



AVDH SIGN

Kristóf HAVASI
MSc THESIS

BUDAPEST UNIVERSITY OF TECHNOLOGY AND ECONOMICS
FACULTY OF MECHANICAL ENGINEERING
DEPARTMENT OF APPLIED MECHANICS



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KRISTÓF HAVASI

BSc Thesis

Writing Abaqus subroutines for advanced hyperelastic models

Supervisor:

Attila KOSSA, PhD

Associate professor

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Budapest University of Technology and Economics

Faculty of Mechanical Engineering

Department of Applied Mechanics

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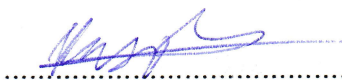
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	The undersigned declares that all prerequisites of the Final Project have been fully accomplished. Otherwise, the present assignment for the Final Project is to be considered invalid.  Havasi Kristóf			

DECLARATIONS

Declaration of individual work

I, *Kristóf Havasi* (EV5HXS), the undersigned, student of the Budapest University of Technology and Economics hereby declare that the present thesis has been prepared by myself without any unauthorized help or assistance such that only the specified sources (references, tools, etc.) were used. All parts taken from other sources word by word or after rephrasing but with identical meaning were unambiguously identified with explicit reference to the sources utilized.

Budapest, 09 Dec 2022

Kristóf Havasi

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Abstract

Nowadays more and more materials are used frequently in engineering practice. This process is becoming more and more usual as both researchers and people working in the industry are searching for optimal solutions, which can be achieved using a wide range of resources, and materials are not an exception to this. In order to support this trend constitutive models have to evolve over the years to support the crucial elements of design and simulation likewise. It is becoming common to use polymers and rubber-like materials in structures as they have advantageous properties compared to metals in different fields of application. On the other hand these materials behave much more different compared to metals, as usually, especially rubbers, are used for large elastic deformations. These have strongly nonlinear behaviours as well. In order to be able to work with these kind of materials the hyperelastic material modelling has been developed. This way we can model materials which are capable to describe large elastic deformations. Although the scientific background is at hand to use these kind of material models, which can be really useful for simulations as well as for structures that can be more easily designed using a wider scale of materials, due to the large amount of different forms of models only some of them are available in commercial use softwares. However we can implement these with some work into the desired environment using user subroutines, and thus be able to work with arbitrary material models. This thesis provides information on the area of hyperelasticity, the general usage of those kind of models and also some custom FORTRAN subroutines for materials. Several different advanced models were analysed from which a few was chosen for implementation for ABAQUS finite element software. Different kind of material modelling subroutines were used throughout the work of this implementation process, and they resulted in easy to use material models that can be introduced to the simulations. This way using the developed codes anyone can use the derived constitutive models for their work without having the problem of creating a unique code for them. The results were also checked using finite element simulations as well, and another, more complex cases were used to ensure the full functionality of the newly provided material subroutines.

Keywords: hyperelasticity, material modelling, finite element simulation, material subroutines, parameter fitting

Kivonat

Manapság egyre több és több anyag kerül gyakori felhasználásra a mérnöki gyakorlatban. Ez a folyamat egyre megszokottabb a kutatók és iparban dolgozó mérnökök körében egyaránt, egy optimális megoldás keresése nyomán. Ehhez széles körű erőforrásokra van szükség, ez alól pedig az anyagok sem kivételek. Annak érdekében, hogy ezen trendet a kutatók támogatni tudják anyagmodellek szükségesek, ezek pedig az évek során fejlődnek is, hiszen egy gyökeres pontját adják mind a tervezésnek, mind pedig a szimulációknak. A polimerek alkalmazása is egyre szélesebb körben terjed el, mint ahogy a gumyszerű anyagok is, különböző szerkezetekben más anyagokkal szemben sokkal előnyösebb tulajdonságokat képesek mutatni mint egyéb anyagok, például a gyakran használt fémek. Másfelől ezen anyagok sokkal másabb viselkedést mutatnak, mint a megszokott szerkezeti anyagok, amik legtöbbször kis alakváltozásra vannak tervezve, hiszen például a gumik gyakorta nagy nyúlásokat szolgálnak ki, ráadásul erősen nemlineáris rugalmas viselkedést is mutatnak. Annak érdekében, hogy ezen anyagokkal megbízhatóan dolgozhassunk a tudományban megjelent a hiperelasztikus anyagok modellezése. Ezzel képesek vagyunk olyan anyagokat leírni, melyek nagy mértékű rugalmas alakváltozást képesek elviselni. Mindazonáltal hogy ezen modellek rendelkezésre állnak és számos lehetőséget nyújtanak, elsődlegesen szimulációs téren, hiszen ez által a tervezés is megbízhatóbb, valamint szélesebb halmazból választhatjuk ki a számunkra megfelelőt, az elérhető programokban nem túl sok lehetőség nyílik ezen modellek használatára. Ennek érdekében, hogy több lehetőségünk legyen, saját modelleket is implementálhatunk felhasználói szubrutinok segítségével. Ezzel már egy általunk választott tetszőleges modellt is képesek vagyunk leírni és használni. Ezen dolgozat többek között a hiperelasztikus viselkedéssel, ezen modellek általános használatával és FORTRAN kódok általi implementálásával foglalkozik. Több különböző összetett modellt elemeztem és ezekből választott modellekre készítettem felhasználói függvényt ABAQUS végelelemes környezethez. Különböző függvény típusokat használtam ezen implementációk megvalósításához. Eredményül könnyen használható anyagmodellek születtek melyek egyszerűen és röviden azonnal használhatóak szimulációkhoz is. Ezáltal a kidolgozott kódokat bárki szabadon használhatja és alkalmazhatja a bennük feldolgozott anyagmodelleket problémamentesen, saját kód fejlesztésének terhe nélkül. Az eredményeket végelelemes szimulációkkal ellenőriztem, és egy további, komplex esetet is megvizsgáltam a széleskörű alkalmazhatóság ellenőrzésére az új kódokat tekintve.

Kulcsszavak: hiperelasztikus anyagok, anyagmodellezés, végelelemes szimuláció, anyagfüggvény, paraméter-illesztés

1 Introduction

In mechanical engineering nowadays more and more areas are getting larger and larger impact in many aspects, let it be design, simulation or production. In order to be able to support this trend many new and specific scientific studies are being published, creating a solid ground for further development and better understanding of processes, structures and materials likewise. Each and every one of these examples requires deep understanding of the world around us, so in order to have the knowledge required to grasp the very essence of nature itself. This is one of the main reasons that today the whole world is aiming to accomplish this goal, no matter the area of science one works in, physicists, chemists, and all kind of engineers or other type of technological background researchers are working towards this goal. Even back in ancient Greece people knew that the world is being governed by several different rules, which eventually got the name the laws of nature. This also tells us that no matter the age or location, humankind always wanted to discover, and eventually create and control life and the environment. In order to do so thorough studies were conducted, many experiments were done and by these several theories arose, many of them proved to be true, others failed, but none without provoking thoughts and bringing in other questions to the subject it described.

In modern days, engineering is getting ever so involved in life which consequently causes the rise of wider usage, considering all sectors it has to suffice let it be big industrial or smaller cases. In order to do so, many areas are evolving into much smaller, specialised groups that form a well developed knowledge on their own, on the contrary they are not general, and much more specific conditions must hold in order for them to prove useful for a given problem. This leads to several smaller study cases, which all are deep and accurate to describe their very own point of interest, with the before mentioned lack of generality. This way, although we have more accurate tools at hand than before, the amount of opportunities are growing such wide that one cannot take into account every aspect of an engineering problem, much more smaller steps must be defined in order to find the optimal solution for each and every problem. This way with the smaller topics specialists can be appointed to them who have vast knowledge of their area of studies compared to others, and thus can find a better solution with their lexical and practical experiences.

The topic of this thesis is also building upon a smaller branch of engineering and science. The main aspect is hyperelasticity, which is a model family used for material modelling. Nowadays it is a common practice to use simulations instead of prototypes to test different structures and machines, the most important advantage being that it is much more economic, the cost is lower by magnitudes. Simulations are crucial indeed, but their reliability is not perfect, as for them to produce actually usable data all aspects of the model built up for the case have to be in accordance to the real world case, which almost never can be satisfied. Every case involves somewhat of an approximation, let it be because of modelling, or numerical errors considering the machine used to calculate

the solutions.

The case we can have an impact on is the modelling step first and foremost. It is general these days that several different materials are used to design and build up machines to use their advantageous properties, for example stiffness or damping effects. It is also becoming more and more general to use different polymer and rubber-like materials by the fact they are much lighter than metal components and also that metal parts require lubrication, while these in most cases are not important for polymer components. The presence of these materials in larger structures also urges engineers to have a deeper understanding on the behaviour of these parts in the structure itself, which eventually can provide them information for simulations and the parameters to describe the physical behaviour of the created mathematical model for them. However these components are almost always used for larger deformation cases, so the small deformation description's bread and butter, Hooke's law is not applicable, nonlinearities have to be taken into account if one wishes to model them in their correct physical sense. This leads a long way to create several models, one of which is the before mentioned hyperelastic model family.

This model family is governed by the continuum concept, which imagines the material as a continuous body, neglecting the fact of atomic gaps and other small changes, but describing the material and its changes quite accurately considering a larger amount of it. Hyperelasticity is used to describe materials that are capable of large deformations while being perfectly elastic, meaning no plasticity or viscous effects are considered in them, on the other hand they can be modified easily to form one that also describe damages or other changes inside it, for example a visco-hyperelastic material model.

With advancing technology these kind of materials start to appear in wider range of applications, and in order to be designable mechanically we must introduce the necessary material models to our simulations. This helps engineers to model more accurately and create complex structures that can prove to be better than former, simpler ones. An example can be seen on Figure 1.1.



Figure 1.1: Examples for elastomer components used in automotive industry (Industries Accessed December 9th, 2022).

1.1 Aims of the thesis

The aim of present thesis is to provide some insight as well as practical cases to material modelling which is an extremely important branch of engineering for designers, modellers and manufacturers likewise. In order to do so I present the reader an introduction to continuum mechanics, as it is the basis upon which hyperelastic material modelling can be built.

I also aim to have numerical test data at hand, in order to do so I reviewed literature to find relevant articles publishing measurement data. Also I am creating and executing my own measurements, thus obtaining a new data set with which another step can be taken to more general material modelling involving a wider range of materials.

Another aim is to develop different subroutines for some chosen models. This way any user can implement these into their simulations easily with providing the necessary parameters for the model. Several subroutine types will be considered that can be useful for this task, and as there is an opportunity to create such models in a subroutine form for both compressible and incompressible cases, I will present different subroutines for both cases considering the models chosen.

One more aim is to compare measurements and numerical results. In order to do so the measurements done for this thesis are to be compared with the help of newly developed subroutines using a numerical finite element solver. This way we can have insight of the behaviour of the model for both analytical and numerical cases, and see if there is any deviation for one another during different calculation methods. The two considered here being the analytically fitted mathematical hyperelastic material model and the numerical modelled coded in the subroutine used by the finite element software.

For more advanced possibilities, I also aim to present a model which uses a different description method compared to the vast majority of mathematical models used in hyperelasticity. With this we can also introduce a self-defined model in the form of a subroutine. This provides us the opportunity to show a generally created model can also be easily implemented using the same steps for code creation for subroutines.

1.2 Structure of the thesis

In section one an introduction is given about the motivation and aims of this thesis, as well as a table containing the most important quantities used throughout the paper.

In the second section a literature review is provided. In that the summarised, short essence of the reviewed articles can be found and in chronological order, considering only the most relevant articles in the area of hyperelasticity and rubber-like materials.

In the third section the theory used in the thesis is introduced to the reader regarding the most important quantities used, the basis of hyperelasticity and different calculations in the are of it. Also several models are presented alongside the ones which are chosen for

further use and implementation into user subroutines. This section contains some information about experimental test cases as well considering the most commonly used ones for this topic. This way one can see the general steps and the theoretical background behind hyperelastic material modelling and the analytical calculations used for the development of them.

In the fourth section the models are used to perform parameter fitting. This way the error of the different models are presented as well as the performance of different test cases'. This section also includes the newly performed experimental results as well, and provides information about the steps used throughout the measurements and post processing steps used for extracting useful data for our cause. Another fitting procedure is shown here for the newly acquired data set as well.

The fifth section provides information on the case of subroutines and the necessary information required of the models in order to create the codes that can be implemented into a finite element software for further use. Here several codes are presented for different models.

The sixth one is about some finite element simulations. Here two different simulations are presented, one for the new subroutines' rectification, and one for further insight into the measurements of two dimensional, equibiaxial tests.

Finally, in the seventh and eighth sections, I summarise the work done for the creation of present thesis in English and Hungarian respectively. Here one can get a general idea of the main points of the paper, as well as have some insight and outlook for the work and possible continuation of it.

1.3 List of notations

The most important notations used throughout the thesis are listed below, without the aim of completeness.

Notation	Name, comment, value	Dimension
t	Time	[s]
$\mathbf{X}(t)$	Lagrangian position vector	[m]
$\mathbf{x}(t)$	Eulerian position vector	[m]
$\lambda_{\mathbf{N}}(\mathbf{X}, t, \mathbf{N})$	Directional stretch ratio in material representation	[1]
$\lambda_{\mathbf{n}}(\mathbf{x}, t, \mathbf{n})$	Directional stretch ratio in spatial representation	[1]
$\chi(\mathbf{X}, t)$	Motion	[m]
$\chi^{-1}(\mathbf{x}, t)$	Inverse motion	[m]
$\mathbf{U}(\mathbf{X}, t)$	Material displacement	[m]
$\mathbf{u}(\mathbf{x}, t)$	Spatial displacement	[m]
$\mathbf{F}(\mathbf{X}, t)$	Deformation gradient	[1]
$\mathbf{F}^{-1}(\mathbf{x}, t)$	Inverse deformation gradient	[1]
$\mathbf{K}(\mathbf{X}, t)$	Material displacement gradient	[1]
$\mathbf{k}(\mathbf{x}, t)$	Spatial displacement gradient	[1]
$\mathbf{C}(\mathbf{X}, t)$	Right Cauchy-Green deformation tensor	[1]
$\mathbf{B}(\mathbf{X}, t)$	Piola deformation tensor	[1]
$\mathbf{c}(\mathbf{x}, t)$	Cauchy deformation tensor	[1]
$\mathbf{b}(\mathbf{x}, t)$	Left Cauchy-Green deformation tensor	[1]
$\mathbf{E}(\mathbf{X}, t)$	Green-Lagrange strain tensor	[1]
$\mathbf{e}(\mathbf{x}, t)$	Euler-Almansi strain tensor	[1]
ε	1 dimensional engineering strain	[1]
$\mathbf{H}(\mathbf{X}, t)$	Material Hencky's strain tensor	[1]
$\mathbf{h}(\mathbf{x}, t)$	Spatial Hencky's strain tensor	[1]
J	Volume ratio	[1]
$\mathbf{U}(\mathbf{X}, t)$	Right stretch tensor	[1]
$\mathbf{V}(\mathbf{x}, t)$	Left stretch tensor	[1]
$\mathbf{R}(\mathbf{X}, t)$	Proper orthogonal tensor	[1]
$\mathbf{t}$	Traction vector	$\left[\frac{\text{N}}{\text{m}^2} \right]$
$\boldsymbol{\sigma}(\mathbf{x}, t)$	Cauchy stress tensor	$\left[\frac{\text{N}}{\text{m}^2} \right]$
$\mathbf{P}(\mathbf{X}, t)$	1st Piola-Kirchhoff stress tensor	$\left[\frac{\text{N}}{\text{m}^2} \right]$
$\mathbf{S}(\mathbf{X}, t)$	2nd Piola-Kirchhoff stress tensor	$\left[\frac{\text{N}}{\text{m}^2} \right]$

Notation	Name, comment, value	Dimension
$\boldsymbol{\tau}(\mathbf{x}, t)$	Kirchhoff stress tensor	$\left[\frac{\text{N}}{\text{m}^2} \right]$
$\mathbf{s}$	Deviatoric stress tensor	$\left[\frac{\text{N}}{\text{m}^2} \right]$
$\mathbf{p}$	Hydrostatic stress tensor	$\left[\frac{\text{N}}{\text{m}^2} \right]$
W	Strain energy density function	$\left[\frac{\text{J}}{\text{m}^3} \right]$
E	Young's modulus	$\left[\frac{\text{N}}{\text{m}^2} \right]$
ν	Poisson's ratio	$[1]$
K	Bulk modulus	$\left[\frac{\text{N}}{\text{m}^2} \right]$
G	Shear modulus	$\left[\frac{\text{N}}{\text{m}^2} \right]$
D	Incompressibility term	$\left[\frac{\text{m}^2}{\text{N}} \right]$
Q	Error function	$[1]$

2 Literature overview

First let us briefly summarize some relevant papers dealing with hyperelasticity, its models and measurement cases without the aim of completeness, dealing with only the ones with high importance and impact on the topic.

The first paper mentioned was published by Treloar in 1943 (Treloar 1944). This one considers that energy is defined as sum of squares of principal stretches with some coefficients and a constant. Several, often used numerical results are involved, regarding extension, shear and combined experiments. Overall this paper created a useful, and applicable model for rubber-like materials discussed in it, mostly considering vulcanized rubbers.

A paper by Flory (Flory 1961) aims to target material analysis on terms of thermodynamic equilibrium for elastic substances, which store energy in both temperature, entropy and elastic internal energy. It also introduces stress-strain relations and thermodynamic state equations regarding the deformation gradient and other thermodynamic state variables.

Beatty's paper (Beatty 1967) considers the stability of finite deformed hyperelastic bodies' using the dead load stability criteria and Korn's equation. Then also introduces two examples, regarding stability of incompressible hyperelastic material, first on Neo-Hookean material model and then on Mooney-Rivlin as well. The calculations include the stress and strain relations as well, using the 4 dimensional elasticity tensor as well.

Next, the article by Valanis (Valanis & Landel 1967) provides information generally about incompressible hyperelastic strain-energy functions, using principal stretches and invariant values of the describing state. Then also provides notes regarding creation of numerical experiments to get the desired coefficients for the model as well. The article concluded that also kinetic theory can be fulfilled, but only for a limited range of deformation, because otherwise the tension and compression cases deviate from each other significantly, that is no longer suitable for application. Some test plots were also provided for experiments.

In 1968 Hill published two articles, (Hill 1968) here several inequalities were introduced that serve as postulates for the material definitions. The material here is defined as Green-elastic material, which is similar to hypoelastic materials this is because a potential can be introduced excluding incompressible cases, and also objective rates are needed to be introduced here, the article applied the Jaumann rate for further analyses of the problem.

The article from Penn (Penn 1970) analyses volumetrically compressible materials, as they are of real-life cases of hyperelastic materials, and also describes a type of measurement for it, with a lot of error sources and how to minimize them. The conclusion of the paper was that the strain-energy function cannot be created by a simple summation of the volumetric and the shear component of the deformation. Also the volumetric strain does not correspond with any stress values, rather with pressure values in the material, but

also for these extrapolation was used as not enough experimental data could be created.

In the article of Tschoegl (Tschoegl 1971) detailed material model creation is described using Mooney-Rivlin model space, to produce one that can work for filled and unfilled elastomers as well. The classic, simple test cases were used to provide both calculations and experimental results about the model, as well as rectification for comparing those. It is noticeable, that high, sometimes even 6th or higher order is required in order to get proper behaviour from the model for this complex problem.

James published two separate articles in 1975 (James et al. 1975) discusses the case of hyperelastic material models. A complex, carefully developed experiment was conducted for a large data pool in order to create enough data points for model regression. Several types of different strain-energy function types were used in order to revisit the topic of correct function choice regarding these kinds of materials. Several application oriented test cases were considered, inflated membranes and tubes, pure material coefficient regression and other design oriented ones. In conclusion, it is clear that higher order terms are indeed necessary to create a good-enough quality model that can be used in both theory and application. While the first article dealt with arbitrary materials, the second one (James & Green 1975) considers the filled ones as its main aim to find answers to. One of the main problems that are visited here is the stress softening, which plays a huge role in these kind of materials. Further application oriented topics are mentioned, like Young's modulus calculation and applied problem solution as well. The paper concludes that these kind of materials show little to no strain in terms of introducing volumetric compressibility, the incompressible case is almost perfectly suitable for application even in finer cases.

Ogden in 1976 published an article (Ogden 1976) about the effects and relevance of using compressibility is discussed here with regards to perturbation theorem, and then applied to a specific kind of strain-energy function. The general form is assumed to be additive once again.

Ball (Ball 1976) dealt with general nonlinear hyperelasticity. Another main topic of this paper is mathematical considerations using quasi-convexity and other higher mathematics in order to create complex constitutive inequalities. Also non-uniqueness can be introduced this way as well, in example of the buckling phenomenon. The conclusions show that the theory does not behave right in all cases, a higher level one may be needed to unite this with the basic one in order to create a sufficient one.

Kawabata (Kawabata & Kawai 1977) worked with general strain-energy density functions and their stress components. Also kinetic effects are taken into account through thermodynamic means. The paper also deals with biaxial extension experiments as well. The conclusions show that the kinetic model does not work yet on rubber materials, a refined one is needed in order to provide sufficiently accurate results.

Kawabata published a paper (Kawabata et al. 1981) where a strain-energy model was created with biaxial extension experiments that form a nice, analytical solution and pro-

vide a wide range of measurement data.

In the article of Ogden (Ogden 1986) mostly his own model is used to view and create models for experiments, which may show minor deviations, but overall prove a nice and close-to-reality model based on used regressed experimental results. For largely compressible materials it is still to be refined, those are critical, this model is not sufficient for them.

Another interesting article was Yeoh's (Yeoh 1990) in which filled rubber material is used. A difference between tension and compression case can be seen on one of the plots. The progressive characteristic is also an important aspect. Overall, a varying shear modulus is shown to be closer to real-life, compared to a constant valued one, and some sign conventions are created for the parameters of models.

Christoffersen (Christoffersen 1991) also worked on the topic using the Jaumann rate, an elastoplastic material model is created which also involves kinematics, this way providing a more complex model for the material. In conclusion, the volumetric strain's effect is negligible for rubbers, while still getting a good enough, but complicated expression as the model's describing equation. Based on the results of this article we consider mostly incompressible models in hyperelasticity as well

In Simo's paper (Simo & Taylor 1991), based on general continuum mechanics principles, a general compressible material model is analysed using the fourth order elasticity tensor. The model is also tested on numerical data with several additional algorithms presented for use. Also a balloon inflation example is provided here.

Van den Bogert (Van den Bogert et al. 1991) provided a analysis of FEM in the article for hyperelasticity, this way a new and improved method can be applied in simulations. The overall goal of this is to create a new method that excludes unwanted behaviour, such as early termination, due to unstable calculations, with the correct number of DoF and internal pressure degrees given for the material piece in the simulation.

Arruda and Boyce published an article (Arruda & Boyce 1993) involving several models' evaluations with the Treloar data for 3d hyperelastic models. The best performing one was the eight chain model.

In Benjeddou's paper (Benjeddou et al. 1993) based on biaxial problems, model fitting was done for FEA models, using the Levenberg-Marquardt-Fletcher algorithm, which proved sufficient for calculating Ogden's model parameters, and also a significant stretch range could be described successfully.

Ogden-like models were used by Charlton (Charlton et al. 1994) to characterize FEA rubber material models, it is a great conclusion that it can be sufficiently done with synthetic data, which if created with care, can be just as good as real measurement. It was also supported by objective rate calculations in the document.

Yeoh (Yeoh 1993) used several hyperelastic models, like the Neo-Hookean and the Mooney-Rivlin models, but the newly proposed model proved to be superior, outper-

forming both and eventually being useful over a wider range of strain values.

In the paper of Liu (Liu et al. 1994) it was shown by using the multiplicative decomposition of the deformation gradient, we can decompose the calculations, and also two variational formulae were presented for stability and criteria calculations. Efficient FEA methods were also presented.

Van den Bogert Van den Bogert & De Borst (1994) evaluated, in both compression and tension cases, several models, the Mooney-Rivlin and the Besseling models proved to be optimal in terms of strain-energy functions. On the other hand, the Ogden model had progression the former models couldn't reproduce, showing another kind of superiority.

Biwa's (Biwa 1995) critical stretch ratios were used to find the point at which cavities can form inside the hyperelastic material itself. A material model similar to Blatz-Ko model was used as well. All models have other results, but they all can be summarized one-by-one into a set of cavitation surfaces.

In Diehl's (Diehl 1995) PhD document several multidimensional material models were used, regarding nonlinear hyperelastic models. Throughout Ogden-Hill type material models were used, and rolling nip mechanism was also accounted for seriously in it.

Holzapfel (Holzapfel 1996), Based on Ogden-type material models, viscoelastic model was created, with also a time integration algorithm at hand. Some application examples were also provided. Overall new rheological model was created accounting for the elastic model as well.

By Lion (Lion 1996), 3 dimensional strong rate dependence was discovered in filled rubber materials using experiments and hyperelastic material models, finally arriving with mathematical formulation for it.

In Wang's (Wang 1996) PhD thesis a Blatz-Ko material model was analysed, it also accounted for several other points, like non-uniqueness, non-existence and other mathematically important aspects of models as well.

Kaliske's paper (Kaliske & Rothert 1997) provides an overview on several material models considering hyperelastic behaviour, and providing their most important characteristics.

In 1997 Yeoh (Yeoh 1997), using Treloar's data, this paper looks into the depth of the Ogden material model, and concludes its main aspects regarding fitting as well, because it is a strongly nonlinear property of that.

The next year a simple derivative representation was gathered based on Valanis-Landel method by Landel (Landel 1998), which proved rather effective looking at experimental data as well, also creating a simplistic mathematical formula, that is able to be used freely and easily.

Doll's paper (Doll & Schweizerhof 2000a) provides a proposition for a strain energy function with two parameters and a volumetric, compressibility part. Also several former

propositions and models are mentioned as other options as well, that have been in use.

Torsional analyses of hyperelastic materials were done by Horgan (Horgan & Saccomandi 1999), in conclusion the Mooney-Rivlin model was concluded as optimal, and was advised for use for these kind of problems.

By Lambert (Lambert-Diani & Rey 1999), new phenomenological methods for hyperelastic materials are provided and compared with former models, overall it is of great use for new and fresh experimental data, as it is highly compatible.

Boyce (Boyce & Arruda 2000) described a hybrid model of low and high stretch models used, providing a brand new chain model in hyperelasticity. The same year several other publications came to existence. By Chiskis (Chiskis & Parnes 2001), his paper visits the possibilities of linear stress-strain formulae for nonlinear hyperelastic materials. Overall, with certain criteria, it is possible to have linear relationship, Dettmar's paper (Dettmar 2000) provides a Mooney-Rivlin material model in ABAQUS, consisting of both its theory, and the FORTRAN code building it up inside the program as well, Doll's paper (Doll & Schweizerhof 2000b) provides a discussion on the hyperelastic materials' strain energy functions, their development, and potential further upgrades. Also this paper points out the problemativeness of the successful foundation of material parameters and Seibert's article (Seibert & Schoche 2000) provides a wide range of new and classical models compared to each other using Treloar's data with a given method of parameter optimisation, that creates a generic and "standard" method for this comparison.

