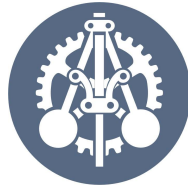


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Students' Scientific Conference (TDK)

Virtual balancing of an inverted pendulum of complex-valued
perturbation

(Inverz inga virtuális egyensúlyozása komplex értékű perturbáció esetén)

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Abstract

In virtual environment in the case of motion control problems, it is easy to modify the parameters of the controlled system. In this way we can realize experiments that cannot be implemented in real world experiments. For example, in virtual environment, we can investigate the effect of employing the Aristotelian dynamics instead of the Newtonian dynamics in inverted pendulum balancing. In my research, I investigate a similar phenomenon: I describe the length of the balanced pendulum as a complex number such that I append an imaginary length component to the real length virtually. This generates a complex-valued dynamics that consist of a real and an imaginary part. In the reality, complex-valued parameter does not exist, so for the subject who performs the balancing task, this can be interpreted as an unknown perturbation.

In my research, I model the human balancing of an inverted pendulum, where the human can be modeled as a PD controller. The Java computer code that can be used for virtual balancing of an inverted pendulum is provided by the Department of Applied Mechanics. I have modify the program code in the way that one can set complex parameter for the length of the pendulum. In the empirical part of my research, I have carried out measurements with the modified program, then I have compared the results with the theoretical results.

As a preliminary expectation, one would think that bigger the imaginary part compared to the real part in the complex perturbation, the more difficult to keep the pendulum in the upper unstable equilibrium. The intuitive reason for this is that we cannot detect the imaginary length during the balancing, so our brain cannot regulate the imaginary part of the dynamics. Theoretical investigation based on the mathematical model showed however that the imaginary part of the pendulum length does not change the difficulty in the task in the sense that the corresponding critical delay is indeendent of the imaginary pendulum length. This was confirmed also by experiments: subjects typically did not found balancing an inverted pendulum of complex-valued length to be more difficult than balancing a pendulum with real valued length.

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Notations

The table contains the names of the notations that occur several times in English, and in the case of physical quantities, its unit of measurement. The designation of each quantity is, where possible, the same as that accepted in the domestic and international literature. An explanation of the rarely used notations can be found at their first location.

Latin letters

Notation	Name, comment, value	Dimension
g	gravitational acceleration [9.81]	m/s ²
l	length of the pendulum	m
L	real length of the pendulum	m
R	imaginary length of the pendulum	m
m	mass of the pendulum	kg
m_0	mass of the cart that is attached to the pendulum	kg
t	time	s

Greek letters

Notation	Name, comment, value	Dimension
λ	characteristic coefficient	-
φ	angle	rad

Indices, exponents

Notation	Name, comment, value
i	imaginary unit ($i^2 = -1$)
Im	imaginary part of a complex number
Re	real part of a complex number

1 Introduction

The human brain is a very complicated system and it is very complicated to investigate its working. In daily life, several motion control task appears (e.g. walking, carrying a cup filled with liquid, balancing on a balance board [4]). The human brain can learn to handle several difficult control task and develop control models and strategies. During these tasks, several unknown perturbation and uncertainty occurs and yet with practice, the brain can learn how to handle these and stabilize the system. The understanding of the working of the brain and the development of these control strategies has important implications.

The human motor control is constrained by the sensory uncertainties and the reaction delay of the neural system [8]. These constraints limit the stabilizability of dynamic systems. Because of this limit, a critical length and a critical time delay can be defined. If the length of the pendulum is below the critical length, or the time delay of the system is above the critical time delay, the system cannot be stabilized. These critical parameters can be measured and used for investigate a system [1].

The advantage of the virtual environment that all of the applied parameters can be easily modified, so measurement can be done which cannot be done in the reality. Virtual environment can be applied for motion control problems. For example in virtual environment investigation of the effect of employing the Aristotelian dynamics instead of the Newtonian dynamics in inverted pendulum balancing can be done [3]. In this study the length of a balanced pendulum is described as a complex number such that the real length is ap-

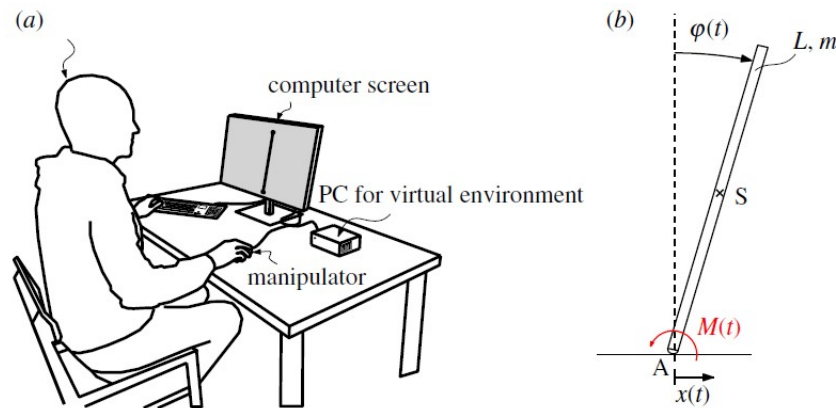


Figure 1.1: The sketch of the virtual test environment (a) and the mechanical model (b). Figure was taken from [3].

pended with an imaginary length virtually. The imaginary length does not exist in reality, so in the virtual environment it was applied in the way that the subject who performs the balancing only sees the real-valued length. In this way the imaginary length can be interpreted as an unknown and invisible complex perturbation. During the experiment, the critical time delay is used to analyze the effect of the added imaginary-valued length for the human balancing of the inverted pendulum.

In the study the human who performs the balancing is modeled as a delayed PD controller. The virtual environment is a computer code in Java environment. The basic code was provided by the Department of Applied Mechanics and it was modified in a way that complex value can be set as a length of the pendulum. The measurement set-up can be seen in Figure 1.1. In the experiment, first the reaction delay of the 5 subject, then the critical time delay for several imaginary lengths for each subjects was measured.

2 Theoretical stability analysis

2.1 Inverted pendulum real-valued length

In the case of inverted pendulum balancing task, the human subject has to keep the inverted pendulum in the upper equilibrium by moving the pivot point of the stick [7]. In this case the human can be modeled as a Proportional-Derivative controller, based on the fact that the human brain can sense position and also velocity [1]. In the human nervous-system a time delay occurs due to the human reaction time, thus in the model a delayed PD controller has to be used. The linearized equation of motion of the controlled inverted pendulum on a cart with uniformly distributed mass can be written in the following form.

$$\ddot{\varphi}(t) - \frac{6g}{cl}\varphi(t) = -\frac{6}{lc(m+m_0)}Q(t) \quad (2.1)$$

where φ is the angular position of the pendulum with $\varphi = 0$ in the upper equilibrium, l is the length of the stick, g is the gravitational acceleration, m is the mass of the pendulum and m_0 is the mas of the cart, $Q(T)$ is the control force and c is a parameter with value

$$c = \left(4 - \frac{3m}{m+m_0}\right) \quad (2.2)$$

The control force in the case of delayed PD controller is the following.

$$Q(t) = K_p\varphi(t-\tau) + K_d\dot{\varphi}(t-\tau) \quad (2.3)$$

where τ is the time delay, K_p is the proportional gain and K_d is the derivative gain of the controller.

2.2 Inverted pendulum complex-valued length

If the length of the pendulum is a complex number, then the parameters in (2.1) becomes complex. A new notation can be implemented, as

$$\ddot{\varphi}(t) - a\varphi(t) = -bQ(t), \quad (2.4)$$

where

$$l = L + Ri \quad a = \alpha + \beta i \quad b = \delta + \kappa i$$

Here, L and R are the real and the imaginary part of the pendulum length. The parameters α , β and δ , κ are the real and imaginary part of parameters a and b . The value of these parameters can be calculated, as

$$\begin{aligned} a &= \frac{6g}{c(L + Ri)} = \frac{6g}{c(L + Ri)} \cdot \frac{L - Ri}{L - Ri} = \frac{6g(L - Ri)}{c(L^2 + R^2)} = \\ &= \underbrace{\frac{6gL}{c(L^2 + R^2)}}_{\alpha} - \underbrace{\frac{6gR}{c(L^2 + R^2)}}_{\beta} i \end{aligned} \quad (2.5)$$

$$\begin{aligned} b &= \frac{6}{c(m + m_0)(L + Ri)} = \frac{6}{c(m + m_0)(L + Ri)} \cdot \frac{L - Ri}{L - Ri} = \frac{6(L - Ri)}{c(m + m_0)(L^2 + R^2)} = \\ &= \underbrace{\frac{6L}{c(m + m_0)(L^2 + R^2)}}_{\delta} - \underbrace{\frac{6R}{c(m + m_0)(L^2 + R^2)}}_{\kappa} i. \end{aligned} \quad (2.6)$$

Note that β and κ involves the negative sign as well. If $R = 0$, than we get the dynamics of a pendulum with real-valued length.

2.3 Stability

In balancing tasks the stability-chart of the system can be derived applying the D-subdivision method. The stability diagram represents the D-curves in the parameter plane where characteristic exponents exist with real part. The D-curves can be derived from the characteristic equation of the equation of motion [5].

$$D(\lambda) = \lambda^2 - (\alpha + \beta i) + (\delta + \kappa i)K_p e^{-\lambda\tau} + (\delta + \kappa i)K_d \lambda e^{-\lambda\tau} = 0 \quad (2.7)$$

$$\lambda = \gamma + \omega i$$

The characteristic exponent λ is a complex number. The system is on the border of the stability when the real part of the characteristic root is zero. So γ is equal to zero and the exponential function can be written as a linear combination of trigonometric functions. The derived equation can be separated to a real part and an imaginary part:

$$\begin{aligned} Re(D(\lambda)) &= -\omega^2 - \alpha + \delta K_p \cos(\omega\tau) + \kappa K_p \sin(\omega\tau) - \\ &\quad - \kappa K_d \omega \cos(\omega\tau) + \delta K_d \omega \sin(\omega\tau) = 0, \\ Im(D(\lambda)) &= -\beta + \kappa K_p \cos(\omega\tau) - \delta K_p \sin(\omega\tau) + \end{aligned} \quad (2.8)$$

$$+\delta K_d \omega \cos(\omega \tau) + \kappa K_d \omega \sin(\omega \tau) = 0. \quad (2.9)$$

Now, this system of equation can be solved for K_p and K_d and two different cases can be separated.

