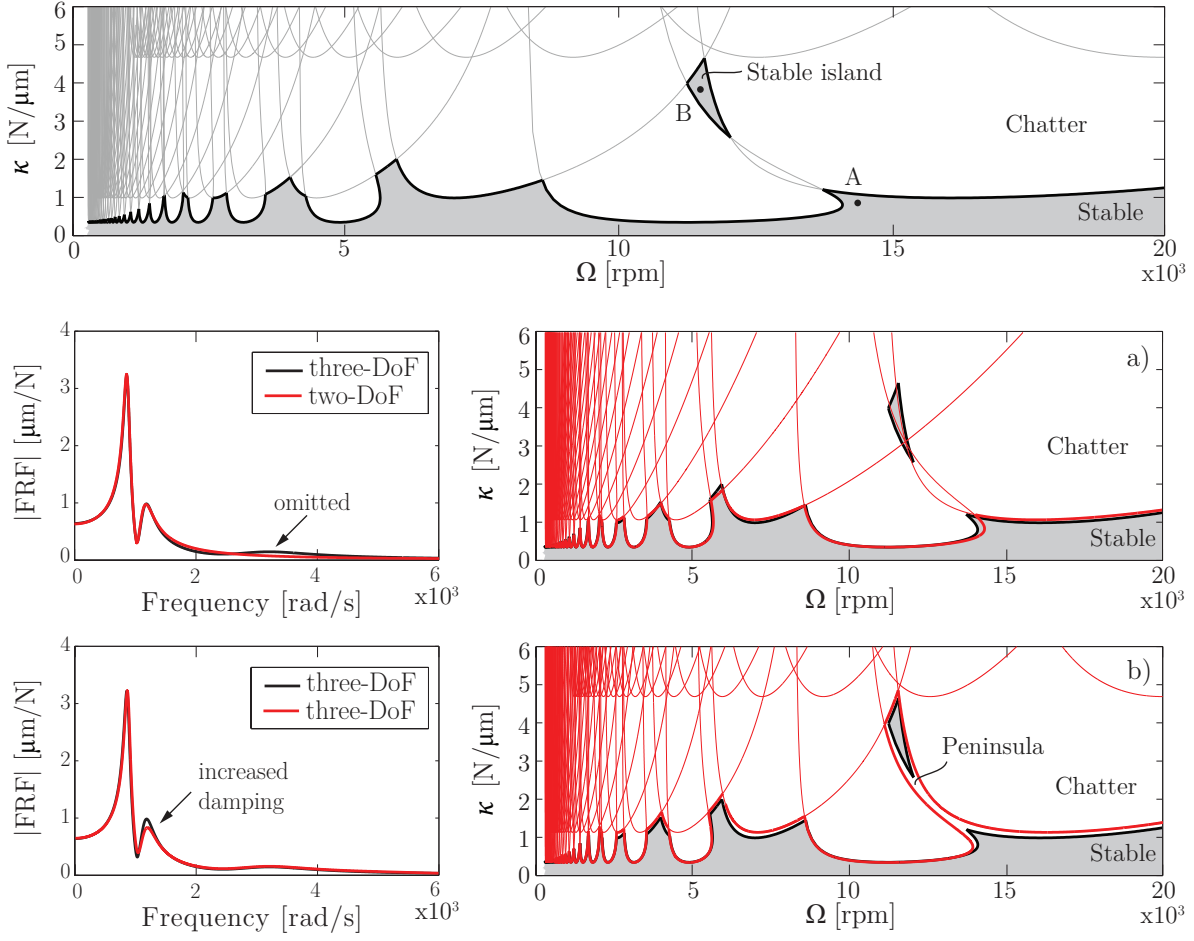


Figure 2.11. Stable island in case of a three-degrees-of-freedom model.

chart was calculated by the frequency technique and by the semi-discretization method). Comparing the parameter points A and B, the material removal rate can be significantly different. Although the revolution is decreased for point B compared to A, the cutting force is nearly four times larger, which means that the depth of cut is also four times larger (see Eq. (2.21)). The material removal rate can be approximately three times larger for point B, the operation time is one third.

The identification of these stable islands can be a difficult task. Usually the islands are quite small and located at high spindle speeds, furthermore they can be really sensitive to

Mode	ω_n [rad/s]	ξ [%]	ψ^2 [s/kg] · 10 ⁻⁶
1	870	6.14	-51.60-169.24 <i>i</i>
2	1150	10.60	60.66-118.14 <i>i</i>
3	3200	21.06	-18.56-75.96 <i>i</i>

Table 2.4. Modal parameters for the three-degrees-of-freedom example with a stable island.

the modal parameters. In Figure 2.11 a), the third mode is neglected, since it does not seem to be significant compared to the others. Grey domains with black boundary indicates the originally stable chart, while red curve indicates the solution of the approximated system. The bifurcation curves show that the stable island disappears if the third mode is neglected, since they intersect each other almost at the same point. The boundary of the stable domain does not change significantly.

In Figure 2.11 b), the damping coefficient of the second mode was increased to 12.6% from 10.6%, and the third mode was not neglected. The stable island increases until it gets connected to the stable area and forms a peninsula. In real cases the fitted modal parameters always contain error and the stable islands like this can easily get lost. In Section 2.1.5, a similar case can be found.

2.1.5 A case study

In this subsection, a measured frequency response function is studied. The experiment was done on a milling machine using an impulse hammer, though here it is assumed to be a turning tool.

In Figure 2.12, the fitted model consists of 7-, 9- and 11-degrees-of-freedom respectively, where the non-proportional equations were used. The corresponding modal parameters can be found in Table 2.5. For the 7-DoF model, the most dominant modes were approximated only, and the waviness of the FRF was neglected. Note, that only the absolute value of the complex function is presented, but the fitting technique uses the real and imaginary part of the complex function. Also note that the scale is logarithmic in the amplitude in order to make visible the entire function (2–3 magnitudes must be presented). On the stability chart the grey domain indicates the theoretically exact stable domain obtained directly from the FRF using the frequency technique, while red curves indicate the result of the fitted functions. As it can be seen for the lowest degrees-of-freedom approximation, the stability chart for 10 000 . . . 25 000 rpm approximates the exact solution, though it is not perfect at all. For higher spindle speeds ($\Omega > 25\,000$ rpm), the difference increases, but the approximation is still acceptable, and does not contain significant error. A stable island also appears if the chart is calculated using directly the measured FRF, but the first approximation does not show it. Although this domain is highly over the practically achievable speeds, since now only the sensitivity of the chart is analyzed, this fact is neglected. Furthermore, if the tool has several cutting edges, like in case of milling, this domain can go down to physically realistic speeds. In order to avoid the errors in the calculation and visualization, each stability chart was checked by frequency method and by the semi-discretization method.

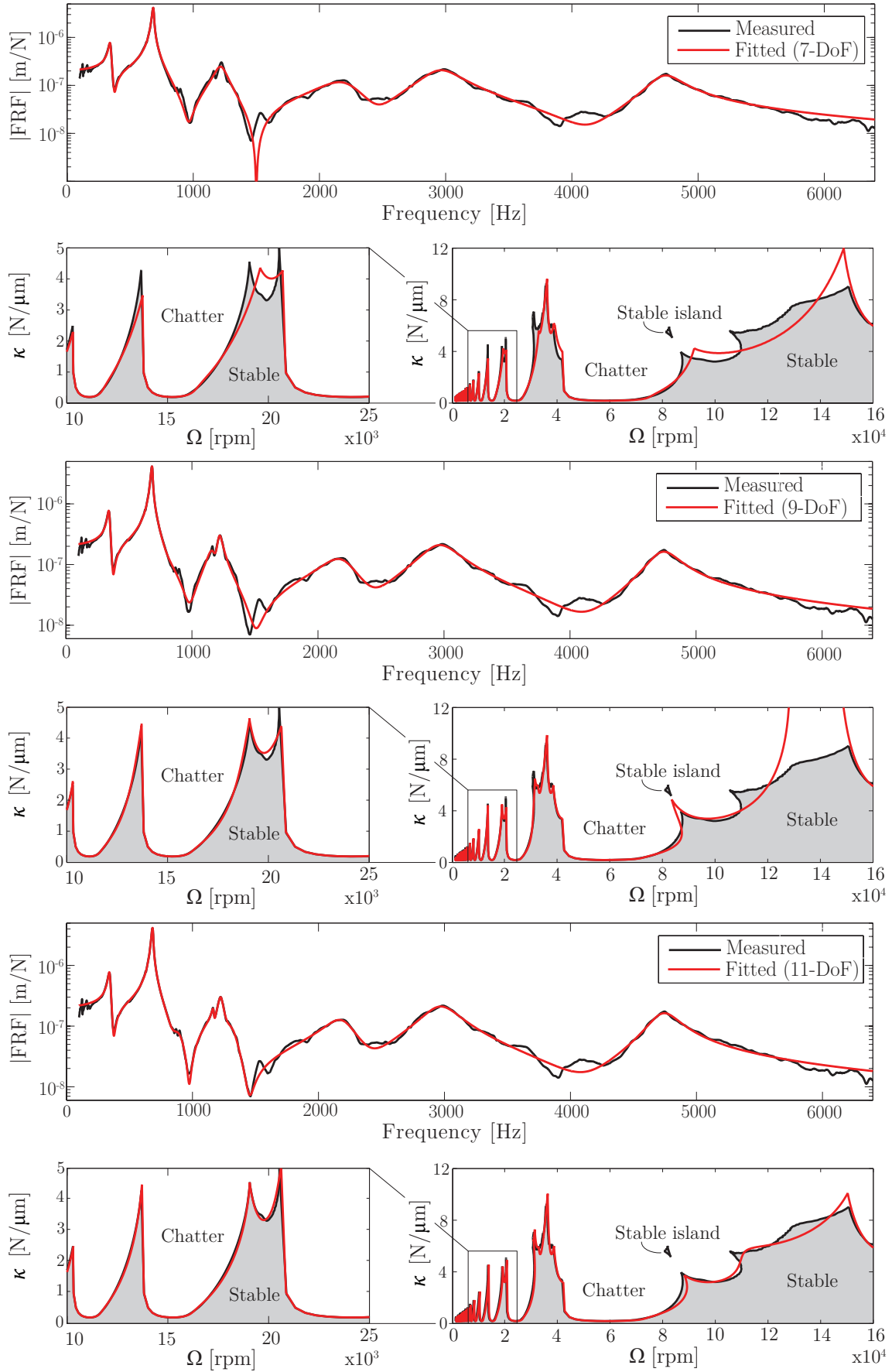


Figure 2.12. Case study: Stability charts for the fitted model with 7-, 9- and 11-DoF.

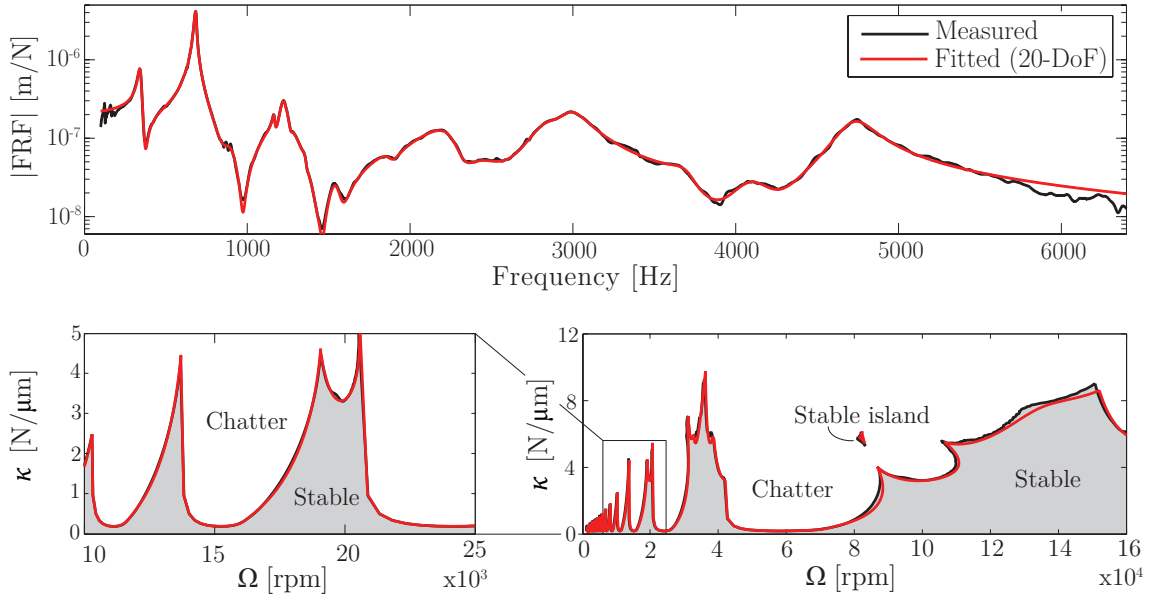


Figure 2.13. Case study: Stability charts for the fitted model with 20-DoF.

7-DoF	ω_n [Hz]	ξ [%]	ψ^2 [s/kg] · 10 ⁻⁶
1	341.92	2.98	0.2279-45.48 <i>i</i>
2	679.94	2.71	122.79-118.69 <i>i</i>
3	684.24	1.00	-129.4-96.34 <i>i</i>
4	1219.90	3.99	-15.35-74.67 <i>i</i>
5	2204.93	7.67	5.088-113.69 <i>i</i>
6	2961.58	5.13	42.63-189.86 <i>i</i>
7	4732.50	2.59	0.747-118.65 <i>i</i>

9-DoF	ω_n [Hz]	ξ [%]	ψ^2 [s/kg] · 10 ⁻⁶
1	342.19	3.00	-0.940-45.98 <i>i</i>
2	681.67	2.06	116.4-196.87 <i>i</i>
3	685.45	0.97	-138.7-22.40 <i>i</i>
4	789.67	12.59	30.29-14.97 <i>i</i>
5	1165.84	3.11	-8.411-30.46 <i>i</i>
6	1216.94	1.77	10.16-35.53 <i>i</i>
7	2190.77	6.22	14.47-90.07 <i>i</i>
8	2967.41	4.87	32.36-180.07 <i>i</i>
9	4734.75	2.50	-1.766-114.23 <i>i</i>

11-DoF	ω_n [Hz]	ξ [%]	ψ^2 [s/kg] · 10 ⁻⁶
1	342.28	2.99	-1.393-45.72 <i>i</i>
2	681.11	2.13	115.1-178.17 <i>i</i>
3	685.19	0.97	-134.0-38.39 <i>i</i>
4	882.28	7.17	-8.729-23.45 <i>i</i>
5	1118.37	20.35	100.25-27.17 <i>i</i>
6	1167.50	0.76	-3.406-3.50 <i>i</i>
7	1225.14	1.74	-13.01-30.74 <i>i</i>
8	1318.95	4.28	-19.72-3.34 <i>i</i>
9	2190.83	5.39	12.71-76.18 <i>i</i>
10	2971.77	4.75	25.39-175.34 <i>i</i>
11	4736.36	2.45	-3.501-111.97 <i>i</i>

20-DoF	ω_n [Hz]	ξ [%]	ψ^2 [s/kg] · 10 ⁻⁶
1	342.28	2.92	-1.347-44.62 <i>i</i>
2	603.03	7.05	5.741-9.35 <i>i</i>
3	681.28	1.91	93.83-205.30 <i>i</i>
4	685.72	0.94	-119.7-7.58 <i>i</i>
5	892.81	5.94	-11.08-13.25 <i>i</i>
6	1165.68	0.70	-1.882-3.73 <i>i</i>
7	1181.70	8.02	-18.04-56.76 <i>i</i>
8	1220.29	1.90	3.115-37.56 <i>i</i>
9	1269.12	1.33	4.615+0.28 <i>i</i>
10	1358.78	0.75	-0.275+0.75 <i>i</i>
11	1416.01	2.19	0.889+0.21 <i>i</i>
12	1543.71	2.76	0.808-5.67 <i>i</i>
13	1881.76	2.81	-5.373-3.98 <i>i</i>
14	2258.38	5.36	-108.7-58.83 <i>i</i>
15	2342.36	9.46	143.5-105.09 <i>i</i>
16	2837.92	3.72	31.63-43.30 <i>i</i>
17	2988.25	3.32	24.76-96.23 <i>i</i>
18	3699.10	2.43	-7.876+2.20 <i>i</i>
19	4094.20	3.29	0.968-12.90 <i>i</i>
20	4727.91	2.43	5.778-112.34 <i>i</i>

Table 2.5. Modal parameters corresponding to the experiment (7-, 9-, 11- and 20-DoF).