Starting from the 21st century the research of this topic accelerated even further. In 2001 Amin's paper (Amin 2001) provides information about microstructural and hysteresis effects as well on high-damping and natural rubber materials. In addition several measurements were conducted in the article. Overall viscoelasticity is also considered in the material models, which may prove useful in constructional cases. The next year he published another one (Amin et al. 2002), where the viscoelastic response of hyperelastic rubber materials are used here, which cause the damping in them as well. Several effects are modelled separately, with different measurements to prove and calculate the necessary coefficients of the material model, also including highly nonlinear characteristics in damping as well.

Zuniga (Zúñiga & Beatty 2002) created a new model for stress softening characteristics, also creating constitutive models for it and a new model based on the Mullins-Tobin damage model. Also experimental data is used to validate, and comparison is done using the Neo-Hookean network model. The new model's accuracy is rather good based on the measurements.

In 2003 a vast amount of papers arose from different authors. Akron's paper (AKRON RUBBER DEVELOPMENT LABORATORY 2003) uses the Prony constant to describe the softening characteristics as well, as also mentioning the statistical hyperelastic models in difference to the continuum mechanics based ones. Amin's (Amin et al. 2003) says some huge volumetric deformations can be done on the rubber material as well, because

of the high damping and natural rubbers used and the lateral deformations applied there. Measurements were done to rectify the theory. Hartmann had two separate articles, (Hartmann & Neff 2003) says polynomial models were analysed, which in conclusion generally are non-polyconvex. Also a new strain energy function was proposed and tested analytically, all for nearly incompressible materials. The second (Hartmann et al. 2003) considers carbon-filled hyperelastic material was used for modelling using optical measurements for parameter identification and then FEA for parameter fitting. The optimization proved to be more cost efficient than beforehand methods. By Besdo (Besdo & Ihlemann 2003) a new constitutive model was proposed representing several material effects, while also numerical examples were solved with this new model. FEA showed that the model is capable for predictions and calculations for the future regarding rubber-like materials. By Marlow (Marlow 2003) new general model for principal invariants were created, this way the model is easily interstrainable. The model is also capable of including other parameters as well for example temperature, field parameters or any other, that may be of importance to our modelling. And Rivlin's article (Rivlin 2003) provides an overlook to the Valanis-Landel model, and concludes its place in the general theory, creating the necessary formulae to place it in the correct equations and show the proof that they are of the same root as any other models in the given area of models.

Garcia's paper (Atlı & Evcin 2020) analyses the hyperelastic model, especially the Mooney-Rivlin one from terms of thermoforming and manufacturing. Also experimental data usage is of importance in the paper.

The same year Ogden had an article (Ogden et al. 2004) which discusses the Ogden material model, while accounting for the parameter optimisation, the strain energy sensitivity, and the several test case data that are required for the parameter identification for the model.

Amin had another paper in 2006 (Amin et al. 2006) where the high-damping material model is used in FEA and for parameter identification in order to create a suitable model for further applications. Still in that year Marckmann's paper (Marckmann & Verron 2006) proposes a way of determining the parameters of a model for on basis of constitutive and phenomenological models also using a set of predefined optimization rules. Also Rivlin's paper (Rivlin 2006) analyses the similarities and differences of the Valais-Landel strain energy function and the classical strain energy function.

In Nowak's article (Nowak 2008) the hyperelastic material model was infused with a material damage property, creating a new model, which also includes the effects of internal degradation. FEA is also used for validation, but as discussed, it may also involve problematic cases and errors as well.

As there were already plenty of articles, Verron (Verron 2008) provided a historical overview of effects and model properties as they appeared in scientific models, showing the development of the models and their use in the world thorough the step-by-step forwarding of the models.

In Zheng's thesis (Zheng 2008) the stability and predictability of the hyperelastic model is discussed, based on a specific FEM software.

Horgan's paper from 2009 (Horgan & Murphy 2009) focuses on the hyperelastic models in FEA, and the moduli that they define physically for slightly compressible rubber materials. In Miroshnychenko's paper (Miroshnychenko & Green 2009) a heuristic search is done based on Treloar's data which results in an eight chain model sufficient as a hyperelastic material model. By Sussman (Sussman & Bathe 2009) a new model using splines was created. The new data seems to be a generalized Ogden-type model. Overall, it only has one great restriction, that the strain energy must be decomposable into two parts.

The first article by Darijani in 2010 (Darijani et al. 2010*b*) was about a Saint-Venant-Kirchhoff type material model was created using Treloar's data, with a nice fitting result in the end. In the second article (Darijani et al. 2010*a*) from the phenomenological point of view a lot of models were compared, grouped into function types they included. And in his third paper (Darijani & Naghdabadi 2010) different sums of independent strain energy function were used as new ones, with great success, the new groups created desirable, new material models overall.

Simple shear was analysed in Horgan's paper (Horgan & Murphy 2010) on incompressible and slightly compressible hyperelastic materials. The results were used on the Neo-Hookean model for parameter identification.

This year Silber (Silber et al. 2010) also worked with highly compressible hyperelastic material model was used, the hyperfoam model, for which mathematical considerations, FEA and experimental results were used in the end for checking the model performance.

By Steinmann (Steinmann et al. 2012*a*) 14 different material models were used in the paper to check for Treloar's data, as well as a continuum mechanics preliminary is given in the paper. Also the simple shear was analysed in Destrade's paper (Destrade et al. 2012) with also considering invertability of the stress-strain relation, while by Hossain (Hossain & Steinmann 2013) the three and eight chain model is analysed in his discussion, also the tangent operators get a point in this article. Pancheri (Pancheri & Dorfmann 2012) done the characterization of natural rubber, based on experimental results as well, with lots of constitutive equations. Rashid (Rashid et al. 2012) worked on brain tissue material model characterization, Ogden's model described it the best. FEA was also used for comparison of results.

By Eshraghi (Eshraghi et al. 2013) rate-based hyperelasticity is created based on Jaumann rate in Eulerian frame. Also this year by Khajehsaeid (Khajehsaeid et al. 2013) a new strain energy function is proposed, which can be used for any advantageous parts of former ones, also built an eight chain model of it and careful experiments were conducted with nice numerical results by Moreira (Moreira & Nunes 2013).

In 2014 by Soe (Soe et al. 2014) Durafoam Flex material model building and using in FEA for creating and designing a functioning system.

By Shahzad (Shahzad et al. 2015a) a specimen for which hyperelastic model was searched for, the best result was given by Yeoh model. In addition the equibiaxial measurements correlated best, which generally prove rather problematic.

Great discussion of moduli, even incremental for nonlinear material models, and the stress types needed for specific material models based on the coordinate frame was discussed by Mihai (Mihai & Goriely 2017).

Many of large volumetric change hyperelastic models, all capable of modelling foam-like materials were done by Moerman (Moerman et al. 2020).

The last few years are not covered in depth, the reason for this is the limit of scope for present thesis, as it is not the main aspect of present paper. For this only the most important articles were covered, and one newer one is to be highlighted, that also served as a base of presented work.

This is an article written by He (He et al. 2022), which, as mention before, also proved as a base ground of present thesis, in it 85 hyperelastic models were covered, all of which are described and grouped, the best of them analysed based on their performance of fitting to Treloar's data.

3 Hyperelasticity

The hyperelastic material model assumes that the material undergoes of large elastic deformation without the presence of any plasticity or other kind of non-elastic behaviour. This model can describe rubber-like materials quite well, the elasticity of those are really high, and plasticity or viscous effects are rather low in them, the elastic behaviour highly dominates the deformation. An interesting addition is that rubbers usually show small to none volumetric strains, meaning that often for the sake of simplicity it is a good assumption to say that these kind of materials are incompressible.

In order to be able to describe the governing equations of hyperelasticity and create the necessary models for constitutive modelling let us first summarize the most important ingredients of continuum mechanics.

3.1 Summary on Continuum mechanics

Continuum mechanics is a large topic in engineering and material modelling (Holzapfel 2002). The most important aspect of it is that it considers the material a continuous substance which does not involve any jumps, holes or cavities, it also disregards the information that the material is built up from atoms and molecules, meaning that it is not completely homogeneous, but small concentrated masses can be found by the fact of these particles. On the other hand, if we are speaking of large amounts of a given material, where the number of these building blocks is really high we can state that the

behaviour of this large group is not dominated anymore by the singular particles, but rather the structure made up of them, and can use the assumption that this material is indeed almost equivalent to a complete mass of the given substance.

It is also a really important advantage of it that this way we can create nonlinear relations more easily and thus describe large deformations as well contrary to the general applicable region of Hooke's law, which works only for small strain cases.

In continuum mechanics we usually use two different frames to describe the state of the body we consider. The first one is the Lagrangian frame, where we take into account the original configuration, and create a correlation between the actual time instance's points back into the original state, the other being the Eulerian description, where the current configuration is used directly to derive further information of our volume of interest. In this thesis let us use a common rule so that the two frames' coordinate systems are identical Cartesian coordinate systems. Also, for quantities defined in the Lagrangian frame the notation is with uppercase letter, while the notations used for the Eulerian frame are lowercase letters. For visualisation of the two frames Figure 3.1 provides some additional information.

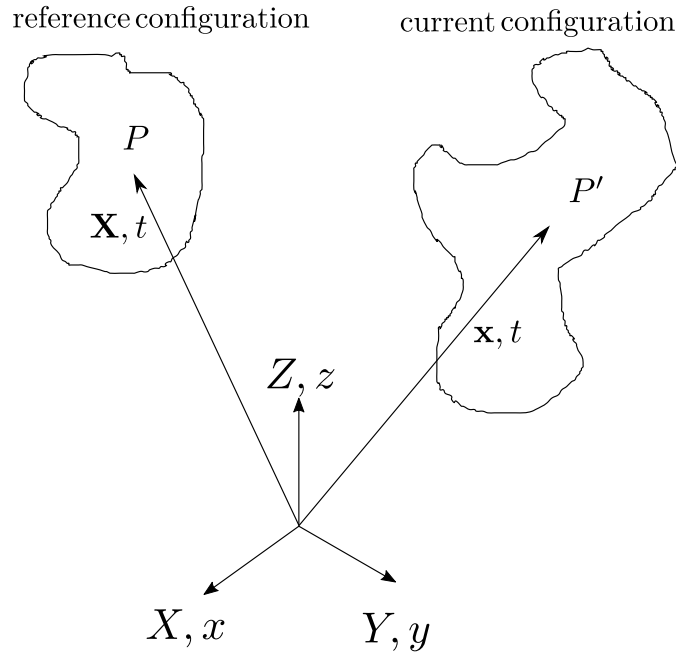


Figure 3.1: Coordinate systems introduced to the frames of continuum mechanics.

Let us start by defining the motion of our volume of interest. This quantity is a projection that tells us the transformation between the current spatial and the initial reference configurations

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t). \quad (3.1)$$

It's inverse is also used for some other considerations, that tells us the inverse transform of this spatial change

$$\mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x}, t). \quad (3.2)$$

Next we can define the displacement. This quantity tells us the spatial distance of a point in the two frames. It can be defined for both the spatial and the reference frame

$$U(\mathbf{X}, t) = \boldsymbol{\chi}(\mathbf{X}, t) - \mathbf{X}, \quad \mathbf{u}(\mathbf{x}, t) = \mathbf{x} - \boldsymbol{\chi}^{-1}(\mathbf{x}, t). \quad (3.3)$$

The next really important quantity is the deformation gradient. This tensor provides information about the deformation at point $(\mathbf{X}, t)$. It can be noted that this quantity is a result of first order Taylor expansion, meaning it is not exact in larger scales, only small linear neighbourhoods can be analysed this way.

Here once again both the deformation gradient and its inverse are frequently used quantities, so let us define both of them

$$\mathbf{F}(\mathbf{X}, t) = \frac{\partial \boldsymbol{\chi}(\mathbf{X}, t)}{\partial \mathbf{X}}, \quad \mathbf{F}^{-1}(\mathbf{x}, t) = \frac{\partial \boldsymbol{\chi}^{-1}(\mathbf{x}, t)}{\partial \mathbf{x}}. \quad (3.4)$$

The inverse connection between the two tensors is not always easy to describe as they may involve the coordinate values as well, which have to be transformed via the motion in order for the desired coordinates to appear in the tensor's formula. The condition for the case when no such transformation is required is the homogeneous deformation, what can be inspected such that if the deformation gradient contains no coordinate values, meaning it does not depend on the point's coordinates, the deformation is homogeneous. This corresponds with the fact that for homogeneous deformations straight line segments remain straight lines throughout the deformation.

Next, with the help of the deformation gradient we can also define the displacement gradients as well.

$$\mathbf{K}(\mathbf{X}, t) = \frac{\partial U(\mathbf{X}, t)}{\partial \mathbf{X}} = \mathbf{F}(\mathbf{X}, t) - \mathbf{I}, \quad (3.5)$$

$$\mathbf{k}(\mathbf{x}, t) = \frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial \mathbf{x}} = \mathbf{i} - \mathbf{F}^{-1}(\mathbf{x}, t), \quad (3.6)$$

where $\mathbf{I}$ and $\mathbf{i}$ are the second-order identity tensors in their corresponding coordinate frames.

With the help of this we can also define the stretch ratio of a material line segment.

For both the spatial and reference frame, we need to define a direction along which we are interested in the stretch. Let these be $\mathbf{N}$ and $\mathbf{n}$. The stretch ratios are defined as

$$\lambda_{\mathbf{N}}(\mathbf{X}, t) = \sqrt{\mathbf{N} \mathbf{F}^T(\mathbf{X}, t) \mathbf{F}(\mathbf{X}, t) \mathbf{N}}, \quad \lambda_{\mathbf{n}}(\mathbf{x}, t) = \frac{1}{\sqrt{\mathbf{n} \mathbf{F}^{-T}(\mathbf{x}, t) \mathbf{F}^{-1}(\mathbf{x}, t) \mathbf{n}}}. \quad (3.7)$$

The following quantities are the deformation tensors. Four of these are that frequently appear in many papers and textbooks, these are the right Cauchy-Green (**C**), the Piola (**B**), the Cauchy (**c**) and the left Cauchy-Green (**b**) deformation tensors. They can be calculated as follows

$$\mathbf{C}(\mathbf{X}, t) = \mathbf{F}^T(\mathbf{X}, t)\mathbf{F}(\mathbf{X}, t), \quad (3.8)$$

$$\mathbf{B}(\mathbf{X}, t) = \mathbf{F}^{-1}(\mathbf{x}, t)\mathbf{F}^{-T}(\mathbf{x}, t), \quad (3.9)$$

$$\mathbf{c}(\mathbf{x}, t) = \mathbf{F}^{-T}(\mathbf{x}, t)\mathbf{F}^{-1}(\mathbf{x}, t), \quad (3.10)$$

$$\mathbf{b}(\mathbf{x}, t) = \mathbf{F}(\mathbf{X}, t)\mathbf{F}^T(\mathbf{X}, t). \quad (3.11)$$

The deformations are defined this way, but we also require some strain measurements in our applications, by this we can also create strain measuring tensor quantities.

Some common strain tensors are the Green-Lagrange **E**, the Euler-Almansi **e**, the Hencky's **H** and the spatial Hencky's strain tensors, which are defined as follows

$$\mathbf{E}(\mathbf{X}, t) = \frac{1}{2}(\mathbf{C}(\mathbf{X}, t) - \mathbf{I}), \quad (3.12)$$

$$\mathbf{e}(\mathbf{x}, t) = \frac{1}{2}(\mathbf{i} - \mathbf{c}(\mathbf{x}, t)), \quad (3.13)$$

$$\mathbf{H}(\mathbf{X}, t) = \ln(\mathbf{U}(\mathbf{X}, t)), \quad (3.14)$$

$$\mathbf{h}(\mathbf{x}, t) = \ln(\mathbf{V}(\mathbf{x}, t)). \quad (3.15)$$

The strains in one dimensional case can similarly be given by multiple measuring quantities, like so with the logarithmic and the engineering strains

$$\varepsilon = \ln(\lambda), \quad \varepsilon^{\text{eng}} = \lambda - 1. \quad (3.16)$$

These quantities are really similar near the zero strain value, that is why the small strain deformation is only defined for once strain case, while for large deformations multiple strain measures can be introduced, which all can describe the same deformation equivalently well.

From the deformation gradient we can also define an important quantity, namely the volume ratio

$$\frac{dv}{dV} = \det \mathbf{F}(\mathbf{X}, t) = J. \quad (3.17)$$

This quantity tells us the ratio of the current and initial volume.

We can also create several decompositions of the deformation gradient. In one approach, the polar decomposition we have a rotation and a stretch tensor, one only describing the rotation, the other the stretching deformation of the whole process

$$\mathbf{F}(\mathbf{X}, t) = \mathbf{U}(\mathbf{X}, t)\mathbf{R}(\mathbf{X}, t) = \mathbf{R}(\mathbf{X}, t)\mathbf{V}(\mathbf{x}, t). \quad (3.18)$$

Another decomposition is the isochoric and volumetric decomposition, this way we can decompose the deformation gradient into two tensors that describe the isochoric part of the deformation, and then apply the other part on that to have the volumetric deformation as well

$$\mathbf{F}(\mathbf{X}, t) = \mathbf{F}_{\text{iso}}(\mathbf{X}, t)\mathbf{F}_{\text{vol}}(\mathbf{X}, t). \quad (3.19)$$

The components can be given as

$$\mathbf{F}_{\text{iso}}(\mathbf{X}, t) = J^{-\frac{1}{3}}\mathbf{F}(\mathbf{X}, t), \quad (3.20)$$

$$\mathbf{F}_{\text{vol}}(\mathbf{X}, t) = J^{\frac{1}{3}}\mathbf{I}. \quad (3.21)$$

This decomposition is really useful for hyperelastic modelling as it can decompose the deformation into an incompressible and a pure compression part.

After giving description about the deformations and strains we must also study the stresses that gives us the opportunity to build constitutive models for the materials.

The first quantity we can define is the traction $\mathbf{t}$, this creates a correspondence between the force acting on a given surface of a cross section.

The Cauchy stress tensor can be defined with this if we take the surface normal vector to the area, $\mathbf{n}$, the Cauchy stress can be calculated from the following equation

$$\mathbf{t} = \boldsymbol{\sigma}(\mathbf{x}, t)\mathbf{n}. \quad (3.22)$$

This stress is an Eulerian quantity.

Other stress measures are also in use, like for strain and deformation measuring tensors, we can define the first Piola-Kirchhoff $\mathbf{P}$, the second Piola-Kirchhoff $\mathbf{S}$ and the Kirchhoff $\boldsymbol{\tau}$ stress tensor. They can be obtained as follows

$$\mathbf{P}(\mathbf{X}, t) = J\boldsymbol{\sigma}(\mathbf{x}, t)\mathbf{F}^{-T}(\mathbf{x}, t), \quad (3.23)$$

$$\mathbf{S}(\mathbf{X}, t) = J\mathbf{F}^{-1}(\mathbf{x}, t)\boldsymbol{\sigma}(\mathbf{x}, t)\mathbf{F}^{-T}(\mathbf{x}, t), \quad (3.24)$$

$$\boldsymbol{\tau}(\mathbf{x}, t) = J\boldsymbol{\sigma}(\mathbf{x}, t). \quad (3.25)$$

Another important decomposition is the deviatoric and hydrostatic component decomposition of the stress tensor. This way we can obtain the hydrostatic stress $\mathbf{p}$ and the deviatoric stress tensor $\mathbf{s}$

$$\boldsymbol{\sigma}(\mathbf{X}, t) = \mathbf{s} + \mathbf{p}. \quad (3.26)$$

The components can be given as

$$\mathbf{p} = \frac{1}{3}\boldsymbol{\sigma}(\mathbf{x}, t), \quad (3.27)$$

$$\mathbf{s} = \text{dev}[\boldsymbol{\sigma}(\mathbf{x}, t)] = \boldsymbol{\sigma}(\mathbf{x}, t) - \mathbf{p}. \quad (3.28)$$

An important measure regarding stress is the equivalent stress, a commonly used measure for that is the von Mises stress, that can give us information about the scalar equivalent stress of the three dimensional stress case

$$\sigma_{\text{equi}} = \sqrt{\frac{3}{2}\mathbf{s} : \mathbf{s}} \quad (3.29)$$

3.2 Different forms of Hooke's law

Let us also take a look at Hooke's law and its different forms, as they give a proper ground to show the evolution of different constitutive models and the common point upon which the final assumptions can be made.

First, we can define Hooke's law in the usual form using the Young's modulus and the Poisson's ratio

$$\boldsymbol{\sigma} = \frac{E}{1 + \nu} \left(\boldsymbol{\varepsilon} + \frac{\nu}{1 - 2\nu} \text{tr} \boldsymbol{\varepsilon} \mathbf{I} \right). \quad (3.30)$$

This equation is a simple representation, but others are also possible. For example, we can take the Lamé parameters and define the law with them as well to create an equivalent definition using another representation

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda \text{tr} \boldsymbol{\varepsilon} \mathbf{I}. \quad (3.31)$$

Also another method is to use the shear and bulk moduli to create the law's mathematical form

$$\boldsymbol{\sigma} = 2G\mathbf{e} + 3K\boldsymbol{\varepsilon}_V, \quad (3.32)$$

where $\mathbf{e}$ is the deviatoric and $\boldsymbol{\varepsilon}_V$ is the spherical component of the strain tensor.

Another method to create the equation is to use a fourth order projection tensor. This way we can introduce a fourth-order tensor connecting term between the stress and strain tensors

$$\boldsymbol{\sigma} = \mathbf{D}^e : \boldsymbol{\varepsilon}. \quad (3.33)$$

In this form

$$\mathbf{D}^e = 2G\mathbf{T} + 3K\mathbf{P}, \quad (3.34)$$

and

$$\mathbf{P} = \frac{1}{3}\mathbf{I} \otimes \mathbf{I}, \quad (3.35)$$

$$\mathbf{T} = \mathbf{I}_{(4)} - \mathbf{P}. \quad (3.36)$$

With this representation, and the fact that there are multiple mathematical representations for the same equation we can suppose that there is a deeper, common ground upon which this is based upon. This is the strain energy density function, which is a scalar valued tensor function such that

$$W = \frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \frac{1}{2}\boldsymbol{\varepsilon} : \mathbf{D}^e : \boldsymbol{\varepsilon}. \quad (3.37)$$

Then the stress tensor can be calculated as

$$\boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{\varepsilon}}. \quad (3.38)$$

3.3 General hyperelasticity

The general base of hyperelastic material modelling (Doghri 2000) is to find a strain energy density function such that it can represent the material as accurate as possible. It is a hard exercise to do so, as no perfectly matching model been found yet, some describe one material almost without error, while for others it is not that applicable, others show different capabilities. Overall we can say that this is the most important point in creating hyperelastic material models that needs refining still in our time although the many papers created on this topic, as we have seen in the paper comparing many models (He et al. 2022).

We can calculate the stress from a given strain energy function with a differentiation in terms of a tensor. Using the right Cauchy-Green deformation tensor the first Piola-Kirchhoff stress tensor can be calculated as

$$\mathbf{P} = 2\mathbf{F} \frac{\partial W}{\partial \mathbf{C}} = \frac{\partial W}{\partial \mathbf{F}}. \quad (3.39)$$

The Cauchy stress tensor can also be calculated as follows

$$\boldsymbol{\sigma} = \frac{2}{J}\mathbf{b} \frac{\partial W}{\partial \mathbf{b}}. \quad (3.40)$$

We can of course create several other formulations due the connectivity between the different stress tensors. Let us also give a form for the Cauchy stress tensor based on the derivatives of the principal invariants of the left (or right) Cauchy-Green deformation tensor

$$\boldsymbol{\sigma} = \frac{2}{J} \left[\left(\frac{\partial W}{\partial I_1} + I_1 \frac{\partial W}{\partial I_2} \right) \mathbf{b} - \frac{\partial W}{\partial I_2} \mathbf{b}^2 + I_3 \frac{\partial W}{\partial I_3} \mathbf{I} \right]. \quad (3.41)$$

We can also consider the rubber-like assumption which takes into account the materials described this way are usually close to incompressible, so we can take this property and account for perfect incompressibility in order to create easier, more compact description.

For the incompressible case, we must also note that the energy function is only capable of describing the isochoric deformations, and the volumetric, real life cases, where small, but still relevant volumetric changes take place the energy must be supplemented with a volumetric additive term as well.

The strain energy can be formulated as

$$W = W_{\text{iso}} + W_{\text{vol}}. \quad (3.42)$$

The deformation gradient's isochoric and volumetric decomposition can be used here, the isochoric term will be the part of the gradient from which the true incompressible hyperelastic behaviours can be calculated, and the volumetric part is modelled separately.

The compressible hyperelastic models can also be defined rather easily with another strain energy density function component.

If we consider compressible hyperelasticity, we must also mention that the principal scalar invariants of the left (or right) Cauchy-Green deformation tensor are not to be used directly, as they also include the volumetric changes as well and the invariants are required only for the isochoric part, while the volume ratio describes the volumetric part.

In order to ensure the separation of the two cases, the decomposition of the deformation gradient is required like presented in equation (3.19).

The isochoric, incompressible part's left Cauchy Green tensor should be obtained like so

$$\bar{\mathbf{b}} = \mathbf{F}_{\text{iso}} \mathbf{F}_{\text{iso}}^T. \quad (3.43)$$

Using this the principal scalar invariants, that are required for the description of the stress state can be calculated as follows

$$\bar{I}_1 = \text{tr}(\bar{\mathbf{b}}), \quad (3.44)$$

$$\bar{I}_2 = \frac{1}{2} \left((\text{tr}(\bar{\mathbf{b}}))^2 - \text{tr}(\bar{\mathbf{b}}^2) \right), \quad (3.45)$$

$$\bar{I}_3 = \det(\bar{\mathbf{b}}). \quad (3.46)$$

This is of course different from the total deformation's scalar values, and must be used when we are considering volumetric changes as well.

The strain energy density function components are dependent on different variables, their dependences are as follows

$$W_{\text{iso}} = W_{\text{iso}}(\bar{I}_1, \bar{I}_2), \quad (3.47)$$

$$W_{\text{vol}} = W_{\text{vol}}(J). \quad (3.48)$$

For the case of simplicity, if not mentioned directly, then for incompressible hyperelastic models the I_1 , I_2 and I_3 values are these bar denoted values, and the compressible case is already decoupled from the deformation gradient, like for the following calculations for the defined incompressible material models.