- If $\omega = 0$ (static D-curve)

The D-curves initiated from the parameter line where $\omega = 0$

$$(2.8) \rightarrow -\alpha + \delta K_p = 0 \rightarrow K_p = \frac{\alpha}{\delta} \quad (2.10)$$

$$(2.9) \rightarrow -\beta + \kappa K_p = 0 \rightarrow K_p = \frac{\beta}{\kappa} \quad (2.11)$$

This seems to be a controversial solution but the two solution is exactly equal to each other, since

$$\frac{\alpha}{\delta} = \frac{\frac{6gL}{c(L^2 + R^2)}}{6L} = g(m + m_0) \quad (2.12)$$

$$\frac{\beta}{\kappa} = \frac{-\frac{6gR}{c(L^2 + R^2)}}{-\frac{6R}{c(m + m_0)(L^2 + R^2)}} = g(m + m_0). \quad (2.13)$$

- If $\omega \neq 0$

$$-\omega^2 - \alpha + K_p(\delta \cos(\omega \tau) + \kappa \sin(\omega \tau)) - K_d(\kappa \omega \cos(\omega \tau) + \delta \omega \sin(\omega \tau)) = 0 \quad (2.14)$$

$$-\beta + K_p(\kappa \cos(\omega \tau) - \delta \sin(\omega \tau)) + K_d(\delta \omega \cos(\omega \tau) + \kappa \omega \sin(\omega \tau)) = 0 \quad (2.15)$$

In order to get K_p and K_d as function of ω , these two equations have to be manipulated as

$$\begin{aligned} (2.8) \times \cos(\omega \tau) - (2.9) \times \sin(\omega \tau) \Rightarrow \\ (-\omega^2 - \alpha) \cos(\omega \tau) + \beta \sin(\omega \tau) + K_p \delta - K_d \kappa \omega = 0 \end{aligned} \quad (2.16)$$

$$\begin{aligned} (2.8) \times \sin(\omega \tau) + (2.9) \times \cos(\omega \tau) \Rightarrow \\ (-\omega^2 - \alpha) \sin(\omega \tau) - \beta \cos(\omega \tau) + K_p \kappa - K_d \delta \omega = 0 \end{aligned} \quad (2.17)$$

By solving this system of equation, the control parameters can be calculated, as

$$\begin{aligned} \Rightarrow K_d(\omega) = \frac{1}{\kappa^2 \omega + \delta^2 \omega} \left(\omega^2 \delta \sin(\omega \tau) + \alpha \delta \sin(\omega \tau) + \beta \delta \cos(\omega \tau) \right) - \\ - \frac{1}{\kappa^2 \omega + \delta^2 \omega} \left(\omega^2 \kappa \cos(\omega \tau) - \alpha \kappa \cos(\omega \tau) + \kappa \beta \sin(\omega \tau) \right), \end{aligned} \quad (2.18)$$

$$\begin{aligned} \Rightarrow K_p(\omega) = & \frac{\omega^2 + \alpha}{\delta} \cos(\omega\tau) - \frac{\beta}{\delta} \sin(\omega\tau) + \frac{1}{\kappa^2 + \delta^2} \left(\omega^2 \kappa \sin(\omega\tau) + \alpha \kappa \sin(\omega\tau) \right) + \\ & + \frac{1}{\kappa^2 + \delta^2} \left(\beta \kappa \cos(\omega\tau) - \frac{\omega^2 \kappa^2}{\delta} \cos(\omega\tau) - \frac{\alpha \kappa^2}{\delta} \cos(\omega\tau) + \frac{\kappa^2 \beta}{\delta} \sin(\omega\tau) \right). \end{aligned} \quad (2.19)$$

Note that the characteristic roots do not appear as complex pairs, since the characteristic equation (2.7) involves complex coefficients. In other words, if $\lambda = \gamma + \omega i$ is a characteristic exponent, then $\lambda = \gamma - \omega i$ is not necessarily a characteristic exponent. This implied that (2.18) and (2.19) gives different dynamic D-curves for positive and negative ω values Therefore, as opposed to the case of a real-valued length, the parameter ω has to be swept over both a negative and a positive domain. This also implies that along the dynamic D-curves only one characteristic exponent changes sign in its real part.

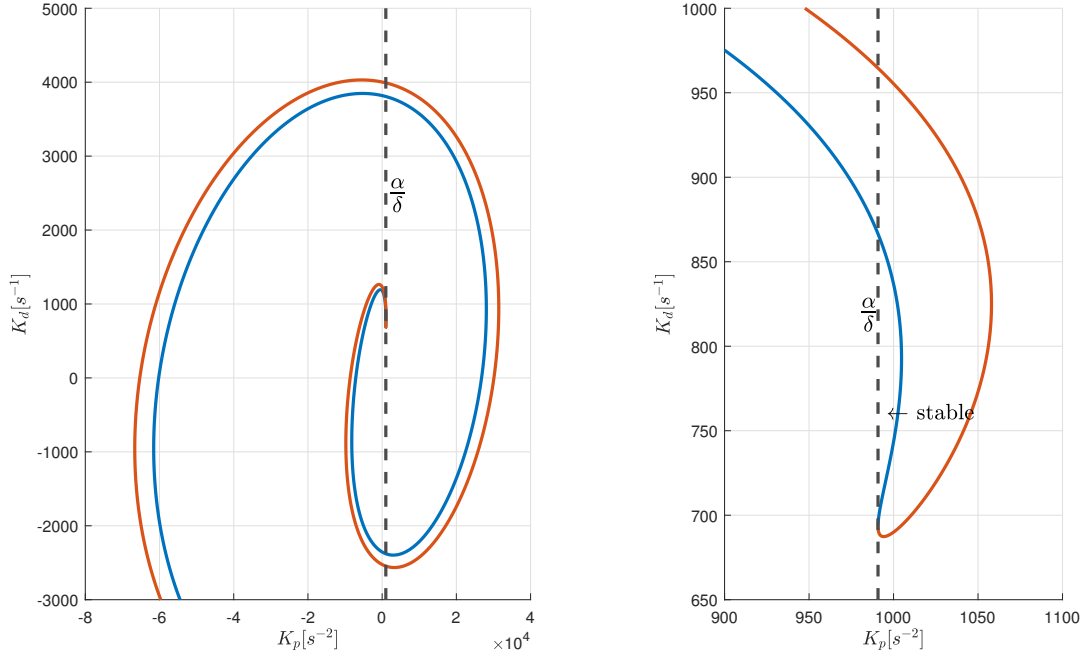


Figure 2.1: Stability chart of a delayed feedback PD controller with the feedback delay $\tau = 0.7$ s and the complex length $l = 5 + 1i$ m. Black line indicates static D-curve with $\lambda = 0$, red and blue lines indicate dynamic D-curves with $\omega > 0$ and $\omega < 0$, respectively.

2.4 Stability chart

The stability chart is shown on Figure 2.1 that in the case of nonzero imaginary length component, two types of D-curves appear. In the case of positive and negative imaginary length R with the same value, the D-curves are symmetric. This can also be seen by parameterizing K_p and K_d with R : $K_p(\omega, R) = K_p(-\omega, -R)$ and $K_d(\omega, R) = K_d(-\omega, -R)$. This is shown in Figure 2.2 The dynamic D-curves touches the static one at the point $(K_p, K_d) = \left(\frac{\alpha}{\delta}, \tau \frac{\alpha}{\delta}\right)$.

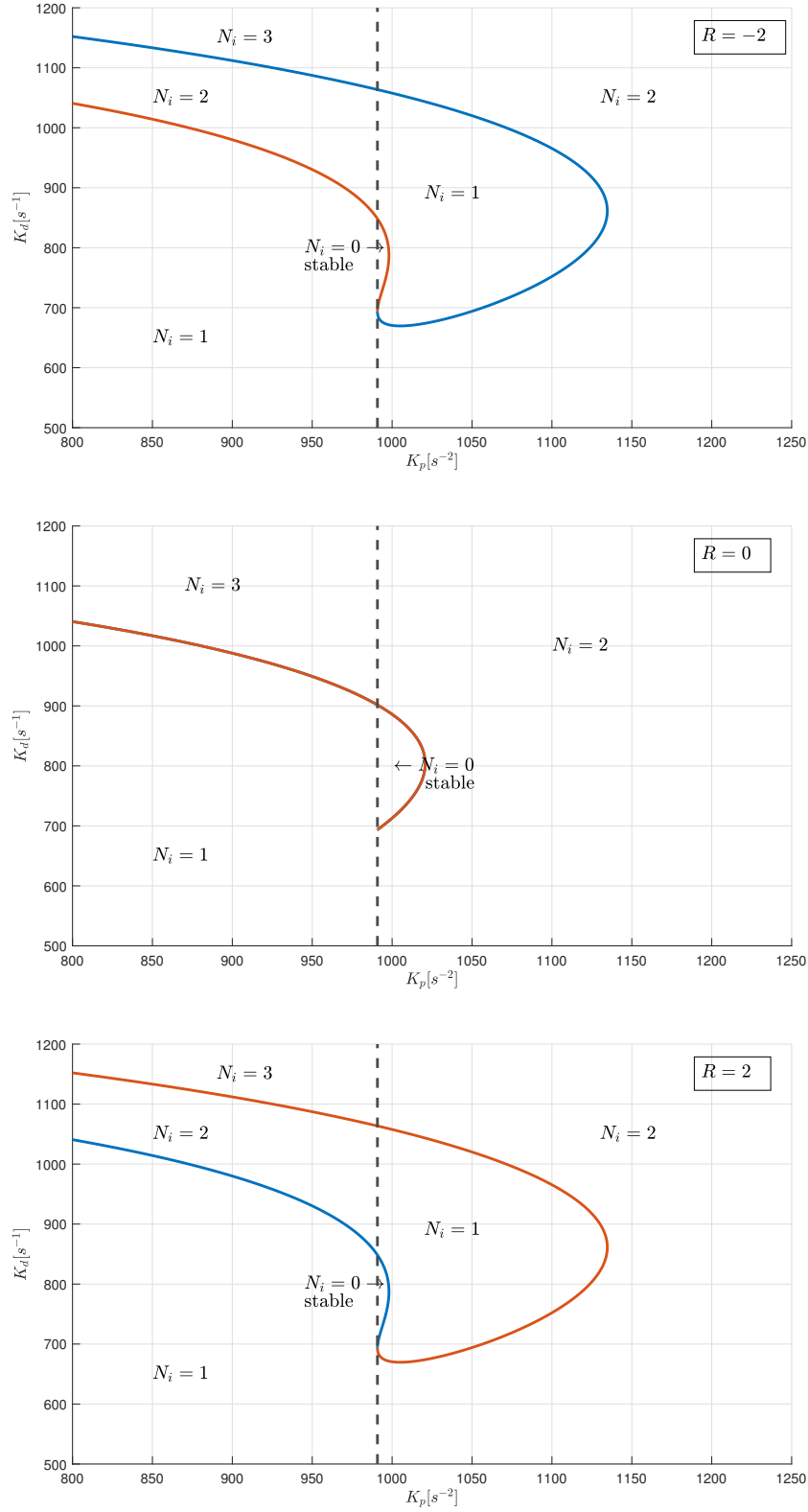


Figure 2.2: Stability chart of a delayed feedback PD controller with the feedback delay $\tau = 0.7$ s, with the real length $L = 5$ m and with imaginary length $R = -2, 0, 2$ m (top, middle, bottom panels).