The number of degrees of freedom was increased to improve the precision, and the FRF was approximated more accurately over the region 700-1200 Hz by taking into account two more modes. The results of the 9-DoF model can also be found in Figure 2.12. For lower speeds, the precision was improved, and below 10 000 rpm, the difference is practically zero. For higher speeds, the error is much larger than in the case of the 7-DoF model, though the estimation of the FRF seems to be better.

The 11-Dof model provides a better result, although only two more modes were taken into account near to 882 Hz and 1318 Hz. Including new modes slightly changes the modal parameters of the already fitted modes, but the approximation of the stability chart get better. Still, the stable island is undiscovered.

A 20-DoF approximation was prepared, where most of the non-dominant modes are taken into account. The results can be seen in Figure 2.13 and the corresponding modal parameters in Table 2.5. The measured FRF is very well covered by the fitted one, the difference is practically under the thickness of the line. For low spindle speeds, the difference is almost unnoticeable. As the speed increases, since the chart becomes more and more sensitive, the error also increases, however, the 20-DoF model gives highly the best approximation. In case of this model, the stable island is also discovered.

As the presented case study shows, in order to approximate properly the exact stability chart obtained directly from the measured frequency response function, very high number of degrees of freedom must be considered, which can lead to difficulties. For low spindle speeds, usually the precise approximation of the dominant modes is sufficient, however, for higher speeds, large variations can be observed which can hardly be predicted in advance. Even for the most precise fit, the chart can show sensitivity for high spindle speeds, but the theoretically exact boundaries can be approximated sufficiently as more modes are taken into account.

2.2 Milling

The regenerative machine tool chatter appears in milling operations too, but the mathematical description is different since the intermittent nature of the cutting teeth has to be taken into account. The surface regeneration is coupled with parametric excitation resulting time-periodic coefficients in the governing equations. The first results regarding the stability analysis of milling operations appeared in the works of Tobias and Thusty where the periodic terms were averaged in time, and the equations reduce to DDEs with constant coefficients similarly to turning. This model can be used in cases, where the radial immersion is large or the number of the cutting edges of the tool is high. Otherwise the intermittent nature of process, which is the result of the periodically entering and exiting cutting edges, cannot be neglected [4].

In the last decade, due to the extensive work of several scientists, many new results

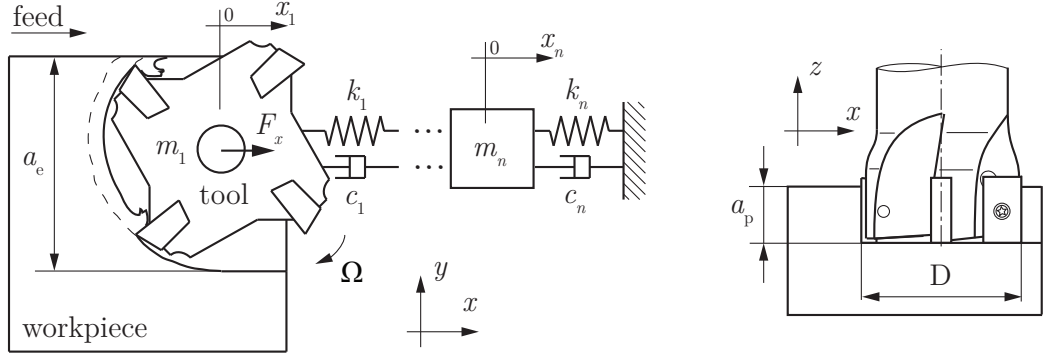


Figure 2.14. One-dimensional model of milling operations including a straight fluted tool.

were published. In this work, the semi-discretization method is presented and used, though there exists other techniques with which the limits of the stability can be determined. Just to mention one, the so-called *extended multi frequency solution* method presented in [13] uses directly the measured frequency response function, and a comparative real case study using the two different techniques can be found in [14].

In order to make the analysis simpler, during the sensitivity analysis a one-dimensional model is considered with multiple-degrees-of-freedom as it can be seen in Figure 2.14. Therefore it is assumed that all of the possible vibration modes of the tool appear purely in direction x , i.e. the system is rigid in direction y . Furthermore the tool has N equally distributed cutting teeth with zero helix angle. The dynamical model and the stability analysis for a single-degree-of-freedom system can be found in [4]. The equation of motion for multiple-degrees-of-freedom model based on Section 2.1 reads

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{f}(t), \quad (2.41)$$

where $\mathbf{f}(t) = (F_x(t) \ 0 \dots 0)^T$, $\mathbf{f} \in \mathbb{R}^n$ and $F_x(t)$ is the x component of the cutting force vector. The geometry of the milling operation can be seen in Figure 2.16. The radial and tangential components of the cutting force at the j^{th} tooth can be given as

$$F_{j,t}(t) = g_j(t) K_t a_p h_j^q(t), \quad (2.42)$$

$$F_{j,r}(t) = g_j(t) K_r a_p h_j^q(t), \quad (2.43)$$

where K_t and K_r are the tangential and radial cutting-force coefficients, a_p is the axial depth of cut, $h_j(t)$ is the chip thickness at the j^{th} tooth at time t , q is the cutting-force exponent and $g_j(t)$ is a screen function, which is 1 if the tooth cuts and 0 if it is out of the workpiece. The actual angular position of the j^{th} tooth can be given as

$$\varphi_j(t) = \frac{2\pi\Omega}{60}t + j\frac{2\pi}{N}, \quad (2.44)$$

where Ω is the spindle speed (given in rpm). The angle where the tooth enters the workpiece is denoted by φ_{en} and the angle of the exit is denoted by φ_{ex} . The screen function

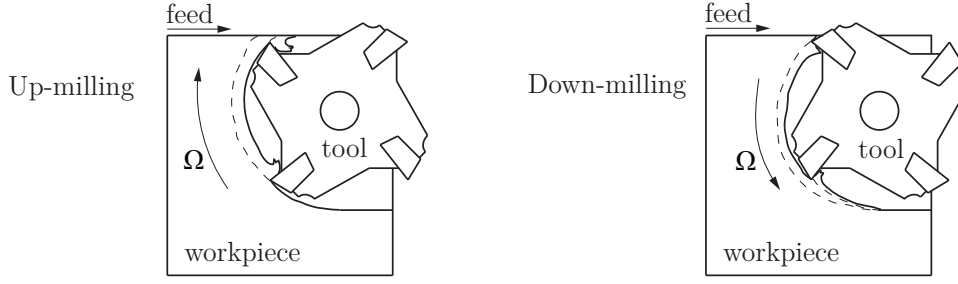


Figure 2.15. Difference between up-milling and down-milling operations.

therefore reads

$$g_j(t) = \begin{cases} 1, & \text{if } \varphi_{\text{en}} < (\varphi_j(t) \bmod 2\pi) < \varphi_{\text{ex}} \\ 0, & \text{otherwise} \end{cases}. \quad (2.45)$$

Two main different milling operations can be distinguished based on the geometrical properties, up-milling and down-milling, which can be seen in Figure 2.15. The entering and exiting angles can be calculated for up-milling as

$$\varphi_{\text{en}} = 0 \quad \text{and} \quad \varphi_{\text{ex}} = \arccos\left(1 - \frac{2a_e}{D}\right), \quad (2.46)$$

or for down-milling as

$$\varphi_{\text{en}} = \arccos\left(\frac{2a_e}{D} - 1\right) \quad \text{and} \quad \varphi_{\text{ex}} = \pi. \quad (2.47)$$

Usually the feed per tooth f_z is given, and the instantaneous chip thickness including the flexibility of the tool is calculated as (see Figure 2.14 and Figure 2.16)

$$A(t) = f_z + x_1(t - \tau) - x_1(t), \quad (2.48)$$

where the delay is $\tau = 60/(N\Omega)$. The approximated chip thickness projected to the j^{th} tooth is

$$h_j(t) = A(t) \sin \varphi_j(t) = (f_z + x_1(t - \tau) - x_1(t)) \sin \varphi_j(t). \quad (2.49)$$

Since eq. (2.42) and eq. (2.43) are defined in tangential and radial directions, the x component of the cutting force (y is omitted since the tool is assumed to be rigid in that direction) reads

$$F_{j,x}(t) = F_{j,t}(t) \cos \varphi_j(t) + F_{j,r}(t) \sin \varphi_j(t). \quad (2.50)$$

Since the tool has several cutting edges, the complete forcing is the sum of the x components of all of the arising forces at each tooth. Substitution into eq. (2.42) and eq. (2.43) gives the forcing vector as

$$F_x(t) = Q(t)(f_z + x_1(t - \tau) - x_1(t))^q, \quad (2.51)$$

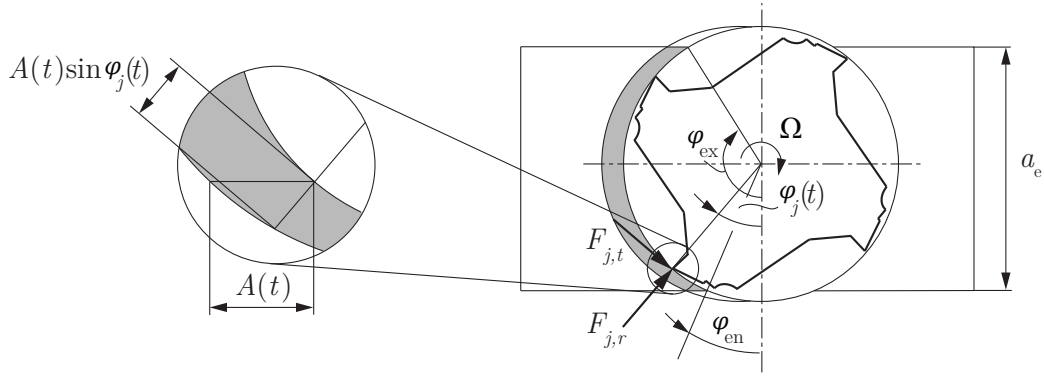


Figure 2.16. Components of the cutting-force vector in case of a one-dimensional model.

where

$$Q(t) = \sum_{j=1}^N a_p g_j(t) \sin^q \varphi_j(t) (K_t \cos \varphi_j(t) + K_r \sin \varphi_j(t)), \quad Q(t) = Q(t + \tau) \quad (2.52)$$

is a τ -periodic function. If the nonlinear equation of motion is formed, its solution can be given as $\mathbf{x}(t) = \mathbf{x}_p(t) + \boldsymbol{\xi}(t)$, where $\mathbf{x}_p(t) = \mathbf{x}_p(t + \tau)$ is a periodic solution and $\boldsymbol{\xi}(t)$ is a small perturbation. After the linearization, the linear equation can be analyzed. The detailed derivations for a single-degree-of-freedom model can be found in [4]. Thus the equation of motion is given as

$$\mathbf{M}\ddot{\boldsymbol{\xi}}(t) + \mathbf{C}\dot{\boldsymbol{\xi}}(t) + \mathbf{K}\boldsymbol{\xi}(t) = \boldsymbol{\kappa}(t)(\boldsymbol{\xi}(t - \tau) - \boldsymbol{\xi}(t)), \quad (2.53)$$

where $\boldsymbol{\kappa}(t) = \boldsymbol{\kappa}(t + \tau)$ is also periodic, so-called directional dynamic cutting-force coefficient, which depends on radial and axial cutting parameters and depth of cut. In case of turning operations, $\boldsymbol{\kappa}(t) = \boldsymbol{\kappa}$ is constant. Since only the tooltip is excited, $\boldsymbol{\kappa}(t)$ contains only one non-zero element, i.e.

$$\boldsymbol{\kappa}(t) = \begin{pmatrix} \kappa(t) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \quad (2.54)$$

and

$$\kappa(t) = a_p q f_z^{q-1} \sum_{j=1}^N g_j(t) \sin^q \varphi_j(t) (K_t \cos \varphi_j(t) + K_r \sin \varphi_j(t)). \quad (2.55)$$

2.2.1 Semi-discretization

Starting from the spatial representation, the corresponding state space equations reads

$$\begin{aligned} \dot{\mathbf{z}}(t) &= \mathbf{A}_s(t)\mathbf{z}(t) + \mathbf{B}_s(t)u(t - \tau), \\ u(t) &= \mathbf{C}_s\mathbf{z}(t), \end{aligned} \quad (2.56)$$

where the system matrix $\mathbf{A}_s(t)$ and the input matrix $\mathbf{B}_s(t)$ are periodic and can be given according to eq. (2.53). Here the size of $\mathbf{B}_s(t)$ and $\mathbf{C}_s$ are $2n \times 1$ and $1 \times 2n$ respectively, since $\boldsymbol{\kappa}(t)$ contains only one non-zero element. The state matrices are therefore

$$\begin{aligned} \mathbf{z}(t) &= \begin{pmatrix} \boldsymbol{\xi}(t) \\ \dot{\boldsymbol{\xi}}(t) \end{pmatrix}, \quad \mathbf{A}_s(t) = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}(\mathbf{K} + \boldsymbol{\kappa}(t)) & -\mathbf{M}^{-1}\mathbf{C} \end{pmatrix}, \\ \mathbf{B}_s(t) &= \begin{pmatrix} 0 & \cdots & 0 & \frac{\kappa(t)}{m_1} & \cdots & 0 \end{pmatrix}^T, \quad \mathbf{C}_s = \begin{pmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}. \end{aligned} \quad (2.57)$$

The equation cannot be solved analytically by the variation of constants because of the periodicity of the coefficients. In these cases the Floquet theory of periodic delay differential equations can be applied. According to the theory, the solution segment of $\mathbf{z}_t$ for a general linear time-periodic DDE of the form

$$\dot{\mathbf{z}}(t) = \mathbf{L}(t, \mathbf{z}_t), \quad \mathbf{L}(t + T, \cdot) = \mathbf{L}(t, \cdot) \quad (2.58)$$

associated with the initial function $\mathbf{z}_0$ can be given as $\mathbf{z}_t = \boldsymbol{\mathcal{U}}(t)\mathbf{z}_0$, where $\mathbf{z}_t(\theta) = \mathbf{z}(t + \theta)$, $\theta \in [-T, 0]$, $\mathbf{L}$ is a linear functional, which is periodic in its first argument, furthermore T is the principal period and $\boldsymbol{\mathcal{U}}(t)$ is the solution operator. The stability of the system is determined by the spectrum of the corresponding monodromy operator $\boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{U}}(T)$ [4]. This operator usually cannot be determined in closed form but can be approximated numerically.