The stress tensor can also be written using this additive decomposition

$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}} = \frac{\partial W_{\text{iso}}}{\partial \mathbf{F}} + \frac{\partial W_{\text{vol}}}{\partial \mathbf{F}} = \mathbf{P}_{\text{iso}} + \mathbf{P}_{\text{vol}}. \quad (3.49)$$

The Cauchy stress tensor can also be reformulated for the modified principal scalar invariants. This way the formula for it will be the following

$$\boldsymbol{\sigma} = \frac{2}{J} \left[\left(\frac{\partial W_{\text{iso}}}{\partial \bar{I}_1} + \bar{I}_1 \frac{\partial W_{\text{iso}}}{\partial \bar{I}_2} \right) \bar{\mathbf{b}} - \frac{\partial W_{\text{iso}}}{\partial \bar{I}_2} \bar{\mathbf{b}}^2 - \frac{1}{3} \left(\bar{I}_1 \frac{\partial W_{\text{iso}}}{\partial \bar{I}_1} + 2\bar{I}_2 \frac{\partial W_{\text{iso}}}{\partial \bar{I}_2} \right) \mathbf{I} \right] + \frac{\partial W_{\text{vol}}}{\partial J} \mathbf{I}. \quad (3.50)$$

The decomposition for this tensor can be taken in the form

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_{\text{iso}} + \boldsymbol{\sigma}_{\text{vol}}, \quad (3.51)$$

the components being

$$\boldsymbol{\sigma}_{\text{iso}} = \frac{2}{J} \left[\left(\frac{\partial W_{\text{iso}}}{\partial \bar{I}_1} + \bar{I}_1 \frac{\partial W_{\text{iso}}}{\partial \bar{I}_2} \right) \bar{\mathbf{b}} - \frac{\partial W_{\text{iso}}}{\partial \bar{I}_2} \bar{\mathbf{b}}^2 - \frac{1}{3} \left(\bar{I}_1 \frac{\partial W_{\text{iso}}}{\partial \bar{I}_1} + 2\bar{I}_2 \frac{\partial W_{\text{iso}}}{\partial \bar{I}_2} \right) \mathbf{I} \right], \quad (3.52)$$

$$\boldsymbol{\sigma}_{\text{vol}} = \frac{\partial W_{\text{vol}}}{\partial J} \mathbf{I}. \quad (3.53)$$

The most common volumetric strain energy density function form to be used for modelling purposes is the following one

$$W_{\text{vol}}^{\text{STD}} = \frac{1}{D}(J - 1)^2. \quad (3.54)$$

This is of course not perfect, as it gives a finite value for the $J = 0$ case, when it should be infinite. Several other models were developed, some more advanced models can be found in a textbook by Drass (Drass 2020), that can even handle the compression case. However the presented model is used most generally among researchers and even softwares alike, so in this thesis only this one will be considered for usage.

Another important thing to mention is the calculation of parameter D for the volumetric part of the strain energy density function. We can use the definition of the bulk modulus, as it describes the relation between the volumetric strain and the hydrostatic stress component. This way we can see that the hydrostatic stress state is defined by the volumetric strain and the bulk modulus of the material

$$\sigma_{\text{vol}} = K\varepsilon_V. \quad (3.55)$$

From this we can see that using (3.53) the relation between the strain energy density function and the bulk modulus is

$$\frac{\partial W_{\text{vol}}}{\partial J} = K\varepsilon_V. \quad (3.56)$$

Here ε_V is the volumetric strain, which for us, in engineering sense can be written as

$$\varepsilon_V = J - 1. \quad (3.57)$$

Considering the energy function given in (3.54), the relation between D and K can be given for the linearised case, with the linearised response's initial bulk modulus K_0 , which is true close to zero strain, as

$$\frac{2(J - 1)}{D} = K_0(J - 1). \quad (3.58)$$

We can also use the formula to replace the bulk modulus with the help of Young's modulus and the Poisson's ratio, all in linearised case, we get

$$K_0 = \frac{E_0}{3(1 - \nu_0)}. \quad (3.59)$$

The parameter is thus defined using this substitution and considering the initial values for them as

$$D = \frac{2}{K_0} = \frac{6(1 - 2\nu_0)}{E_0}, \quad (3.60)$$

here E_0 and ν_0 are parameters describing the linearised behaviour of the material model around zero strain.

The initial Young modulus E_0 can be calculated from the stress solution for a case, by taking its first order Taylor expansion, neglecting higher order terms. With this the initial modulus can be defined for the material model.

During calculations and simulations we want to control the initial Poisson's ratio, because the initial Poisson's ratio can also describe the volumetric changes, this is commonly used as the direct parameter used to define the volumetric term in the energy function and D being the implicit one. For perfectly incompressible materials it is $\nu_0 = 0.5$, and for nearly incompressible materials it is close to this. To define the rubber-like materials that are considered in hyperelastic material modelling it is general practice to use $\nu_0 = 0.4999$, or even six consecutive nines in numerical softwares for incompressible materials, as those cannot handle total incompressibility, that results in numerical instability.

3.4 Incompressible case

The incompressible case tells us that the volume ratio is always one. In case of incompressible material, the Cauchy stress can be written as

$$\boldsymbol{\sigma} = \text{dev} \left[2 \left(\frac{\partial W}{\partial I_1} + I_1 \frac{\partial W}{\partial I_2} \right) \mathbf{b} - 2 \frac{\partial W}{\partial I_2} \mathbf{b}^2 \right] + \mathbf{p}. \quad (3.61)$$

The hydrostatic stress tensor $\mathbf{p}$ can be calculated from the given stress state assumption as the stress tensor's general form is known for the necessary test cases.

Using this we can calculate the experimental data fitting solutions to which the fitting can be completed.

3.5 Standard incompressible models of hyperelasticity

The simplest model is the Neo-Hookean model, which from its name already tells us the origin of this model, the basic Hooke's law. This model defines the strain energy density function as follows

$$W^{\text{Neo-Hookean}} = C_1(I_1 - 3). \quad (3.62)$$

This model is really simple and easy to use, however cannot describe the behaviour of hyperelastic materials quite well.

Another important model is the Mooney-Rivlin model (Mooney 1940), this has the form

$$W^{\text{Mooney-Rivlin}} = C_1(I_1 - 3) + C_2(I_2 - 3). \quad (3.63)$$

The Ogden's model (Ogden 1972), which uses the principal stretches to describe the strain energy density function is written as

$$W^{\text{Ogden}} = \sum_{i=1}^N \frac{2\mu_i}{\alpha_i^2} (\lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i} - 3). \quad (3.64)$$

Here N is the order of the model.

As mentioned before, there are several other hyperelastic models as well that appeared over time, for some of these this thesis offers numerical implementations. The chosen models were the following.

The Arruda-Boyce model (Arruda & Boyce 1993), which was published in 1993 by the two authors. It contains two parameters and is quite unique considering it contains the inverse Langevin function. This function has the following form

$$\mathcal{L}(x) = \coth(x) - \frac{1}{x}. \quad (3.65)$$

However the inverse of this function cannot be given in closed form, by this fact it is approximated, most commonly with the following formula

$$W^{\text{A-B}} = \mu N \left(\beta_{\text{chain}} \lambda_{\text{chain}} + \ln \frac{\beta_{\text{chain}}}{\sinh \beta_{\text{chain}}} \right). \quad (3.66)$$

The two parameters are scalars containing general properties of polymer and their chains. They can be defined as follows

$$\lambda_{\text{chain}} = \sqrt{\frac{I_1}{3}}, \quad \beta_{\text{chain}} = \mathcal{L}^{-1} \left(\frac{\lambda_{\text{chain}}}{\sqrt{N}} \right). \quad (3.67)$$

A numerical application is also complicated due to the involvement of the inverse Langevin function, so an expansion is used thus from equation (3.66) using a fifth order expansion we get the following strain energy density function

$$W^{\text{A-B}} = \mu \left(\frac{1}{2} (I_1 - 3) + \frac{1}{20 \lambda_{\text{chain}}^2} (I_1^2 - 3^2) + \frac{11}{1050 \lambda_{\text{chain}}^4} (I_1^3 - 3^3) + \right. \\ \left. + \frac{19}{7050 \lambda_{\text{chain}}^6} (I_1^4 - 3^4) + \frac{519}{673750 \lambda_{\text{chain}}^8} (I_1^5 - 3^5) \right). \quad (3.68)$$

This material model lays on a physically sophisticated base as the two describing parameters (3.67) are directly derived from the natural properties of polymer chains and the behaviour connected to them, the model is also considering these by result of using these parameters as variables.

Next up is the Polynomial model, which involves the first two scalar invariants of the Cauchy-Green deformation tensors as a series of polynomials. This model is not a clearly

defined one, as it is more a larger group containing several individual models all relating to the base, with the difference of the order of the model. The model contain each and every polynomial up until the exponent that is the order of our unique desire. This way, for example in first order case we only have two, in third order case we already have five different parameters that can be fitted to experimental data. For an order of N , the model can be written as

$$W^{\text{Pol}} = \sum_{i+j=1}^N C_{ij}(I_1 - 3)^i(I_2 - 3)^j. \quad (3.69)$$

In second order case the exact form is the following

$$W^{\text{Pol}} = C_{10}(I_1 - 3) + C_{01}(I_2 - 3)^1 + C_{20}(I_1 - 3)^2 + C_{11}(I_1 - 3)(I_2 - 3) + C_{02}(I_2 - 3)^2. \quad (3.70)$$

Another model is the Gent-Thomas model (Gent & Thomas 1958), which uses a logarithmic term. This term is useful for capturing the hardening effects in the stress characteristics, thus providing a nice behaviour at larger strains for the model. This model also contains two parameters to describe the behaviour of the material.

The strain energy density function can be given in closed form

$$W^{\text{G-T}} = C_1(I_1 - 3) + C_2 \ln \left(\frac{I_2}{3} \right). \quad (3.71)$$

The fourth model this thesis considers is the Humphrey-Yin model (Humphrey & Yin 1987). This model was published in 1987 and also operates with two hyperelastic model parameters to be fitted onto experimental test data set. The model involves an exponential term, which base on previous results can capture the softening characteristics of the stress model. This way this model also has a specific range, namely the lower strain values, where it is advantageous to use compared to some other models.

The energy function can be given as follows

$$W^{\text{H-Y}} = C_1 \left(e^{C_2(I_1-3)} - 1 \right). \quad (3.72)$$

Finally one other model is to be introduced to this group. This model was inspired by the so called BAM model (Bergstrom 2015). The new model is built of the previously mentioned two models, as a summation between them. The new model this way is already involving both exponential and logarithmic terms, meaning that optimally the effects of these can benefit the model in their respective region, while containing a moderate amount of parameters, namely four.

Thus the new model, the H-K model can be given by the following strain energy density function

$$W^{\text{H-K}} = C_1 \left(e^{C_2(I_1-3)} - 1 \right) + C_1(I_1 - 3) + C_2 \ln \left(\frac{I_2}{3} \right). \quad (3.73)$$

3.6 Homogeneous loading cases

For experimental cases several different base cases are used generally in different papers and for application purposes. The main assumption for these are that the stress state of the material purely homogeneous. This assumption is true considering Saint-Venant's principle, that relatively far from the constraints the stress state is homogeneous regardless of the type of fixtures used throughout the experiments. In order to have proper knowledge about the behaviour of the material itself we need measurement data to fit our model to the real world case, thus we need several experimental results from which these homogeneous cases tell us the most about the material's general behaviour.

Let us consider the three main cases for these stress states that are the most common to acquire test data for further application purposes.

3.6.1 Uniaxial tension

One of the easiest and most frequent loading used to acquire information of a given material is the uniaxial tensile test. This can be done with simple tensile test machines. One aspect that could introduce errors that some experimental devices are built that the test specimen is lying horizontal in the experimental plane, thus gravity introduces a small error, however it is really small compared to the main measurement range, thus it can be neglected.

The basis of this experiment is a single tension along a principle axis of the test specimen, the other directions are unconstrained, they can deform freely, this way contraction can be observed along the other two principal axes. These other axes must fulfill the constraint that, the stress along these are zero, as they deform freely, they show no tensile effects in their dedicated directions. The main concept of this test can be observed in Figure 3.2

Another important factor is that for our purposes considering rubber-like materials, the material itself most commonly is assumed to be completely incompressible, the volume strain is assumed to be zero. By this we can also tell that principal stretches' product must be exactly one, this way we can assume the form of the deformation tensor as well for future calculations.

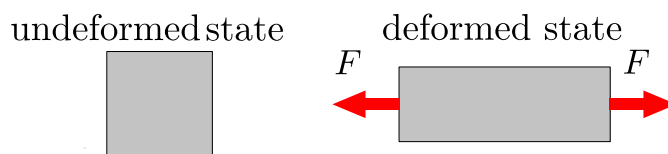


Figure 3.2: Uniaxial tensile test's deformations and excitations.

In order to be able to use the presented models, we also have to derive the stress solutions. The test cases solutions must be obtained considering incompressible case throughout, as we aim to find the parameters of the isochoric part of the strain energy density functions.

Let us start with the uniaxial loading test. Based on the information provided and analysed in a former section, the deformation gradient of this case will be the following:

$$\mathbf{F} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\lambda}} \end{bmatrix}. \quad (3.74)$$

In this case the first principal direction is the controlled one, while the other two can deform freely. The incompressibility constraint implies that

$$\det(\mathbf{F}) = \lambda \cdot \frac{1}{\sqrt{\lambda}} \cdot \frac{1}{\sqrt{\lambda}} = \frac{\lambda}{\lambda} = 1. \quad (3.75)$$

With the help of this tensor we can obtain the left Cauchy-Green deformation tensor's matrix for this case

$$\mathbf{b} = \mathbf{F}\mathbf{F}^T = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & \frac{1}{\lambda} \end{bmatrix}. \quad (3.76)$$

The principal scalar invariants are expressed as

$$I_1 = \lambda^2 + \frac{2}{\lambda}, \quad I_2 = 2\lambda + \frac{1}{\lambda^2}, \quad I_3 = 1. \quad (3.77)$$

The assumed stress state has the following matrix form

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (3.78)$$

3.6.2 Equibiaxial tension test

The equibiaxial tension is a bit more complicated, as this requires two simultaneously actuated axes in order to create a stress state that involves two excitation directions with the same force values applied along these two axes. To be able to support this kind of homogeneous stress state we must have a test device that supports the simultaneous motion of two axes this way maintaining this homogeneous state throughout the whole actuation period.

As mentioned, the two axes along which we apply the excitations are equivalent, they must exert the same amount of force this way the strain values and thus deformation are

the same. It is important that this true only if the material we are working is isotropic, so the deformation's principal axes are not affecting the resulting strain energy state of the given specimen. The test case is visualised in Figure 3.3 for a governing concept.

It is important to mention that all these statements can only be made for a perfect homogeneous case, which is not entirely true for the real life case, however for a small region of the specimen we can state that this assumption is valid, meaning that for calculations we use a smaller segment of the complete geometry and generate useful data about the material.

Along the third axis we can once again observe a free deformation, meaning that this contraction will result in a stress free case, also these kind of tests also assume the incompressibility of the material, so the previously mentioned assumptions made by this fact are also present in this kind of test's descriptions.

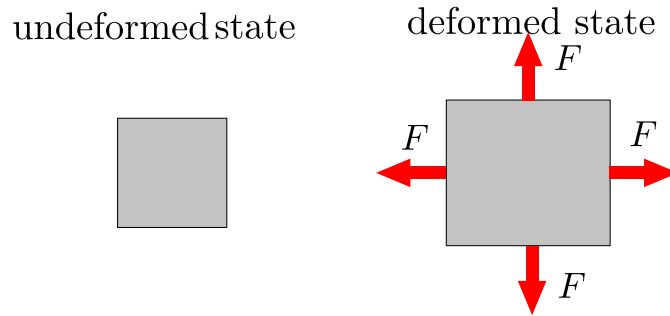


Figure 3.3: Equibiaxial test case, its deformation and excitations.

Next, let us consider this test case's deformation gradient, it has the form

$$\mathbf{F} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \frac{1}{\lambda^2} \end{bmatrix}. \quad (3.79)$$

With this quantity the left Cauchy-Green deformation tensor can be given, which is

$$\mathbf{b} = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \frac{1}{\lambda^4} \end{bmatrix}. \quad (3.80)$$

From this we can get the necessary principal scalar invariants

$$I_1 = 2\lambda^2 + \frac{1}{\lambda^4}, \quad I_2 = \lambda^4 + \frac{2}{\lambda^2}, \quad I_3 = 1. \quad (3.81)$$

With this we have the deformation data, but we also have to make the assumption on the form of the stress matrix. The main form will be the following, assuming that the

first two directions are equally stressed and the third one is free

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (3.82)$$

3.6.3 Planar tension test

Among the three basic test cases, the planar tension test is the most difficult to implement. This case requires a uniaxial loading of the specimen while simultaneously constraining another principal axis in their deformation restricting all deformation that would cause contraction (or expansion) so the plane in along that axis is undisturbed. This fixture along a line is the greatest problem to implement when the measurement is built, if the constraint is not assured then the experiment will provide false results. To create such experiments one must have a uniaxial loading, most commonly provided by such an experimental machine, and the corresponding constraints, that deny the deformation of the specimen in two of the three main directions, and only allows the load directional displacement. If these conditions are met we can safely conduct the experiments, however these kind of tests are done rarely as of the difficulties of the structure and setup.

It is noteworthy that for moderately large deformations this case is similar in nature to the pure shear test case, the constraints introduce such forces that the stress case is in accordance with the before mentioned planar tension's homogeneous stress state.

This kind of unidirectional loading and planar constraint is indeed really difficult to implement, however they can provide crucial information about the material's behaviour in more general stress states, so it is advisory to fit the model to these measurements as well if experimental data is available. This stress state involves a controlled deformation along the uniaxial loading direction, a deformation free axis due to the constraints with a unique governing stress and a stress free third direction perpendicular to the other two's plane along which the specimen may deform freely causing a stress free case for that principal axis. For better understanding of this test case Figure 3.4 illustrates two steps of an experiment.

Once again in practice the incompressibility constraint is assumed for the material for most hyperelastic materials behave so, thus allowing a simpler calculation scheme for stress states and model fitting to the experimental data provided by the different test cases conducted on the material's specimens.

We can calculate the required quantities for this case as well. The deformation gradient tensor will have the following form assuming homogeneous stress state

$$\mathbf{F} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\lambda} \end{bmatrix}. \quad (3.83)$$

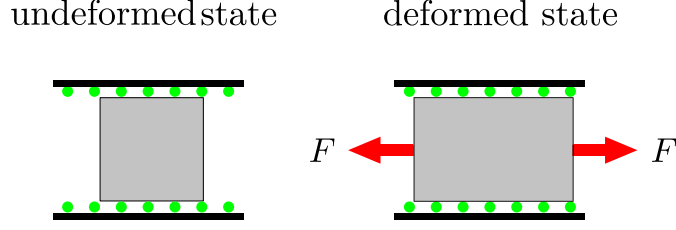


Figure 3.4: Planar tension test case's deformation and excitations.

With this at hand the left Cauchy-Green deformation tensor once again can be given, this time its form is

$$\mathbf{b} = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\lambda^2} \end{bmatrix}. \quad (3.84)$$

With the help of these the scalar invariants for this case will be the following

$$I_1 = \lambda^2 + 1 + \frac{1}{\lambda^2}, \quad I_2 = \lambda^2 + 1 + \frac{1}{\lambda^2}, \quad I_3 = 1. \quad (3.85)$$

Knowing this information we once again arrived with a set deformation state, with the stress state still to be defined. Assuming the necessary boundary conditions the stress tensor's matrix must be of the following form

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma^C & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (3.86)$$

For all test cases the hydrostatic stress component has to be subtracted from the stress tensor as shown in (3.61), because the models used here are incompressible hyperelastic material models that can only describe isochoric changes.

With all these at hand we can finally derive the stress solutions for our models that are to be used.

3.7 Stress solutions of the particular models

Firstly, the Arruda-Boyce model provides with the following derivatives using the fifth order expansion, otherwise no closed form solution could be obtained.

$$\frac{\partial W^{A-B}}{\partial I_1} = C_1 \left(\frac{1}{2} + \frac{I_1}{10C_2^2} + \frac{11I_1^2}{350C_2^4} + \frac{38I_1^3}{3525C_2^6} + \frac{519I_1^4}{134750C_2^8} \right), \quad (3.87)$$

$$\frac{\partial W^{A-B}}{\partial I_2} = 0. \quad (3.88)$$

After this let us take the three basic test cases, and solve them to arrive with the following engineering stress results based on the hyperelastic model's differentiation

$$P_U^{A-B} = \frac{C_1(-1 + \lambda^3)(9499875C_2^8\lambda^4 + 1899975C_2^6\lambda^3(2 + \lambda^3))}{9499875C_2^8\lambda^6} + \frac{C_1(-1 + \lambda^3)(597135C_2^4\lambda^2(2 + \lambda^3)^2 + 204820C_2^2\lambda(2 + \lambda^3)^3 + 73179(2 + \lambda^3)^4)}{9499875C_2^8\lambda^6}, \quad (3.89)$$

$$P_B^{A-B} = \frac{C_1(-1 + \lambda^6)(9499875C_2^8\lambda^{16} + 597135C_2^4\lambda^8(1 + 2\lambda^6)^2)}{9499875C_2^8\lambda^{21}} + \frac{C_1(-1 + \lambda^6)(204820C_2^2\lambda^4(1 + 2\lambda^6)^3 + 73179(1 + 2\lambda^6)^4 + 1899975C_2^6(\lambda^{12} + 2\lambda^{18}))}{9499875C_2^8\lambda^{21}}, \quad (3.90)$$

$$P_P^{A-B} = -\frac{C_1}{\lambda^3} + C_1\lambda + \frac{C_1\lambda(1 + 1/\lambda^2 + \lambda^2)}{5C_2^2} + \frac{11C_1\lambda(1 + 1/\lambda^2 + \lambda^2)^2}{175C_2^4} + \frac{76C_1\lambda(1 + 1/\lambda^2 + \lambda^2)^3}{3525C_2^6} + \frac{519C_1\lambda(1 + 1/\lambda^2 + \lambda^2)^4}{67375C_2^8} - \frac{C_1(1 + \lambda^2 + \lambda^4)}{5C_2^2\lambda^5} - \frac{11C_1(1 + \lambda^2 + \lambda^4)^2}{175C_2^4\lambda^7} - \frac{76C_1(1 + \lambda^2 + \lambda^4)^3}{3525C_2^6\lambda^9} - \frac{519C_1(1 + \lambda^2 + \lambda^4)^4}{67375C_2^8\lambda^{11}}. \quad (3.91)$$

Using the formula given in (3.60), we can calculate the initial Young's modulus for the uniaxial tension case with the help of linearisation. With that, assuming the initial Poisson's ratio is known (0.5 for incompressibility, for near incompressible cases it is slightly lower) we can define the incompressibility term for a given model

$$D^{A-B} = \frac{18999750C_2^8(1 - 2\nu_0)}{C_1(5927499 + 5530140C_2^2 + 5374215C_2^4 + 5699925C_2^6 + 9499875C_2^8)}. \quad (3.92)$$

Next up is the Polynomial model, which also contains the second scalar invariant as well. The derivatives for the N order model are the following

$$\frac{\partial W^{\text{Pol}}}{\partial I_1} = \sum_{i+j=1}^N C_{ij}(I_1 - 3)^{i-1}(I_2 - 3)^j, \quad (3.93)$$

$$\frac{\partial W^{\text{Pol}}}{\partial I_2} = \sum_{i+j=1}^N C_{ij}(I_1 - 3)^i(I_2 - 3)^{j-1}. \quad (3.94)$$

Considering a second order model with all five parameters, the derivatives will transform to

$$\frac{\partial W^{\text{Pol}}}{\partial I_1} = C_{10} + 2C_{20}(I_1 - 3) + C_{11}(I_2 - 3), \quad (3.95)$$

$$\frac{\partial W^{\text{Pol}}}{\partial I_2} = C_{01} + 2C_{02}(I_2 - 3) + C_{11}(I_1 - 3). \quad (3.96)$$

The stress results can once again be given like for the previous model, the formulae are the following

$$P_U^{\text{Pol}} = \frac{2(-1 + \lambda^3)(2C_{02}(-1 + \lambda)^2(1 + 2\lambda) + \lambda(3C_{11}(-1 + \lambda)^2(1 + \lambda)))}{\lambda^5} + \frac{2(-1 + \lambda^3)\lambda^2(C_{01} + C_{10}\lambda + 2C_{20}(2 - 3\lambda + \lambda^3))}{\lambda^5}, \quad (3.97)$$

$$P_B^{\text{Pol}} = \frac{2(-1 + \lambda^6)(3C_{11}\lambda^2(-1 + \lambda^2)^2(1 + \lambda^2) + C_{20}(2 - 6\lambda^4 + 4\lambda^6))}{\lambda^9} + \frac{2(-1 + \lambda^6)(\lambda^4(C_{10} + C_{01}\lambda^2 + 2C_{02}(2 - 3\lambda^2 + \lambda^6)))}{\lambda^9}, \quad (3.98)$$

$$P_P^{\text{Pol}} = \frac{2(-1 + \lambda^4)(2C_{20} + C_{01}\lambda^2 + C_{10}\lambda^2 - 4C_{20}\lambda^2 + 2C_{20}\lambda^4)}{\lambda^5} + \frac{2(-1 + \lambda^4)(2C_{02}(-1 + \lambda^2)^2 + 2C_{11}(-1 + \lambda^2)^2)}{\lambda^5}. \quad (3.99)$$

The incompressibility term once again can be calculated following the same steps, it will be

$$D^{\text{Pol}} = \frac{1 - 2\nu_0}{C_1 + C_2}. \quad (3.100)$$

The next model to be derive is the Gent-Thomas model. The derivatives regarding the scalar invariants are going to be the following

$$\frac{\partial W^{\text{G-T}}}{\partial I_1} = C_1, \quad (3.101)$$

$$\frac{\partial W^{\text{G-T}}}{\partial I_2} = \frac{C_2}{I_2}. \quad (3.102)$$

The resulting stress solutions will be of the following form for the three test cases

$$P_U^{\text{G-T}} = \frac{2(-1 + \lambda^3)(C_1 + C_2\lambda + 2C_1\lambda^3)}{\lambda^2 + 2\lambda^5}, \quad (3.103)$$

$$P_B^{\text{G-T}} = \frac{2(-1 + \lambda^6)(C_2\lambda^4 + C_1(2 + \lambda^6))}{\lambda^5(2 + \lambda^6)}, \quad (3.104)$$

$$P_P^{\text{G-T}} = \frac{2(-1 + \lambda^4)(C_2\lambda^2 + C_1(1 + \lambda^2 + \lambda^4))}{\lambda^3 + \lambda^5 + \lambda^7}. \quad (3.105)$$

For volumetric changes the incompressibility terms will be fro this model

$$D^{\text{G-T}} = \frac{3(1 - 2\nu_0)}{3C_1 + C_2}. \quad (3.106)$$

Next let us also derive the necessary formulae for the Humphrey-Yin model as well. The derivatives for this case are

$$\frac{\partial W^{\text{H-Y}}}{\partial I_1} = C_1 C_2 e^{C_2(I_1-3)}, \quad (3.107)$$

$$\frac{\partial W^{\text{H-Y}}}{\partial I_2} = 0. \quad (3.108)$$

The stress solutions will be the following

$$P_{\text{U}}^{\text{H-Y}} = \frac{2(-1 + \lambda^3) C_1 C_2 e^{C_2(-3+2/\lambda+\lambda^2)}}{\lambda^2}. \quad (3.109)$$

$$P_{\text{B}}^{\text{H-Y}} = \frac{2(-1 + \lambda^6) C_1 C_2 e^{C_2(-3+1/\lambda^4+2\lambda^2)}}{\lambda^5}. \quad (3.110)$$

$$P_{\text{P}}^{\text{H-Y}} = \frac{2(-1 + \lambda^4) C_1 C_2 e^{C_2(-2+1/\lambda^2+\lambda^2)}}{\lambda^3}. \quad (3.111)$$

The incompressibility term in this case

$$D^{\text{H-Y}} = \frac{1 - 2\nu_0}{C_1 \cdot C_2}. \quad (3.112)$$

Finally for the introduced H-K model, we can also give the closed form solution for the derivatives,

$$\frac{\partial W^{\text{H-K}}}{\partial I_1} = C_1 C_2 e^{C_2(I_1-3)} + C_3, \quad (3.113)$$

$$\frac{\partial W^{\text{H-K}}}{\partial I_2} = \frac{C_4}{I_2}, \quad (3.114)$$

and for the stress solutions as well. These will be the following for our case starting with the uniaxial tensile solution

$$P_{\text{U}}^{\text{H-K}} = \frac{2(-1 + \lambda^3) \left(C_3 + C_4 \lambda + 2C_3 \lambda^3 + C_1 C_2 e^{C_2(-3+2/\lambda+\lambda^2)} (1 + 2\lambda^3) \right)}{\lambda^2 + 2\lambda^5}, \quad (3.115)$$

the equibiaxial tensile solution,

$$P_{\text{B}}^{\text{H-K}} = \frac{2(-1 + \lambda^6) \left(C_4 \lambda^4 + C_3(2 + \lambda^6) + C_1 C_2 e^{C_2(-3+1/\lambda^4+2\lambda^2)} (2 + \lambda^6) \right)}{\lambda^5(2 + \lambda^6)}. \quad (3.116)$$

and finally the planar stress case's stress solution

$$P_P^{H-K} = \frac{2(-1 + \lambda^4) \left(C_4 \lambda^2 + C_3(1 + \lambda^2 + \lambda^4) + C_1 C_2 e^{C_2(-2+1/\lambda^2+\lambda^2)}(1 + \lambda^2 + \lambda^4) \right)}{\lambda^3 + \lambda^5 + \lambda^7}. \quad (3.117)$$

The incompressibility term for the H-K model is

$$D^{H-K} = \frac{3(1 - 2\nu_0)}{3C_1 \cdot C_2 + 3C_3 + C_4}. \quad (3.118)$$

Let us also derive the stress solutions for the second order Ogden model for later use as comparison in fitting processes.