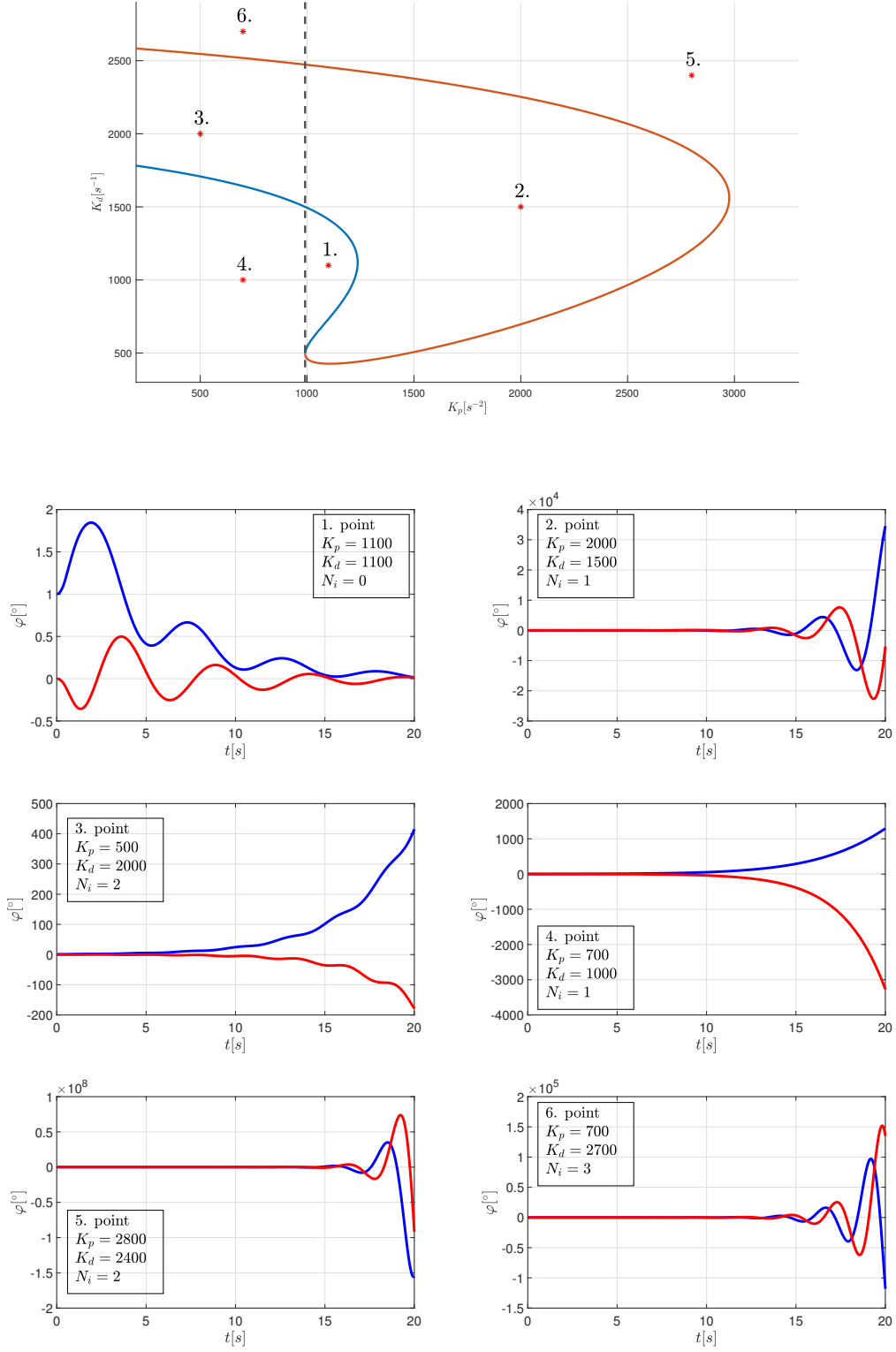


Figure 2.3: The stability chart of a delayed PD controlled inverted pendulum can be seen in the top panel with complex-valued length $l = 8 + 4i$ m and time delay $\tau = 0.5$ s. On the bottom panels, the time history of the real (blue) and imaginary (red) pendulum angle is shown for different points in the stability chart.

2.5 Number of unstable roots in the stability chart

In the stability chart the only stable region is between the vertical line $K_p = \frac{\alpha}{\delta}$ and the closest dynamic D-curve. Here the number of unstable roots is zero. Going from the inside to the outside of this region, the number of the unstable roots is increasing by one when passing through a dynamic D-curve or the vertical parametric line as it can be seen in Figure 2.2.

The complex-valued length inverted pendulum can be interpreted as a coupled real and an imaginary-valued length pendulum, which has effect on each other. With this interpretation, the effect of the number of unstable roots for the convergence of a PD controlled real and imaginary-valued length pendulum can be investigated. On Figure 2.3 a simulation can be seen to investigate this effect using semi-discretization [6]. It is clearly visible that with zero unstable roots, the angle of the real-valued and imaginary-valued length pendulums converges and with increasing unstable roots, the two pendulum's angles diverge with increasing values.

2.6 Critical length and time delay

During controlling tasks the delay of the controller limits the stabilizability of the controlled system. In the case of stick balancing, a critical length l_{crit} can be defined for a given τ delay. If the length of the stick is below this critical length, the system cannot be stabilized because the stick will fall down faster than the controller could intervene. Equivalently a critical time delay τ_{crit} can be defined for a given stick length l . If the time delay of the controller is above this critical time delay, then the system cannot be stabilized because of the same reason. A series of stability diagrams are shown in Figure 2.4, where it can be clearly seen that the stable region shrinks when the imaginary part of the stick length is increased. This observation suggests that the critical delay is smaller when the stick length is a complex number. However, as will be shown in the next subsection, this is not the case.

2.6.1 Pendulum with real-valued length

The relation of the critical length and the critical time delay can be calculated from the Taylor series of the control parameters. In a case of real-valued length pendulum the Taylor series reads

$$K_p(\omega) = a + \left(1 - \frac{1}{2}a\tau^2\right)\omega^2 + \dots \quad (2.20)$$

where

$$a = \frac{3g}{2l}.$$

If the coefficient of the second-order term ω^2 is bigger than zero, then there is a stable region. This gives

$$1 - \frac{1}{2} \frac{3g}{2l} \tau^2 > 0 \quad (2.21)$$

so

$$l > \frac{3}{4}g\tau^2 \quad (2.22)$$

$$\tau < \sqrt{\frac{4l}{3g}}. \quad (2.23)$$

So the critical parameters are the following

$$l_{\text{crit}} = \frac{3}{4}g\tau^2 \quad (2.24)$$

$$\tau_{\text{crit}} = \sqrt{\frac{4l}{3g}}. \quad (2.25)$$

2.6.2 Complex-valued length pendulum

The Taylor series of the control parameter can be calculated in the case of complex-valued length pendulum too, as

$$K_p(\omega) = \frac{\kappa\beta + \alpha\delta}{\kappa^2 + \delta^2} + \underbrace{\left(\frac{\kappa\alpha\tau - \beta\delta\tau}{\kappa^2 + \delta^2}\right)}_{C_1} \omega + \underbrace{\left(\frac{-\frac{1}{2}\alpha\delta\tau^2 + \delta - \frac{1}{2}\kappa\beta\tau^2}{\kappa^2 + \delta^2}\right)}_{C_2} \omega^2 + \dots \quad (2.26)$$

The first dominant part is the linear part, but its coefficient is equal to zero.

$$C_1 = \frac{\kappa\alpha\tau - \beta\delta\tau}{\kappa^2 + \delta^2} = \frac{-\frac{36gLR}{c^2(m+m_0)(L^2+R^2)^2} + \frac{36gLR}{c^2(m+m_0)(L^2+R^2)^2}}{\kappa^2 + \delta^2} \tau = 0 \quad (2.27)$$

The next dominant term is the quadratic one with coefficient C_2 . From this, the relation between the critical length and the critical time delay can be calculated.

$$C_2 = \frac{\delta}{\kappa^2 + \delta^2} - \frac{-\frac{1}{2}\alpha\delta\tau^2 - \frac{1}{2}\kappa\beta\tau^2}{\kappa^2 + \delta^2} > 0 \quad (2.28)$$

$$\Rightarrow \frac{\delta}{\kappa^2 + \delta^2} = \frac{-\frac{1}{2}\alpha\delta - \frac{1}{2}\kappa\beta}{\kappa^2 + \delta^2} \tau_{crit}^2 \quad (2.29)$$

$$\tau_{crit} = \sqrt{\frac{\delta}{\frac{1}{2}\alpha\delta + \frac{1}{2}\kappa\beta}} \quad (2.30)$$

The symbolic value of the pendulum parameters can be substituted. After substituting the pendulum length as $l = L + Ri$, one get

$$\begin{aligned} &\Rightarrow \tau_{crit} = \\ &= \sqrt{\frac{\frac{6L}{c(m+m_0)(L^2+R^2)}}{\frac{1}{2}\frac{6gL}{c(L^2+R^2)}\frac{6L}{c(m+m_0)(L^2+R^2)} + \frac{1}{2}\frac{6gR}{c(L^2+R^2)}\frac{6R}{c(m+m_0)(L^2+R^2)}} = \\ &= \sqrt{\frac{6Lc(L^2+R^2)}{\frac{1}{2}6gL6L + \frac{1}{2}6gR6R}} = \sqrt{\frac{Lc(L^2+R^2)}{3g(L^2+R^2)}} = \sqrt{\frac{Lc}{3g}} \end{aligned} \quad (2.31)$$

$$\tau_{crit} = \sqrt{\frac{cL_{crit}}{3g}} \quad (2.32)$$

A surprising result is that this critical length is independent of R . This means that extending the stick length with an imaginary part does not effect the critical reaction delay. Still, as can be seen in Figure 2.4, the stable region shrinks with increasing imaginary pendulum length. This suggest that a kind of robust critical delay does depend on the imaginary part. This robust critical delay can be defined based on the size of the stable region, actually based on the largest rectangle that can be entirely placed within the stable region.