Based on the semi-discretization method, the periodic coefficients are approximated by piece-wise constant terms, i.e.

$$\mathbf{A}_{s,i} = \frac{1}{h} \int_{t_i}^{t_{i+1}} \mathbf{A}_s dt, \quad \mathbf{B}_{s,i} = \frac{1}{h} \int_{t_i}^{t_{i+1}} \mathbf{B}_s dt \quad (2.59)$$

where $i = 1, 2 \dots p$, $\tau = ph$ and p is the number of the discretization steps. The approximated monodromy matrix in case of turning is the $\mathbf{G}_i$ coefficient matrix itself. For milling operations the monodromy operator is approximated as

$$\boldsymbol{\mathcal{M}} \approx \boldsymbol{\Pi} = \mathbf{G}_{p-1} \mathbf{G}_{p-2} \cdots \mathbf{G}_0, \quad (2.60)$$

furthermore $\boldsymbol{\Pi} \in \mathbb{R}^{2n+r \times 2n+r}$ and

$$\mathbf{G}_i = \left(\begin{array}{c|ccc} \mathbf{P}_i & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{Q}_i \\ \mathbf{C}_s & 0 & \cdots & 0 & 0 \\ \mathbf{0} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & 0 & \cdots & 1 & 0 \end{array} \right), \quad \mathbf{P}_i := e^{\mathbf{A}_{s,i}h}, \quad \mathbf{Q}_i := \int_0^h e^{\mathbf{A}_{s,i}(h-s)} \mathbf{B}_{s,i} ds. \quad (2.61)$$

The system is stable if all of the complex eigenvalues of $\boldsymbol{\Pi}$ are located inside the unit circle, it is unstable otherwise.

Semi-discretization for non-proportional damping

Similarly to turning operations, the equation of motion can be transformed into the modal space. Here only the non-proportional representation is given, but the proportional description can also be formulated. The already presented first-order form can be given as

$$\begin{pmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \dot{\boldsymbol{\xi}}(t) \\ \ddot{\boldsymbol{\xi}}(t) \end{pmatrix} + \begin{pmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{pmatrix} \begin{pmatrix} \boldsymbol{\xi}(t) \\ \dot{\boldsymbol{\xi}}(t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\kappa}(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \left(\begin{pmatrix} \boldsymbol{\xi}(t-\tau) \\ \dot{\boldsymbol{\xi}}(t-\tau) \end{pmatrix} - \begin{pmatrix} \boldsymbol{\xi}(t) \\ \dot{\boldsymbol{\xi}}(t) \end{pmatrix} \right) \quad (2.62)$$

then introducing the notation

$$\mathbf{A}\dot{\mathbf{v}}(t) + \mathbf{B}\mathbf{v}(t) = \tilde{\boldsymbol{\kappa}}(t) (\mathbf{v}(t-\tau) - \mathbf{v}(t)), \quad (2.63)$$

the equation of motion in the modal space reads

$$\boldsymbol{\Psi}^T \mathbf{A} \boldsymbol{\Psi} \dot{\mathbf{q}}_{\mathbf{v}}(t) + \boldsymbol{\Psi}^T \mathbf{B} \boldsymbol{\Psi} \mathbf{q}_{\mathbf{v}}(t) = \boldsymbol{\Psi}^T \tilde{\boldsymbol{\kappa}}(t) \boldsymbol{\Psi} (\mathbf{q}_{\mathbf{v}}(t-\tau) - \mathbf{q}_{\mathbf{v}}(t)). \quad (2.64)$$

This formula can be simplified, which yields

$$\dot{\mathbf{q}}_{\mathbf{v}}(t) - \boldsymbol{\Lambda} \mathbf{q}_{\mathbf{v}}(t) = \tilde{\boldsymbol{\kappa}}_m(t) (\mathbf{q}_{\mathbf{v}}(t-\tau) - \mathbf{q}_{\mathbf{v}}(t)), \quad (2.65)$$

where $\tilde{\boldsymbol{\kappa}}_m(t) = \boldsymbol{\Psi}^T \tilde{\boldsymbol{\kappa}}(t) \boldsymbol{\Psi}$. Since $\tilde{\boldsymbol{\kappa}}(t)$ has only one non-zero element, the matrix condensation gives

$$\tilde{\boldsymbol{\kappa}}_m(t) = \dots = \kappa(t) \begin{pmatrix} \psi_{11} \\ \bar{\psi}_{11} \\ \vdots \\ \psi_{n1} \\ \bar{\psi}_{n1} \end{pmatrix} \begin{pmatrix} \psi_{11} & \bar{\psi}_{11} & \dots & \psi_{n1} & \bar{\psi}_{n1} \end{pmatrix} \quad (2.66)$$

from which $\mathbf{B}_s(t)$ and $\mathbf{C}_s$ can be recognized. The matrices of the state space model are

$$\begin{aligned} \mathbf{z}(t) &= \mathbf{q}_{\mathbf{v}}(t), \quad \mathbf{A}_s(t) = \boldsymbol{\Lambda} - \tilde{\boldsymbol{\kappa}}_m(t), \\ \mathbf{B}_s(t) &= \kappa(t) \begin{pmatrix} \psi_{11} & \bar{\psi}_{11} & \dots & \psi_{n1} & \bar{\psi}_{n1} \end{pmatrix}^T, \quad \mathbf{C}_s = \begin{pmatrix} \psi_{11} & \bar{\psi}_{11} & \dots & \psi_{n1} & \bar{\psi}_{n1} \end{pmatrix}. \end{aligned} \quad (2.67)$$

As it can be seen, using the fitted modal parameters the stability charts can be calculated.

2.2.2 Sensitivity analysis of milling

Similarly to turning, the sensitivity of milling operations is investigated. The simple one-dimensional system is analyzed assuming a two-degrees-of-freedom model. Note, that this model is just a simplification. When the vibrations cannot be neglected in the perpendicular direction y , a two-dimensional model must be considered which extends the analysis.

Error in the estimation of modal parameters

The sensitivity is analyzed from the same points of view. First, the modal parameters are distorted manually and the effects are evaluated. Then the two-degrees-of-freedom model is approximated by a single-degree-of-freedom one when the second mode is neglected or the two modes are merged. The analyzed system agrees with the one presented in Section 2.1.3, the parameters can be found in Table 2.1 (page 34). The selected operation is a 50% up-milling process ($a_e/D = 0.5$) with a straight fluted tool where the number of the cutting edges is $N = 2$. The machining parameters are $q = 0.75$, $K_t = 133 \cdot 10^6 \text{ N/m}^{1+q}$, $K_r = 40 \cdot 10^6 \text{ N/m}^{1+q}$ and $f_z = 0.1 \text{ mm/tooth}$.

First, the sensitivity of the estimation of the natural angular frequencies is analyzed. The frequencies are shifted by $\pm 50 \text{ rad/s}$ ($\pm 5\%$), while the other parameters are fixed. The results can be seen in Figure 2.17. If the frequency of the first mode is approximated incorrectly (Cases a) and b)), similarly to turning processes, the stability chart is almost unaffected for low spindle speeds, but the boundaries shift for higher speeds. If the second mode at higher frequency is fitted incorrectly (Cases c) and d)) the chart gets affected and the boundaries change both for lower and higher spindle speeds. For multiple errors the wandering of the boundaries is approximately the sum of errors of the two modes.

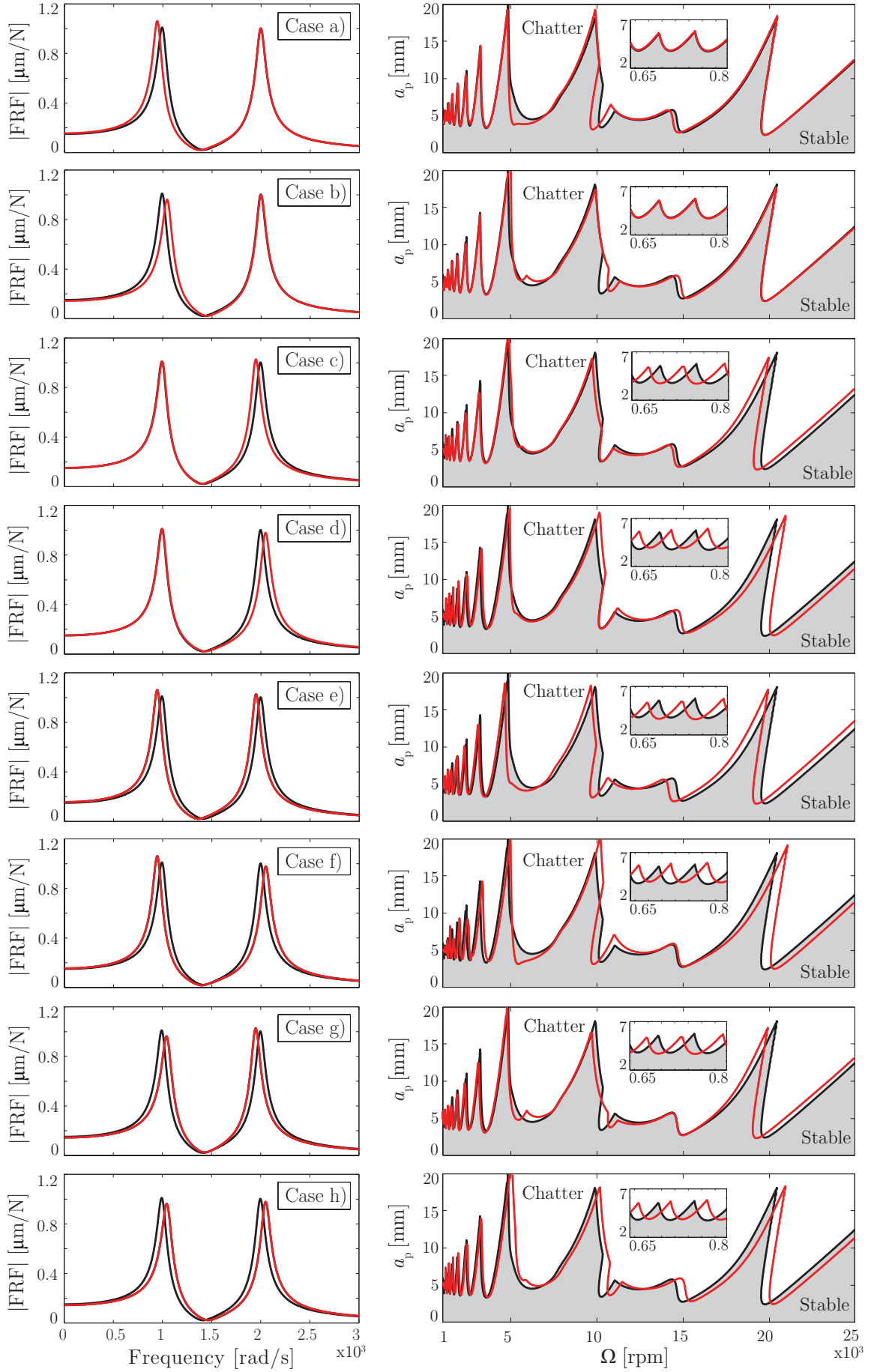
In Figure 2.18, the effect of the change of the damping coefficient ξ is analyzed. In many cases the approximation of this parameter is more uncertain than the fit of the frequency or the mode shape. Cases a) and b) present the examples where damping of the first mode is fitted incorrectly, while Cases c) and d) show how the chart is affected if the second mode has fitting error. It can be seen that in this example the chart is more sensitive to the estimation of the second mode, since the boundaries change for lower and higher frequencies too. The second mode seems to be more dominant. Note that this is not true in general, though in this specific example it is visible. If both of the two modes contain fitting error (Cases e)–h)), the wandering of the boundaries are determined by the error of each fitted mode.

If the estimated ψ^2 contains error only, the results are similar to previously presented ones. Cases a) and b) show how the boundaries change due to estimation of the first mode and Cases c) and d) correspond to the fitting error of the second mode. It can be seen that the more dominant mode is the second one, which affects the chart for lower and higher speeds too. Cases e)–h) present the results of multiple errors.

In this specific example the second mode appears to be more dominant, the chart is more sensitive to the error of the estimation of its modal parameters.

Neglected modes

As it was presented for turning operations, the neglected modes in some cases can have significant effect on the stability chart. In Figure 2.20 some typical cases are given. The original and the fitted modal parameters are listed in Table 2.2.

Figure 2.17. Sensitivity of the stability chart w.r.t. the estimation of ω_n (± 50 rad/sec).