The derivatives of the 2nd order Ogden model in terms of the scalar invariants are the following

$$\frac{\partial W}{\partial \lambda_i} = \sum_{k=1}^2 \frac{2\mu_k}{\alpha_k} \lambda_i^{\alpha_k-1}. \quad (3.119)$$

Finally the stress solutions for this model we can derive and the following resulting analytical solutions can be achieved

$$P_U^{OG} = \sum_{k=1}^2 \frac{2\mu_k}{\alpha_k} \left(\lambda^{\alpha_k-1} - \lambda^{-\frac{\alpha_k}{2}-1} \right) \quad (3.120)$$

$$P_B^{OG} = \sum_{k=1}^2 \frac{2\mu_k}{\alpha_k} \left(\lambda^{\alpha_k-1} - \lambda^{-2\alpha_k-1} \right) \quad (3.121)$$

$$P_P^{OG} = \sum_{k=1}^2 \frac{2\mu_k}{\alpha_k} \left(\lambda^{\alpha_k-1} - \lambda^{-\alpha_k-1} \right). \quad (3.122)$$

With these at hand we can work out some further applications also providing fitting data to experimental data sets.

4 Parameter fitting

Parameter fitting is a crucial element of creating hyperelastic model for a material, as the model itself contains multiple free parameters that need to be fitted to the given measurement data. In order to do so, we first need to have this data ready to use.

On the other hand, it is not easy to find many test data in the literature. Most researchers are using the Treloar data set, which is quite old to this time, but up until today not many other experimental results of rubber and rubber-like materials have been

published. There are some papers considering different experimental results from different materials but most of them are not structural materials.

Rubber results can be found (Jones & Treloar 1975), (Shahzad et al. 2015b), and flexible adhesives are also considered in some articles (Crocker & Duncan 2001), as well as some propellant material (Shabbir et al. 2020). The other results are usually of biomechanical source, like porcine adipose tissue (Comley & Fleck 2012), or the brain tissue, which is a highly popular subject of the topic regarding multiple articles are working with them (Moran et al. 2014), (Miller & Chinzei 2002), (Miller & Chinzei 1997).

Overall we can say that not many, and especially not many structural material experimental data is available in the literature, thus came the opportunity to create some custom measurements as well, which I will go into details in a later section.

During parameter fitting for a given strain energy density function we can perform the optimisation for the derived stress solution of the test cases.

For this we also require a measuring number accounting for the error of the fitting, which we wish to minimise during the fitting process. This error function will be our key to the task. It can be constructed several ways, no objective optimum is available with them. It can use absolute or relative error as well as being quadratic or not. One advantage to using quadratic forms is that the errors do not compensate each other when opposite signs are present. It can be practical to use an error functions that is bounded, this way the resulting number can carry more information to us as well.

For my calculation the root mean square error was used, this type of error consist of the relative errors summed up quadratically and normalised for the length of the data set. After that we take the square root of the result, norming everything down once more.

This error function was the following throughout the calculations of fitting

$$Q = 100 \cdot \sqrt{\frac{1}{n} \sum_{i=1}^n \left(\frac{P_i^m - P_i}{P_i^m} \right)^2}. \quad (4.1)$$

This formula calculates the error for a dataset of length n , where P_i^m is the i th measured engineering stress value and P_i is the i th model solution. For a better understanding of the fitting process the following figure, Figure 4.1 illustrates the error calculation's steps.

```
param = NMinimize[{(1/ 3 (QU[UA[[A11, 2]], UA[[A11, 1]]] +
QB[BA[[A11, 2]], BA[[A11, 1]]] +
QP[PA[[A11, 2]], PA[[A11, 1]]]))*100},
C1, C2, C3, C4, MaxIterations -> 10000, Method -> "RandomSearch"]
```

Here in this code *param* contains the total error value as well as the parameter values, *UA*, *BA*, and *PA* are the uniaxial, equibiaxial and planar experimental test data respec-

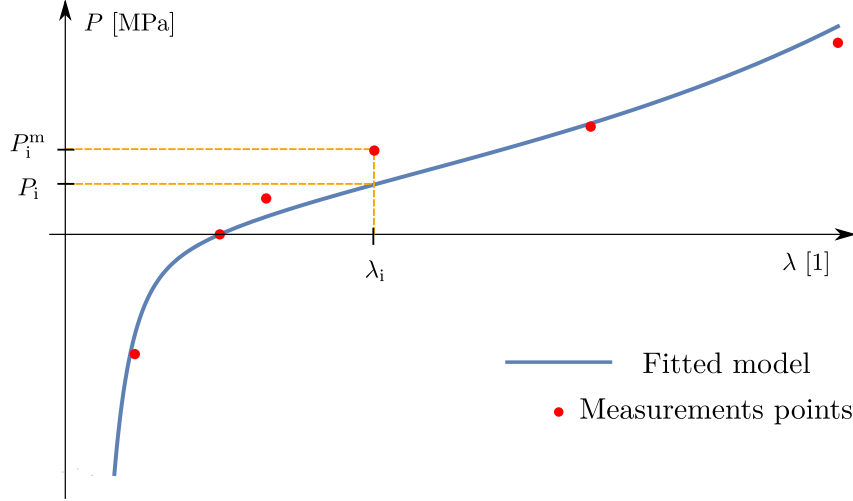


Figure 4.1: Fitting error calculation's geometrical illustration for a given measurement point.

tively, QU , QB and QP are the error functions, while $C1$, $C2$, $C3$ and $C4$ are the model parameters.

Based on the figure a significant difference is present, to obtain the relative error we must normalise this with the accurate value being the measurement point. For each and every point we can calculate these errors and sum them up and perform the desired additional normalising operations necessary.

This can be used for all solutions one by one, considering the uniaxial, equibiaxial and planar tensile test cases.

It is important that the fitting is done for the incompressible model, the compressible part can be modelled separately with the additional terms in the strain energy density function according to the initial Poisson's ratio.

4.1 Treloar's data

Treloar's data is the most frequently used measurement data in the area of hyperelastic material modelling. The models are most commonly tested with this set of measurement data, as it provides a common point to compare with others as well. This way we not only get an error to validate our model, but also comparison opportunities, that can also enable us for further development based on the performance of a given model.

In an article published in 2012 (Steinmann et al. 2012b), the authors summarised the Treloar data set and published it in a table with numerical values. The measurements can be inspected in Figure 4.2.

These data points were used during the fitting process. Table 4.1 contains the exact numerical values

This set of data contains three experimental results, one for uniaxial, equibiaxial and planar test case as well.

Upon these data points we can fit our analytical model like shown with the error function. Four models were used during the fitting for a general comparison to be made as well.

In order to calculate and fit, the error functions must be supplemented with one more fact, that the unique error are added and the arithmetic mean value is taken of them. This way all test cases equally have an effect on the fitting process.

For the fitting calculation the software Wolfram Mathematica was used, with its *NMinimize* function (Research (Accessed December 9th, 2022), the *RandomSearch* method proved the most successful for these calculations. This way we could find an optimum for our material data fitting in terms of the error.

Table 4.1: The Treloar data set's numerical values from the 2012 paper (Steinmann et al. 2012b).

Uniaxial data		Equibiaxial data		Planar data	
λ [1]	P_U [MPa]	λ [1]	P_B [MPa]	λ [1]	P_P [MPa]
1.00	0.00	1.00	0.00	1.00	0.00
1.01	0.03	1.04	0.09	1.06	0.07
1.12	0.14	1.08	0.16	1.14	0.16
1.24	0.23	1.12	0.24	1.21	0.24
1.39	0.32	1.14	0.26	1.32	0.33
1.61	0.41	1.20	0.33	1.46	0.42
1.89	0.50	1.31	0.44	1.87	0.59
2.17	0.58	1.42	0.51	2.40	0.76
2.42	0.67	1.69	0.65	2.98	0.93
3.01	0.85	1.94	0.77	3.48	1.11
3.58	1.04	2.49	0.96	3.96	1.28
4.03	1.21	3.03	1.24	4.36	1.46
4.76	1.58	3.43	1.45	4.69	1.62
5.36	1.94	3.75	1.45	4.96	1.79
5.76	2.29	4.03	1.96		
6.16	2.67	4.26	2.22		
6.40	3.02	4.44	2.43		
6.62	3.39				
6.87	3.75				
7.05	4.12				
7.16	4.47				
7.27	4.85				
7.43	5.21				
7.50	5.57				
7.61	6.30				

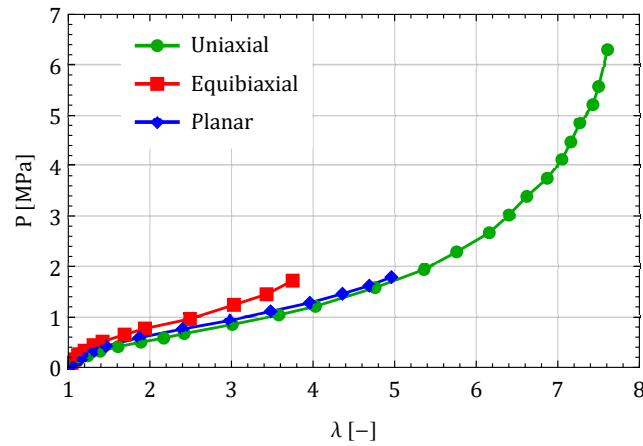


Figure 4.2: The Treloar data visualised in a common coordinate system.

4.1.1 Ogden model

The Ogden model provided the following data for the fitting that can be inspected below.

$$\begin{aligned} C_1 &= 0.293184 \text{ MPa}, & C_2 &= 2.31922, \\ C_3 &= -0.059281 \text{ MPa}, & C_4 &= -1.20074. \end{aligned} \quad (4.2)$$

The error values

$$\begin{aligned} Q &= \frac{Q_U + Q_B + Q_P}{3} = 18.595\%, \\ Q_U &= 28.326\%, & Q_B &= 5.137\%, & Q_P &= 22.322\%. \end{aligned} \quad (4.3)$$

The fitted curves are present in Figures 4.3-4.5.

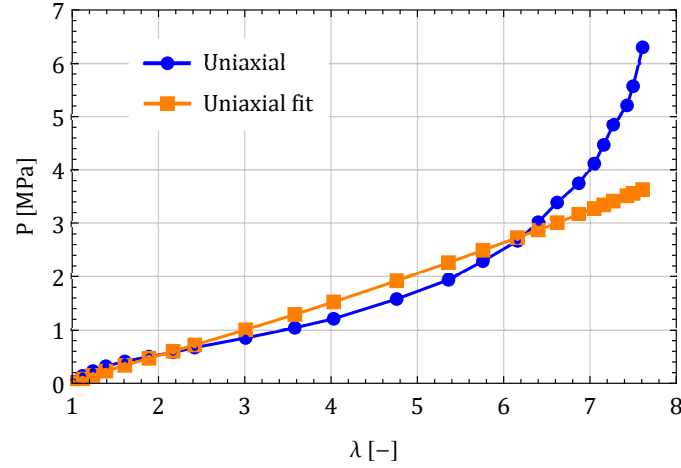


Figure 4.3: Fitting to uniaxial data with the Ogden model of the Treloar data.

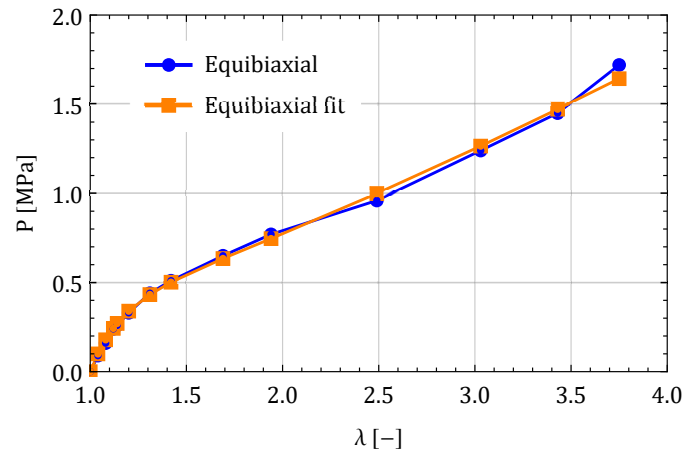


Figure 4.4: Fitting to equibiaxial data with the Ogden model of the Treloar data.

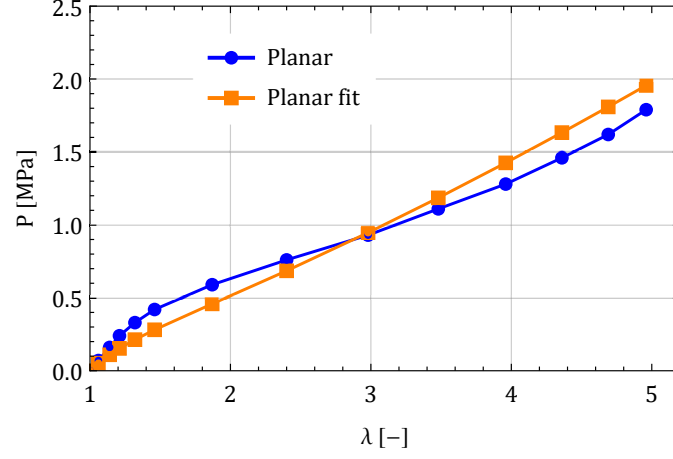


Figure 4.5: Fitting to planar data with the Ogden model of the Treloar data.

4.1.2 Arruda-Boyce model

In case of the Arruda-Boyce model we can similarly conduct the fitting procedure and get the following results.

$$C_1 = 0.333411 \text{ MPa}, \quad C_2 = -5.53107. \quad (4.4)$$

The error values are the following

$$Q = 15.332\%, \quad Q_U = 20.241\%, \quad Q_B = 14.647\%, \quad Q_P = 11.109\%. \quad (4.5)$$

Fitted data is plotted on Figures 4.6-4.8.

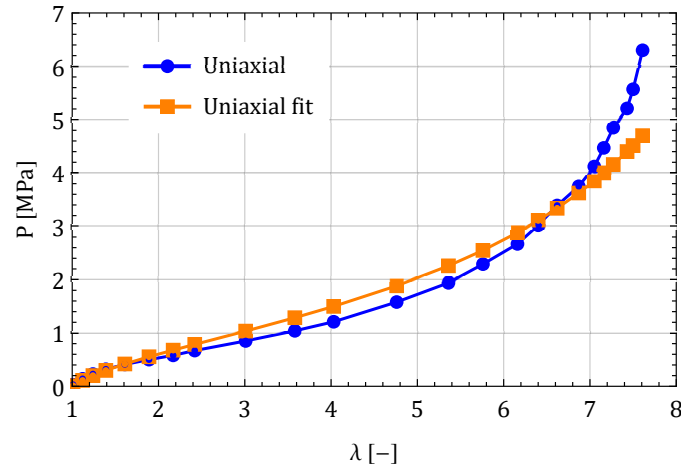


Figure 4.6: Fitting to uniaxial data with the Arruda-Boyce model of the Treloar data.

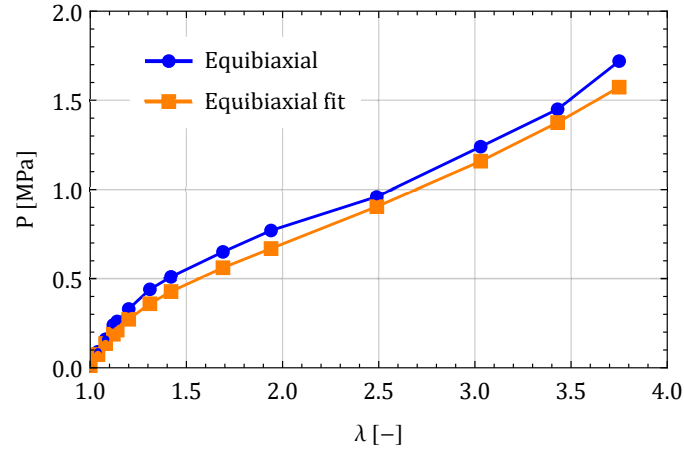


Figure 4.7: Fitting to equibiaxial data with the Arruda-Boyce model of the Treloar data.

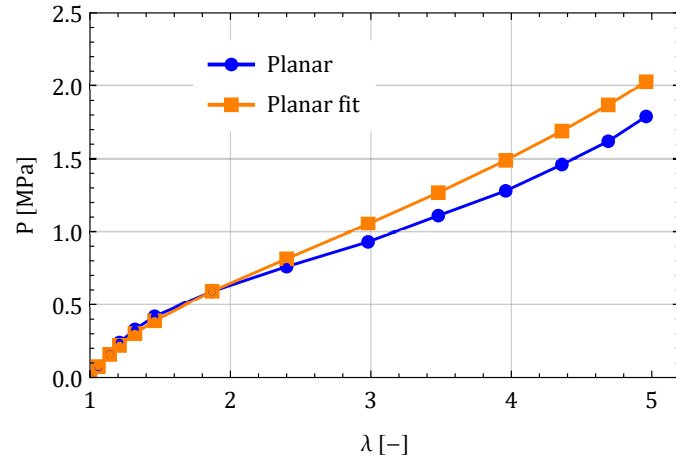


Figure 4.8: Fitting to planar data with the Arruda-Boyce model of the Treloar data.

4.1.3 Polynomial model

Next is the Polynomial model, that returned with the following fitted results.

$$\begin{aligned} C_{10} &= 0.14792 \text{ MPa}, & C_{01} &= 0.028313 \text{ MPa}, & C_{20} &= 0.001753 \text{ MPa}, \\ C_{11} &= -0.002175 \text{ MPa}, & C_{02} &= 0.0001456 \text{ MPa}. \end{aligned} \quad (4.6)$$

The error values are the following

$$Q = 11.685\%, \quad Q_U = 19.4853\%, \quad Q_B = 9.3672\%, \quad Q_P = 6.2014\%. \quad (4.7)$$

Fitted curves are on Figures 4.9-4.11.

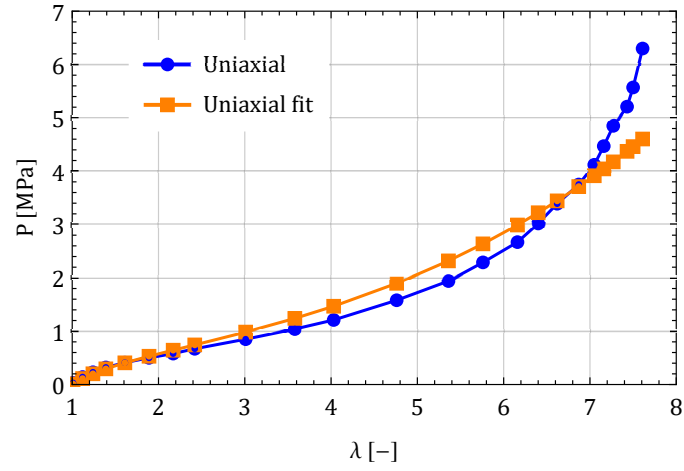


Figure 4.9: Fitting to uniaxial data with the Polynomial model of the Treloar data.

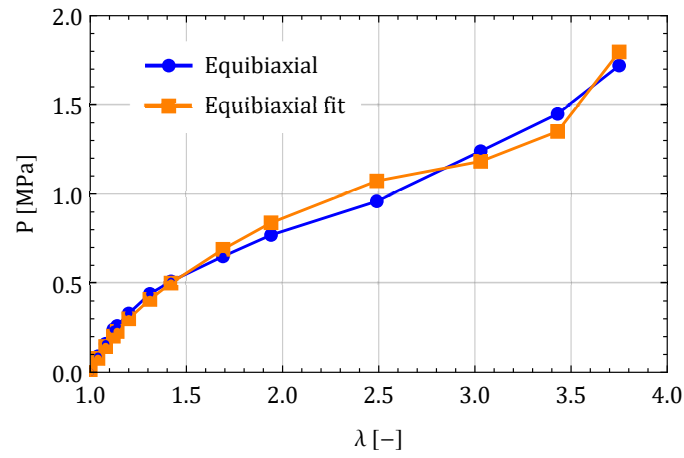


Figure 4.10: Fitting to equibiaxial data with the Polynomial model of the Treloar data.

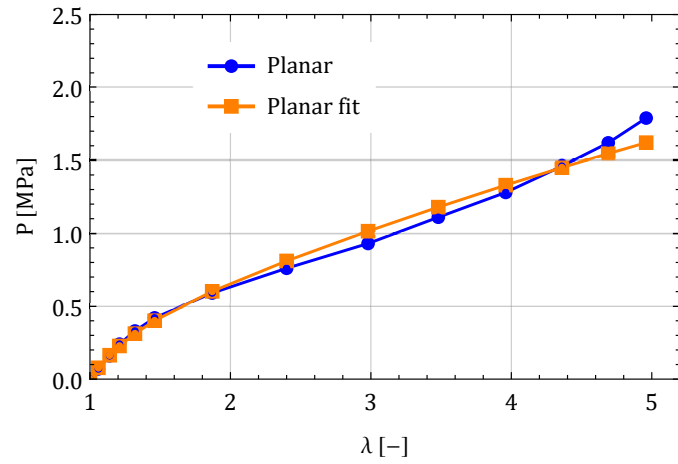


Figure 4.11: Fitting to planar data with the Polynomial model of the Treloar data.

4.1.4 H-K model

For this model the fitting was also done with the Treloar data, the results being the following parameters

$$\begin{aligned} C_1 &= 2.47263 \text{ MPa}, & C_2 &= 0.0274833, \\ C_3 &= 0.0560254 \text{ MPa}, & C_4 &= 0.205664. \end{aligned} \quad (4.8)$$

The error function values

$$Q = 9.19207\%, \quad Q_U = 14.202\%, \quad Q_B = 6.22778\%, \quad Q_P = 7.14639\%. \quad (4.9)$$

As it can be seen this model can perform really good based on the error values of the fitting. The results can be inspected on Figures 4.12-4.14.

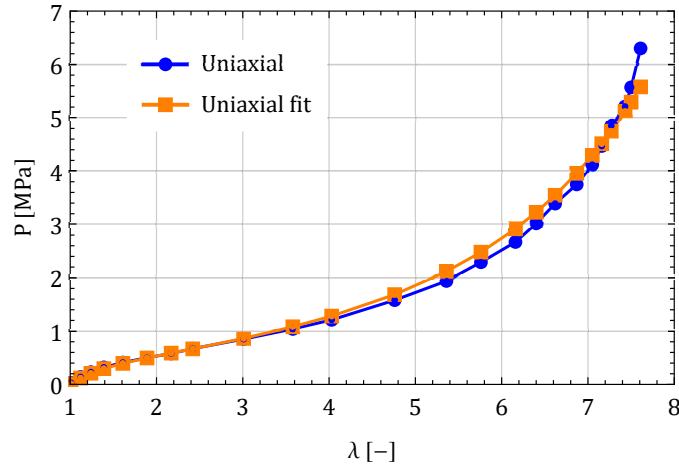


Figure 4.12: Fitting to uniaxial data with the H-K model of the Treloar data.

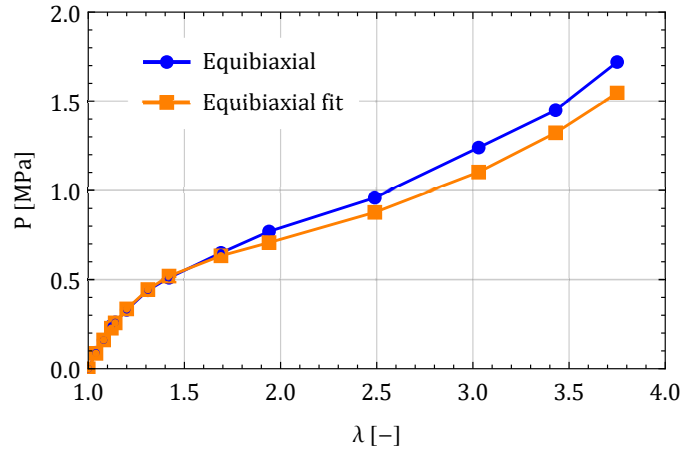


Figure 4.13: Fitting to equibiaxial data with the H-K model of the Treloar data.

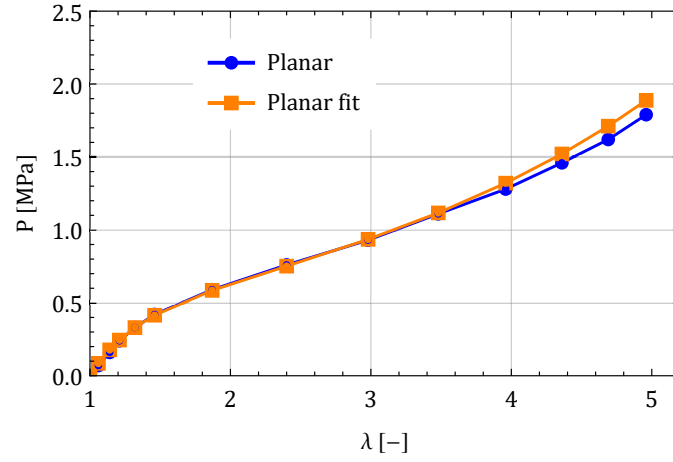


Figure 4.14: Fitting to planar data with the H-K model of the Treloar data.