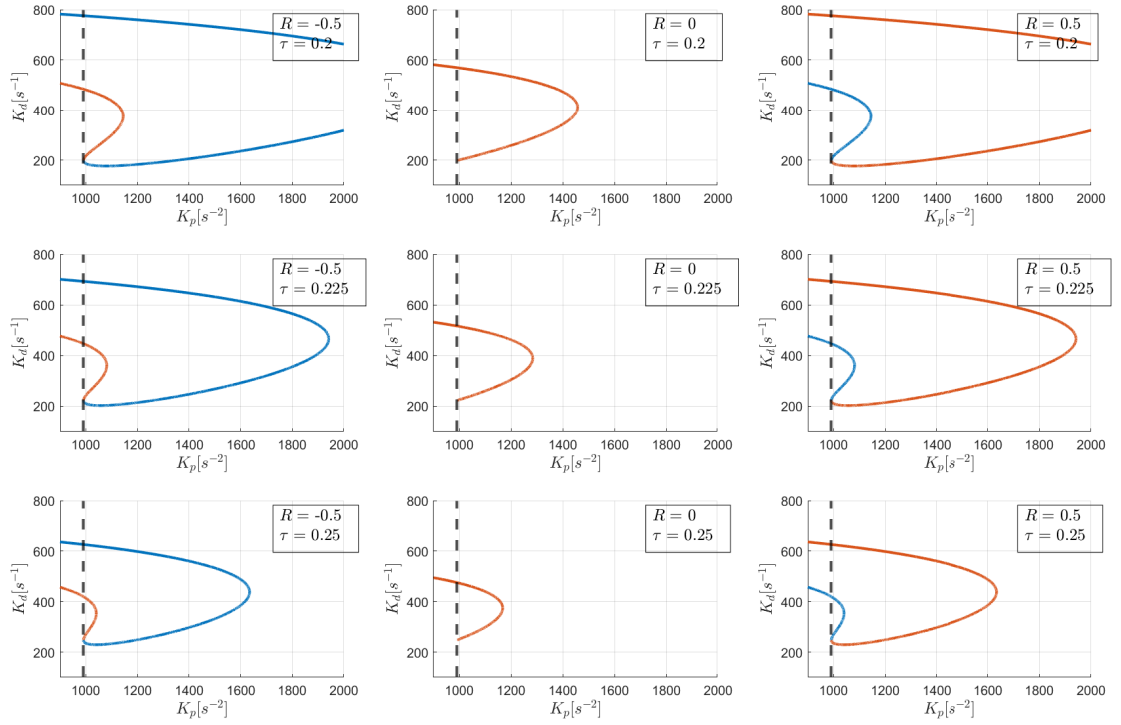


Figure 2.4: Stability charts of a delayed feedback PD controller with real length $L = 8$ m and with several feedback delay time and imaginary length.

2.7 Concept of robust critical delay

In the case of real-valued length pendulum, increasing time delay results in decreasing stable region. It can be seen from equation (2.32) that the value of the critical time delay is only affected by the real part of the length and not by the imaginary part. However from the figure 2.4 it can be seen that the imaginary length affects the size of the stable region. Increasing R results in decreasing the size of the stable region. This means that the robustness of the system is decreasing with increasing imaginary length. When the stable region is smaller, it is more difficult for the human subject to balance the inverted pendulum because it is more difficult to select proper derivative and proportional gains. The decreasing of the robustness can be investigated by introducing a variable ε as a measure of tolerance of the system [8]. In the middle of the stable region the parameters are p^* and d^* . Proportionally with these parameters a rectangle can be drawn within the stable region with sides $2\varepsilon p^*$ and $2\varepsilon d^*$ as shown in Figure 2.5 For a given stick length and tolerance, the critical time delay can be calculated in the way that the time delay

is iterated with smaller and smaller values. The case when the rectangle fits within the stable region and with the next iteration step of τ value the rectangle does not fit in the region, gives the critical time delay.

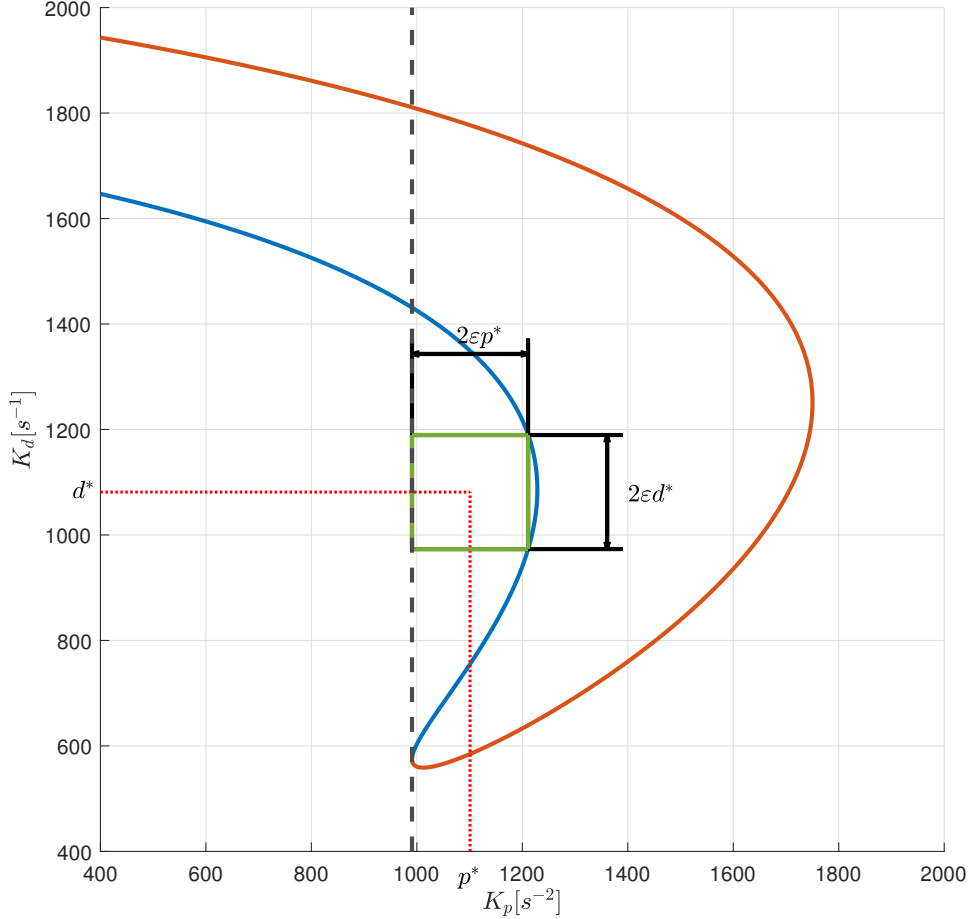


Figure 2.5: Stability chart of a delayed feedback PD controller with the complex length $l = 8 + 2i$ m and with 10% tolerance rectangle, so the $\varepsilon = 0.1$. This is the iteration step where the tolerance rectangle just fit in the inner D-curve. With this method, the critical time delay can be calculated. In this case $\tau_{\text{crit}} = 0.58$ s.

In Figure 2.6 the critical time delay values can be seen as a function of imaginary length for different tolerances ε . It can be seen that as the tolerance is decreasing, the τ_{crit} tends to a horizontal line. This is because as the tolerance approaches to zero, the rectangle approaches to be a point and in this case the disappearing of the stable region

is examined without tolerance and without tolerance the value of the critical time delay does not depend on the value of the imaginary length.

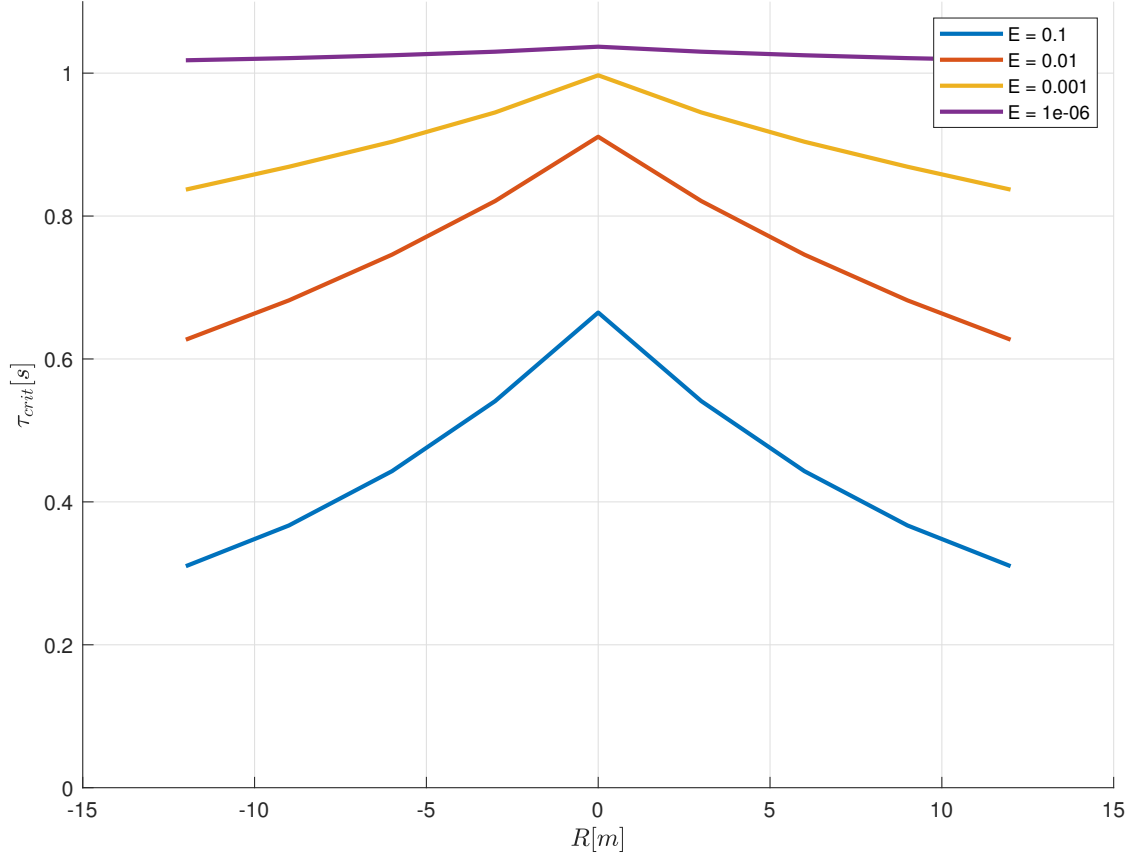


Figure 2.6: The critical time delay as a function of the imaginary length with several tolerance parameter ε . The real length in this example is $L = 8$ m.