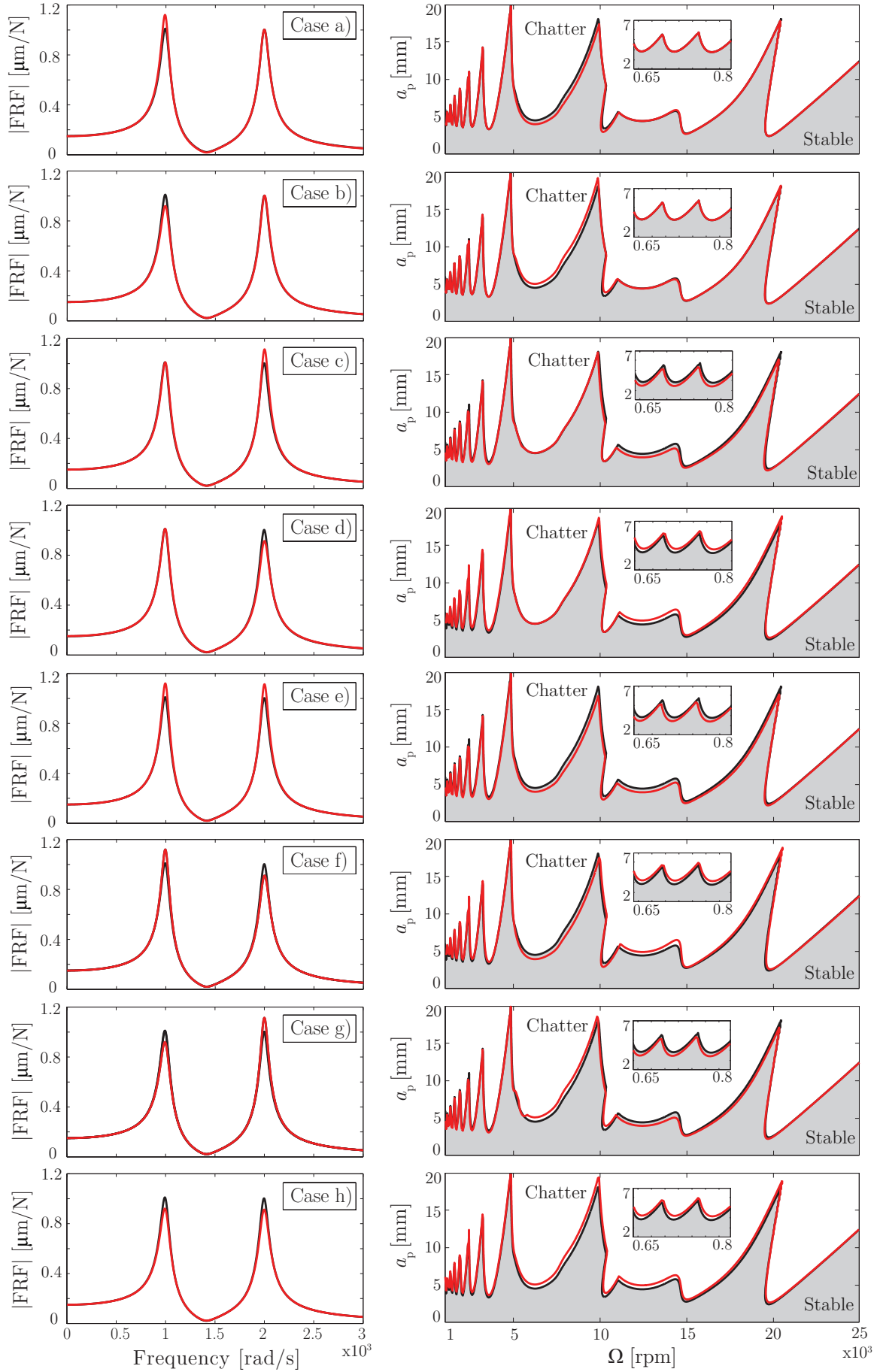
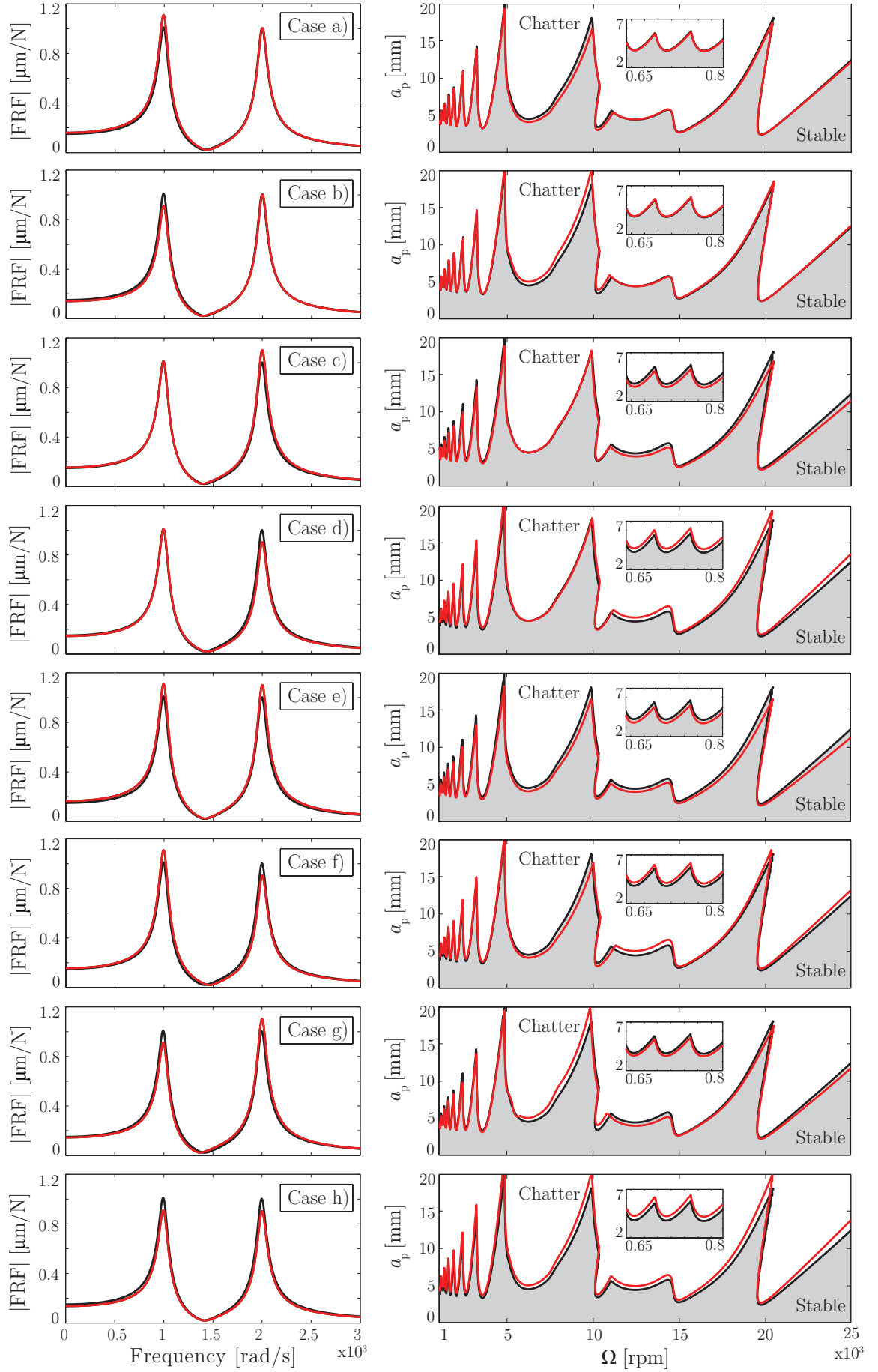


Figure 2.18. Sensitivity of the stability chart w.r.t. the estimation of ξ ($\pm 10\%$).

Figure 2.19. Sensitivity of the stability chart w.r.t. the estimation of ψ^2 ($\pm 10\%$).

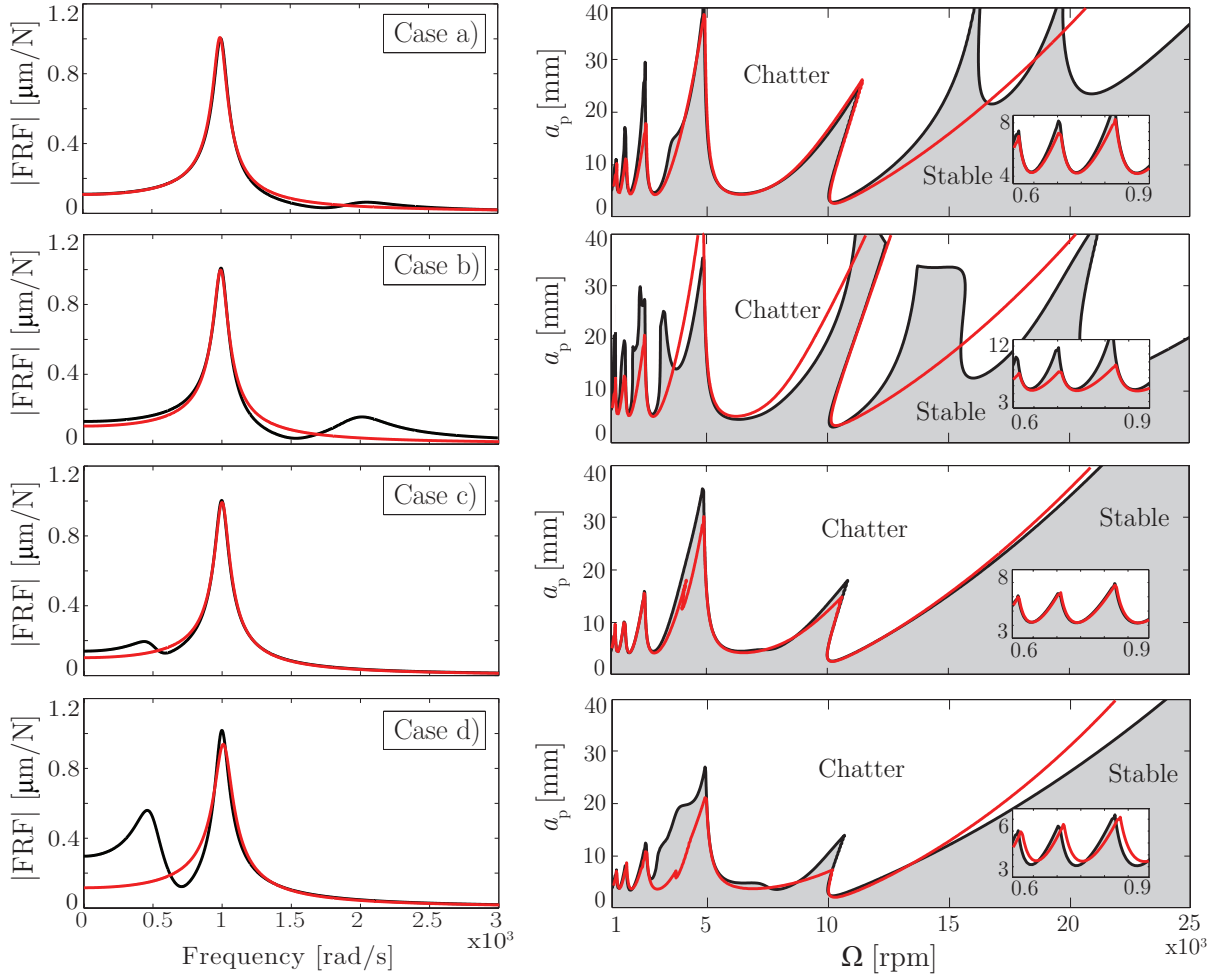


Figure 2.20. Results of the sensitivity analysis in case of neglected mode.

For Cases a) and b), when the neglected mode appears at higher frequency compared to the dominant mode, the stability boundaries significantly change for high spindle speeds. For lower speeds, if the neglected mode is small, the difference is not significant, but as the neglected mode gets larger, the difference also increases. In real studies, similar fitting errors can occur, therefore the accurate approximation of the frequency response function is necessary.

If the neglected mode is at lower frequency compared to the dominant mode, as it can be seen in Cases c) and d), the stability chart is not so sensitive, compared to the previous cases. Since the fitted single-degree-of-freedom model cannot approximate properly the original FRF, there will be difference between the stability charts. However it can be seen that it is not as significant as in Cases a) and b). This phenomenon between the different cases is already explained in Figure 2.9.

It can be observed that the stability chart is more sensitive at higher spindle speeds and the estimation of modes at higher frequencies can highly affect the stable boundaries.

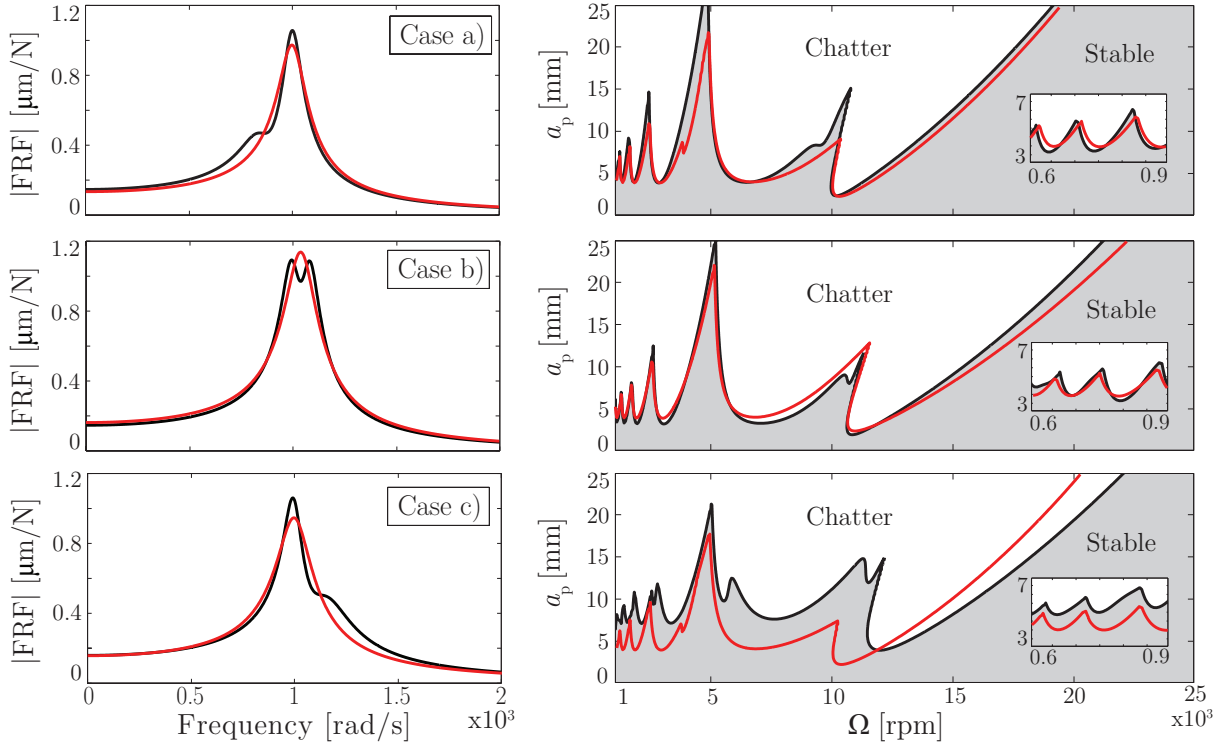


Figure 2.21. Results of the sensitivity analysis in case of merged modes.

Merged modes

In a similar manner, the effect of merged modes, which always occurs in real cases, can be analyzed. The original and fitted modal parameters can be found in Table 2.3, and the corresponding stability charts in Figure 2.21. If the neglected mode is located exactly before the dominant mode (Case a)), then the effect is not significant, though some differences can be observed for low and high speeds. Similarly to Case b), where the two modes are closely the same, the single-degree-of-freedom approximation gives acceptable approximation. In contrast, when the neglected mode is at higher frequency (Case c)), the stability chart changes drastically. Note, that in these figures the error is enlarged, but these cases clearly show that the chart is more sensitive to the estimation of modes at higher frequencies.

2.3 Mode interaction in milling operations

In most of the cases, the tool-workpiece system cannot be characterized by a one-dimensional model as it was presented in the previous sections. Usually, the tool is the most flexible part of the assembly, which can be modeled as a cantilever beam. In this case, due to the symmetry, the system has symmetric (modal) parameters [4]. Therefore, the measured FRFs in directions x and y are very similar. Some real measured case studies can be found in [2, 8, 13, 18]). In this section, the dynamical model and derivations

are presented for a milling operation with helical tools [4, 16, 17] including the mode interaction effect [15].

$$\mathbf{M}_y \ddot{\mathbf{y}}(t) + \mathbf{C}_y \dot{\mathbf{y}}(t) + \mathbf{K}_y \mathbf{y}(t) = \mathbf{f}_y(t), \quad (2.69)$$
$$\varphi_j(t, z) = \frac{2\pi\Omega}{60}t + j\frac{2\pi}{N} - z\frac{2\pi}{Nl_p}, \quad (2.70)$$
$$\mathrm{d}F_{j,r}(t, z) = g_j(t, z)K_r h_j^q(t, z)\mathrm{d}z, \quad (2.72)$$

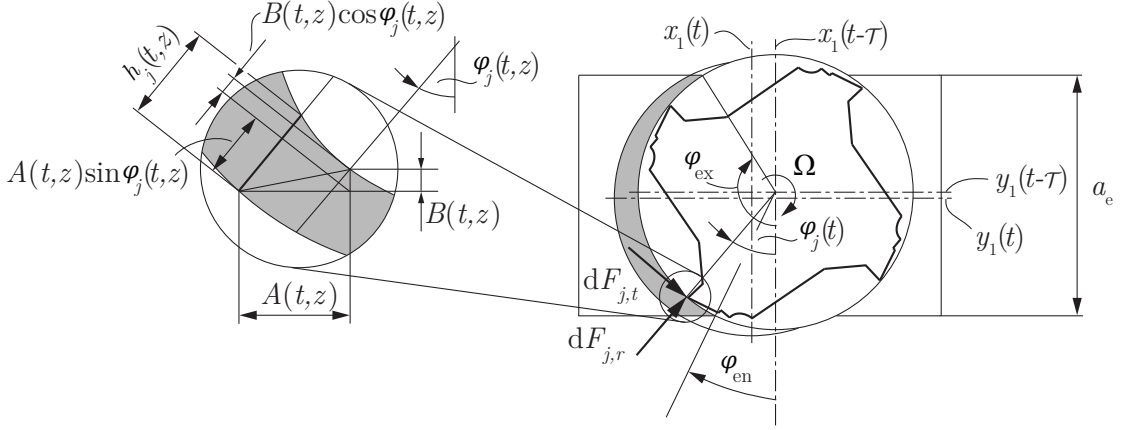


Figure 2.23. Chip thickness calculation for the two-dimensional milling model.