Another important aspect is the Drucker stability (Drucker et al. 1963) of the material model itself. In order to have a stable material model the stress must increase together with strain increments, this way the the derivative of the stress solution is always positive.

The derivative of the solutions can be seen on the following Figures 4.15-4.17.

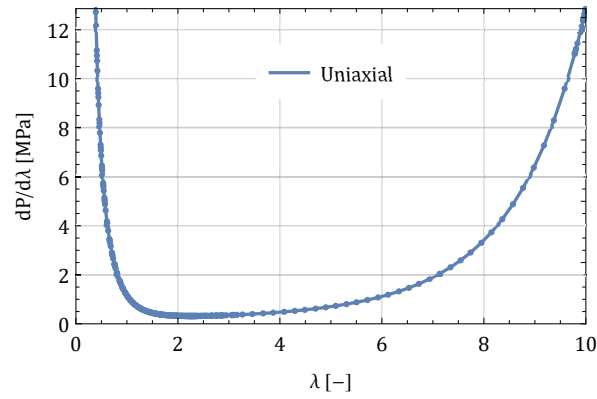


Figure 4.15: Drucker stability for uniaxial data with the H-K model of the Treloar data.

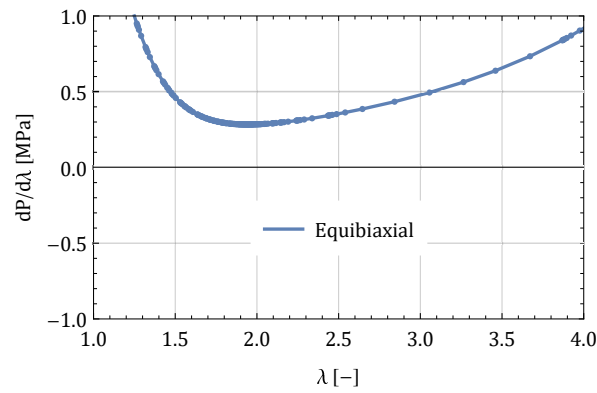


Figure 4.16: Drucker stability for equibiaxial data with the H-K model of the Treloar data.

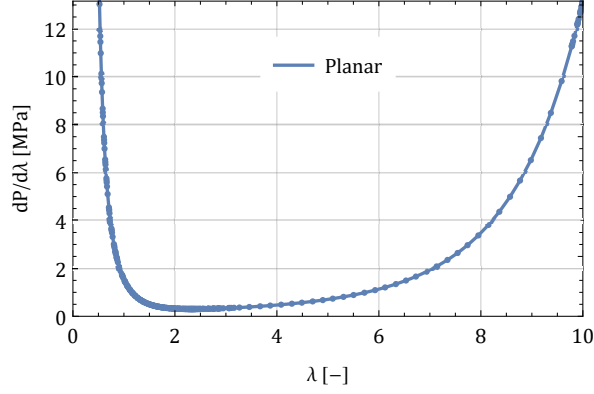


Figure 4.17: Drucker stability for planar data with the H-K model of the Treloar data.

All of the solutions' derivatives are always positive thorough the experimental range, so the Drucker stability is ensured for the model.

4.2 New measurement results

Treloar's data is a widely accepted and commonly used data that was measured with vulcanised rubber. Based on this fact and the goal to create another material's model, I also have completed a set of measurements on my own.

During the measurements the material used was a silicon lamina, 600x800 mm, with a thickness of 2 mm. The material is a vulcanised HTV silicon processed under high temperatures (Bondex (Accessed December 9th, 2022)). This silicon's main property is the it can suffer large deformations, even for 3-4 times its original size, with a real high elastic deformation and almost no other. Additionally it has a moderate tensile strength as well, thus the resulting stiffness of the measurement chain in the experimental machine will prove to be sufficiently large, the specimens will suffer the deformations instead of the machine itself. For the following part this material will be analysed in depth.

During the experiments the test specimens' geometry was an important aspect. The uniaxial tensile tests were done on the classical tensile geometry used in these kind of tests, while the cross model used at biaxial measurements has no standardised form, nor optimal one, based on this we must choose one that can provide us with a homogeneous stress state of the specimen. By these facts in order to choose a well behaving one, I analysed several options to find one that can help us in our work. Many forms are commonly used in measurements, usual features are roundings, cuts near the arms so that the geometry is dominated in its behaviour by the middle section (Admet (Accessed December 9th, 2022)), or even when the arms are only kept wider for proper fixture with a head element at the end (Wikikwand (Accessed December 9th, 2022)). These are also used in metals from time to time (Montalvão & Wren 2017). In finite element calculations they are also commonly used, from them we can see that larger rounding radii can produce a better homogeneity in the stress state of the test specimen (Zhao et al. 2019) . Based on experience, it has been found that a middle, thinner part can also be of great use in

the experiments, however for present measurements this method was not suitable as we only had a fixed thickness piece of material. By this the chosen geometry was a simple rounded model, which was cut out with the uniaxial tensile test bodies as well. Another important rule is that the effects of the constraints have to be negligible, but based on Saint-Venant's principal (Saint-Venant et al. 1855) it is acceptable.

The final geometries drawings and cut-out versions can be seen in Figures 4.18-4.19.

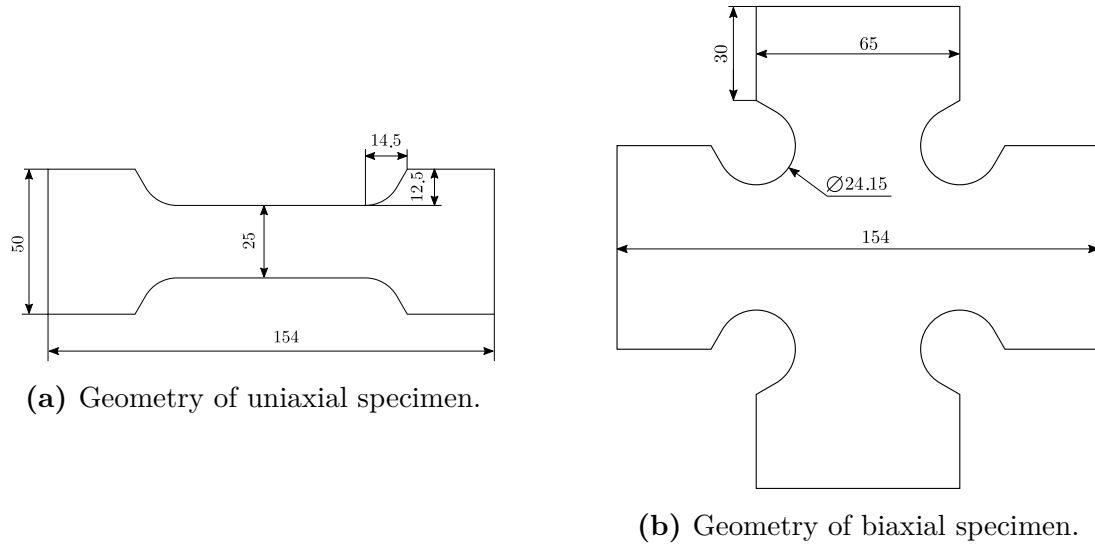


Figure 4.18: The test specimens were cut out from the silicon laminas with the help of a laser cutter.

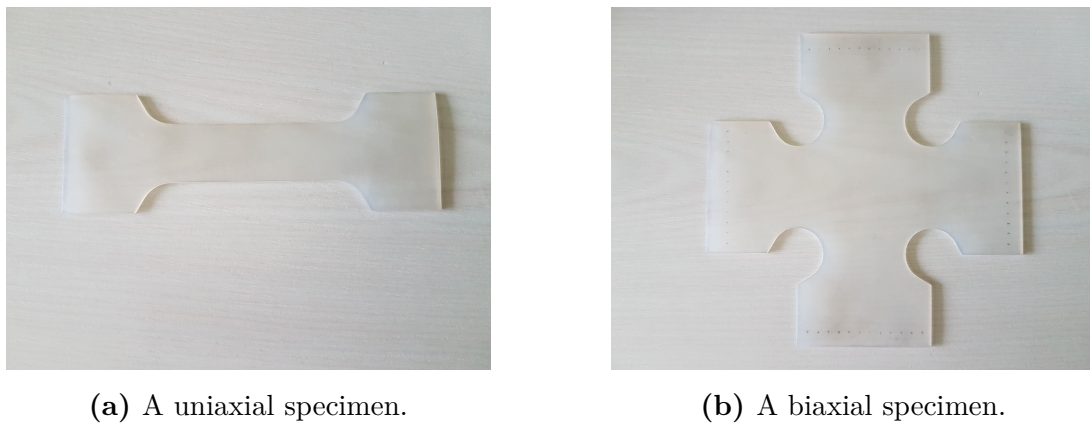
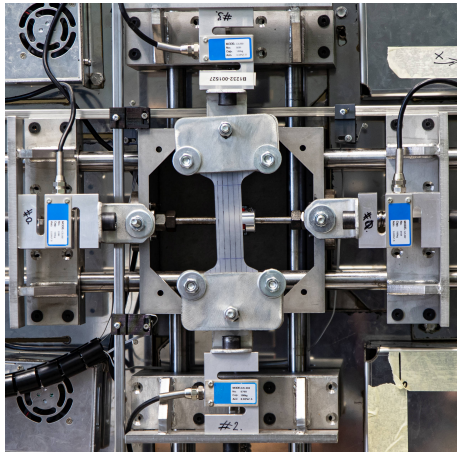


Figure 4.19: Photos of the cut-out specimens.

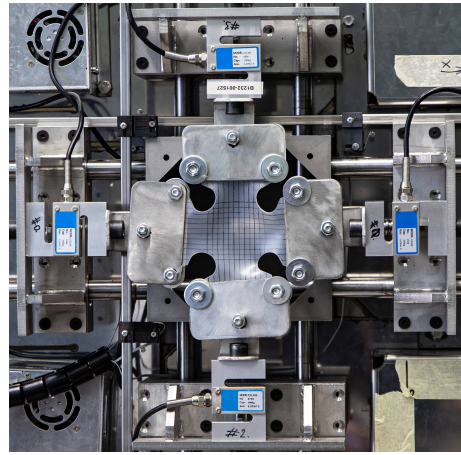
The measurements were done on a two axes tensile measurement device, designed by the Department of Applied Mechanics . The specimens were mounted via clamping jaws on two, independent motion capable axes. These were position controlled, and also had all in all four load cells mounted for force measurements. This way for known and controlled displacements we could measure the corresponding forces. One aspect of this measurement is that we measure the complete force and strain values, so for the homogeneous part to be define we must introduce some corrections.

Two base cases were done, uniaxial and equibiaxial measurements. For the first case the test specimen was elongated by 400 mm, while the equibiaxial was withstanding 200 mm. Both experiments were done so that viscous effects would not take place in them, so the application of the loads were done under ten minutes each. This way the time dependant reactions of the material can be mostly eliminated, the static behaviour is the dominating one. After removing the load the materials became really similar to their starting geometry, very little permanent deformation was present in them. This deformation could be inspected in the biaxial specimen more visibly than the uniaxial one. The experiments' initial and end states can be inspected on Figures 4.20-4.21.

These effects, as mentioned are really small, so during calculations I dismissed them and dealt with the purely elastic case as hyperelastic materials.

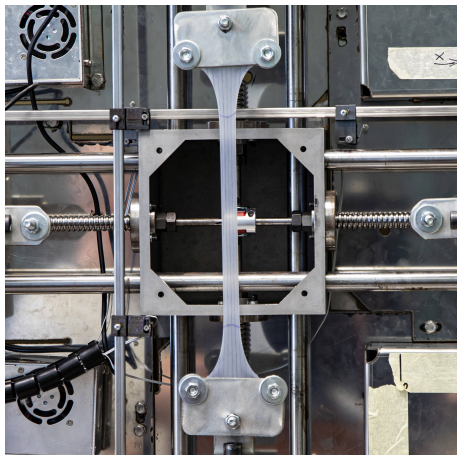


(a) Initial state of uniaxial specimen.

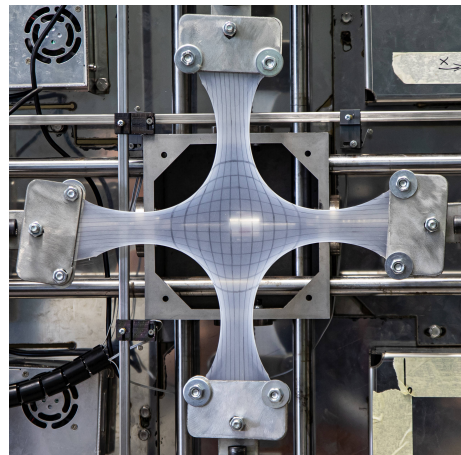


(b) Initial state of biaxial specimen.

Figure 4.20: Initial state.



(a) Final state of uniaxial specimen.



(b) Final state of biaxial specimen.

Figure 4.21: Final state.

Some effects were studied first in order to have a sense of the behaviour of the material for different loading directions and cycles.

First, an anisotropy test was done. this involved two uniaxial specimens, but they were cut out perpendicular to each other from the silicon base. This way we can see if the direction we apply the load onto the specimen has any significant effect. The results are on Figure 4.22.

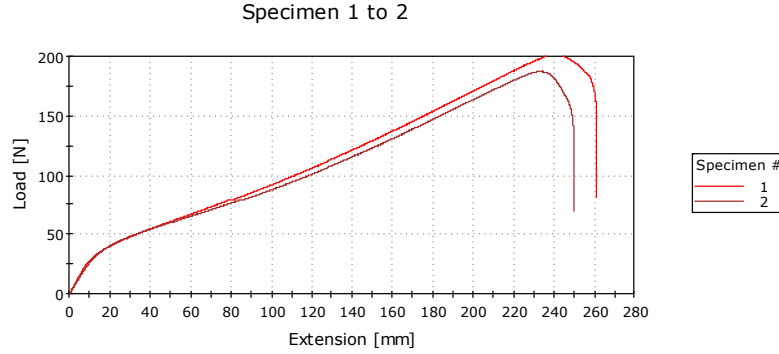


Figure 4.22: Anisotropy measurement of uniaxial test specimen.

We can see that the stresses are in correspondence to each other, the anisotropy is not present in the material, we can neglect the fact which direction we are loading the material.

The other test case was for the Mullins effect (Kossa 2021). This effect is caused by the structural change of a polymer. After applying a certain load the connections between the chains break, and later they are not restored (unless speaking of self regenerative materials), but for larger loads the material behaves like we expect it to, only the used strain region is changed regarding stress response. This overall means that for later loadings the results are weaker compared to the initial one, if the strain value has already been reached before by the specimen.

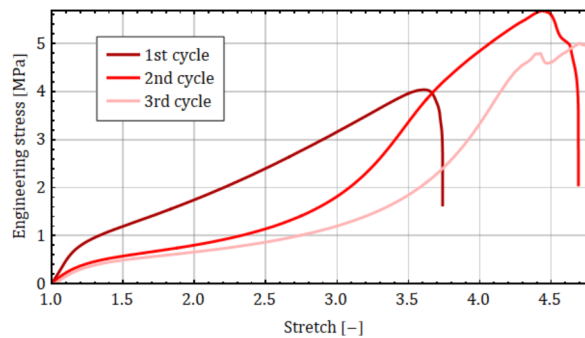


Figure 4.23: The Mullins effect on the test specimen.

In our case the material did show the Mullins effect, so in order to be sure the experiments are successful, one specimen could only be loaded once, and no pretension could be allowed on the material before mounting it onto the machine and testing it. This was ensured throughout our investigations, the specimens were only taken out for measurements and were left undisturbed, also the used ones were marked so they would not be

used again by mistake. The visible proof of the Mullins effect for the measurements can be inspected on Figure 4.23.

During the measurements the time, force and displacement values were sampled with 50000 Hz. On the other hand, the results were sampled down to 100 Hz as the vast amount of data proved irrelevant, fewer were sufficient. Another reason is to reduce the noise in the system, as high sampling frequency results in a noisy signal, which is unfavourable for postprocessing purposes. On a single axis the load cells' results were averaged, this way we only had to work with two force signals. The originally undeformed test specimens at the beginning of the experiment start to deform inhomogeneously, thus we also have to apply a correction for this kind of error.

For this the so-called slack correction was used, as it takes into account the inflexion point at which the transition in deformation takes place. The first point we want to use is the one that is in the region of the inflexion point's tangent line's right hand side domain. The end point is more easily available, it can be said that it is of the highest force value during the measurement. This can only be stated if the experiment is done slowly, so non-elastic effects are not important, which is true in present case.

The slack correction algorithm's most important steps are the following. The inflexion point can be found using numerical differentiations. The tangent line at this point cuts off the unnecessary part of the data, leaving us with the useful part only. The inflexion point can be identified as a change in the second derivative of the signal, and the tangent line was constructed as a chord between the previous and the next points in my calculations. The formulae used in the calculations were the following

$$\ddot{F}(t) = \frac{F^{i+1} - 2F^i + F^{i-1}}{\Delta t^2}, \quad \dot{F}(t) = \frac{F^{i+1} - F^{i-1}}{2\Delta t}. \quad (4.10)$$

We can visualise the slack corrections data modification in Figure 4.24.

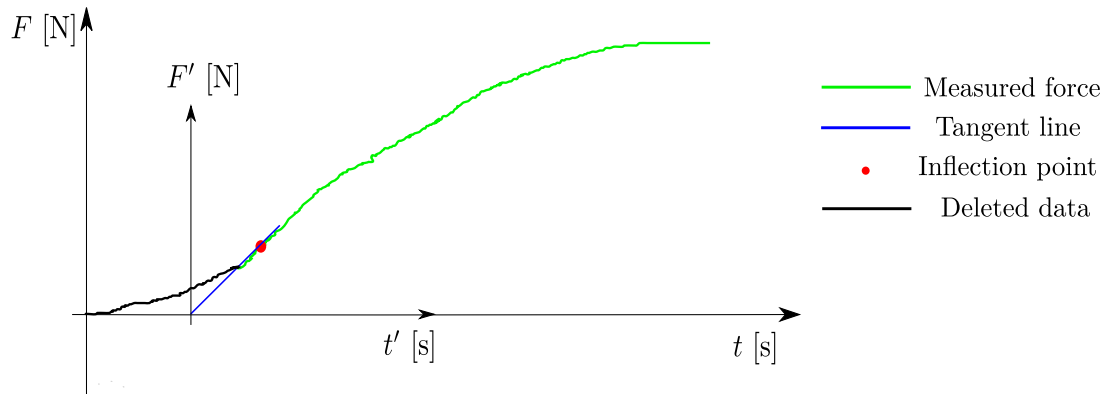


Figure 4.24: The concept of slack correction visualised.

The code used for the slack correction can be written in Mathematica as follows.

```

bit = 0; m = 0; i = 2501;
While[i < Length[F] && bit == 0,
If[(F[[i + 1]] - F[[i]])/2 - (F[[i]] - F[[i - 1]])/2)/ 2 <= 0,
m = (F[[i + 1]] - F[[i - 1]])/2; hatar = i; bit = 1, i++]]
limit = 0; bit = 0; i = hatar - 1;
While[bit == 0 && i > 1, If[F[[hatar]] - m*(hatar - i) <= F[[i]],
limit = i; bit = 1, i-]]

```

In the code the inflection point's position is searched in *hatar* variable from a starting point in the signal, the *limit* variable stores the point that notes the initial point of the useful signal, F being the force signal.

After the measurements one can post-process the data in the useful region, this way we can arrive with data that also carry physically sensible data to our experimental test cases.

During the experiments, at fixed instances photos were taken of the setup, this way we could analyse the displacements more precisely, as the total deformation and the homogeneous one are not the same. We can also find the corresponding force values so the stresses can be calculated based on the initial cross section resulting in the engineering stress. The final result of this is a data set of stress-strain points, to which we can fit our designated models.

From the measurements we can conclude that the homogeneous deformations are really important, so in order to find that we must analyse the experiments in depth. This was the reason the pictures were taken, as they provide the opportunity to do some further calculations.

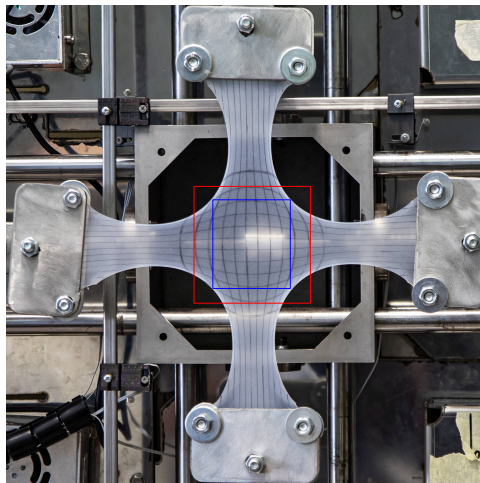


Figure 4.25: The method used to calculate strains in the biaxial case.

The biaxial body had a mesh drawn on it, so the second to outer lines are touching the

complete cross section, meaning the complete force can be measured along them, not just a part of it. Based on this I measured the displacement of these lines, applied an outer and inner box to approximate the real value, and calculated the average of them. This way we have the strain values for a given time instance, doing so for both axes and taking the force values' average as well we could arrive to the final values for our engineering stress-engineering strain curve. A time instance's calculation's visualised on Figure 4.25.

Form the calculations the following stress and deformation measures were calculated like follows

$$P = \frac{F}{A_0}, \quad \lambda = 1 + \frac{u}{L_0}, \quad \varepsilon = \frac{u}{L_0}. \quad (4.11)$$

As mentioned before, the engineering values take into account the initial case's measures, let it be cross section, initial length, the instantaneous values used are the force and homogeneous displacements calculated by the detailed method.

The presented quantities could be measured from the pictures, the displacement values can be read from them, that are the base points of the stains. We can also find the homogeneous area of the specimens, and thus take into account only these for the calculations as well. During these the middle line was used to have unbiased results for both the uni- and biaxial cases. This way we can have a more accurate calculation as well as a standard to which all measurements can be done. The visuals provided can also prove the assumption that the homogeneous case is a good approximation for the experiments. We can also create a strain-time correlation, based upon which the homogeneous strain and stress results could also be checked.

The resultant engineering strain and engineering stress values are visualised on Figure 4.26 for both experiments in a common coordinate system.

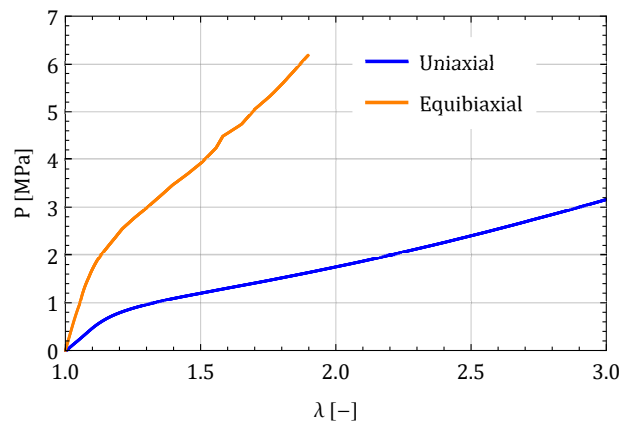


Figure 4.26: The data from the newly done measurements.

This data was used for the following set of fittings.

4.3 Fitting to the new experimental data

For the presented measurement results fittings were done once again for the detailed silicon specimens.

The numerical fitting was done for only two error functions this time, as only two experiments were done, this way the arithmetic mean is built up only from the uniaxial and equibiaxial cases.

4.3.1 Ogden model

The second order Ogden model has the following parameters for the fitting

$$\begin{aligned} C_1 &= 1.41387 \text{ MPa}, & C_2 &= 0.508764, \\ C_3 &= 0.0141873 \text{ MPa}, & C_4 &= 6.25645. \end{aligned} \tag{4.12}$$

The error values are the following

$$Q = 10.1411\%, \quad Q_U = 10.7929\%, \quad Q_B = 9.48923\%. \tag{4.13}$$

The fitted curves are visible on Figures 4.27-4.28.

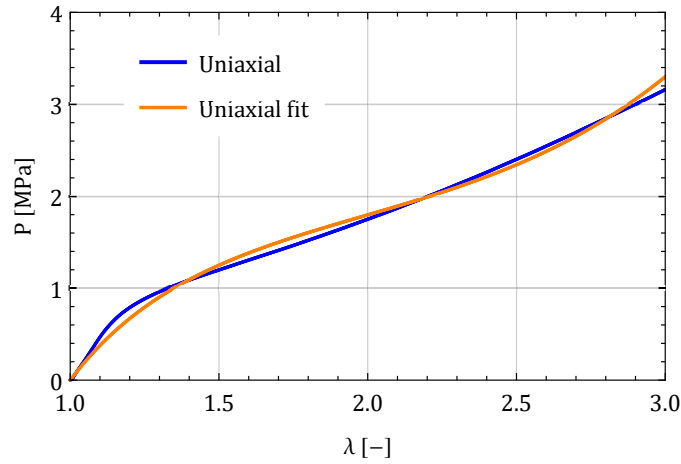


Figure 4.27: Fitting to uniaxial data with the Ogden model of own experimental data.

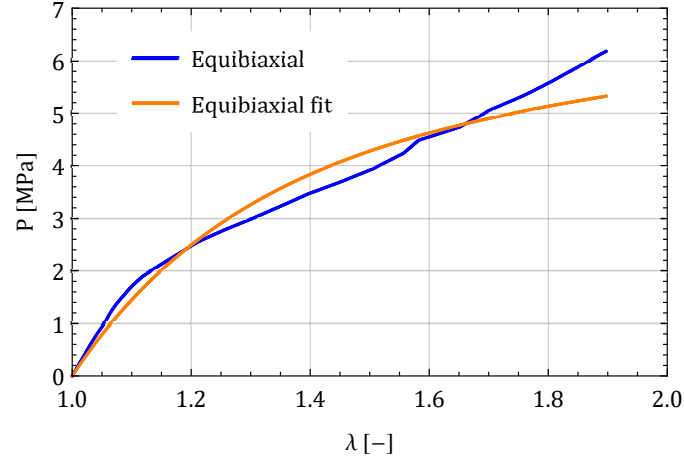


Figure 4.28: Fitting to biaxial data with the Ogden model of own experimental data.

4.3.2 Arruda-Boyce model

With the Arruda-Boyce model we can get the following

$$C_1 = 1.13891 \text{ MPa}, \quad C_2 = -104156. \quad (4.14)$$

$$Q = 39.402\%, \quad Q_U = 15.3803\%, \quad Q_B = 63.4238\%. \quad (4.15)$$

This model can be seen fitted on Figures 4.29-4.30.

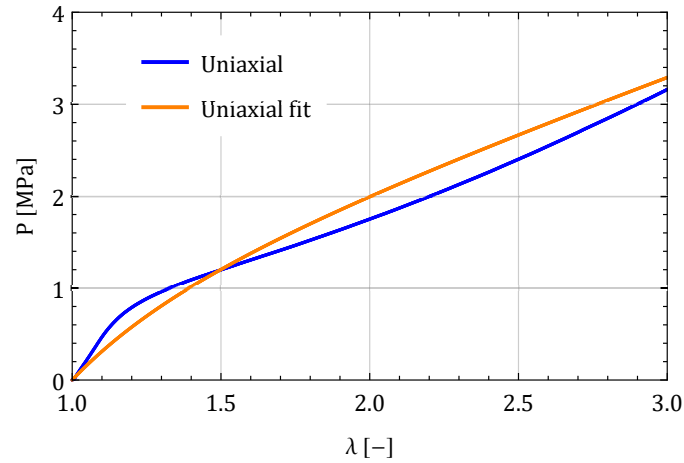


Figure 4.29: Fitting to uniaxial data with the Arruda-Boyce model of own experimental data.