3 Methods

3.1 Mathematical model

In the software the non-linearized form of the equation of motion of the inverted pendulum can be used, where the pivot point of the pendulum is attached to a cart and this cart is driven by the human subject with the help of the computer mouse. The governing equation reads

$$\ddot{\varphi}(t) + \frac{3g}{2l} \sin(\varphi(t)) = - \underbrace{\frac{3}{2l} \cos(\varphi(t)) \ddot{x}}_u, \quad (3.1)$$

where $\ddot{x}$ is the acceleration of the cart which is fixed to the pendulum. Two unknown variable is in this equation, but during the application of the virtual balancing software, the $\ddot{x}$ is equal to the acceleration of the mouse. Thus the equation is numerically solvable. In this task, the length of the stick is a complex number, so the control action

$$u = -\frac{3}{2l} \cos(\varphi(t)) \ddot{x} \quad (3.2)$$

has to be decomposed into real and imaginary part, as

$$\begin{aligned} u &= -\frac{3}{2L + 2Ri} \cos(\varphi(t)) \ddot{x} = \frac{-6L + 6Ri}{4L^2 + 4R^2} \cos(\varphi(t)) \ddot{x} \rightarrow \\ u &= \underbrace{-\frac{6L}{4L^2 + 4R^2} \cos(\varphi(t)) \ddot{x}}_{Real} + \underbrace{\frac{6R}{4L^2 + 4R^2} \cos(\varphi(t)) \ddot{x}}_{Imaginary} i. \end{aligned} \quad (3.3)$$

Hence, the angular acceleration reads

$$\ddot{\varphi} = \frac{3g}{2l} \sin(\varphi(t)) + u \quad (3.4)$$

$$\ddot{\varphi} = \frac{3g}{2L + 2Ri} \sin(\varphi(t)) + u = \frac{6gL - 6gRi}{4L^2 + 4R^2} \sin(\varphi(t)) + u$$

Thus, the whole equation can be separated into a real and an imaginary part.

Real part:

$$\text{Re}(\ddot{\varphi})(t) = \frac{6gL}{4L^2 + 4R^2} \sin(\varphi(t)) - \frac{6L}{4L^2 + 4R^2} \cos(\varphi(t)) \ddot{x} \quad (3.5)$$

Imaginary part:

$$\text{Im}(\ddot{\varphi})(t) = \left(-\frac{6gRi}{4L^2 + 4R^2} \sin(\varphi(t)) + \frac{6R}{4L^2 + 4R^2} \cos(\varphi(t)) \ddot{x} \right) i \quad (3.6)$$

These two equations were implemented into the software. The software therefore accounts with a real and an imaginary dynamics of the pendulum and both affected by the acceleration of the computer mouse. Both pendulum contain the real and the imaginary length too, so they affect each other too.

3.2 Virtual measurement environment

The virtual balancing task was implemented by a software developed in a JAVA environment. The original program code was provided by Balázs Kovács from the department of Applied Mechanics [3] [1] and was developed further to handle complex numbers in the case of the inverted pendulum length. The governing differential equation (3.1) is solved by using a fourth-order Runge-Kutta method with an adaptive time step, which corresponds to the screen refresh rate 60 Hz [3]. The input device during the measurement was a wireless mouse with a receiver. The mouse was controlled by the subjects' hands. The velocity and the acceleration of the input device was measured and the actuation was determined by using of a numerical derivation scheme based on the pixel position of the cursor on the screen [2]. Because of the numerical derivation, the resulted signal was noisy, so a sampling filter was applied. The inverted pendulum was seen on the computer screen and the subjects were instructed to try to focus on the top of the pendulum and try to move the computer mouse continuously to approach a PD controller more precisely. The test environment can be seen in Figure 1.1 and the virtual environment can be seen in Figure 3.1.

3.3 Machine delay

During a measurement, several types of machine delay was appeared. The measurement is sampled, so a sampling delay was introduced with the value of $\tau_{\text{sampling}} = 8.33$ ms. Also, a filtering delay was appeared with the approximate value of $\tau_{\text{filtering}} = 50$ ms. The computer also needs a processing time, while the input (e.g. a mouse click) is processed and appeared on the computer screen. This can called as the input-to-output lag of the machine and it introduces the response delay of the machine τ_{response} . The machine delay is the sum of these delays and its value can vary for different configurations.

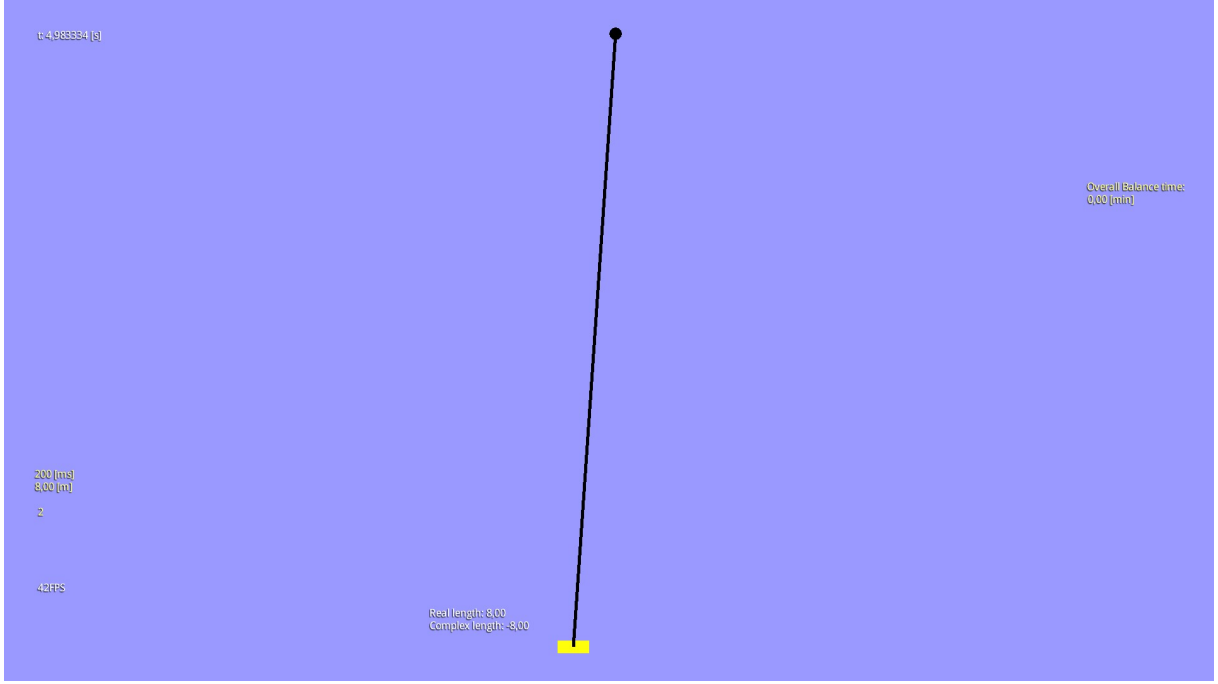


Figure 3.1: The virtual environment. This was the screen that was seen by the subjects.

For this measurement its value was approximated as 125 ms

$$\tau_m = \tau_{\text{sampling}} + \tau_{\text{filtering}} + \tau_{\text{response}} \approx 125[ms]. \quad (3.7)$$

The overall reaction delay in the measurement therefore

$$\tau_r = \tau_m + \tau_n + \tau_{\text{added}}, \quad (3.8)$$

where τ_n is the reaction delay of the subject's neural system and τ_{added} is the delay that is manually added in the system to measure the critical time delay. The added delay was only used during the balancing measurements, in the case of the reaction time measurements the added delay was zero $\tau_{\text{added}} = 0$.

3.4 Subject's reaction measurement

3.4.1 Instant reaction test

The reaction delay of the subjects was measured with two methods. The first method was the instant reaction test. In this test the subject had to push the mouse button when a red rectangle flashed randomly on the computer screen. The subjects had to repeat this measurement 10 times and the average of these counted as the instant reaction time.

This method also involves the input-to-output lag of the screen, which is the time from the moment the subject clicked with the mouse to the moment the screen displayed it. During this method the human subject detect the red flesh and immediately react so it is not necessary to take any decision. However during inverted pendulum balancing after detection, the subjects have to decide which direction they have to intervene with how large force.

3.4.2 Blank-out test

A better way to estimate the reaction time of the subjects is the blank-out test. In this method the subjects balance an inverted pendulum and at a random time moment, the stick disappears for 0.5 s. After the reappearance of the stick, the subjects intervene in the system after their reaction time. The overall reaction time is the time between the moment, when the stick reappeared and the moment, when the subject intervened. This moment of intervention is the moment, when the cart started to move, therefore the moment, when the acceleration of the cart is started to change in either direction as it can be seen in Figure 3.2.