where $h_j(t, z)$ is the chip thickness cut by the j^{th} tooth at axial immersion z , and the screen function reads

$$g_j(t, z) = \begin{cases} 1, & \text{if } \varphi_{\text{en}} < (\varphi_j(t, z) \bmod 2\pi) < \varphi_{\text{ex}}, \\ 0, & \text{otherwise.} \end{cases} \quad (2.73)$$

Based on Figure 2.23, the actual chip thickness at tooth j can be calculated approximately as

$$\begin{aligned} h_j(t, z) &\approx A(t, z) \sin \varphi_j(t, z) + B(t, z) \sin \varphi_j(t, z) \\ &\approx (f_z + x_1(t - \tau) - x_1(t)) \sin \varphi_j(t, z) + (y_1(t - \tau) - y_1(t)) \cos \varphi_j(t, z), \end{aligned} \quad (2.74)$$

where $x_1(t)$ and $y_1(t)$ are the displacements of the center of the tool in direction x and y , and $\tau = 60/(N\Omega)$ is the tooth-passing period. The components of the elementary cutting force acting on tooth j reads

$$dF_{j,x}(t, z) = dF_{j,t}(t, z) \cos \varphi_j(t, z) + dF_{j,r}(t, z) \sin \varphi_j(t, z), \quad (2.75)$$

$$dF_{j,y}(t, z) = -dF_{j,t}(t, z) \sin \varphi_j(t, z) + dF_{j,r}(t, z) \cos \varphi_j(t, z), \quad (2.76)$$

from which the resultant cutting forces can be calculated as

$$F_{j,x}(t) = \sum_{j=1}^N \int_0^{a_p} dF_{j,x}(t, z) dz = \sum_{j=1}^N \int_0^{a_p} g_j(t, z) (K_t \cos \varphi_j(t, z) + K_r \sin \varphi_j(t, z)) h_j^q(t) dz, \quad (2.77)$$

$$F_{j,y}(t) = \sum_{j=1}^N \int_0^{a_p} dF_{j,y}(t, z) dz = \sum_{j=1}^N \int_0^{a_p} g_j(t, z) (-K_t \sin \varphi_j(t, z) + K_r \cos \varphi_j(t, z)) h_j^q(t) dz. \quad (2.78)$$

The equation of motion can be represented in the form

$$\mathbf{M}\ddot{\mathbf{v}}(t) + \mathbf{C}\dot{\mathbf{v}}(t) + \mathbf{K}\mathbf{v}(t) = \mathbf{f}_v(t), \quad (2.79)$$

where

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_x & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_y \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} \mathbf{C}_x & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_y \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} \mathbf{K}_x & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_y \end{pmatrix}, \quad \mathbf{f}_v(t) = \begin{pmatrix} \mathbf{f}_x(t) \\ \mathbf{f}_y(t) \end{pmatrix}. \quad (2.80)$$

The general coordinate vector is introduced as $\mathbf{v}(t) = \mathbf{v}_p(t) + \boldsymbol{\epsilon}(t)$, where

$$\mathbf{v}_p(t) = \begin{pmatrix} \mathbf{x}_p(t) \\ \mathbf{y}_p(t) \end{pmatrix} \quad (2.81)$$

is the τ -periodic component of the tool motion and

$$\boldsymbol{\epsilon}(t) = \begin{pmatrix} \boldsymbol{\xi}(t) \\ \boldsymbol{\eta}(t) \end{pmatrix} \quad (2.82)$$

is the perturbation around the $\mathbf{v}_p(t) = \mathbf{v}_p(t + \tau)$ periodic solution. From the above introduced equations, the variational system is finally obtained as

$$\mathbf{M}\ddot{\boldsymbol{\epsilon}}(t) + \mathbf{C}\dot{\boldsymbol{\epsilon}}(t) + \mathbf{K}\boldsymbol{\epsilon}(t) = \boldsymbol{\kappa}(t)(\boldsymbol{\epsilon}(t - \tau) - \boldsymbol{\epsilon}(t)), \quad (2.83)$$

where the specific directional factor $\boldsymbol{\kappa}(t)$ reads

$$\boldsymbol{\kappa}(t) = \left(\begin{array}{ccc|ccc} \overbrace{\kappa_{xx}(t)}^{n_x} & \cdots & 0 & \overbrace{\kappa_{xy}(t)}^{n_y} & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \kappa_{yx}(t) & \cdots & 0 & \kappa_{yy}(t) & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{array} \right) \left. \begin{array}{l} \left. \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right\} n_x \\ \left. \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right\} n_y \end{array} \right\} \quad (2.84)$$

and

$$\kappa_{xx}(t) = \sum_{j=1}^N q f_z^{q-1} \int_0^{a_p} g_j(t, z) (K_t \cos \varphi_j(t, z) + K_r \sin \varphi_j(t, z)) \sin^q \varphi_j(t, z) dz, \quad (2.85)$$

$$\kappa_{xy}(t) = \sum_{j=1}^N q f_z^{q-1} \int_0^{a_p} g_j(t, z) (K_t \cos \varphi_j(t, z) + K_r \sin \varphi_j(t, z)) \cos \varphi_j(t, z) \sin^{q-1} \varphi_j(t, z) dz, \quad (2.86)$$

$$\kappa_{yx}(t) = \sum_{j=1}^N q f_z^{q-1} \int_0^{a_p} g_j(t, z) (-K_t \sin \varphi_j(t, z) + K_r \cos \varphi_j(t, z)) \sin^q \varphi_j(t, z) dz, \quad (2.87)$$

$$\kappa_{yy}(t) = \sum_{j=1}^N q f_z^{q-1} \int_0^{a_p} g_j(t, z) (-K_t \sin \varphi_j(t, z) + K_r \cos \varphi_j(t, z)) \cos \varphi_j(t, z) \sin^{q-1} \varphi_j(t, z) dz. \quad (2.88)$$

The detailed derivations, linearization, a single-degree-of-freedom example for helical milling and a two-degrees-of-freedom model with straight flutes can be found in [4].

2.3.1 Semi-discretization in case of decoupled modes

Based on the model presented in Figure 2.22, the excitation in direction x does not induce vibration in direction y (and vice versa). Furthermore, it can be shown, that the modal transformation matrix and the diagonal modal matrices of (2.79) in case of proportional damping read

$$\Phi = \begin{pmatrix} \Phi_x & \mathbf{0} \\ \mathbf{0} & \Phi_y \end{pmatrix}, \quad \mathbf{C}_m = \begin{pmatrix} (2\xi\omega_{n,x}) & \mathbf{0} \\ \mathbf{0} & (2\xi\omega_{n,y}) \end{pmatrix}, \quad \mathbf{K}_m = \begin{pmatrix} (\omega_{n,x}^2) & \mathbf{0} \\ \mathbf{0} & (\omega_{n,y}^2) \end{pmatrix}, \quad (2.89)$$

since the system matrices in (2.80) contain no cross terms. Therefore the FRFs can also be fitted independently. Similarly, omitting the derivation, the modal matrices of the non-proportional description can be written as

$$\Psi = \begin{pmatrix} \Psi_x & \mathbf{0} \\ \mathbf{0} & \Psi_y \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \Lambda_x & \mathbf{0} \\ \mathbf{0} & \Lambda_y \end{pmatrix}. \quad (2.90)$$

The semi-discretization method is only presented for the non-proportional case, but the proportional representation can also be easily formulated.

Let us define the linearized equation of motion in the first-order form as

$$\mathbf{A}\dot{\mathbf{v}}(t) + \mathbf{B}\mathbf{v}(t) = \tilde{\mathbf{\kappa}}(t) (\mathbf{v}(t - \tau) - \mathbf{v}(t)) \quad (2.91)$$

where

$$\mathbf{A} = \begin{pmatrix} 2n_x & 2n_y \\ \mathbf{A}_x & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_y \end{pmatrix} \begin{matrix} 2n_x \\ 2n_y \end{matrix}, \quad \mathbf{B} = \begin{pmatrix} 2n_x & 2n_y \\ \mathbf{B}_x & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_y \end{pmatrix} \begin{matrix} 2n_x \\ 2n_y \end{matrix}, \quad \mathbf{v}(t) = \begin{pmatrix} 1 \\ \mathbf{v}_x(t) \\ \mathbf{v}_y(t) \end{pmatrix} \begin{matrix} 2n_x \\ 2n_y \end{matrix}. \quad (2.92)$$

Furthermore

$$\mathbf{A}_\bullet = \begin{pmatrix} \mathbf{C}_\bullet & \mathbf{M}_\bullet \\ \mathbf{M}_\bullet & \mathbf{0} \end{pmatrix}, \quad \mathbf{B}_\bullet = \begin{pmatrix} \mathbf{K}_\bullet & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_\bullet \end{pmatrix}, \quad \mathbf{v}_x(t) = \begin{pmatrix} \xi(t) \\ \dot{\xi}(t) \end{pmatrix}, \quad \mathbf{v}_y(t) = \begin{pmatrix} \eta(t) \\ \dot{\eta}(t) \end{pmatrix}, \quad (2.93)$$

and $\tilde{\mathbf{\kappa}}(t)$ has to be extended as

$$\tilde{\mathbf{\kappa}}(t) = \left\{ \begin{array}{cc} \overbrace{\begin{pmatrix} \kappa_{xx}(t) & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}}^{2n_x} & \overbrace{\begin{pmatrix} \kappa_{xy}(t) & \cdots & 0 \\ \vdots & & \vdots \\ \kappa_{yx}(t) & \cdots & 0 \end{pmatrix}}^{2n_y} \\ \overbrace{\begin{pmatrix} \kappa_{yy}(t) & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}}^{2n_y} & \end{array} \right\} \begin{matrix} 2n_x \\ 2n_y \end{matrix}. \quad (2.94)$$

Using the $\Psi \mathbf{z}(t) = \mathbf{v}(t)$ modal transformation presented for non-proportional systems, the forced linear equation in the modal space reads

$$\dot{\mathbf{z}}(t) - \Lambda \mathbf{z}(t) = \tilde{\kappa}_m(t) (\mathbf{z}(t - \tau) - \mathbf{z}(t)), \quad (2.95)$$

where $\tilde{\kappa}_m(t) = \Psi^T \tilde{\kappa}(t) \Psi$, moreover Ψ and Λ are given in eq. (2.90). The state space equations can be formulated according to

$$\begin{aligned} \dot{\mathbf{z}}(t) &= \mathbf{A}_s(t) \mathbf{z}(t) + \mathbf{B}_s(t) \mathbf{u}(t - \tau), \\ \mathbf{u}(t) &= \mathbf{C}_s \mathbf{z}(t). \end{aligned} \quad (2.96)$$

It was already presented for several examples, that due to the zero elements of matrix $\tilde{\kappa}(t)$ (since only the first coordinates in direction x and y are excited), the multiplication $\tilde{\kappa}_m(t) = \Psi^T \tilde{\kappa}(t) \Psi$ can be drastically reduced by eliminating the unnecessary rows of Ψ . Omitting the derivation, the final state space matrices are

$$\begin{aligned} \mathbf{A}_s(t) &= \Lambda - \tilde{\kappa}_m(t), \\ \mathbf{B}_s(t) &= \begin{pmatrix} \psi_{11}^x & \bar{\psi}_{11}^x & \cdots & \psi_{n_x 1}^x & \bar{\psi}_{n_x 1}^x & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \psi_{11}^y & \bar{\psi}_{11}^y & \cdots & \psi_{n_y 1}^y & \bar{\psi}_{n_y 1}^y \end{pmatrix}^T \begin{pmatrix} \kappa_{xx}(t) & \kappa_{xy}(t) \\ \kappa_{yx}(t) & \kappa_{yy}(t) \end{pmatrix}, \\ \mathbf{C}_s &= \begin{pmatrix} \psi_{11}^x & \bar{\psi}_{11}^x & \cdots & \psi_{n_x 1}^x & \bar{\psi}_{n_x 1}^x & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \psi_{11}^y & \bar{\psi}_{11}^y & \cdots & \psi_{n_y 1}^y & \bar{\psi}_{n_y 1}^y \end{pmatrix}. \end{aligned} \quad (2.97)$$

Note, that only two rows of the transformation matrix must be determined, which correspond to the directions x and y respectively. The resulting coefficient matrix of the discrete mapping is

$$\mathbf{G}_i = \begin{pmatrix} 2n & 2 & 2 & 2 \\ \mathbf{P}_i & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{Q}_i \\ \mathbf{C}_s & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} & \mathbf{0} \end{pmatrix} \begin{matrix} 2n \\ 2 \\ 2 \\ 2 \\ 2 \end{matrix}, \quad \mathbf{P}_i := e^{\mathbf{A}_{s,i} h}, \quad \mathbf{Q}_i := \int_0^h e^{\mathbf{A}_{s,i}(h-s)} \mathbf{B}_{s,i} ds, \quad (2.98)$$

where $n = n_x + n_y$ is the number of degrees of freedom. Finally the monodromy matrix is approximated as $\Pi = \mathbf{G}_{p-1} \mathbf{G}_{p-2} \cdots \mathbf{G}_0$.

2.3.2 Semi-discretization in case of coupled modes

In Section 2.3.1, the formulation of the state equations in the modal space for decoupled modes was presented. It is known, that in many cases, the cross vibrations cannot be

neglected. In reality, the mode shapes of the vibration do not appear purely in direction x or y . When the structure is excited in direction x , the response includes displacement in direction y (and vice versa). The theoretical background was already discussed in Section 1.4.1. The mechanical model is presented in Figure 2.24. Therefore in practice, the machine tool is measured in the main directions and cross directions too, resulting four FRFs. Theoretically, the system matrices can be given in the spatial form as

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_x & \mathbf{M}_c \\ \mathbf{M}_c & \mathbf{M}_y \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} \mathbf{C}_x & \mathbf{C}_c \\ \mathbf{C}_c & \mathbf{C}_y \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} \mathbf{K}_x & \mathbf{K}_c \\ \mathbf{K}_c & \mathbf{K}_y \end{pmatrix}, \quad (2.99)$$

where each matrix is symmetric, and $(\bullet)_c$ refers to the cross matrices expressing the coupling between the main directions x and y . The description in the modal space simplifies the notation. Here, only the non-proportional formulation is presented, but the proportional equations can be similarly formed. Let us have the governing equation in the form

$$\mathbf{A}\dot{\mathbf{v}}(t) + \mathbf{B}\mathbf{v}(t) = \mathbf{f}_v(t), \quad \mathbf{v}(t) = \begin{pmatrix} \mathbf{x}(t) & \mathbf{y}(t) & \dot{\mathbf{x}}(t) & \dot{\mathbf{y}}(t) \end{pmatrix}^T \quad (2.100)$$

where $\mathbf{A} = \mathbf{A}^T \in \mathbb{R}^{2n \times 2n}$ and $\mathbf{B} = \mathbf{B}^T \in \mathbb{R}^{2n \times 2n}$, furthermore $n = n_x + n_y$ is the number of degrees of freedom. In this representation, since the modes can have arbitrary direction, the parametric expression for the FRFs is the same for the one-dimensional model, i.e.

$$H_{ij}(\omega) = \sum_{r=1}^n \left(\frac{\psi_{ri}\psi_{rj}}{\omega i - \lambda_r} + \frac{\bar{\psi}_{ri}\bar{\psi}_{rj}}{\omega i - \bar{\lambda}_r} \right). \quad (2.101)$$