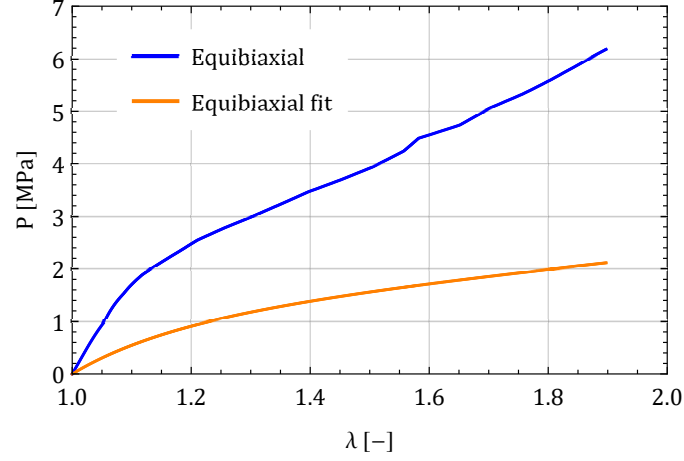


Figure 4.30: Fitting to biaxial data with the Arruda-Boyce model of own experimental data.

4.3.3 Polynomial model

The Polynomial model's fitting is possible as well, this will result in

$$\begin{aligned} C_{10} &= 0.0290109 \text{ MPa}, & C_{01} &= 0.926484 \text{ MPa}, & C_{20} &= 0.0895543 \text{ MPa}, \\ C_{11} &= -0.181826 \text{ MPa}, & C_{02} &= 0.0281267 \text{ MPa}. \end{aligned} \quad (4.16)$$

The error function values for this fitting

$$Q = 16.8864\%, \quad Q_U = 15.0046\%, \quad Q_B = 18.7683\%. \quad (4.17)$$

Plotting the fitted curves onto them the results are on Figures 4.31-4.32.

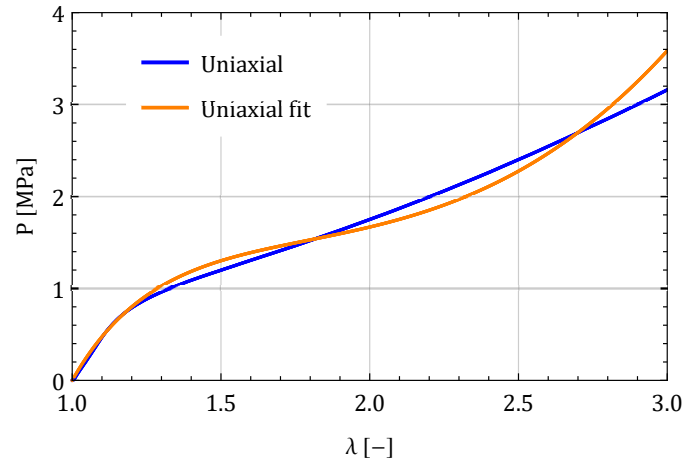


Figure 4.31: Fitting to uniaxial data with the Polynomial model of own experimental data.

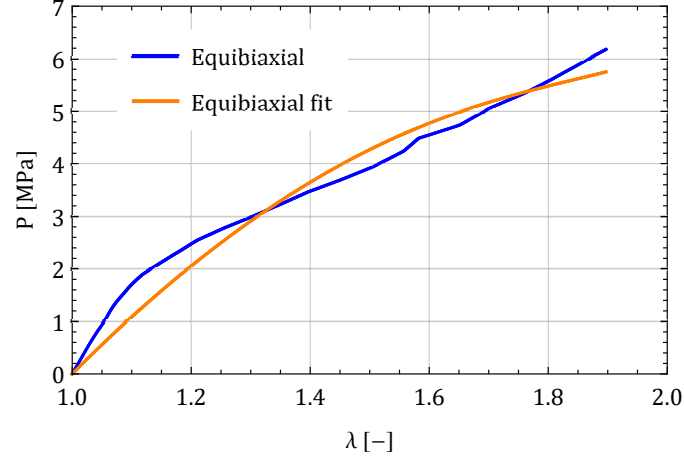


Figure 4.32: Fitting to biaxial data with the Polynomial model of own experimental data.

4.3.4 H-K model

Finally we can also fit our own advance model, this way we can get

$$C_1 = 2.13625 \text{ MPa}, \quad C_2 = -0.38046, \quad C_3 = 0.362662 \text{ MPa}, \quad C_4 = 4.3547. \quad (4.18)$$

The error values in this case are

$$Q = 15.5993\%, \quad Q_U = 14.9227\%, \quad Q_B = 16.2759\%. \quad (4.19)$$

With the fitted parameters we can get the results shown on Figures 4.33-4.34.

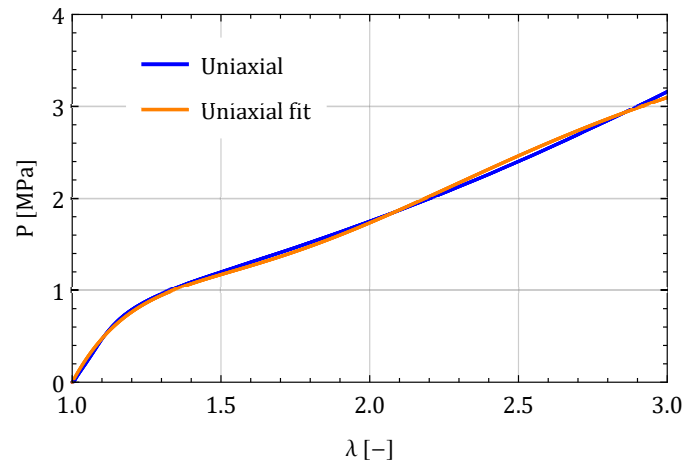


Figure 4.33: Fitting to uniaxial data with the H-K model of own experimental data.

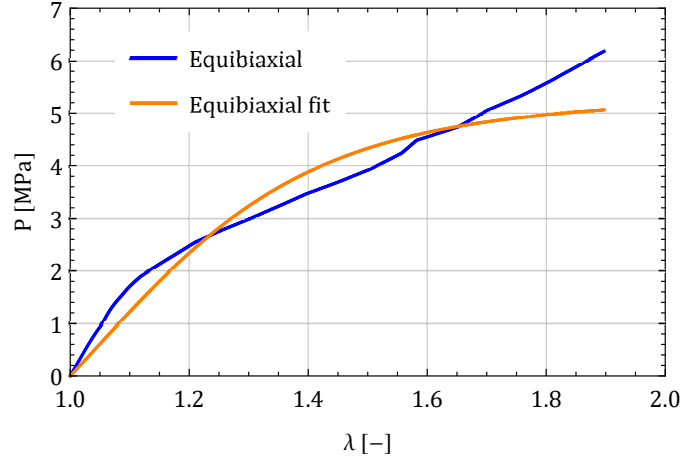


Figure 4.34: Fitting to biaxial data with the H-K model of own experimental data.

5 Abaqus user material subroutines

In order to be able to use advanced material models in finite element analyses, we must also be able to define subroutines that can be implemented into a software. For this we can see some examples in Boulbes’s textbook (Boulbes 2020). This way we can create an extension to the base capabilities of the software and be able to use arbitrary models in our simulations. This can be useful when newly developed models are to be tested and for simulations involving complex material models, let it be a hardening material, a coordinate dependent material model or an arbitrary hyperelastic material model.

5.1 Different implementations

This thesis is built upon the capabilities of ABAQUS CAE, so the material model subroutines can be implemented accordingly. In this software there are multiple ways to implement custom user material models, although they have one common property, they have to be defined in FORTRAN language in order to be executable and compilable by the software.

For compiling the software also needs to have an extension via some code editor and a FORTRAN compiler attached to it, the base software is not capable of doing this.

The first kind of material definition that ABAQUS can use is the UMAT subroutine. This subroutine is the most general, it requires the stress update procedure and the definition of the corresponding tangent matrix. This subroutine can not only define hyperelastic models, but general arbitrary ones, meaning that for any purpose of material modelling is can prove useful. It should be also noted that this one is an implicit method of model implementation.

Another type of material subroutine is the VUMAT subroutine. This differs from the

UMAT by only a little, generally that this method is an explicit method of defining constitutive laws. This way we only need to derive the stress values for the solver, that can be easily done by the calculations shown beforehand. An important aspect is that we must use a compressible simulation case, as it can only work with generally applicable models, thus we must also decouple the deformation gradient in order to be able to calculate the purely isochoric behaviour described by our given hyperelastic model, and the volumetric part is calculated by the extension of the strain energy density function. This subroutine also requires the decomposition of the deformation gradient so the isochoric and volumetric parts of the deformation and energy can be analysed separately, with the according modified ($\bar{}$) principal scalar invariants.

This code, as mentioned requires the stress update step, so the stress values need to be calculated in the subroutine. The code requires a main header that can be accessed from the ABAQUS user manual (Dassault (Accessed December 9th, 2022)).

A VUMAT header provided is the following 5.1.

```

subroutine vumat(
C Read only (unmodifiable) variables -
1  nblock, ndir, nshr, nstatev, nfieldv, nprops, lanneal,
2  stepTime, totalTime, dt, cmname, coordMp, charLength,
3  props, density, strainInc, relSpinInc,
4  tempOld, stretchOld, defgradOld, fieldOld,
5  stressOld, stateOld, enerInternOld, enerInelasOld,
6  tempNew, stretchNew, defgradNew, fieldNew,
C Write only (modifiable) variables -
7  stressNew, stateNew, enerInternNew, enerInelasNew )
C
    include 'vaba_param.inc'
C
    dimension props(nprops), density(nblock), coordMp(nblock,*),
1  charLength(nblock), strainInc(nblock,ndir+nshr),
2  relSpinInc(nblock,nshr), tempOld(nblock),
3  stretchOld(nblock,ndir+nshr),
4  defgradOld(nblock,ndir+nshr+nshr),
5  fieldOld(nblock,nfieldv), stressOld(nblock,ndir+nshr),
6  stateOld(nblock,nstatev), enerInternOld(nblock),
7  enerInelasOld(nblock), tempNew(nblock),
8  stretchNew(nblock,ndir+nshr),
8  defgradNew(nblock,ndir+nshr+nshr),
9  fieldNew(nblock,nfieldv),
1  stressNew(nblock,ndir+nshr), stateNew(nblock,nstatev),
2  enerInternNew(nblock), enerInelasNew(nblock),
C

```

```

      character*80 cmname
C
      do 100 km = 1,nblock
        user coding
100 continue
C
      return
      end

```

Figure 5.1: General VUMAT code structure.

Last, but not least we can also talk about the UHYPER subroutine, which is a dedicated one for hyperelastic material modelling. This is also really useful as for a given strain energy density function we can generate the complete behaviour of the model. The decoupling is handled by the software this way, and both compressible and completely incompressible cases can be defined as well.

The required quantities that need to be defined for the application of such subroutine is the calculation of the strain energy density function's derivatives with respect to the volume ratio and the modified principal scalar invariants. An example can be seen below in Figure 5.2.

```

      SUBROUTINE UHYPER(BI1,BI2,AJ,U,UI1,UI2,UI3,TEMP,NOEL,
1 CMNAME,INCMPLFLAG,NUMSTATEV,STATEV,NUMFIELDV,FIELDV,
2 FIELDVINC,NUMPROPS,PROPS)
C
      INCLUDE 'ABA_PARAM.INC'
C
      CHARACTER*80 CMNAME
      DIMENSION U(2),UI1(3),UI2(6),UI3(6),STATEV(*),FIELDV(*),
2 FIELDVINC(*),PROPS(*)
C
      user coding to define U,UI1,UI2,UI3,STATEV
      RETURN
      END

```

Figure 5.2: General UHYPER code structure.

The required derivatives to be defined are the following

$$\begin{aligned}
& \frac{\partial W}{\partial \bar{I}_1}, \quad \frac{\partial W}{\partial \bar{I}_2}, \quad \frac{\partial W}{\partial J}, \\
& \frac{\partial^2 W}{\partial \bar{I}_1^2}, \quad \frac{\partial^2 W}{\partial \bar{I}_2^2}, \quad \frac{\partial^2 W}{\partial J^2}, \quad \frac{\partial^2 W}{\partial \bar{I}_1 \partial \bar{I}_2}, \quad \frac{\partial^2 W}{\partial \bar{I}_1 \partial J}, \quad \frac{\partial^2 W}{\partial \bar{I}_2 \partial J}, \\
& \frac{\partial^3 W}{\partial \bar{I}_1^2 \partial J}, \quad \frac{\partial^3 W}{\partial \bar{I}_2^2 \partial J}, \quad \frac{\partial^3 W}{\partial \bar{I}_1 \partial \bar{I}_2 \partial J}, \quad \frac{\partial^3 W}{\partial \bar{I}_1 \partial J^2}, \quad \frac{\partial^3 W}{\partial \bar{I}_2 \partial J^2}, \quad \frac{\partial^3 W}{\partial J^3}.
\end{aligned} \tag{5.1}$$

All of these subroutines are capable of defining well working, useful material models for our purpose.

For the implementation, I chose to use two of the above mentioned three, these are the UHYPER and VUMAT subroutines, with the help of these the material models can be successfully implemented with user defined FORTRAN subroutines.

The volumetric strain energy density functions for the compressible codes are defined in the standard form given in equation (3.54).

5.2 The codes developed models for use in ABAQUS CAE

I developed several codes for the five listed advanced models, with these one can implement the given material model as pasting the code into the solver and running it, without any trouble.

For the five selected models chosen for implementation two UHYPER codes and a VUMAT code was developed. This way we have, for all before mentioned models, an incompressible UHYPER, a compressible UHYPER and a compressible VUMAT subroutine. The codes were designed with the aim of defining the necessary quantities shown in the previous section.

The necessary parameters must be given in the software. for the codes to be complete the number of parameters required is the sum of isochoric and compressible material parameters present in the strain energy density function. Generally speaking the required data are the isochoric parameters C_i and the initial Poisson's ratio ν_0 , in this order.

The codes are given in this thesis as well, presented in Figures 5.3-5.17.

5.2.1 Subroutines developed for the Arruda-Boyce model

```

SUBROUTINE UHYPER(BI1,BI2,AJ,U,UI1,UI2,UI3,TEMP,NOEL,CMNAME,
1  INCMPLFLAG,NUMSTATEV,STATEV,NUMFIELDV,
2  FIELDV,FIELDVINC,NUMPROPS,PROPS)

C
  INCLUDE 'ABA_PARAM.INC'

C
  CHARACTER*80 CMNAME
  DIMENSION UI1(3),UI2(6),UI3(6),STATEV(*),FIELDV(*),
1           FIELDVINC(*),PROPS(*)

C
  PARAMETER (ZERO=0.0D0,ONE=1.0D0,TWO=2.0D0,EIGHT=8.0D0,P1=18999750.0D0,
1  P2=5927499.0D0,P3=5530140.0D0,P4=5374215.0D0,P5=5699925.0D0,
2  P6=9499875.0D0,SIX=6.0D0,M1=20.0D0,M2=1050.0D0,M3=11.0D0,
3  M4=7050.0D0,M5=19.0D0,M6=519.0D0,M7=673750.0D0,FIVE=5.0D0)

C
  C1=PROPS(1)
  C2=PROPS(2)

C
  U=C1*((BI1-THREE)/TWO+(BI1**2-THREE**2)/(M1*C2**2)+M3*(BI1**3-THREE**3)
1  /(M2*C2**4)+M5*(BI1**4-THREE**4)/(M4*C2**6)+M6*(BI1**5-THREE**5)
2  /(M7*C2**8))
  UI1(1)=C1*(ONE/TWO+TWO*BI1/(M1*C2**2)+THREE*M3*BI1**2
1  /(M2*C2**4)+TWO**2*M5*BI1**3/(M4*C2**6)+FIVE*M6*BI1**3
2  *TWO**2/(M7*C2**8))
  UI1(2)=ZERO
  UI1(3)=ZERO
  UI2(1)=C1*(TWO/(M1*C2**2)+THREE*TWO*M3*BI1
1  /(M2*C2**4)+TWO**2*THREE*M5*BI1**2/(M4*C2**6)+FIVE*M6*BI1**4
2  /(M7*C2**8))
  UI2(2)=ZERO
  UI2(3)=ZERO
  UI2(4)=ZERO
  UI2(5)=ZERO
  UI2(6)=ZERO
  UI3(1)=ZERO
  UI3(2)=ZERO
  UI3(3)=ZERO
  UI3(4)=ZERO
  UI3(5)=ZERO
  UI3(6)=ZERO
  RETURN
  END

```

Figure 5.3: Fortran subroutine for the Arruda-Boyce model using incompressible UHYPER code.

```

SUBROUTINE UHYPER(BI1,BI2,AJ,U,UI1,UI2,UI3,TEMP,NOEL,CMNAME,
1 INCMPLFLAG,NUMSTATEV,STATEV,NUMFIELDV,
2 FIELDV,FIELDVINC,NUMPROPS,PROPS)

C
  INCLUDE 'ABA_PARAM.INC'
  CHARACTER*80 CMNAME
  DIMENSION UI1(3),UI2(6),UI3(6),STATEV(*),FIELDV(*),
1           FIELDVINC(*),PROPS(*)

C
  PARAMETER (ZERO=0.0D0,ONE=1.0D0,TWO=2.0D0,EIGHT=8.0D0,P1=18999750.0D0,
1 P2=5927499.0D0,P3=5530140.0D0,P4=5374215.0D0,P5=5699925.0D0,
2 P6=9499875.0D0,SIX=6.0D0,M1=20.0D0,M2=1050.0D0,M3=11.0D0,
3 M4=7050.0D0,M5=19.0D0,M6=519.0D0,M7=673750.0D0,FIVE=5.0D0)

C
  C1=PROPS(1)
  C2=PROPS(2)

C
  D1=P1*C2**8*(1-2*PROPS(3))/(C1*(P2+P3*C2**2+P4*C2**4+P5*C2**6+P6
1 *C2**8))
  U=C1*((BI1-THREE)/TWO+(BI1**2-THREE**2)/(M1*C2**2)+M3*(BI1**3-THREE**3)
1 /(M2*C2**4)+M5*(BI1**4-THREE**4)/(M4*C2**6)+M6*(BI1**5-THREE**5)
2 /(M7*C2**8))+((AJ-ONE)**2)/D1
  UI1(1)=C1*(ONE/TWO+TWO*BI1/(M1*C2**2)+THREE*M3*BI1**2
1 /(M2*C2**4)+TWO**2*M5*BI1**3/(M4*C2**6)+FIVE*M6*BI1**3
2 *TWO**2/(M7*C2**8))
  UI1(2)=ZERO
  UI1(3)=TWO*(AJ-ONE)/D1
  UI2(1)=C1*(TWO/(M1*C2**2)+THREE*TWO*M3*BI1
1 /(M2*C2**4)+TWO**2*THREE*M5*BI1**2/(M4*C2**6)+FIVE*M6*BI1**4
2 /(M7*C2**8))
  UI2(2)=ZERO
  UI2(3)=TWO/D1
  UI2(4)=ZERO
  UI2(5)=ZERO
  UI2(6)=ZERO
  UI3(1)=ZERO
  UI3(2)=ZERO
  UI3(3)=ZERO
  UI3(4)=ZERO
  UI3(5)=ZERO
  UI3(6)=ZERO
  RETURN
  END

```

Figure 5.4: Fortran subroutine for the Arruda-Boyce model using compressible UHYPER code

```

SUBROUTINE VUMAT(NBLOCK, NDI, NSHR, NSTATEV, NFIELDV, NPROPS,
1 LANNEAL, STEPTIME, TOTTIME, DT, CMNAME, COORDMP,
2 CHARLEN, PROPS, DENSITY, DSTRAIN, RSPININC, TEMPO,
3 UO, FO, FIELD0, STRESSVECO, STATE0,
4 INTENEO, INELAENO, TEMP1, U1,
5 F1, FIELD1, STRESSVEC1, STATE1, INTENE1, INELAEN1)
IMPLICIT NONE
INTEGER NBLOCK, NDI, NSHR, NSTATEV, NFIELDV, NPROPS, LANNEAL
REAL STEPTIME, TOTTIME, DT
CHARACTER*80 CMNAME
PARAMETER (ONE=1.0D0, HALF=0.5D0, TENTH=0.1D0, TWO=2.0D0,
1 THREE=3.0D0, ELE=11.0D0, THH=350.0D0, TEI=38.0D0,
2 THT=3525.0D0, FIH=519.0D0, HTH=134750.0D0)
REAL COORDMP(NBLOCK,*)
REAL CHARLEN, PROPS(NPROPS), DENSITY(NBLOCK),
1 DSTRAIN(NBLOCK,NDI+NSHR), RSPININC(NBLOCK,NSHR),
2 TEMPO(NBLOCK), UO(NBLOCK,NDI+NSHR),
3 FO(NBLOCK,NDI+NSHR+NSHR), FIELD0(NBLOCK,NFIELDV),
4 STRESSVECO(NBLOCK,NDI+NSHR), STATE0(NBLOCK,NSTATEV),
5 INTENEO(NBLOCK), INELAENO(NBLOCK), TEMP1(NBLOCK),
6 U1(NBLOCK,NDI+NSHR), F1(NBLOCK,NDI+NSHR+NSHR),
7 FIELD1(NBLOCK,NFIELDV), STRESSVEC1(NBLOCK,NDI+NSHR),
8 STATE1(NBLOCK,NSTATEV), INTENE1(NBLOCK), INELAEN1(NBLOCK)
C LOCAL VARIABLES
REAL F(3,3), B(3,3), J, T1, T2, T3, C1,
1 C2, D1, I1, I2, I3
INTEGER I
C1 =PROPS(1)
C2 =PROPS(2)
D1 =P1*C2**8*(1-2*PROPS(3))/(C1*(P2+P3*C2**2+P4*C2**4+P5*C2**6+P6
1 *C2**8))
C LOOP THROUGH ALL BLOCKS
DO I = 1, NBLOCK
C SETUP F (UPPER DIAGONAL PART)
F(1,1) = U1(I,1)
F(2,2) = U1(I,2)
F(3,3) = U1(I,3)
F(1,2) = U1(I,4)

```

```

      IF (NSHR .EQ. 1) THEN
        F(2,3) = 0.0
        F(1,3) = 0.0
      ELSE
        F(2,3) = U1(I,5)
        F(1,3) = U1(I,6)
      END IF
C  CALCULATE J
      T1 = F(1,1) * (F(2,2)*F(3,3) - F(2,3)**2)
      T2 = F(1,2) * (F(2,3)*F(1,3) - F(1,2)*F(3,3))
      T3 = F(1,3) * (F(1,2)*F(2,3) - F(2,2)*F(1,3))
      J = T1 + T2 + T3
      T1 = J**(-2.0/3.0)
      B(1,1) = T1*(F(1,1)**2 + F(1,2)**2 + F(1,3)**2)
      B(2,2) = T1*(F(1,2)**2 + F(2,2)**2 + F(2,3)**2)
      B(3,3) = T1*(F(1,3)**2 + F(2,3)**2 + F(3,3)**2)
      B(1,2) = T1*(F(1,1)*F(1,2)+F(1,2)*F(2,2)+F(1,3)*F(2,3))
      B(2,3) = T1*(F(1,2)*F(1,3)+F(2,2)*F(2,3)+F(2,3)*F(3,3))
      B(1,3) = T1*(F(1,1)*F(1,3)+F(1,2)*F(2,3)+F(1,3)*F(3,3))
C
      I1=(B(1,1) + B(2,2) + B(3,3))
      I2=B(1,1)*B(2,2)+B(2,2)*B(3,3)+B(1,1)*B(3,3)
1  -B(1,2)**2-B(2,3)**2-B(1,3)**2
C
      T1 = B(1,1) * (B(2,2)*B(3,3) - B(2,3)**2)
      T2 = B(1,2) * (B(2,3)*B(1,3) - B(1,2)*B(3,3))
      T3 = B(1,3) * (B(1,2)*B(2,3) - B(2,2)*B(1,3))
      I3=T1+T2+T3
      T1 = (B(1,1) + B(2,2) + B(3,3)) / 3.0
      T2 = TWO*C1/J*(HALF+I1*TENTH/C2**2+I1**2*ELE/THH/C2**4
1  +I1**3*TEI/THH/C2**6+I1**4*FIH/HTH/C2**8)
      T3 = (J-1)*TWO/D1
      STRESSVEC1(I,1) = T2 * (B(1,1) - T1) + T3
      STRESSVEC1(I,2) = T2 * (B(2,2) - T1) + T3
      STRESSVEC1(I,3) = T2 * (B(3,3) - T1) + T3
      STRESSVEC1(I,4) = T2 * B(1,2)
      IF (NSHR .EQ. 3) THEN
        STRESSVEC1(I,5) = T2 * B(2,3)
        STRESSVEC1(I,6) = T2 * B(1,3)
      END IF
      END DO
      RETURN

```

Figure 5.5: Fortran subroutine for the Arruda-Boyce model using VUMAT code.