Beside the input-to-output lag, this method also takes account the decision process before the intervention and the filtering and sampling delays. Therefore this type of reaction test make a better approximation for the human neural reaction delay and for the machine delay in the case of inverted pendulum balancing.

3.5 Participants

5 subject was recruited for the measurement, 1 female and 4 male. The age of the subjects was 15, 18, 22, 23 and 46. All subject was free of any neurological or musculoskeletal impairment that could effect virtual stick balancing. Two of the subjects (Subject 1 and Subject 5) are experienced with virtual stick balancing and for the other three subjects, the measurement was the first time to see this program.

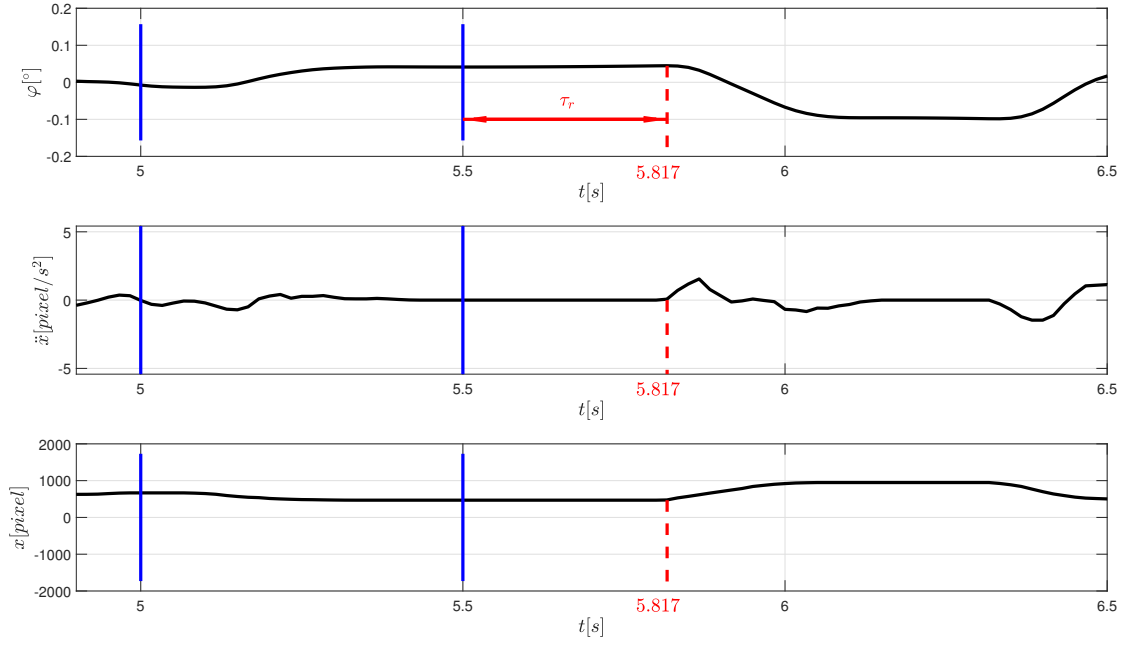


Figure 3.2: Example for a blank-out test with a stick length $l = 8 + 0i$ m. The angle φ of the pendulum, the acceleration $\ddot{x}$ of the cart and the position x of the cart is depicted in the figure. The acceleration of the cart is starting to change at time $t = 5.817$ s, so the reaction time of the subject is $\tau_r = 0.317$ s.

3.6 Balancing protocol

Each measurement was performed in the following way:

1. Practice with a 8 m long pendulum without adding artificial time delay: 15-20 min. The practice was lasted until the subject performed a balancing without any difficulty for 20 s.
2. Instant reaction time test: 10 trials.
3. Blank-out reaction time test: 15 well balanced trials.
4. Critical time delay measurement. The real length L of the stick was 8 m during the measurement. The initial imaginary length $R = 0$ m. The critical time delay was measured by using the halving method. First, the subject had to balance the stick with $\tau_{\text{added}} = 200$ ms added delay for 5 times. The balance was successful if 20 s balance time was reached without the stick's angle exceeded the $\pm 20^\circ$ for at least

once out of the five trials. If the balance was successful the added delay was changed to 300 ms, if it was not successful it was changed to 100 ms. The five balance trials was performed again and if it was successful or unsuccessful the added delay was increased or decreased with 50 ms and so on. After four steps the added delay was changed by 25 ms. The critical time delay was the last time delay wherewith the subject could successfully balance the pendulum.

5. Step 4 was repeated for imaginary length $R = 12, 8, 4, 0, -4, -8, -12$ m with real length $L = 8$.

With this method the critical time delay was determined with 25 ms precision for the given imaginary lengths. The whole measurement was typically took 60 min to perform.

3.7 Sequence of measured lengths

During the measurement the subjects balanced the inverted pendulum with different sequence of imaginary lengths, but with the same values. The sequence of the imaginary length R can be read from Table 3.1.

Sequence of R	Subject 1	Subject 2	Subject 3	Subject 4	Subject 5
1.	$0m$	$-4m$	$-4m$	$4m$	$0m$
2.	$4m$	$-8m$	$-8m$	$8m$	$-4m$
3.	$8m$	$-12m$	$-12m$	$12m$	$4m$
4.	$12m$	$12m$	$12m$	$-12m$	$-8m$
5.	$-12m$	$8m$	$8m$	$-8m$	$8m$
6.	$-8m$	$4m$	$4m$	$-4m$	$-12m$
7.	$-4m$	$0m$	$0m$	$0m$	$12m$

Table 3.1: The sequence of the set imaginary lengths during the measurement for each subjects.

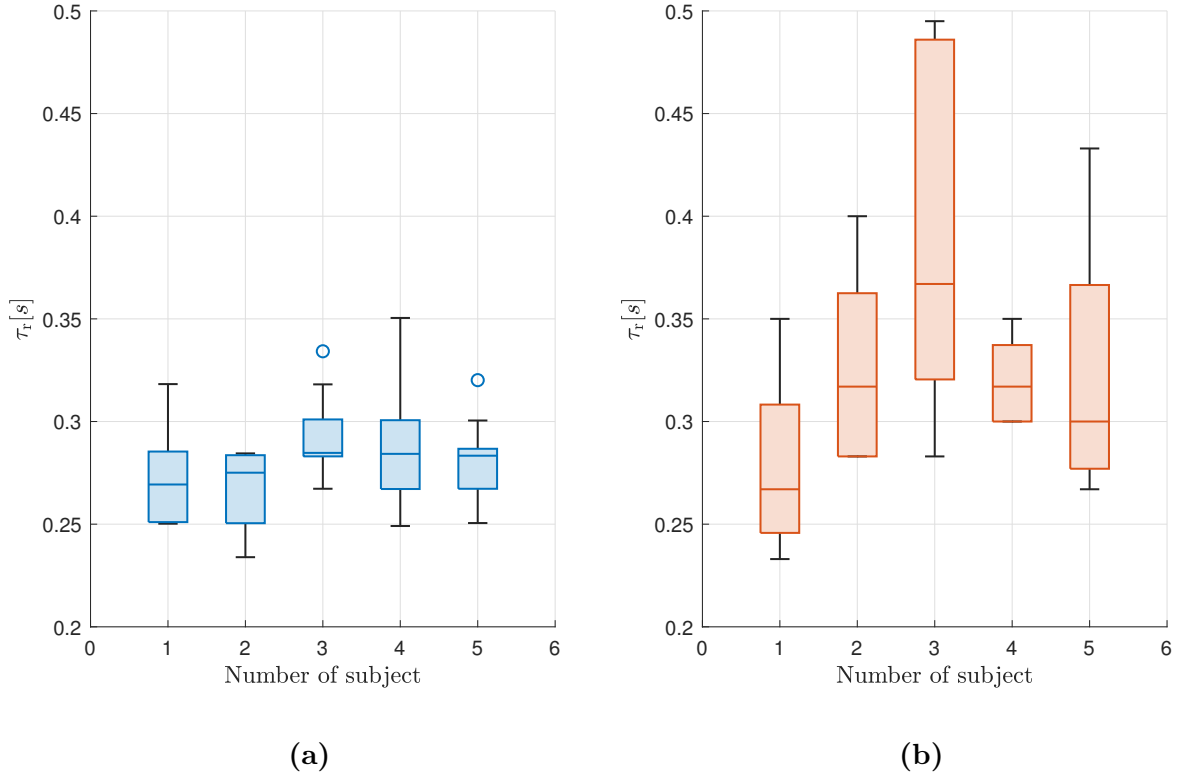


Figure 4.1: The results of the two types of reaction tests. The instant reaction test (a) and the blank-out reaction test (b). Blue/red lines boxes: interquartile range (IQR) with median, whiskers: min/max values, circles: outliers.

4 Results

4.1 Reaction time measurement

The reaction time of the subjects was measured by two different methods. The results of the two type of reaction measurement can be seen in Figure 4.1. The first reaction measuring method was the instant reaction test. This was the less difficult method because the subjects only had to push the computer mouse button as a response of a visual input, they did not have to make any decision. The blank-out test is the more difficult method because the subjects had to make a decision that they intervene in the system in what direction and with how large force. The control loop in this case also involves the machine delay. The difficulty level of the reaction tests, as expected, can be clearly seen in Figure 4.1. The reaction time results of the blank-out tests are higher, which means that for the subjects it has taken more time to react. It also can be seen in the figure that the

deviation of the blank-out test results is larger than the deviation of the instant reaction test results. This is because of the difficulty level of the tasks, the deviation of the instant reaction test is smaller because it is more controllable. It can also be seen on the figure that the two more experienced subjects (Subject 1 and Subject 5) have smaller difference between the instant reaction test and the blank-out test.

The average result of the instant reaction test with 95% confidence was 293.81 ± 71.79 ms and the result of the blank-out test with the same confidence was 321 ± 63.95 ms.