The difference between the two representation is the meaning of the mode shape vectors. In the original formulation, it represented the vibration in one dimension, here the cross terms are also included, and the mode shapes are two-dimensional. Each vibration mode can appear in any direction. If it vibrates only in direction x , the mode shape vector contains zero elements corresponding to direction y . Furthermore, since the system is excited in two dimension, the modal behavior must be determined by fitting four FRFs (the cross terms are theoretically symmetric) simultaneously, this means that two rows of the modal transformation matrix Ψ can be determined. Let us have the transformed linearized forced equation of motion corresponding to the two-dimensional milling model in the form

$$\dot{\mathbf{z}}(t) - \mathbf{\Lambda}\mathbf{z}(t) = \tilde{\boldsymbol{\kappa}}_m(t) (\mathbf{z}(t - \tau) - \mathbf{z}(t)), \quad (2.102)$$

where $\tilde{\boldsymbol{\kappa}}_m(t) = \Psi^T \tilde{\boldsymbol{\kappa}}(t) \Psi$. The only nonzero elements of $\tilde{\boldsymbol{\kappa}}(t)$ are $\kappa_{xx}(t)$, $\kappa_{xy}(t)$, $\kappa_{yx}(t)$ and $\kappa_{yy}(t)$, just like before. Though coordinates x and y are undefined, since the physically measured and excited coordinates are the same, the multiplication $\Psi^T \tilde{\boldsymbol{\kappa}}(t) \Psi$ can be simplified according to

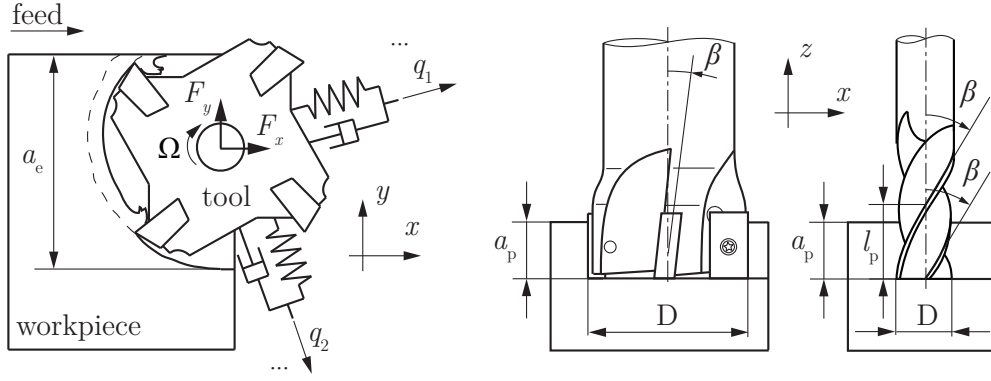


Figure 2.24. Two-dimensional dynamical model in the modal space.

$$\begin{aligned}
 \mathbf{\Psi}^T \tilde{\mathbf{\kappa}}(t) \mathbf{\Psi} &= \begin{pmatrix} \cdots & \psi_{1i} & \cdots & \psi_{1j} & \cdots \\ \cdots & \bar{\psi}_{1i} & \cdots & \bar{\psi}_{1j} & \cdots \\ & \vdots & & \vdots & \\ \cdots & \psi_{ni} & \cdots & \psi_{nj} & \cdots \\ \cdots & \bar{\psi}_{ni} & \cdots & \bar{\psi}_{nj} & \cdots \end{pmatrix} \begin{pmatrix} \vdots & \vdots \\ \cdots & \kappa_{ii}(t) & \cdots & \kappa_{ij}(t) & \cdots \\ \vdots & \vdots \\ \cdots & \kappa_{ji}(t) & \cdots & \kappa_{jj}(t) & \cdots \\ \vdots & \vdots \end{pmatrix} \begin{pmatrix} \vdots & \vdots & \vdots & \vdots \\ \psi_{1i} & \bar{\psi}_{1i} & \cdots & \psi_{ni} & \bar{\psi}_{ni} \\ \vdots & \vdots & & \vdots & \vdots \\ \psi_{1j} & \bar{\psi}_{1j} & \cdots & \psi_{nj} & \bar{\psi}_{nj} \\ \vdots & \vdots & & \vdots & \vdots \end{pmatrix} \\
 &= \begin{pmatrix} \psi_{1i} & \psi_{1j} \\ \bar{\psi}_{1i} & \bar{\psi}_{1j} \\ \vdots & \vdots \\ \psi_{ni} & \psi_{nj} \\ \bar{\psi}_{ni} & \bar{\psi}_{nj} \end{pmatrix} \begin{pmatrix} \kappa_{ii}(t) & \kappa_{ij}(t) \\ \kappa_{ji}(t) & \kappa_{jj}(t) \end{pmatrix} \begin{pmatrix} \psi_{1i} & \bar{\psi}_{1i} & \cdots & \psi_{ni} & \bar{\psi}_{ni} \\ \psi_{1j} & \bar{\psi}_{1j} & \cdots & \psi_{nj} & \bar{\psi}_{nj} \end{pmatrix}, \quad (2.103)
 \end{aligned}$$

where $i = x$, and $j = y$. It can be seen, that only the fit of two rows of $\mathbf{\Psi}$ is necessary, the others are eliminated during the multiplications. These are the elements that can be determined by modal analysis according to (1.76). The corresponding state space equations are given as

$$\begin{aligned}
 \dot{\mathbf{z}}(t) &= \mathbf{A}_s(t) \mathbf{z}(t) + \mathbf{B}_s(t) \mathbf{u}(t - \tau), \\
 \mathbf{u}(t) &= \mathbf{C}_s \mathbf{z}(t). \quad (2.104)
 \end{aligned}$$

where the state matrices are

$$\begin{aligned}
 \mathbf{A}_s(t) &= \mathbf{\Lambda} - \tilde{\mathbf{\kappa}}_m(t), \quad \mathbf{B}_s(t) = \begin{pmatrix} \psi_{1i} & \bar{\psi}_{1i} & \cdots & \psi_{ni} & \bar{\psi}_{ni} \\ \psi_{1j} & \bar{\psi}_{1j} & \cdots & \psi_{nj} & \bar{\psi}_{nj} \end{pmatrix}^T \begin{pmatrix} \kappa_{xx}(t) & \kappa_{xy}(t) \\ \kappa_{yx}(t) & \kappa_{yy}(t) \end{pmatrix}, \\
 \mathbf{C}_s &= \overbrace{\begin{pmatrix} \psi_{1i} & \bar{\psi}_{1i} & \cdots & \psi_{ni} & \bar{\psi}_{ni} \\ \psi_{1j} & \bar{\psi}_{1j} & \cdots & \psi_{nj} & \bar{\psi}_{nj} \end{pmatrix}}^{2n}. \quad (2.105)
 \end{aligned}$$

Note, that the formulation of the state matrices in eq. (2.97) and eq. (2.105) differs in $\mathbf{B}_s(t)$ and $\mathbf{C}_s$. If the modes are uncoupled, the FRFs in the main directions ($H_{xx}(\omega)$ and $H_{yy}(\omega)$) can be fitted independently, and the matrices can be constructed by merging them. The modal transformation matrix has zero elements, expressing the uncoupled behavior. In case of coupled modes, all of the FRFs must be fitted simultaneously ($H_{xx}(\omega)$, $H_{xy}(\omega) = H_{yx}(\omega)$, $H_{yy}(\omega)$), and the transformation matrix can be constructed, which generally does not contain zero elements. This expresses the direction of the mode shapes. Also note, that in the first case the number of degrees of freedom is $n_x + n_y$, while in the latter case it is n .

2.3.3 Semi-discretization in case of non-symmetric system matrices

When the mathematical models are formulated, several physical phenomena are neglected in order to make the analysis simpler and feasible, since the real effects are too complicated to handle efficiently. It is the same for modal analysis. Nonlinearities, gyroscopic effect, fluid-structure interactions etc. are neglected, because it can make the analysis extremely complicated and the measurement is uncertain also. When the modal behavior of a real structure is determined, several points must be excited and the vibrations has to be measured. In case of machine tool vibrations, the simplest analysis includes the measurement of the tool-tip from different directions. Usually four FRFs are measured in perpendicular directions.

According to Section 1.4.1, the cross FRFs ($H_{xy}(\omega)$ and $H_{yx}(\omega)$) are the same or similar. Many cases, the amplitude of the vibration in these cross directions are much smaller, in some cases it can be neglected, but sometimes it can have significant effect on the stability chart [18]. It is also known that against the theory, the cross FRFs can sometimes slightly or significantly differ in real cases. Nonlinearities, gyroscopic effect or magnetic field can cause this behavior [6, 12]. The corresponding mathematical model was introduced in Section 1.4.2.

If the system matrices are non-symmetric, then the modal transformation matrices constructed from the left and right eigenvectors can be used. If the linearized equation of motion is given as

$$\mathbf{A}\dot{\mathbf{v}}(t) + \mathbf{B}\mathbf{v}(t) = \tilde{\mathbf{\kappa}}(t) (\mathbf{v}(t - \tau) - \mathbf{v}(t)), \quad (2.106)$$

then using the modal transformation $\mathbf{v}(t) = \mathbf{\Psi}_R \mathbf{z}(t)$, and multiplying by $\mathbf{\Psi}_L^T$ from the left, one can obtain

$$\mathbf{\Psi}_L^T \mathbf{A} \mathbf{\Psi}_R \dot{\mathbf{z}}(t) + \mathbf{\Psi}_L^T \mathbf{B} \mathbf{\Psi}_R \mathbf{z}(t) = \mathbf{\Psi}_L^T \tilde{\mathbf{\kappa}}(t) \mathbf{\Psi}_R (\mathbf{z}(t - \tau) - \mathbf{z}(t)). \quad (2.107)$$

The simplification of the equation above according to Section 1.4.2 gives

$$\dot{\mathbf{z}}(t) - \mathbf{\Lambda}\mathbf{z}(t) = \tilde{\mathbf{\kappa}}_m(t) (\mathbf{z}(t - \tau) - \mathbf{z}(t)), \quad (2.108)$$

where $\tilde{\mathbf{\kappa}}_m(t) = \mathbf{\Psi}_L^T \tilde{\mathbf{\kappa}}(t) \mathbf{\Psi}_R$. This expression is analogous to the previous cases. The state space equation reads

$$\begin{aligned} \dot{\mathbf{z}}(t) &= \mathbf{A}_s(t)\mathbf{z}(t) + \mathbf{B}_s(t)\mathbf{u}(t - \tau), \\ \mathbf{u}(t) &= \mathbf{C}_s\mathbf{z}(t), \end{aligned} \quad (2.109)$$

where the state matrices (after the same simplification as before) can be given as

$$\begin{aligned} \mathbf{A}_s(t) &= \mathbf{\Lambda} - \tilde{\mathbf{\kappa}}_m(t), \quad \mathbf{B}_s(t) = \begin{pmatrix} \psi_{1i}^L & \bar{\psi}_{1i}^L & \cdots & \psi_{ni}^L & \bar{\psi}_{ni}^L \\ \psi_{1j}^L & \bar{\psi}_{1j}^L & \cdots & \psi_{nj}^L & \bar{\psi}_{nj}^L \end{pmatrix}^T \begin{pmatrix} \kappa_{xx}(t) & \kappa_{xy}(t) \\ \kappa_{yx}(t) & \kappa_{yy}(t) \end{pmatrix}, \\ \mathbf{C}_s &= \begin{pmatrix} \psi_{1i}^R & \bar{\psi}_{1i}^R & \cdots & \psi_{ni}^R & \bar{\psi}_{ni}^R \\ \psi_{1j}^R & \bar{\psi}_{1j}^R & \cdots & \psi_{nj}^R & \bar{\psi}_{nj}^R \end{pmatrix}. \end{aligned} \quad (2.110)$$

Note, that for symmetric matrices in case of non-proportional damping, only six real parameters must be determined for each degree of freedom (ψ_{ij} is complex), while for non-symmetric matrices, this goes up to ten because of the left and right eigenvectors. It is also important to see that even if the unknowns are increased, the dimension of the state matrices does not change.

2.3.4 A case study

Similarly to Section 2.1.5, a case study is presented based on a real measurement ¹. The measured FRFs can be seen in Figure 2.25. It can be seen that the cross functions (denoted by dashed line) are in the magnitude of the diagonal FRFs, therefore they cannot be neglected. The parameters of the analyzed machining process: $q = 1$, $K_t = 500 \cdot 10^6 \text{ N/m}^{1+q}$, $K_r = 200 \cdot 10^6 \text{ N/m}^{1+q}$ and $f_z = 0.05 \text{ mm/tooth}$. Different fitted FRFs were prepared using the theories presented previously. First, it is assumed that the cross FRFs can be neglected and the main functions are fitted independently using 8-8 degrees of freedom in each direction (referred as ‘No cross FRFs’). The FRFs can be seen in Figure 2.26. Note, that the scale is logarithmic in order to make visible the least dominant modes too, which are the most difficult to fit. As the next step, the symmetric formulation was used (referred as ‘Sym. cross FRFs’). The fitted functions of the 16-DoF model can be seen in Figure 2.27. Since the cross terms are significantly different, the fitted cross function cannot approximate any of the them properly. Finally, the non-symmetry of the FRFs was taken into account according to the method presented in Section 2.3.3. The fitted result consists of 16 degrees of freedom here also, which can be seen in Figure 2.28.

¹The used FRFs are measured by Zoltán Dombóvári, and his results are presented in [2].

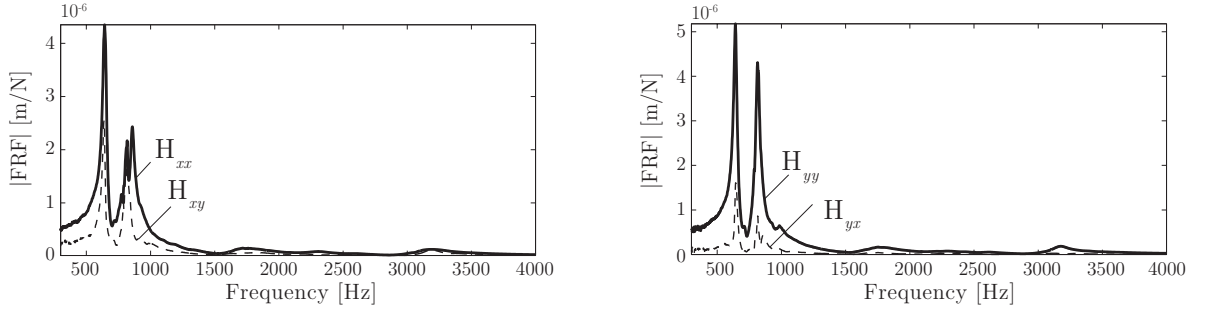


Figure 2.25. Case study: Measured frequency response functions [2].

The fitted natural frequencies and damping ratios for the different models can be found in Table 2.6, the normalized mode shapes are not listed here, since it would result too many parameters, which are difficult to compare. As it can be seen, the fitted natural frequencies are usually close to each other, although there are some modes which are not always found. In contrast, the fitted damping ratios show notable difference. Note, that although the modal parameters slightly differ, the fitted curve precisely approximates the measurement for each case (Figure 2.26–2.28).

Once the measured FRFs are fitted by any of the methods presented in this work, the semi-discretization technique can be used. To reduce the computational time and improve the precision during the calculation of the stability charts, the first-order semi-discretization method was utilized (see Appendix A.3).