5.2.2 Subroutines developed for the Polynomial model

```

SUBROUTINE UHYPER(BI1,BI2,AJ,U,UI1,UI2,UI3,TEMP,NOEL,CMNAME,
1 INCOMPFLAG,NUMSTATEV,STATEV,NUMFIELDV,
2 FIELDV,FIELDVINC,NUMPROPS,PROPS)
C
  INCLUDE 'ABA_PARAM.INC'
C
  CHARACTER*80 CMNAME
  DIMENSION UI1(3),UI2(6),UI3(6),STATEV(*),FIELDV(*),
1           FIELDVINC(*),PROPS(*)
C
  PARAMETER (ZERO=0.0D0,ONE=1.0D0,TWO=2.0D0,THREE=3.0D0)
C
  C1=PROPS(1)
  C2=PROPS(2)
  C3=PROPS(3)
  C4=PROPS(4)
  C5=PROPS(5)
C
  U=C1*(BI1-3)+C2*(BI2-3)+C3*(BI1-3)**2+
1 C4*(BI1-3)*(BI2-3)+C5*(BI2-3)**2
  UI1(1)=C1+TWO*C3*(BI1-3)+C4*(BI2-3)
  UI1(2)=C2+TWO*C5*(BI2-3)+C4*(BI1-3)
  UI1(3)=ZERO
  UI2(1)=TWO*C3
  UI2(2)=TWO*C5
  UI2(3)=ZERO
  UI2(4)=C4
  UI2(5)=ZERO
  UI2(6)=ZERO
  UI3(1)=ZERO
  UI3(2)=ZERO
  UI3(3)=ZERO
  UI3(4)=ZERO
  UI3(5)=ZERO
  UI3(6)=ZERO
  RETURN
END

```

Figure 5.6: Fortran subroutine for the Polynomial model using incompressible UHYPER code.

```

SUBROUTINE UHYPER(BI1,BI2,AJ,U,UI1,UI2,UI3,TEMP,NOEL,CMNAME,
1 INCOMPFLAG,NUMSTATEV,STATEV,NUMFIELDV,
2 FIELDV,FIELDVINC,NUMPROPS,PROPS)

```

C

```

INCLUDE 'ABA_PARAM.INC'

```

C

```

CHARACTER*80 CMNAME

```

```

DIMENSION UI1(3),UI2(6),UI3(6),STATEV(*),FIELDV(*),

```

```

1 FIELDVINC(*),PROPS(*)

```

C

```

PARAMETER (ZERO=0.0D0,ONE=1.0D0,TWO=2.0D0,THREE=3.0D0)

```

C

```

C1=PROPS(1)

```

```

C2=PROPS(2)

```

```

C3=PROPS(3)

```

```

C4=PROPS(4)

```

```

C5=PROPS(5)

```

```

D1=(ONE-TWO*PROPS(6))/(C1+C2)

```

C

```

U=C1*(BI1-3)+C2*(BI2-3)+C3*(BI1-3)**2+

```

```

1 C4*(BI1-3)*(BI2-3)+C5*(BI2-3)**2

```

```

UI1(1)=C1+TWO*C3*(BI1-3)+C4*(BI2-3)

```

```

UI1(2)=C2+TWO*C5*(BI2-3)+C4*(BI1-3)

```

```

UI1(3)=TWO*(AJ-ONE)/D1

```

```

UI2(1)=TWO*C3

```

```

UI2(2)=TWO*C5

```

```

UI2(3)=TWO/D1

```

```

UI2(4)=C4

```

```

UI2(5)=ZERO

```

```

UI2(6)=ZERO

```

```

UI3(1)=ZERO

```

```

UI3(2)=ZERO

```

```

UI3(3)=ZERO

```

```

UI3(4)=ZERO

```

```

UI3(5)=ZERO

```

```

UI3(6)=ZERO

```

```

RETURN

```

```

END

```

Figure 5.7: Fortran subroutine for the Polynomial model using compressible UHYPER code

```

SUBROUTINE VUMAT(NBLOCK, NDI, NSHR, NSTATEV, NFIELDV, NPROPS,
1 LANNEAL, STEPTIME, TOTTIME, DT, CMNAME, COORDMP,
2 CHARLEN, PROPS, DENSITY, DSTRAIN, RSPININC, TEMPO,
3 UO, FO, FIELD0, STRESSVECO, STATE0,
4 INTENEO, INELAENO, TEMP1, U1,
5 F1, FIELD1, STRESSVEC1, STATE1, INTENE1, INELAEN1)
IMPLICIT NONE
INTEGER NBLOCK, NDI, NSHR, NSTATEV, NFIELDV, NPROPS, LANNEAL
REAL STEPTIME, TOTTIME, DT
CHARACTER*80 CMNAME
PARAMETER (ZERO=0.0D0, ONE=0.1D0, TWO=2.0D0, THREE=3.0D0)
REAL COORDMP(NBLOCK,*)
REAL CHARLEN, PROPS(NPROPS), DENSITY(NBLOCK),
1 DSTRAIN(NBLOCK,NDI+NSHR), RSPININC(NBLOCK,NSHR),
2 TEMPO(NBLOCK), UO(NBLOCK,NDI+NSHR),
3 FO(NBLOCK,NDI+NSHR+NSHR), FIELD0(NBLOCK,NFIELDV),
4 STRESSVECO(NBLOCK,NDI+NSHR), STATE0(NBLOCK,NSTATEV),
5 INTENEO(NBLOCK), INELAENO(NBLOCK), TEMP1(NBLOCK),
6 U1(NBLOCK,NDI+NSHR), F1(NBLOCK,NDI+NSHR+NSHR),
7 FIELD1(NBLOCK,NFIELDV), STRESSVEC1(NBLOCK,NDI+NSHR),
8 STATE1(NBLOCK,NSTATEV), INTENE1(NBLOCK), INELAEN1(NBLOCK)
C LOCAL VARIABLES
REAL F(3,3), B(3,3), B2(3,3), J, T1, T2, T3, T4, T5, C1,
1 C2, C3, C4, C5, D1, I1, I2, I3
INTEGER I
C1 =PROPS(1)
C2 =PROPS(2)
C3 =PROPS(3)
C4 =PROPS(4)
C5 =PROPS(5)
D1 =(ONE-TWO*PROPS(6))/(C1+C2)
C LOOP THROUGH ALL BLOCKS
DO I = 1, NBLOCK
C SETUP F (UPPER DIAGONAL PART)
F(1,1) = U1(I,1)
F(2,2) = U1(I,2)
F(3,3) = U1(I,3)
F(1,2) = U1(I,4)
IF (NSHR .EQ. 1) THEN
F(2,3) = 0.0
F(1,3) = 0.0
ELSE
F(2,3) = U1(I,5)
F(1,3) = U1(I,6)
END IF

```

C CALCULATE J

```

T1 = F(1,1) * (F(2,2)*F(3,3) - F(2,3)**2)
T2 = F(1,2) * (F(2,3)*F(1,3) - F(1,2)*F(3,3))
T3 = F(1,3) * (F(1,2)*F(2,3) - F(2,2)*F(1,3))
J = T1 + T2 + T3
T1 = J**(-2.0/3.0)
B(1,1) = T1*(F(1,1)**2 + F(1,2)**2 + F(1,3)**2)
B(2,2) = T1*(F(1,2)**2 + F(2,2)**2 + F(2,3)**2)
B(3,3) = T1*(F(1,3)**2 + F(2,3)**2 + F(3,3)**2)
B(1,2) = T1*(F(1,1)*F(1,2)+F(1,2)*F(2,2)+F(1,3)*F(2,3))
B(2,3) = T1*(F(1,2)*F(1,3)+F(2,2)*F(2,3)+F(2,3)*F(3,3))
B(1,3) = T1*(F(1,1)*F(1,3)+F(1,2)*F(2,3)+F(1,3)*F(3,3))

```

C

```

B2(1,1) = (B(1,1)**2 + B(1,2)**2 + B(1,3)**2)
B2(2,2) = (B(1,2)**2 + B(2,2)**2 + B(2,3)**2)
B2(3,3) = (B(1,3)**2 + B(2,3)**2 + B(3,3)**2)
B2(1,2) = (B(1,1)*B(1,2)+B(1,2)*B(2,2)+B(1,3)*B(2,3))
B2(2,3) = (B(1,2)*B(1,3)+B(2,2)*B(2,3)+B(2,3)*B(3,3))
B2(1,3) = (B(1,1)*B(1,3)+B(1,2)*B(2,3)+B(1,3)*B(3,3))

```

C

```

I1=(B(1,1) + B(2,2) + B(3,3))
I2=B(1,1)*B(2,2)+B(2,2)*B(3,3)+B(1,1)*B(3,3)
1 -B(1,2)**2-B(2,3)**2-B(1,3)**2

```

C

```

T1 = B(1,1) * (B(2,2)*B(3,3) - B(2,3)**2)
T2 = B(1,2) * (B(2,3)*B(1,3) - B(1,2)*B(3,3))
T3 = B(1,3) * (B(1,2)*B(2,3) - B(2,2)*B(1,3))
I3=T1+T2+T3
T1 = (B(1,1) + B(2,2) + B(3,3)) / 3.0
T2 = TWO/J*(C1+TWO*C3*(I1-THREE)+I1*(C2+C4*(I1-THREE)
1 +TWO*C5*(I2-THREE))+C4*(I2-THREE))
T3 = (J-1)*TWO/D1
T4=TWO/J*(-C2-C4*(I1-THREE)-TWO*C5*(I2-THREE))
T5=(B2(1,1) + B2(2,2) + B2(3,3)) / 3.0
STRESSVEC1(I,1) = T2 * (B(1,1) - T1) + T3+T4*(B2(1,1)-T5)
STRESSVEC1(I,2) = T2 * (B(2,2) - T1) + T3+T4*(B2(2,2)-T5)
STRESSVEC1(I,3) = T2 * (B(3,3) - T1) + T3+T4*(B2(3,3)-T5)
STRESSVEC1(I,4) = T2 * B(1,2)+T4*B2(1,2)
IF (NSHR .EQ. 3) THEN
STRESSVEC1(I,5) = T2 * B(2,3)+T4*B2(2,3)
STRESSVEC1(I,6) = T2 * B(1,3)+T4*B2(1,3)
END IF
END DO
RETURN

```

Figure 5.8: Fortran subroutine for the Polynomial model using VUMAT code.

5.2.3 Subroutines developed for the Gent-Thomas model

```

SUBROUTINE UHYPER(BI1,BI2,AJ,U,UI1,UI2,UI3,TEMP,NOEL,CMNAME,
1 INCOMPFLAG,NUMSTATEV,STATEV,NUMFIELDV,
2 FIELDV,FIELDVINC,NUMPROPS,PROPS)
C
  INCLUDE 'ABA_PARAM.INC'
C
  CHARACTER*80 CMNAME
  DIMENSION UI1(3),UI2(6),UI3(6),STATEV(*),FIELDV(*),
1           FIELDVINC(*),PROPS(*)
C
  PARAMETER (ZERO=0.0D0,ONE=1.0D0,TWO=2.0D0,THREE=3.0D0)
C
  C1=PROPS(1)
  C2=PROPS(2)
C
  U=C1*(BI1-THREE)+C2*LOG(BI2/THREE)
  UI1(1)=C1
  UI1(2)=C2/BI2
  UI1(3)=ZERO
  UI2(1)=ZERO
  UI2(2)=-C2/BI2/BI2
  UI2(3)=ZERO
  UI2(4)=ZERO
  UI2(5)=ZERO
  UI2(6)=ZERO
  UI3(1)=ZERO
  UI3(2)=ZERO
  UI3(3)=ZERO
  UI3(4)=ZERO
  UI3(5)=ZERO
  UI3(6)=ZERO
  RETURN
END

```

Figure 5.9: Fortran subroutine for the Gent-Thomas model using incompressible UHYPER code.

```

SUBROUTINE UHYPER(BI1,BI2,AJ,U,UI1,UI2,UI3,TEMP,NOEL,CMNAME,
1 INCMPLFLAG,NUMSTATEV,STATEV,NUMFIELDV,
2 FIELDV,FIELDVINC,NUMPROPS,PROPS)

```

C

```

INCLUDE 'ABA_PARAM.INC'

```

C

```

CHARACTER*80 CMNAME

```

```

DIMENSION UI1(3),UI2(6),UI3(6),STATEV(*),FIELDV(*),

```

```

1 FIELDVINC(*),PROPS(*)

```

C

```

PARAMETER (ZERO=0.0D0,ONE=1.0D0, TWO=2.0D0, THREE=3.0D0)

```

C

```

C1=PROPS(1)

```

```

C2=PROPS(2)

```

```

D1=THREE*(ONE-TWO*PROPS(3))/(THREE*C1+C2)

```

C

```

U=C1*(BI1-THREE)+C2*LOG(BI2/THREE)+((AJ-ONE)**2)/D1

```

```

UI1(1)=C1

```

```

UI1(2)=C2/BI2

```

```

UI1(3)=TWO*(AJ-ONE)/D1

```

```

UI2(1)=ZERO

```

```

UI2(2)=-C2/BI2/BI2

```

```

UI2(3)=TWO/D1

```

```

UI2(4)=ZERO

```

```

UI2(5)=ZERO

```

```

UI2(6)=ZERO

```

```

UI3(1)=ZERO

```

```

UI3(2)=ZERO

```

```

UI3(3)=ZERO

```

```

UI3(4)=ZERO

```

```

UI3(5)=ZERO

```

```

UI3(6)=ZERO

```

```

RETURN

```

```

END

```

Figure 5.10: Fortran subroutine for the Gent-Thomas model using compressible UHYPER code

```

SUBROUTINE VUMAT(NBLOCK, NDI, NSHR, NSTATEV, NFIELDV, NPROPS,
1 LANNEAL, STEPTIME, TOTTIME, DT, CMNAME, COORDMP,
2 CHARLEN, PROPS, DENSITY, DSTRAIN, RSPININC, TEMPO,
3 UO, FO, FIELD0, STRESSVECO, STATEO,
4 INTENEO, INELAENO, TEMP1, U1,
5 F1, FIELD1, STRESSVEC1, STATE1, INTENE1, INELAEN1)
IMPLICIT NONE
INTEGER NBLOCK, NDI, NSHR, NSTATEV, NFIELDV, NPROPS, LANNEAL
REAL STEPTIME, TOTTIME, DT
CHARACTER*80 CMNAME
PARAMETER (ZERO=0.0D0, ONE=0.1D0, TWO=2.0D0, THREE=3.0D0)
REAL COORDMP(NBLOCK,*)
REAL CHARLEN, PROPS(NPROPS), DENSITY(NBLOCK),
1 DSTRAIN(NBLOCK,NDI+NSHR), RSPININC(NBLOCK,NSHR),
2 TEMPO(NBLOCK), UO(NBLOCK,NDI+NSHR),
3 FO(NBLOCK,NDI+NSHR+NSHR), FIELD0(NBLOCK,NFIELDV),
4 STRESSVECO(NBLOCK,NDI+NSHR), STATEO(NBLOCK,NSTATEV),
5 INTENEO(NBLOCK), INELAENO(NBLOCK), TEMP1(NBLOCK),
6 U1(NBLOCK,NDI+NSHR), F1(NBLOCK,NDI+NSHR+NSHR),
7 FIELD1(NBLOCK,NFIELDV), STRESSVEC1(NBLOCK,NDI+NSHR),
8 STATE1(NBLOCK,NSTATEV), INTENE1(NBLOCK), INELAEN1(NBLOCK)
C LOCAL VARIABLES
REAL F(3,3), B(3,3), B2(3,3), J, T1, T2, T3, T4, T5, C1,
1 C2, D1, I1, I2, I3
INTEGER I
C1 =PROPS(1)
C2 =PROPS(2)
D1 =THREE*(ONE-TWO*PROPS(3))/(THREE*C1+C2)
C LOOP THROUGH ALL BLOCKS
DO I = 1, NBLOCK
C SETUP F (UPPER DIAGONAL PART)
F(1,1) = U1(I,1)
F(2,2) = U1(I,2)
F(3,3) = U1(I,3)
F(1,2) = U1(I,4)
IF (NSHR .EQ. 1) THEN
F(2,3) = 0.0
F(1,3) = 0.0
ELSE
F(2,3) = U1(I,5)
F(1,3) = U1(I,6)
END IF

```

C CALCULATE J

```
T1 = F(1,1) * (F(2,2)*F(3,3) - F(2,3)**2)
T2 = F(1,2) * (F(2,3)*F(1,3) - F(1,2)*F(3,3))
T3 = F(1,3) * (F(1,2)*F(2,3) - F(2,2)*F(1,3))
J = T1 + T2 + T3
T1 = J**(-2.0/3.0)
B(1,1) = T1*(F(1,1)**2 + F(1,2)**2 + F(1,3)**2)
B(2,2) = T1*(F(1,2)**2 + F(2,2)**2 + F(2,3)**2)
B(3,3) = T1*(F(1,3)**2 + F(2,3)**2 + F(3,3)**2)
B(1,2) = T1*(F(1,1)*F(1,2)+F(1,2)*F(2,2)+F(1,3)*F(2,3))
B(2,3) = T1*(F(1,2)*F(1,3)+F(2,2)*F(2,3)+F(2,3)*F(3,3))
B(1,3) = T1*(F(1,1)*F(1,3)+F(1,2)*F(2,3)+F(1,3)*F(3,3))
```

C

```
B2(1,1) = (B(1,1)**2 + B(1,2)**2 + B(1,3)**2)
B2(2,2) = (B(1,2)**2 + B(2,2)**2 + B(2,3)**2)
B2(3,3) = (B(1,3)**2 + B(2,3)**2 + B(3,3)**2)
B2(1,2) = (B(1,1)*B(1,2)+B(1,2)*B(2,2)+B(1,3)*B(2,3))
B2(2,3) = (B(1,2)*B(1,3)+B(2,2)*B(2,3)+B(2,3)*B(3,3))
B2(1,3) = (B(1,1)*B(1,3)+B(1,2)*B(2,3)+B(1,3)*B(3,3))
```

C

```
I1=(B(1,1) + B(2,2) + B(3,3))
I2=B(1,1)*B(2,2)+B(2,2)*B(3,3)+B(1,1)*B(3,3)
1 -B(1,2)**2-B(2,3)**2-B(1,3)**2
```

C

```
T1 = B(1,1) * (B(2,2)*B(3,3) - B(2,3)**2)
T2 = B(1,2) * (B(2,3)*B(1,3) - B(1,2)*B(3,3))
T3 = B(1,3) * (B(1,2)*B(2,3) - B(2,2)*B(1,3))
I3=T1+T2+T3
T1 = (B(1,1) + B(2,2) + B(3,3)) / 3.0
T2 = TWO/J*(C1+C2*I1/I2)
T3 = (J-1)*TWO/D1
T4=-TWO/J*C2/I2
T5=(B2(1,1) + B2(2,2) + B2(3,3)) / 3.0
STRESSVEC1(I,1) = T2 * (B(1,1) - T1) + T3+T4*(B2(1,1)-T5)
STRESSVEC1(I,2) = T2 * (B(2,2) - T1) + T3+T4*(B2(2,2)-T5)
STRESSVEC1(I,3) = T2 * (B(3,3) - T1) + T3+T4*(B2(3,3)-T5)
STRESSVEC1(I,4) = T2 * B(1,2)+T4*B2(1,2)
IF (NSHR .EQ. 3) THEN
STRESSVEC1(I,5) = T2 * B(2,3)+T4*B2(2,3)
STRESSVEC1(I,6) = T2 * B(1,3)+T4*B2(1,3)
END IF
END DO
RETURN
```

Figure 5.11: Fortran subroutine for the Gent-Thomas model using VUMAT code.

5.2.4 Subroutines developed for the Humphrey-Yin model

```

SUBROUTINE UHYPER(BI1,BI2,AJ,U,UI1,UI2,UI3,TEMP,NOEL,CMNAME,
1 INCOMPFLAG,NUMSTATEV,STATEV,NUMFIELDV,
2 FIELDV,FIELDVINC,NUMPROPS,PROPS)
C
  INCLUDE 'ABA_PARAM.INC'
C
  CHARACTER*80 CMNAME
  DIMENSION UI1(3),UI2(6),UI3(6),STATEV(*),FIELDV(*),
1           FIELDVINC(*),PROPS(*)
C
  PARAMETER (ZERO=0.0D0,ONE=1.0D0,TWO=2.0D0,THREE=3.0D0)
C
  C1=PROPS(1)
  C2=PROPS(2)
C
  U=C1*(EXP(C2*(BI1-THREE))-ONE)
  UI1(1)=C1*C2*EXP(C2*(BI1-THREE))
  UI1(2)=ZERO
  UI1(3)=ZERO
  UI2(1)=C1*C2*C2*EXP(C2*(BI1-THREE))
  UI2(2)=ZERO
  UI2(3)=ZERO
  UI2(4)=ZERO
  UI2(5)=ZERO
  UI2(6)=ZERO
  UI3(1)=ZERO
  UI3(2)=ZERO
  UI3(3)=ZERO
  UI3(4)=ZERO
  UI3(5)=ZERO
  UI3(6)=ZERO
  RETURN
END

```

Figure 5.12: Fortran subroutine for the Humphrey-Yin model using incompressible UHYPER code.

```

SUBROUTINE UHYPER(BI1,BI2,AJ,U,UI1,UI2,UI3,TEMP,NOEL,CMNAME,
1 INCMPLFLAG,NUMSTATEV,STATEV,NUMFIELDV,
2 FIELDV,FIELDVINC,NUMPROPS,PROPS)
C
  INCLUDE 'ABA_PARAM.INC'
C
  CHARACTER*80 CMNAME
  DIMENSION UI1(3),UI2(6),UI3(6),STATEV(*),FIELDV(*),
1           FIELDVINC(*),PROPS(*)
C
  PARAMETER (ZERO=0.0D0,ONE=1.0D0, TWO=2.0D0, THREE=3.0D0)
C
  C1=PROPS(1)
  C2=PROPS(2)
  D1=(ONE-TWO*PROPS(3))/(C1*C2)
C
  U=C1*(EXP(C2*(BI1-THREE))-ONE)+((AJ-ONE)**2)/D1
  UI1(1)=C1*C2*EXP(C2*(BI1-THREE))
  UI1(2)=ZERO
  UI1(3)=TWO*(AJ-ONE)/D1
  UI2(1)=C1*C2*C2*EXP(C2*(BI1-THREE))
  UI2(2)=ZERO
  UI2(3)=TWO/D1
  UI2(4)=ZERO
  UI2(5)=ZERO
  UI2(6)=ZERO
  UI3(1)=ZERO
  UI3(2)=ZERO
  UI3(3)=ZERO
  UI3(4)=ZERO
  UI3(5)=ZERO
  UI3(6)=ZERO
  RETURN
END

```

Figure 5.13: Fortran subroutine for the Humphrey-Yin model using compressible UHYPER code

```

SUBROUTINE VUMAT(NBLOCK, NDI, NSHR, NSTATEV, NFIELDV, NPROPS,
1 LANNEAL, STEPTIME, TOTTIME, DT, CMNAME, COORDMP,
2 CHARLEN, PROPS, DENSITY, DSTRAIN, RSPININC, TEMPO,
3 UO, FO, FIELD0, STRESSVECO, STATEO,
4 INTENEO, INELAENO, TEMP1, U1,
5 F1, FIELD1, STRESSVEC1, STATE1, INTENE1, INELAEN1)
IMPLICIT NONE
INTEGER NBLOCK, NDI, NSHR, NSTATEV, NFIELDV, NPROPS, LANNEAL
REAL STEPTIME, TOTTIME, DT
CHARACTER*80 CMNAME
PARAMETER (ZERO=0.0D0, ONE=0.1D0, TWO=2.0D0, THREE=3.0D0)
REAL COORDMP(NBLOCK,*)
REAL CHARLEN, PROPS(NPROPS), DENSITY(NBLOCK),
1 DSTRAIN(NBLOCK,NDI+NSHR), RSPININC(NBLOCK,NSHR),
2 TEMPO(NBLOCK), UO(NBLOCK,NDI+NSHR),
3 FO(NBLOCK,NDI+NSHR+NSHR), FIELD0(NBLOCK,NFIELDV),
4 STRESSVECO(NBLOCK,NDI+NSHR), STATEO(NBLOCK,NSTATEV),
5 INTENEO(NBLOCK), INELAENO(NBLOCK), TEMP1(NBLOCK),
6 U1(NBLOCK,NDI+NSHR), F1(NBLOCK,NDI+NSHR+NSHR),
7 FIELD1(NBLOCK,NFIELDV), STRESSVEC1(NBLOCK,NDI+NSHR),
8 STATE1(NBLOCK,NSTATEV), INTENE1(NBLOCK), INELAEN1(NBLOCK)
C LOCAL VARIABLES
REAL F(3,3), B(3,3), J, T1, T2, T3, C1,
1 C2, D1, I1, I2, I3
INTEGER I
C1 =PROPS(1)
C2 =PROPS(2)
D1 =(ONE-TWO*PROPS(3))/(C1*C2)
C LOOP THROUGH ALL BLOCKS
DO I = 1, NBLOCK
C SETUP F (UPPER DIAGONAL PART)
F(1,1) = U1(I,1)
F(2,2) = U1(I,2)
F(3,3) = U1(I,3)
F(1,2) = U1(I,4)
IF (NSHR .EQ. 1) THEN
F(2,3) = 0.0
F(1,3) = 0.0
ELSE
F(2,3) = U1(I,5)
F(1,3) = U1(I,6)
END IF

```

C CALCULATE J

```

T1 = F(1,1) * (F(2,2)*F(3,3) - F(2,3)**2)
T2 = F(1,2) * (F(2,3)*F(1,3) - F(1,2)*F(3,3))
T3 = F(1,3) * (F(1,2)*F(2,3) - F(2,2)*F(1,3))
J = T1 + T2 + T3
T1 = J**(-2.0/3.0)
B(1,1) = T1*(F(1,1)**2 + F(1,2)**2 + F(1,3)**2)
B(2,2) = T1*(F(1,2)**2 + F(2,2)**2 + F(2,3)**2)
B(3,3) = T1*(F(1,3)**2 + F(2,3)**2 + F(3,3)**2)
B(1,2) = T1*(F(1,1)*F(1,2)+F(1,2)*F(2,2)+F(1,3)*F(2,3))
B(2,3) = T1*(F(1,2)*F(1,3)+F(2,2)*F(2,3)+F(2,3)*F(3,3))
B(1,3) = T1*(F(1,1)*F(1,3)+F(1,2)*F(2,3)+F(1,3)*F(3,3))

```

C

```

I1=(B(1,1) + B(2,2) + B(3,3))
I2=B(1,1)*B(2,2)+B(2,2)*B(3,3)+B(1,1)*B(3,3)
1 -B(1,2)**2-B(2,3)**2-B(1,3)**2

```

C

```

T1 = B(1,1) * (B(2,2)*B(3,3) - B(2,3)**2)
T2 = B(1,2) * (B(2,3)*B(1,3) - B(1,2)*B(3,3))
T3 = B(1,3) * (B(1,2)*B(2,3) - B(2,2)*B(1,3))
I3=T1+T2+T3
T1 = (B(1,1) + B(2,2) + B(3,3)) / 3.0
T2 = TWO/J*C1*C2*EXP(C2*(I1-THREE))
T3 = (J-1)*TWO/D1
STRESSVEC1(I,1) = T2 * (B(1,1) - T1) + T3
STRESSVEC1(I,2) = T2 * (B(2,2) - T1) + T3
STRESSVEC1(I,3) = T2 * (B(3,3) - T1) + T3
STRESSVEC1(I,4) = T2 * B(1,2)
IF (NSHR .EQ. 3) THEN
STRESSVEC1(I,5) = T2 * B(2,3)
STRESSVEC1(I,6) = T2 * B(1,3)
END IF
END DO
RETURN

```

Figure 5.14: Fortran subroutine for the Humphrey-Yin model using VUMAT code.