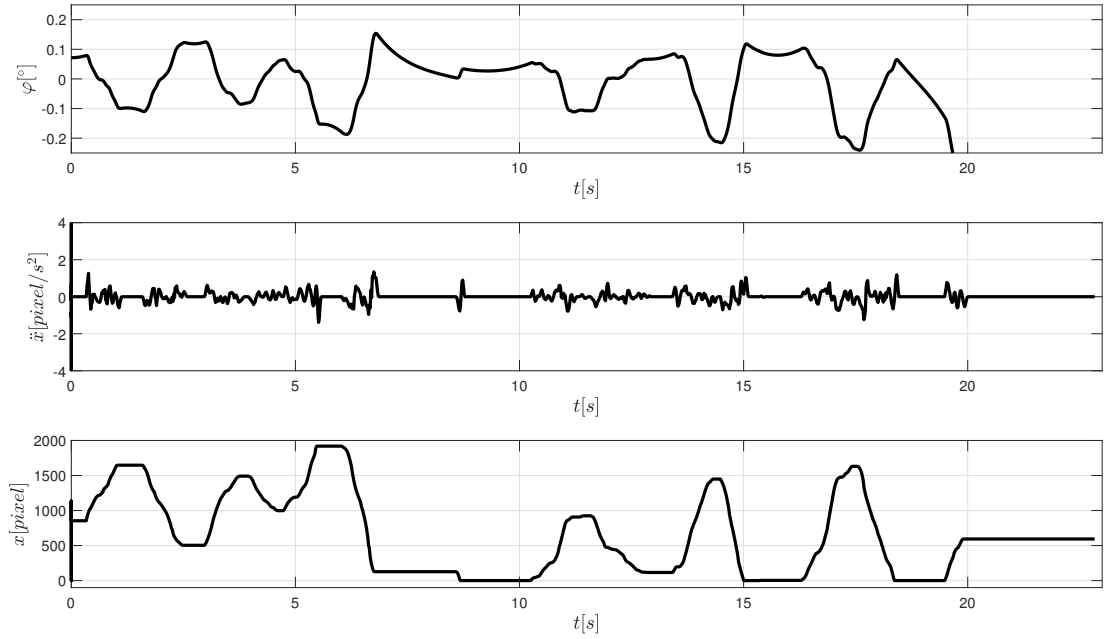


Figure 4.2: Balancing test with complex length $l = 8+0i$ m with added delay $\tau_{\text{added}} = 200$ ms. The imaginary length is zero, therefore the actuations are small and fast movements. Panels show the angular position φ of the stick in degrees (top), the acceleration $\ddot{x}$ of the cart (middle) and the position x of the cart (bottom).

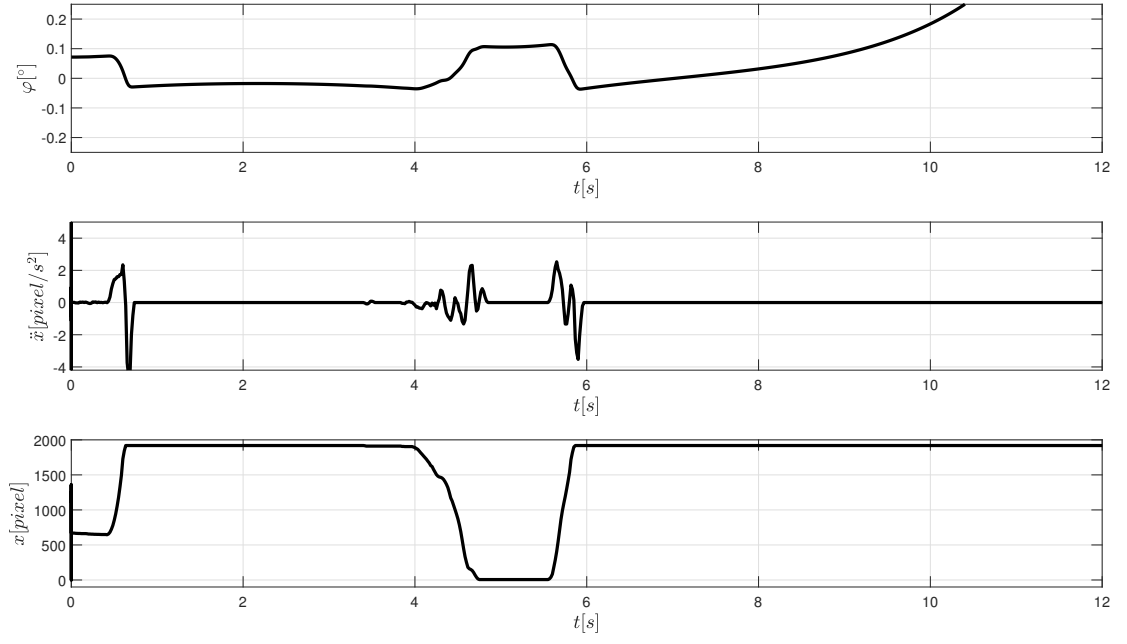


Figure 4.3: Balancing test with complex length $l = 8 - 12i$ m with added delay $\tau_{\text{added}} = 200$ ms.

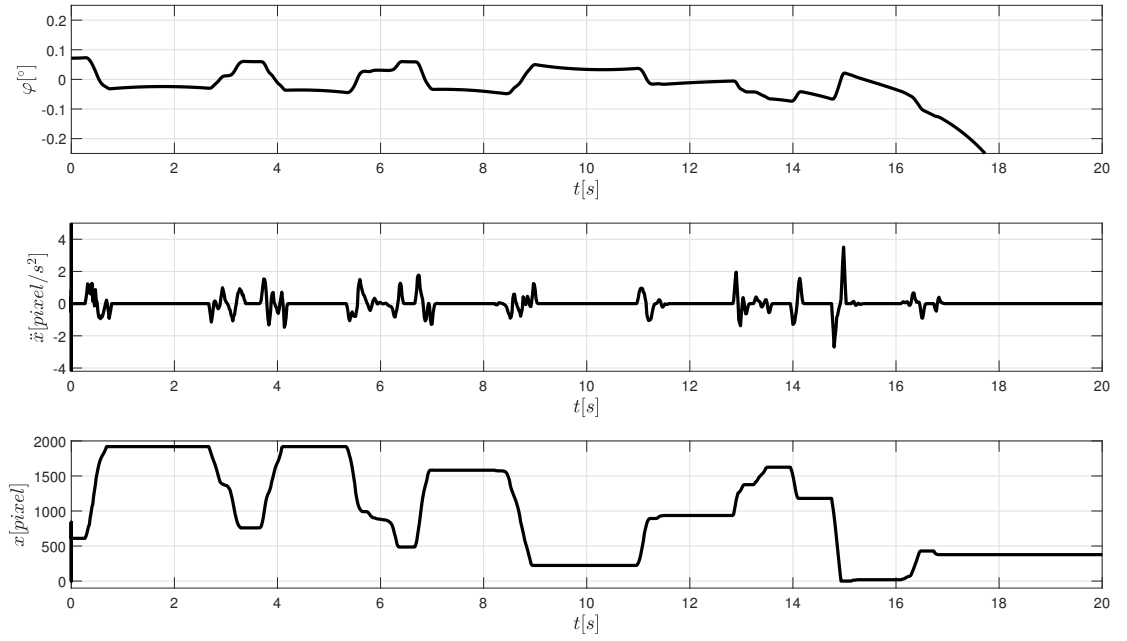


Figure 4.4: Balancing test with complex length $l = 8 - 12i$ m with added delay $\tau_{\text{added}} = 100$ ms. The same parameters as in Figure 4.3, just with smaller added delay.

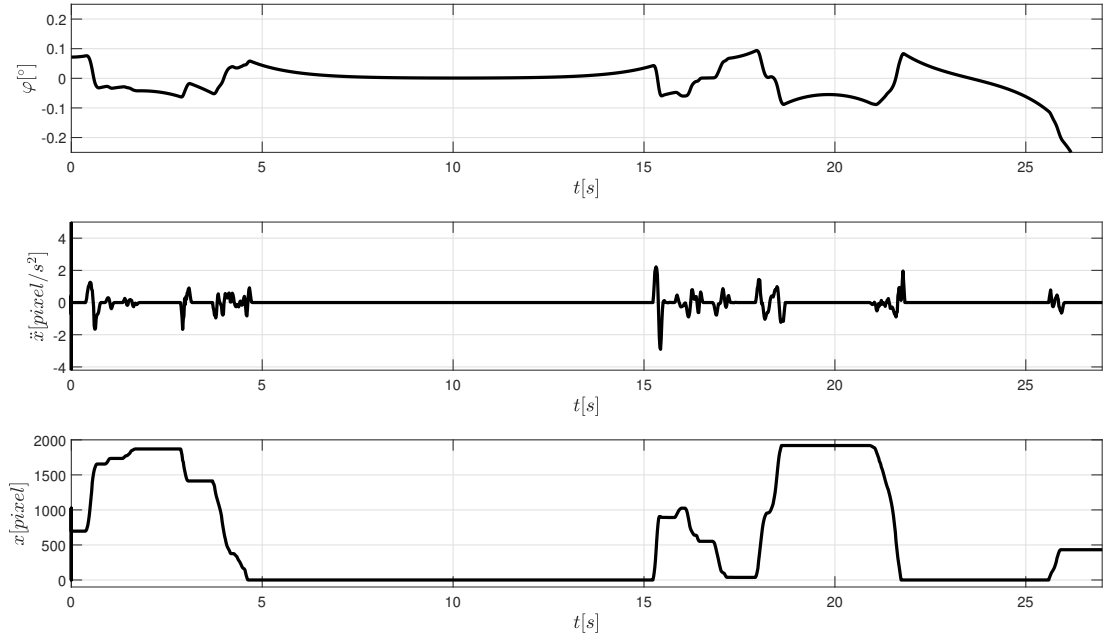


Figure 4.5: Balancing test with complex length $l = 8+9i$ m with added delay $\tau_{\text{added}} = 250$ ms.