The stability charts of the different models in case of 50%- and 10% down-milling process can be seen in Figure 2.29. The boundaries corresponding to the non-symmetric fitting is indicated by thin black line, while thick black line shows the results without and with symmetric cross functions respectively. In case of the 50% down-milling operations (*a*) and *b*), the difference is not significant, although it is important that the decoupled and symmetric results are closer to each other, than any of them to the non-symmetric one. In other words, the stability chart corresponding to the non-symmetric cross terms differ the most from the others. Therefore the chart seems to be more sensitive to the non-symmetry. If the radial immersion decreases, the difference increases. In case of a 10% down-milling processes (*c*) and *d*), the stable domain changes drastically. Note, that here it is also true, that the charts calculated without the cross functions or with symmetric cross functions are closer to each other than to results of the non-symmetric functions. It can be said that in this case, the stability is sensitive to the approximation of the non-symmetry in the frequency response function matrix. It has significant practical relevance, since during the cutting process the spindle rotates. If the gyroscopic effect cannot be neglected, then even if the measured static structure has symmetric FRFs, during the operation the symmetry fails and the stability chart changes.

In Figure 2.30, the same comparison is presented in case of 50%- and 10% up-milling process. The 50% up-milling (*a*) and *b*)) seems to be more sensitive to the estimation of the

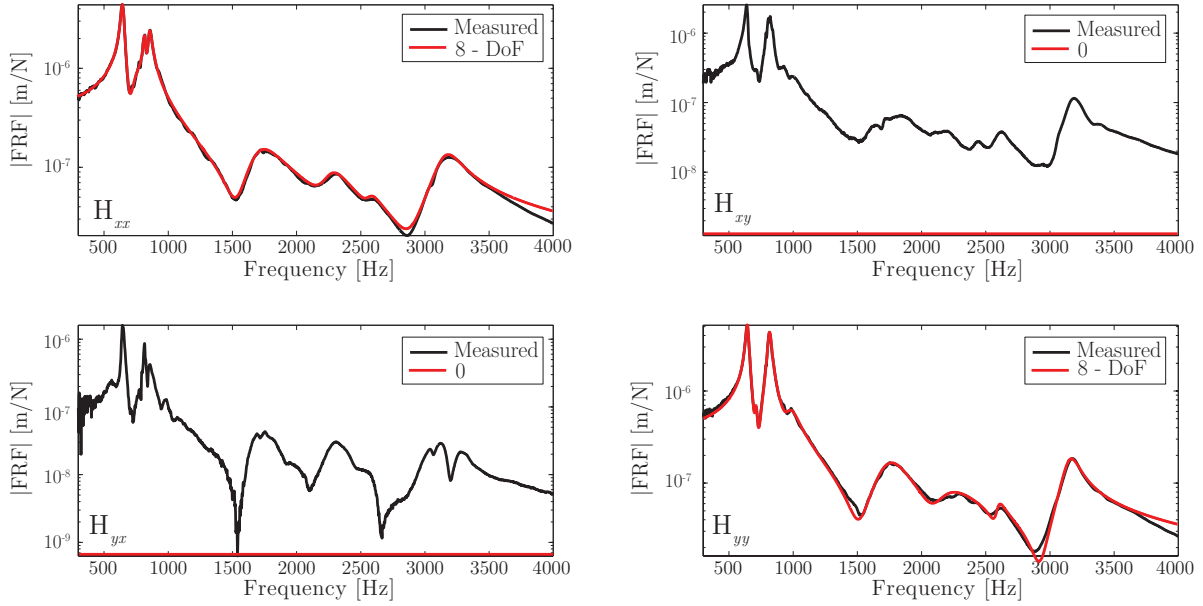


Figure 2.26. Case study: Fitted FRFs without cross functions.

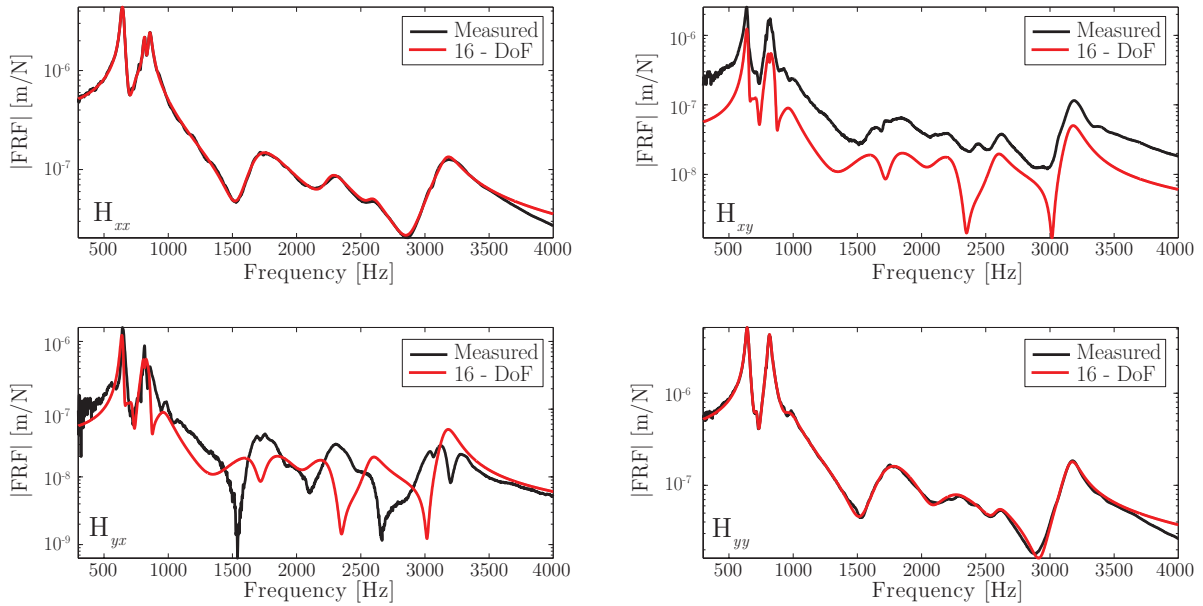


Figure 2.27. Case study: Fitted FRFs with symmetric cross functions.

cross functions than the down-milling process. The boundaries slightly change in general, except for the large pocket around 20 000 rpm. Note, that it is still true, that the results of the decoupled and symmetric models are closer to each other, than to the non-symmetric model. However, the 10% up-milling processes show surprisingly large sensitivity (*c*) and *d*)), all of the three different models give very different result. Though the stable pocket can be roughly identified, the error in subfigure *c*) is not acceptable.

This case study shows that the inaccurate approximation of the cross frequency re-

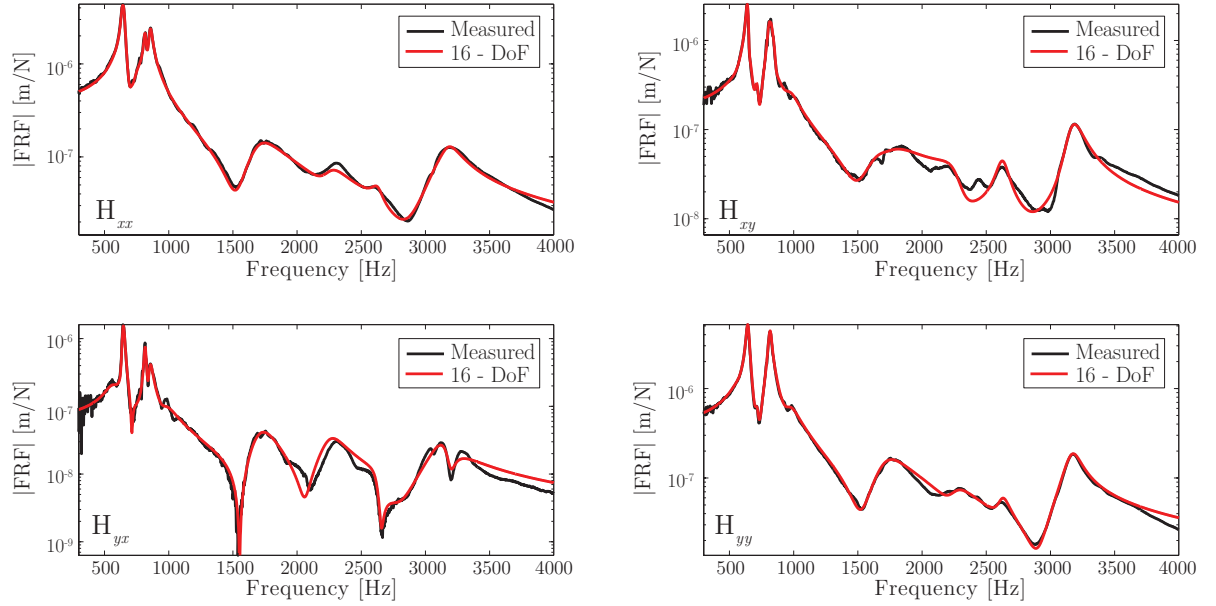


Figure 2.28. Case study: Fitted FRFs with non-symmetric cross functions.

No cross FRFs			Sym. cross FRFs			Non-sym. cross FRFs		
Mode	ω_n [Hz]	ξ [%]	Mode	ω_n [Hz]	ξ [%]	Mode	ω_n [Hz]	ξ [%]
1	643.26	1.86	1	638.46	1.76	1	568.83	9.68
2	645.19	2.00	2	639.30	1.82	2	637.96	1.65
3	774.38	9.05	3	647.86	1.87	3	642.25	2.07
4	815.54	1.82	4	715.51	2.39	4	648.25	1.86
5	815.76	1.44	5	775.74	10.02	5	714.39	2.21
6	853.96	1.90	6	815.44	2.04	6	779.90	10.55
7	710.46	1.54	7	815.93	1.11	7	815.01	1.96
8	974.66	5.53	8	854.53	1.84	8	815.87	1.19
9	1706.17	8.11	9	959.53	6.60	9	854.71	1.85
10	1746.80	7.95	10	1702.76	7.49	10	965.87	6.19
11	2176.83	8.19	11	1768.56	9.78	11	1699.65	10.19
12	2273.32	4.60	12	2135.17	8.03	12	1711.19	7.08
13	2574.92	2.41	13	2264.53	3.99	13	2256.59	4.32
14	2598.08	1.34	14	2585.73	2.55	14	2626.33	1.82
15	3159.51	2.10	15	3140.98	4.19	15	3150.30	3.89
16	3162.28	3.48	16	3163.44	2.13	16	3170.38	2.10

Table 2.6. Natural frequencies and damping ratios corresponding to the different fitting models.

sponse function can significantly affect the stable domains, moreover the sensitivity is generally larger if processes with small radial immersions are studied. Furthermore, it has to be highlighted that the non-symmetry of the frequency response function matrix also shows sensitivity, which is important from the operational point of view. During the cutting process the gyroscopic effect of the rotating elements of the spindle can be significant, and therefore the operational modal behavior and the corresponding stability charts can be substantially different.

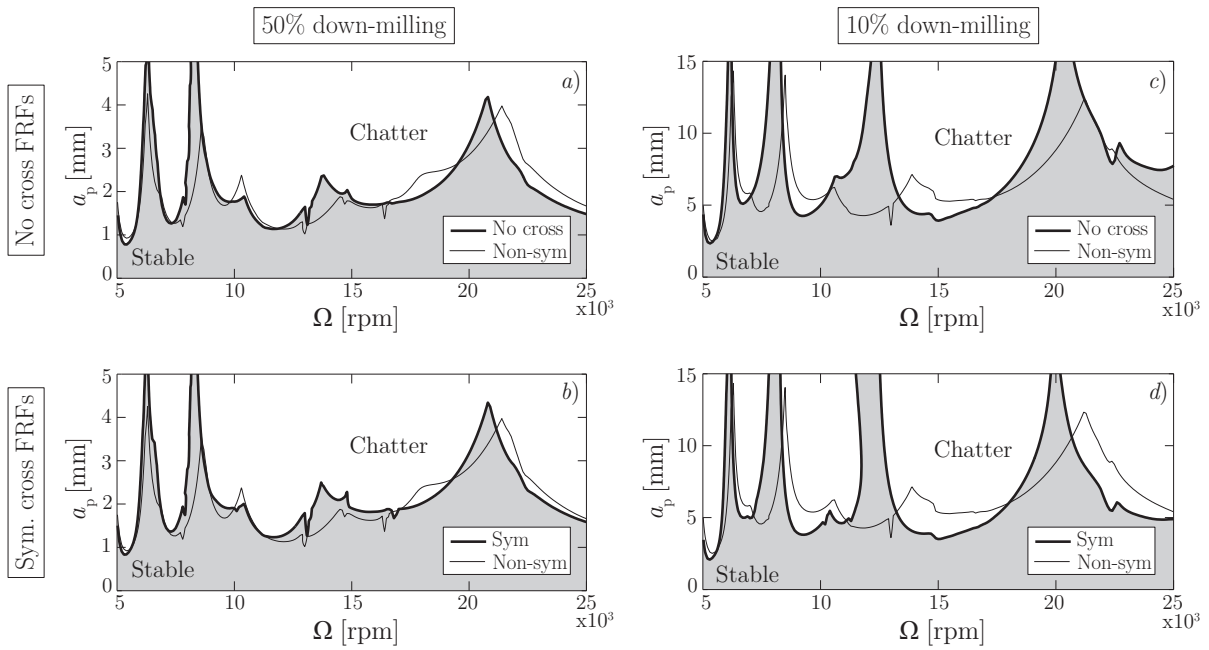


Figure 2.29. Case study: sensitivity of milling operations in case of down-milling process.

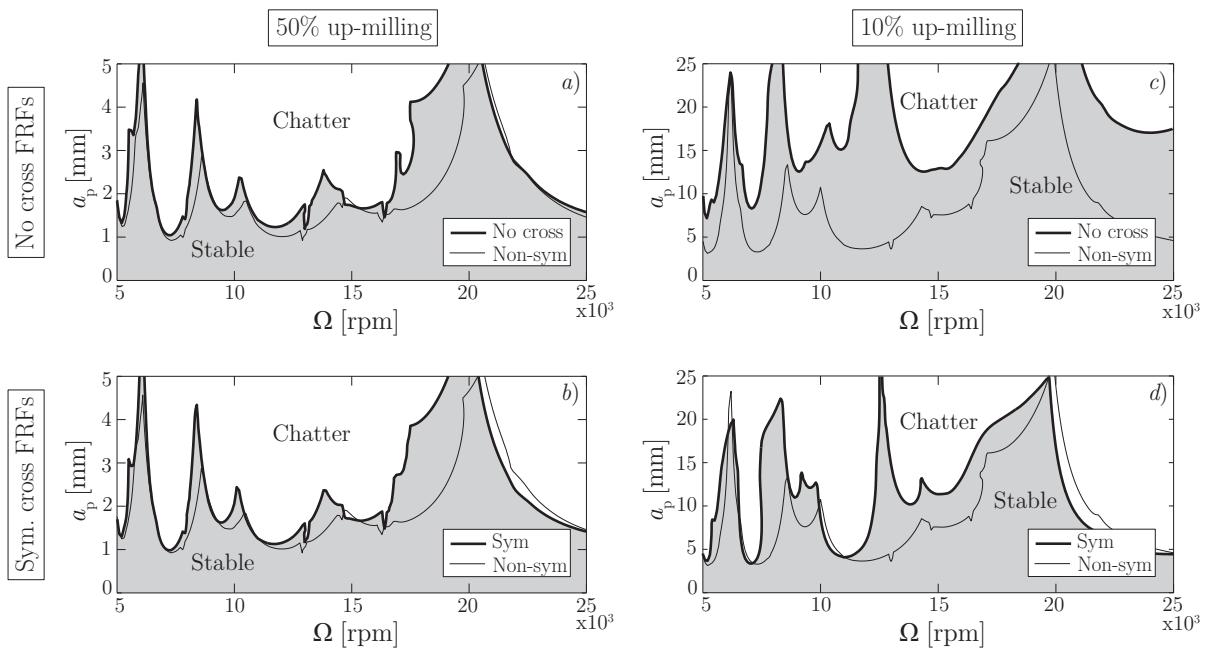


Figure 2.30. Case study: sensitivity of milling operations in case of up-milling process.