5.2.5 Subroutines developed for the H-K model

```

SUBROUTINE UHYPER(BI1,BI2,AJ,U,UI1,UI2,UI3,TEMP,NOEL,CMNAME,
1  INCOMPFLAG,NUMSTATEV,STATEV,NUMFIELDV,
2  FIELDV,FIELDVINC,NUMPROPS,PROPS)
C
  INCLUDE 'ABA_PARAM.INC'
C
  CHARACTER*80 CMNAME
  DIMENSION UI1(3),UI2(6),UI3(6),STATEV(*),FIELDV(*),
1           FIELDVINC(*),PROPS(*)
C
  PARAMETER (ZERO=0.0D0,ONE=1.0D0,TWO=2.0D0,THREE=3.0D0)
C
  C1=PROPS(1)
  C2=PROPS(2)
  C3=PROPS(3)
  C4=PROPS(4)
C
  U=C1*(EXP(C2*(BI1-THREE))-ONE)+C3*(BI1-THREE)+C4*LOG(BI2/THREE)
  UI1(1)=C1*C2*EXP(C2*(BI1-THREE))+C3
  UI1(2)=C4/BI2
  UI1(3)=ZERO
  UI2(1)=C1*C2*C2*EXP(C2*(BI1-THREE))
  UI2(2)=-C4/BI2/BI2
  UI2(3)=ZERO
  UI2(4)=ZERO
  UI2(5)=ZERO
  UI2(6)=ZERO
  UI3(1)=ZERO
  UI3(2)=ZERO
  UI3(3)=ZERO
  UI3(4)=ZERO
  UI3(5)=ZERO
  UI3(6)=ZERO
  RETURN
END
```

Figure 5.15: Fortran subroutine for the H-K model using incompressible UHYPER code.

```

SUBROUTINE UHYPER(BI1,BI2,AJ,U,UI1,UI2,UI3,TEMP,NOEL,CMNAME,
1 INCMPLFLAG,NUMSTATEV,STATEV,NUMFIELDV,
2 FIELDV,FIELDVINC,NUMPROPS,PROPS)

C
INCLUDE 'ABA_PARAM.INC'

C
CHARACTER*80 CMNAME
DIMENSION UI1(3),UI2(6),UI3(6),STATEV(*),FIELDV(*),
1 FIELDVINC(*),PROPS(*)

C
PARAMETER (ZERO=0.0D0,ONE=1.0D0,TWO=2.0D0,THREE=3.0D0)

C
C1=PROPS(1)
C2=PROPS(2)
C3=PROPS(3)
C4=PROPS(4)
D1=THREE*(ONE-TWO*PROPS(5))/(THREE*C1*C2+THREE*C3+C4)

C
U=C1*(EXP(C2*(BI1-THREE))-ONE)+C3*(BI1-THREE)+C4*LOG(BI2/THREE)
1 +((AJ-ONE)**2)/D1
UI1(1)=C1*C2*EXP(C2*(BI1-THREE))+C3
UI1(2)=C4/BI2
UI1(3)=TWO*(AJ-ONE)/D1
UI2(1)=C1*C2*C2*EXP(C2*(BI1-THREE))
UI2(2)=-C4/BI2/BI2
UI2(3)=TWO/D1
UI2(4)=ZERO
UI2(5)=ZERO
UI2(6)=ZERO
UI3(1)=ZERO
UI3(2)=ZERO
UI3(3)=ZERO
UI3(4)=ZERO
UI3(5)=ZERO
UI3(6)=ZERO
RETURN
END

```

Figure 5.16: Fortran subroutine for the H-K model using compressible UHYPER code.

```

SUBROUTINE VUMAT(NBLOCK, NDI, NSHR, NSTATEV, NFIELDV, NPROPS,
1 LANNEAL, STEPTIME, TOTTIME, DT, CMNAME, COORDMP,
2 CHARLEN, PROPS, DENSITY, DSTRAIN, RSPININC, TEMPO,
3 UO, FO, FIELD0, STRESSVECO, STATE0,
4 INTENEO, INELAENO, TEMP1, U1,
5 F1, FIELD1, STRESSVEC1, STATE1, INTENE1, INELAEN1)
IMPLICIT NONE
INTEGER NBLOCK, NDI, NSHR, NSTATEV, NFIELDV, NPROPS, LANNEAL
REAL STEPTIME, TOTTIME, DT
CHARACTER*80 CMNAME
PARAMETER (ZERO=0.0D0, ONE=0.1D0, TWO=2.0D0, THREE=3.0D0)
REAL COORDMP(NBLOCK,*)
REAL CHARLEN, PROPS(NPROPS), DENSITY(NBLOCK),
1 DSTRAIN(NBLOCK,NDI+NSHR), RSPININC(NBLOCK,NSHR),
2 TEMPO(NBLOCK), UO(NBLOCK,NDI+NSHR),
3 FO(NBLOCK,NDI+NSHR+NSHR), FIELD0(NBLOCK,NFIELDV),
4 STRESSVECO(NBLOCK,NDI+NSHR), STATE0(NBLOCK,NSTATEV),
5 INTENEO(NBLOCK), INELAENO(NBLOCK), TEMP1(NBLOCK),
6 U1(NBLOCK,NDI+NSHR), F1(NBLOCK,NDI+NSHR+NSHR),
7 FIELD1(NBLOCK,NFIELDV), STRESSVEC1(NBLOCK,NDI+NSHR),
8 STATE1(NBLOCK,NSTATEV), INTENE1(NBLOCK), INELAEN1(NBLOCK)
C LOCAL VARIABLES
REAL F(3,3), B(3,3), J, T1, T2, T3, T4, T5, C1,
1 C2, C3, C4, D1, I1, I2, I3
INTEGER I
C1 =PROPS(1)
C2 =PROPS(2)
C3 =PROPS(3)
C4 =PROPS(4)
D1=THREE*(ONE-TWO*PROPS(5))/(THREE*C1*C2+THREE*C3+C4)
C LOOP THROUGH ALL BLOCKS
DO I = 1, NBLOCK
C SETUP F (UPPER DIAGONAL PART)
F(1,1) = U1(I,1)
F(2,2) = U1(I,2)
F(3,3) = U1(I,3)
F(1,2) = U1(I,4)
IF (NSHR .EQ. 1) THEN
F(2,3) = 0.0
F(1,3) = 0.0
ELSE
F(2,3) = U1(I,5)
F(1,3) = U1(I,6)
END IF

```

C CALCULATE J

```

T1 = F(1,1) * (F(2,2)*F(3,3) - F(2,3)**2)
T2 = F(1,2) * (F(2,3)*F(1,3) - F(1,2)*F(3,3))
T3 = F(1,3) * (F(1,2)*F(2,3) - F(2,2)*F(1,3))
J = T1 + T2 + T3
T1 = J**(-2.0/3.0)
B(1,1) = T1*(F(1,1)**2 + F(1,2)**2 + F(1,3)**2)
B(2,2) = T1*(F(1,2)**2 + F(2,2)**2 + F(2,3)**2)
B(3,3) = T1*(F(1,3)**2 + F(2,3)**2 + F(3,3)**2)
B(1,2) = T1*(F(1,1)*F(1,2)+F(1,2)*F(2,2)+F(1,3)*F(2,3))
B(2,3) = T1*(F(1,2)*F(1,3)+F(2,2)*F(2,3)+F(2,3)*F(3,3))
B(1,3) = T1*(F(1,1)*F(1,3)+F(1,2)*F(2,3)+F(1,3)*F(3,3))

```

C

```

B2(1,1) = (B(1,1)**2 + B(1,2)**2 + B(1,3)**2)
B2(2,2) = (B(1,2)**2 + B(2,2)**2 + B(2,3)**2)
B2(3,3) = (B(1,3)**2 + B(2,3)**2 + B(3,3)**2)
B2(1,2) = (B(1,1)*B(1,2)+B(1,2)*B(2,2)+B(1,3)*B(2,3))
B2(2,3) = (B(1,2)*B(1,3)+B(2,2)*B(2,3)+B(2,3)*B(3,3))
B2(1,3) = (B(1,1)*B(1,3)+B(1,2)*B(2,3)+B(1,3)*B(3,3))

```

C

```

I1=(B(1,1) + B(2,2) + B(3,3))
I2=B(1,1)*B(2,2)+B(2,2)*B(3,3)+B(1,1)*B(3,3)
1 -B(1,2)**2-B(2,3)**2-B(1,3)**2

```

C

```

T1 = B(1,1) * (B(2,2)*B(3,3) - B(2,3)**2)
T2 = B(1,2) * (B(2,3)*B(1,3) - B(1,2)*B(3,3))
T3 = B(1,3) * (B(1,2)*B(2,3) - B(2,2)*B(1,3))
I3=T1+T2+T3
T1 = (B(1,1) + B(2,2) + B(3,3)) / 3.0
T2 = TWO/J*(C3+C4*I1/I2+C1*C2*EXP(C2*(I1-THREE))
T3 = (J-1)*TWO/D1
T4=-TWO/J*C4/I2
T5=(B2(1,1) + B2(2,2) + B2(3,3)) / 3.0
STRESSVEC1(I,1) = T2 * (B(1,1) - T1) + T3+T4*(B2(1,1)-T5)
STRESSVEC1(I,2) = T2 * (B(2,2) - T1) + T3+T4*(B2(2,2)-T5)
STRESSVEC1(I,3) = T2 * (B(3,3) - T1) + T3+T4*(B2(3,3)-T5)
STRESSVEC1(I,4) = T2 * B(1,2)+T4*B2(1,2)
IF (NSHR .EQ. 3) THEN
STRESSVEC1(I,5) = T2 * B(2,3)+T4*B2(2,3)
STRESSVEC1(I,6) = T2 * B(1,3)+T4*B2(1,3)
END IF
END DO
RETURN

```

Figure 5.17: Fortran subroutine for the H-K model using VUMAT code.

6 Numerical simulation examples

With the codes we can also test their performance and the behaviour of the material model for simulation purposes. We can see some simulations regarding these in the following sections. The material model used for the simulations is the incompressible H-K model.

6.1 Homogeneous test cases

First some simulations were run with a single element, for uniaxial and equibiaxial tensile stress state defined on a unit square element with the necessary boundary conditions. With this we can check the analytical and numerical models behaviours and compare them as well, considering mainly a general comparison, and not in depth analysis of the case. During the simulation an incompressible material was used with the new measurements used for fitting and parameter calculation based on 4.18 parameters. The model used a single hybrid element for the incompressible case.

The meshing this way is trivial, the single element creates the mesh for the whole volume, it can be seen in Figure 6.1. The exact element type used for the simulation was C3D8H.

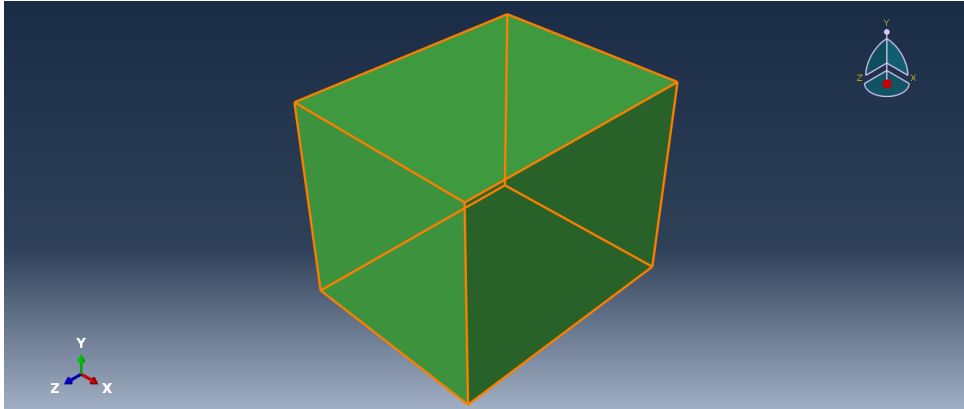
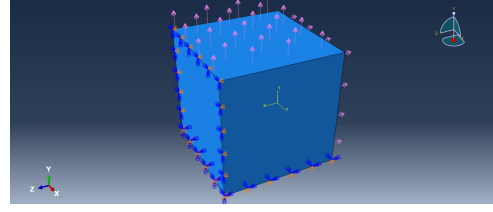
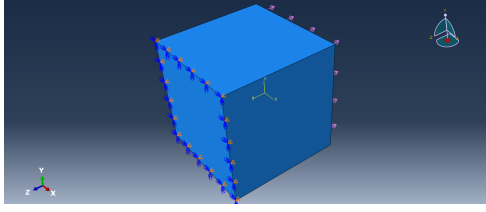


Figure 6.1: Mesh of the body.

For the uniaxial model the load was 5 MPa stress along a single direction, while for biaxial case 4.5 MPa loading was used in two different perpendicular directions. These case can be seen on Figure 6.2. The opposite sides for the stress applications all were constrained to have zero displacement in their loading direction.

Running the simulation we must post-process the results, these can be done with the *XY plot* command. For the uniaxial loading we got the following results. The strains were measured in engineering strain values and the stresses as Cauchy stress, so they had to be transformed back into engineering stress in order to have a common frame with the experimental data, and thus the fitted data as well

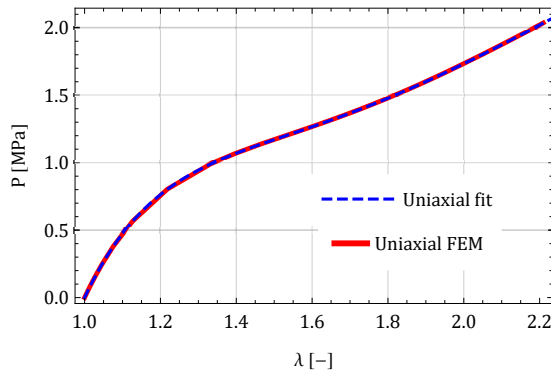
$$\sigma = P(\lambda) \cdot \lambda = P(1 + \varepsilon)\varepsilon. \quad (6.1)$$



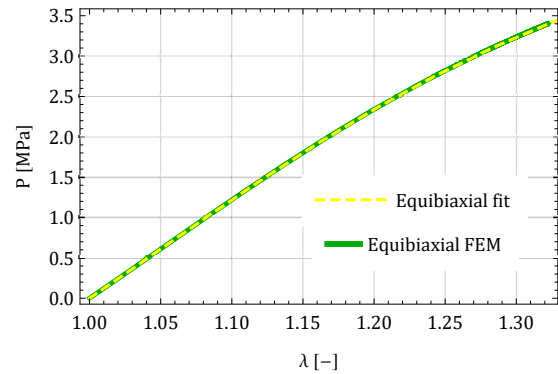
(a) Boundary conditions for uniaxial loading. (b) Boundary conditions for biaxial loading.

Figure 6.2: Loading cases for model validation.

The results with the fitted curves are visible of Figures 4.33 and 4.34, they can show us the accuracy the numerical software can handle the given model. Finally we can see the comparison on Figures 6.3a and 6.3b



(a) Validation of uniaxial behaviour.



(b) Validation of biaxial behaviour.

Figure 6.3: Validation of stresses for the FEM.

It is clear that the two cases are rather similar to the measurement results, naturally with some small differences, however these are not problematic, as the source of these are mainly the precision used of the hyperelastic model's parameters, also the numerical simulations have some precision errors, but considering all these, the two results are matching, the numerical implementation can indeed be used to simulate cases for this material with the implemented model.

6.2 Non-homogeneous loading

For the biaxial case a relatively complex geometry must be analysed, so another simulation was done for that as well. This way we can also get information about the homogeneous zone and the stress field of the material.

For modelling the geometry was defined as two dimensional elements, however we could also use further simplifications, as the symmetries present in the system could be used to reduce the model into an eighth of the original, but for robustness the complete geometry was used throughout the simulation

After meshing the two dimensional geometry the mesh took the form of the following

structure, which can be seen in Figure 6.4. Overall the model involves 10714 elements in the complete mesh. The meshing elements were CPS4 elements, the material model was an incompressible H-K material model fitted for the new measurement results.

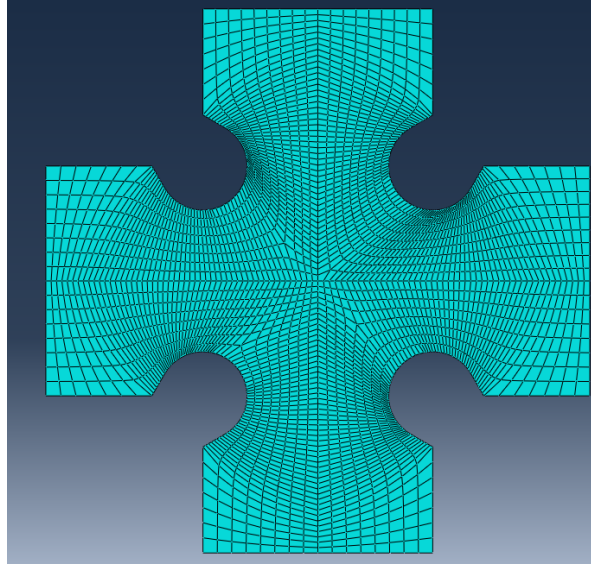


Figure 6.4: Mesh of the cross specimen.

As loads for the experiments, displacement values were used, so here again I applied displacements on the heads of the geometry for all arms, 50 mm for all so along a single axis 100 mm displacement is present. The boundary conditions are shown in Figure 6.5.

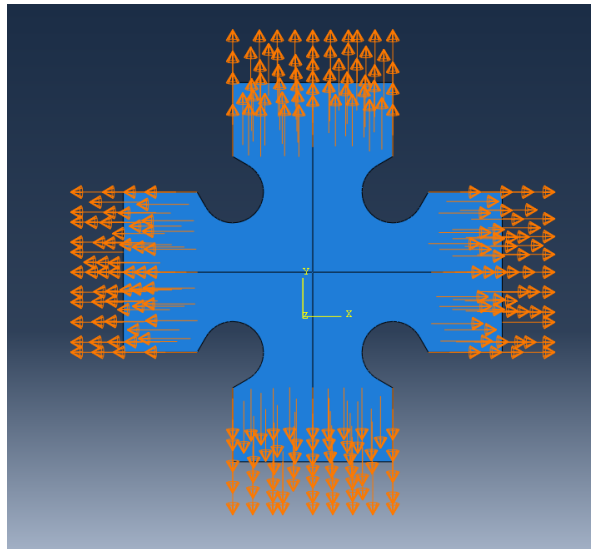


Figure 6.5: Boundary conditions of the model.

The results of this simulation became the following. The results are visible in Figure 6.6.

We can inspect the logarithmic strains as well, these are visible in Figure 6.7.

It is clear from the simulation that the stress and strain values are nearly homoge-

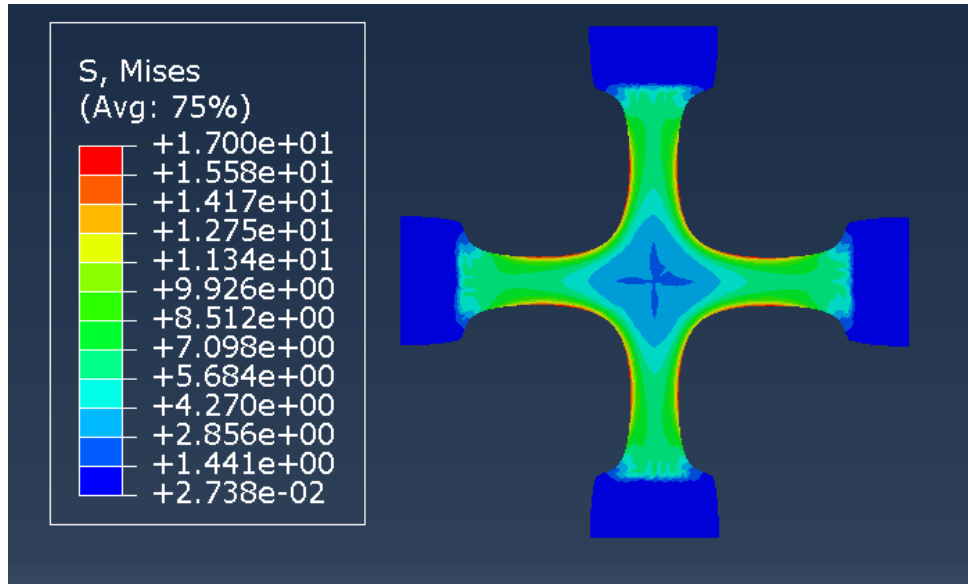


Figure 6.6: The von Mises stress in the structure

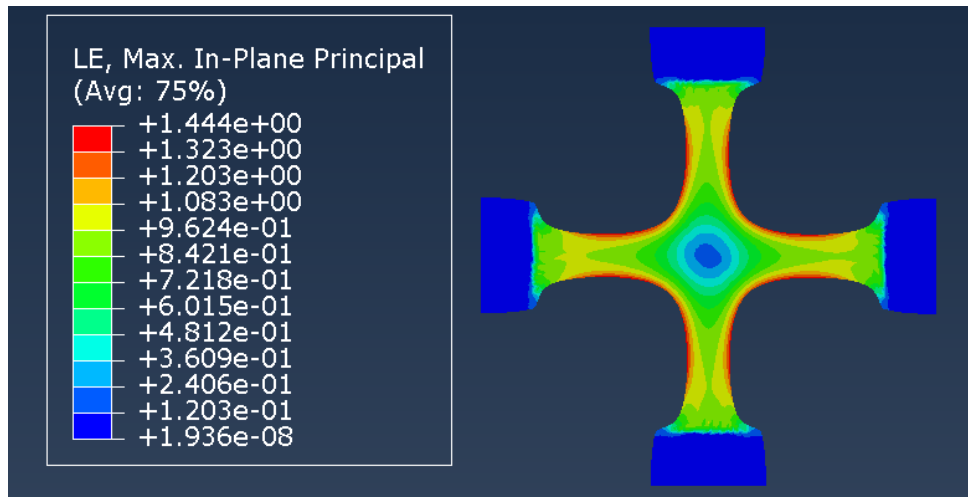


Figure 6.7: The largest logarithmic strain values in the structure

neous in the centre section of the specimen, while near the roundings the stress are much higher, it concentrates the stresses. Based on this information further experiments and simulations could be done, and perhaps a more homogeneous, optimised geometry could be defined to be used for later measurements. This could be a large iterative problem, so for current thesis it is not done.

7 Summary

The primary goals of the thesis were one-by-one taken into account and presented throughout the document. The literature review contains several articles that can provide an insight to the reader who is new to the topic of hyperelasticity. The most important aspect of the hyperelastic material models and continuum mechanics part was the introduced H-K model, which presents a simple yet powerful method for creating new incompressible hyperelastic formulation.

The parameter fitting provided accurate numerical results. The quality of the fitting can be inspected on both numerical data provided and via the figures involving the experimental data alongside the model with the fitted parameters used.

The measurements provided useful data as well as experience about the production of these kind of measurement results. The first thing I wish to note here is the complexity of creating such test cases considering the homogeneous constraints and preserving them. In order to do so the fixture of the test specimen has to be preformed with great care. It is a great disadvantage that larger contractions can take place at large stretches of the material, and if the fixing of the body is done with force constraints, for low pressing force the specimen may get loose and thus create an unsuccessful measurements.

Another important aspect for materials described with hyperelastic models is that they are polymers and thus Mullins effect can take place during their loading cycles. In order to get reliable data we have to be sure that the part was unloaded so the true characteristic of the material can be measured instead of a modified, weakened one after some previous loads that destroy a part of the bonds inside the material.

The subroutines developed also were successful, several models have been implemented into ABAQUS finite element software with the help of different user subroutines developed in FORTRAN. There are a few methods to do this, considering the available templates provided by the software's user manual, such as VUMAT and UHYPER that are analysed in depth of this thesis as well.

For a wider case of application both incompressible and compressible material models were developed, this way ideal and more realistic cases can be simulated as well, and thus further knowledge can be gathered regarding the topic of incompressibility and compressibility differences.

Finally, the finite element simulations were also of great use for checking the correspondence of the analytically obtained model and the numerically implemented one, which resulted in an almost perfect fit, meaning that the implemented material model can really imitate the one programmed in the subroutine.

Some outlooks are also to be mentioned here, such as creating a better working support for the experiments, such as that it does not allow failures in the middle of experiments due to large deformation of test subjects. Another development is that more finite element

analyses should be done for ensuring the proper behaviour of simulations with the defined user material subroutines.

Finally an iterative experiment-simulation cycle could also be helpful providing a better understanding of the exact equibiaxial stress distribution, as well as giving an opportunity for a more suitable test geometry to be designed.

8 Összefoglaló

Összességében a fő célok teljesítésre kerültek a dolgozat kidolgozása során. Az irodalmi összefoglaló válogatott forrásokat elemezve biztosít rálátást a hiperelasztikus anyagmodellezés történetére, annak fejlődésére és jelenlegi állására. Ezen felül az elméleti összefoglalóban kiemeltek közül a legfontosabb kiemelendő az új anyagmodell, a H-K modell bevezetése volt, mely egy praktikus, mégis mély alapokat figyelembevevő modell, valamint képes jó közelítéssel leírni az általunk modellezni kívánt alapanyagot.

A paraméter-illesztés sok számadattal szolgált, melyek mutatják a különböző modellek pontosságát és hatékonyságát is. Ezek mind numerikusan, mind pedig grafikusan, képek formájában megjelennek a dolgozatban.

Hasznos adatokat tudtam kinyerni a kivitelezett mérésekből, ezzel párhuzamosan pedig számos gyakorlati tapasztalatra is szert tettem a kísérletek során. Egy fontos kiemelendő a rögzítés kialakításának komplex problémája. Ennek megfelelőnek kell lenni annak érdekében, hogy a homogén esetet maximálisan biztosítani tudjuk. Ezt a célt nehezíti a kontrakció, hiszen nagy nyúlások hatására a keresztirányú deformáció is jelentős mértékű lesz, ezzel pedig a rögzítés hatása csökkenhet, amely kísérleteim során meg is történt, és a kísérlet kudarcra zárult, mert a próbadarab kicsúszott a rögzítőfejből.

Egy másik fontos figyelembe veendő hatás, melyre ügyelnünk kell, a Mullins-féle hatás. Ennek oka, hogy az anyagok, melyeket hiperelasztikus modellekkel írunk le polimerek, ez pedig magával hordozza a tényt, hogy a bennük lévő kötések az említett hatás alapján elszakadhatnak, és általában nem öngyógyulnak. Ez azt eredményezi, hogy többszöri fel- és leterhelés során az anyag viselkedése megváltozik, tehát annak érdekében, hogy az anyag eredeti viselkedését vizsgálhassuk a próbadarab nem lehet előzetesen megfeszítve, mert az hamis adatokat generál, és kisebb teherbírást mutat.

Sikeresen létrehoztam számos anyagmodell szubrutint is, ezek különböző típusú FORTRAN kódok segítségével kerültek megírásra. Az ABAQUS végeelem szoftverben definiált VUMAT valamint UHYPER kódok segítségével mélyreható, és változatos anyagmodell implementálásra kerülhetett sor. A modellek széleskörben való alkalmazhatósága érdekében összenyomható és összenyomhatatlan modelleket is létrehoztam, ezáltal ideális és valós esetek egyaránt modellezhetőek, szimulálhatóak.

Végül végeselemes szimulációkat is futtatam, melyek ellenőrzést biztosítottak az analitikusan levezetett és illesztett modellel, és összehasonlíthatóvá tették a numerikus és analitikus megoldások karakterisztikáit nem homogén terhelés esetén. Ez alapján a létrehozott model majdnem tökéletesen illeszkedik a numerikusan számítotttra, és fordítva, vagyis a kettő megfeleltethető egymásnak.

Tegyünk néhány kitekintést is a későbbi lehetőségekre vonatkozóan. A kísérletek további fejlesztésre szorulnak, ennek érdekében a befogók áttervezése lenne a logikus lépés, ezzel elkerülve a sikertelen kísérleteket. Egy további lehetőség ezen felül egyéb szimulációk futtatása is, ezzel bizonyosságot nyerve a modell és a valóság minél részletesebb egyezésének ügyében az újonnan kidolgozott szubrutinok segítségével.

Végezetül egy iteratív ciklust is definiálhatnánk a kísérletek és szimulációk között, ezzel elősegítve a kéttengelyű kísérlet homogenitását és a próbatest részletekbe menő tervezését, hisz ezáltal egy optimális geometriát is létre lehet hozni, melyen a homogén feszültségeloszlás is könnyebben biztosítható.

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