Summary

One of the main optimization goals in manufacturing is to increase the material removal rate and maintain the quality of the product, which are sometimes contradictory requirements. The required time for the manufacturing process can be decreased if the speed of the spindle is increased and therefore the material removal time is also increased. The harmful vibrations arising during the cutting operation can damage the tool, the workpiece or the machine itself. The most commonly used explanation for the machine tool chatter is the so-called *regenerative effect* described by sophisticated mechanical and mathematical models.

In the first part of the present work, the fundamentals of modal analysis and the characteristics of multiple-degrees-of-freedom systems were presented based on the proportional and non-proportional damping models. The frequency response functions defined by the modal parameters are explained in details for both cases. In order to study efficiently the stability analysis of machine tool chatter, a simple iterative technique was utilized with which low number of degrees of freedom can be handled.

In the second part of the work, the sensitivity of the stability charts corresponding to the different manufacturing processes was investigated. Based on a measured noisy frequency response function, the extraction of the real modal parameters is a challenging task. Due to the fitting errors, the stable domains on the stability chart can significantly change. The mechanical models of turning and milling operations including the stability analysis of self-excited vibrations were surveyed. During the sensitivity analysis, the simple models were assumed and its modal parameters were tuned. The results were analyzed and explained in details. Moreover two real case studies are investigated, where the analysis of the turning operation showed that the stability chart can be sensitive to the number of fitted modes, and the milling operation showed that it is also sensitive to the estimation of the fitted modal parameters of the cross frequency response functions. The results also show that the higher degree-of-freedom models can provide better approximation of the stability chart, even for the domains at higher spindle speeds, where the chart is more sensitive.

Based on the presented results, the stability chart is sensitive to the estimation of modes at higher frequencies, especially in cases when a non-dominant mode is neglected or two adjacent modes are approximated by a common one. In extreme cases, the chart changes qualitatively and important domains can get lost or false results can appear. The effect of the fitted cross FRFs for milling operations is also analyzed in a case study. The results show that the stable domains are sensitive to the approximation of the cross functions, whose precise measurement is itself a difficult task.

Összefoglalás

Az ipar által támasztott egyik legfontosabb követelmény a megmunkálási idő csökkentése és a minőség javítása, melyek bizonyos szempontból ellentmondó elvárások. A megmunkálási idő csökkenthető, ha a szerszámgép fordulatszámát, így az anyagleválasztás sebességét növeljük. A folyamat során azonban olyan káros rezgések keletkezhetnek, amelyek nemcsak rontják a minőséget, de tönkreteszhetik a munkadarabot vagy magát a szerszámgépet is. A keletkező rezgések magyarázatának legelfogadottabb elmélete a *regeneratív hatás*, amely összetett matematikai és mechanikai modellekkel írható le.

A dolgozat első részében bemutatásra kerültek a több szabadsági fokú dinamikai rendszerek vizsgálatának módjai, az arányos és nemarányos csillapítás matematikai és mechanikai tulajdonságai valamint a modális analízis alapjai. Utóbbi témájában részletesen elemeztük az arányos és a nemarányos rendszer frekvenciatartománybeli tulajdonságait, valamint a frekvenciaátviteli függvény modális paraméterekkel történő alakjának levezetését. A téma kidolgozása közben egy könnyen alkalmazható iteratív módszeren alapuló paraméterillesztés is alkalmazásra került, amellyel gyorsan és hatékonyan lehet alacsony fokszámú illesztést végezni.

A téma második felében a szerszámgéprezgések esetén gyakran alkalmazott stabilitási térképek érzékenységét vizsgáltuk. Egy valós mért frekvenciaátviteli függvényből nehéz a pontos modális paraméterek meghatározása, a zaj miatt az illesztett paraméterek hibával terheltek. Ennek hatására a stabilitási térképek megváltoznak. Áttekintettük az esztergáláshoz és maráshoz használt dinamikai modelleket és elemeztük az öngerjesztett rendszer stabilitását. Az érzékenységi vizsgálat során egyszerű modelleket feltételeztünk, melynek paramétereit változtattuk. Az eredményeket részletesen elemeztük és magyarázatokkal szolgáltunk. Ezen felül két esettanulmányt is bemutatunk, ahol esztergálásnál vizsgáltuk az illesztés pontosságának hatását a stabilitási térképekre, valamint marásnál láthattuk, hogy a keresztátviteli függvények jelentős hatással lehetnek a stabilitás hatáira. A vizsgált esettanulmány azonban azt is mutatja, hogy kellően magas fokszámú és pontos illesztés esetén megfelelően jól közelíthetők a stabil tartományok még magas fordulatszámoknál is, ahol a térkép kifejezetten érzékeny.

A vizsgálat eredményei alapján a stabilitási térkép érzékenyen reagál a magas frekvenciájú módusok modális paramétereinek hibás illesztésére. Külön kiemelhetők itt azokat az eseteket, amikor egy dominánsnak nem tűnő sajátfrekvenciát hanyagolunk el, vagy közeli módusokat illesztünk egyetlen közzel. Extrém esetekben minőségileg is megváltozhat a stabilitási térkép és akár jelentős területeket veszíthetünk vagy hamisakat nyerhetünk. Marás esetén külön vizsgált a keresztátviteli függvények hatása, amelynek a mérése nehézkes és pontatlan, ugyanakkor jelentősen befolyásolhatja a kapott eredményeket.

Appendix A

Mathematical background

A.1 Relation of proportional and non-proportional FRF

Although the two representations have differences, the proportionally damped parametric form of the frequency response function can be defined using the non-proportional one. Let us start with this form defined for one degree of freedom, i.e.

$$H(\omega) = \frac{\psi^2}{i\omega - \lambda} + \frac{\bar{\psi}^2}{i\omega - \bar{\lambda}}. \quad (\text{A.1})$$

Expressing the terms with the common denominator gives

$$H(\omega) = \frac{\psi^2(i\omega - \bar{\lambda}) + \bar{\psi}^2(i\omega - \lambda)}{(i\omega - \lambda)(i\omega - \bar{\lambda})} = \frac{i\omega(\psi^2 + \bar{\psi}^2) - \psi^2\bar{\lambda} - \bar{\psi}^2\lambda}{-\omega^2 - (\lambda + \bar{\lambda})i\omega + \lambda\bar{\lambda}}. \quad (\text{A.2})$$

The denominator can be rewritten if $\lambda = -\xi\omega_n + \sqrt{1 - \xi^2}\omega_n i$ definition is used, which simplifies the equation to

$$H(\omega) = \frac{i\omega(\psi^2 + \bar{\psi}^2) - \psi^2(-\xi\omega_n - \sqrt{1 - \xi^2}\omega_n i) - \bar{\psi}^2(-\xi\omega_n + \sqrt{1 - \xi^2}\omega_n i)}{-\omega^2 + 2\xi\omega_n\omega i + \omega_n^2}. \quad (\text{A.3})$$

The FRF defined for the proportional system reads

$$H(\omega) = \frac{\phi^2}{-\omega^2 + 2\xi\omega_n\omega i + \omega_n^2}, \quad (\text{A.4})$$

where the nominator does not depend on ω . The two form can lead to the same expression if $\psi^2 + \bar{\psi}^2 = 0$, which is possible if the real part of ψ^2 is zero, i.e. it can be written as $\psi^2 = Vi$. Finally, in case of the single-degree-of-freedom system, the modal parameters can be calculated according to

$$\begin{aligned} \omega_n &= \sqrt{\frac{k}{m}}, & \xi &= \frac{c}{2m\omega_n}, \\ \phi^2 &= \frac{1}{m}, & \psi^2 &= -\frac{i}{\sqrt{4km - c^2}}. \end{aligned} \quad (\text{A.5})$$

A.2 Solution of linear inhomogeneous ODEs

¹ Let us consider the linear inhomogeneous ordinary differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}(t), \quad (\text{A.6})$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{b}(t) \in \mathbb{R}^n$ is a continuous function. Using the method called variation of constants, the solution can be searched in the form

$$\mathbf{x}(t) = e^{\mathbf{A}t} \mathbf{g}(t), \quad (\text{A.7})$$

where $\mathbf{g}(t) \in \mathbb{R}^n$, is a differentiable function and $e^{\mathbf{A}t}$ is the matrix exponential which can be calculated in different ways, for instance, using a transformation with the eigenvectors of $\mathbf{A}$ or by the Taylor series

$$e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{1}{k!} \mathbf{A}^k t^k. \quad (\text{A.8})$$

Differentiating (A.7) yields

$$\dot{\mathbf{x}}(t) = \mathbf{A}e^{\mathbf{A}t} \mathbf{g}(t) + e^{\mathbf{A}t} \dot{\mathbf{g}}(t). \quad (\text{A.9})$$

Substitution into (A.6), and simplification gives

$$e^{\mathbf{A}t} \dot{\mathbf{g}}(t) = \mathbf{b}(t), \quad (\text{A.10})$$

which implies

$$\mathbf{g}(t) = \int_{t_0}^t e^{-\mathbf{A}s} \mathbf{b}(s) ds + \mathbf{K}, \quad (\text{A.11})$$

where $\mathbf{K} \in \mathbb{R}^n$ is a constant vector. The solution of (A.6) reads

$$\mathbf{x}(t) = e^{\mathbf{A}t} \int_{t_0}^t e^{-\mathbf{A}s} \mathbf{b}(s) ds + e^{\mathbf{A}t} \mathbf{K}, \quad (\text{A.12})$$

and $\mathbf{K}$ can be determined from the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$, which gives

$$\mathbf{K} = e^{\mathbf{A}t_0} \mathbf{x}_0. \quad (\text{A.13})$$

Thus the final solution can be given as

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)} \mathbf{x}_0 + \int_{t_0}^t e^{\mathbf{A}(t-s)} \mathbf{b}(s) ds. \quad (\text{A.14})$$

¹Taken from [4]

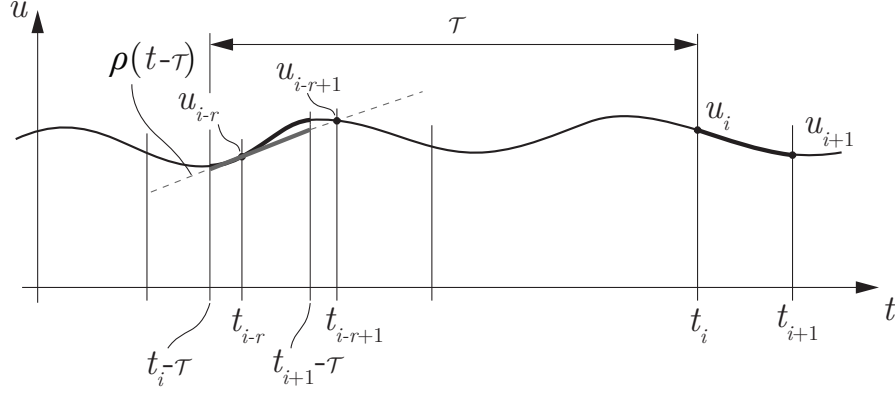


Figure A.1. First-order approximation of the delay term $u(t - \tau)$ by $\rho(t - \tau)$ shown by the dashed line [4].

A.3 First-order semi-discretization

During the numerical calculations, in order to reduce the time of computation and to improve the precision of the stability chart, instead of the zeroth-order semi-discretization, the so-called first-order method was used. The method in details is presented in [4].

For the first-order case, the approximation polynomial $\rho(t - \tau)$ for single point delay is given in the form

$$\rho(t - \tau) = \beta_0(t)u(t_{i-r}) + \beta_1(t)u(t_{i-r+1}) \quad (\text{A.15})$$

with

$$\beta_0 = \frac{\tau + (i - r + 1)h - t}{h}, \quad \beta_1 = \frac{t - (i - r)h - \tau}{h}, \quad (\text{A.16})$$

and $r = \tau/h + 1/2$. The scheme can be seen in Figure A.1. As t evolves from t_i to t_{i+1} , the delayed term will be between u_{i-r} and u_{i-r+1} , therefore these limit point will define the approximation polynomial. In this case, the approximated system reads

$$\dot{\mathbf{x}}(t) = \mathbf{A}_s \mathbf{x}(t) + \mathbf{B}_s (\beta_0(t)u(t_{i-r}) + \beta_1(t)u(t_{i-r+1})), \quad t \in [t_i, t_{i+1}] \quad (\text{A.17})$$

$$u(t_i) = \mathbf{C}_s \mathbf{x}(t_i). \quad (\text{A.18})$$

The solution over one discretization step (similarly to the zeroth-order approximation), can be formulated as

$$\mathbf{x}_{i+1} = \mathbf{P} \mathbf{x}_i + \mathbf{R}_0 u_{i-r} + \mathbf{R}_1 u_{i-r+1}, \quad (\text{A.19})$$

where

$$\mathbf{P} = e^{\mathbf{A}_s h}, \quad (\text{A.20})$$

$$\mathbf{R}_0 = \int_0^h \frac{\tau - (r - 1)h - s}{h} e^{\mathbf{A}_s(h-s)} ds \mathbf{B}_s, \quad (\text{A.21})$$

A Mathematical background

$$\mathbf{R}_1 = \int_0^h \frac{s - \tau + rh}{h} e^{\mathbf{A}_s(h-s)} ds \mathbf{B}_s. \quad (\text{A.22})$$

If $\mathbf{A}_s^{-1}$ exists, then the integration gives

$$\mathbf{R}_0 = \left(\mathbf{A}_s^{-1} + \frac{1}{h} \left(\mathbf{A}_s^{-2} - (\tau - (r-1)h) \mathbf{A}_s^{-1} \right) (\mathbf{I} - e^{\mathbf{A}_s h}) \right) \mathbf{B}_s \quad (\text{A.23})$$

$$\mathbf{R}_1 = \left(-\mathbf{A}_s^{-1} + \frac{1}{h} \left(-\mathbf{A}_s^{-2} + (\tau - rh) \mathbf{A}_s^{-1} \right) (\mathbf{I} - e^{\mathbf{A}_s h}) \right) \mathbf{B}_s. \quad (\text{A.24})$$

The coefficient matrix $\mathbf{G}$ finally reads

$$\mathbf{G} = \begin{pmatrix} \mathbf{P} & \begin{matrix} 1 & & & r-1 & r \end{matrix} \\ \hline \mathbf{C} & \begin{matrix} \mathbf{0} & \dots & \mathbf{0} & \mathbf{R}_1 & \mathbf{R}_0 \end{matrix} \\ \mathbf{0} & \begin{matrix} 0 & \dots & 0 & 0 & 0 \end{matrix} \\ \vdots & \begin{matrix} \vdots & \ddots & \vdots & \vdots & \vdots \end{matrix} \\ \mathbf{0} & \begin{matrix} 0 & \dots & 1 & 0 & 0 \end{matrix} \\ \mathbf{0} & \begin{matrix} 0 & \dots & 0 & 1 & 0 \end{matrix} \end{pmatrix}, \quad (\text{A.25})$$

where $\mathbf{R}_0$ and $\mathbf{R}_1$ are located at the $(r-1)^{\text{th}}$ and r^{th} elements (according to u_i) respectively.

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