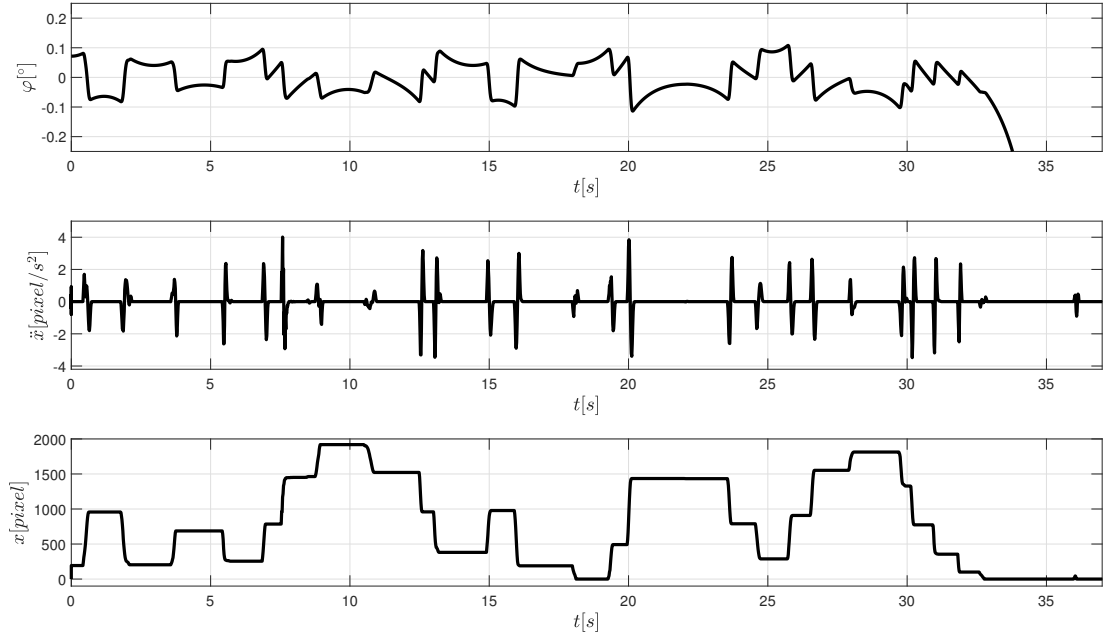


Figure 4.6: Balancing test with complex length $l = 8+4i$ m with added delay $\tau_{\text{added}} = 50$ ms.

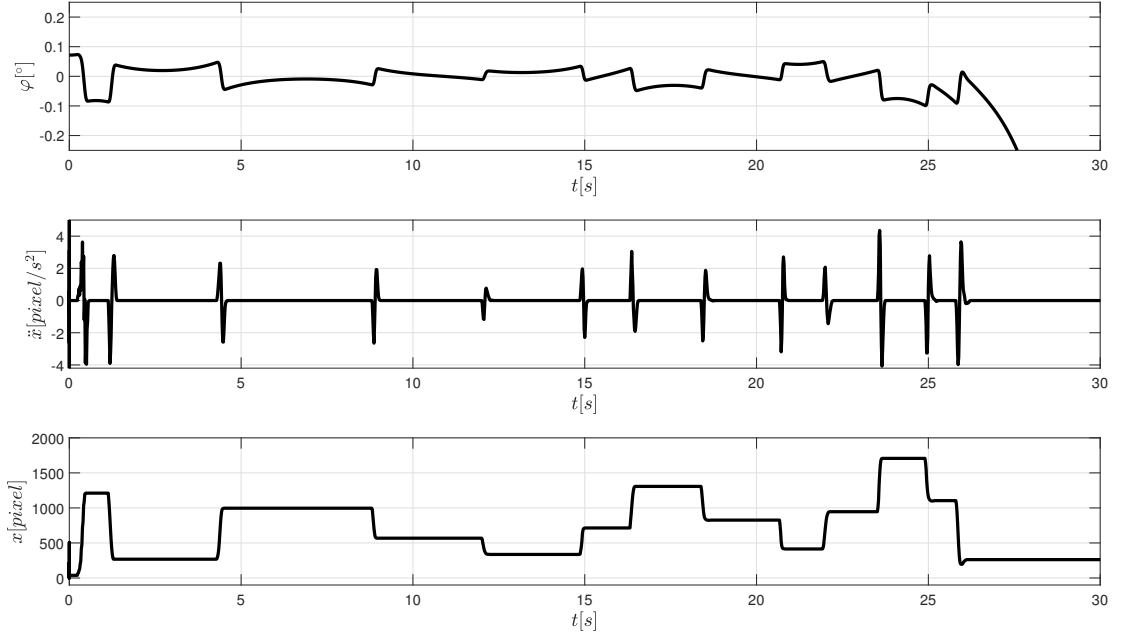


Figure 4.7: Balancing test with complex length $l = 8 - 8i$ m with added delay $\tau_{\text{added}} = 100$ ms.

4.2 Balancing trials

During the balancing trials, several observations were drawn. It was visible that without imaginary length, the pendulum has faster dynamics. In this case the subjects continuously controlled the stick by moving the computer mouse and with this the cart, with smaller, more frequent movements. An example for this can be seen in Figure 4.2. As the imaginary length was increased, the dynamics of the inverted pendulum was slower and slower. Therefore the subjects controlled these less continuously with bigger and less frequent movements. An example for this can be seen in Figure 4.3 and in Figure 4.4 with a smaller added delay. Because of the slower dynamics, the stick leans slower what can be seen if we compare the steepness of the time scale of the angle signals, for example in Figure 4.2 and Figure 4.4.

With increasing imaginary-valued length the unknown perturbation increased too, so it was more difficult for the subjects to restabilized the pendulum if it leaned too much from the upper equilibrium position. This means that the subjects managed to stabilize a stick with smaller imaginary-valued length with a larger angle. Therefore it can be seen in the figures that the subjects tried to keep the larger imaginary length pendulum closer to the

upper equilibrium. This can be seen if we compare for example Figure 4.6 and Figure 4.7, where the lengths are $l = 8 + 4i$ m and $l = 8 - 8i$ m.

Two other interesting figures can be seen in Figure 4.5 and 4.6. In Figure 4.5, it can be seen that the pendulum was stood in the upper equilibrium for a long time. This can be explained with the slow dynamics. In Figure 4.6 it can be seen that the subject managed to balance the pendulum for a long time. With small added delay, it was easier to balance because the overall delay was far from the critical delay.

4.3 Imaginary length effect on balancing

The results of the balancing measurement for all subject can be seen in Figure 4.8. These results are the critical delay for different imaginary lengths in the case of all subject. The balancing measurement resulted the added delay τ_{added} and the blank-out test resulted the sum of the machine delay and the neural time delay of the subject. This sum was chosen to be the average of the blank-out test time delay in the case of each subject. In Figure 4.8, the critical time delay is the sum of the added delay, the neural delay and the machine delay $\tau_{\text{crit}} = \tau_{\text{added}} + \tau_{\text{n}} + \tau_{\text{m}}$.

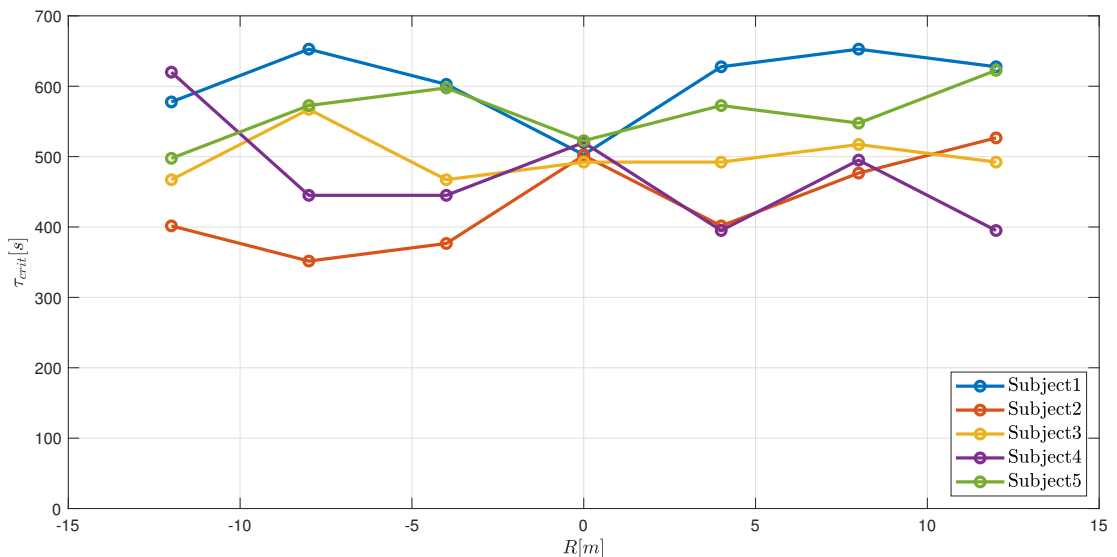


Figure 4.8: The measurement results of the balancing test. The critical time delay can be seen as a function of the imaginary length. The real length of the stick is $L = 8$ m.

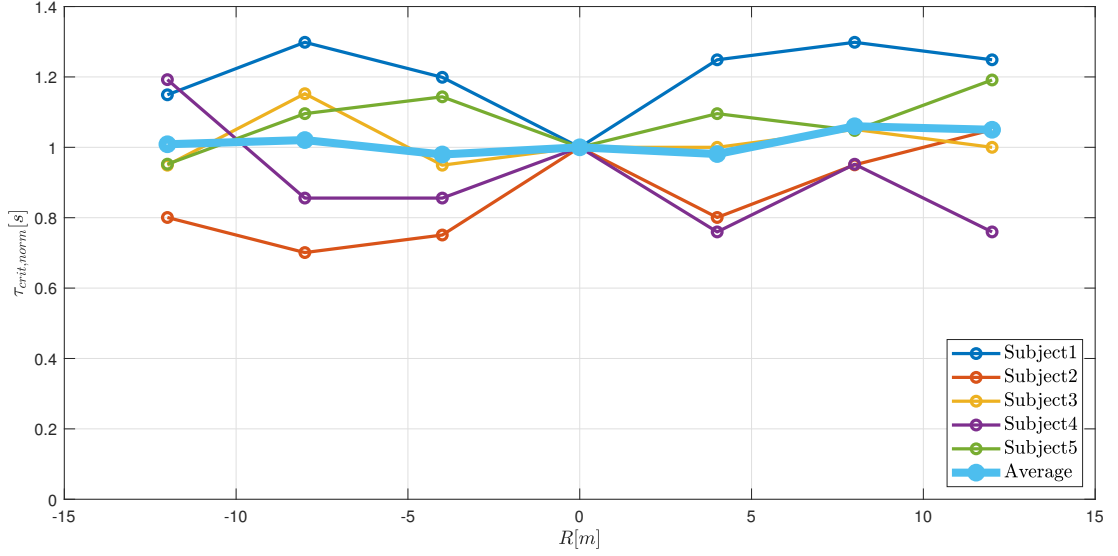


Figure 4.9: The nominal values of the balancing test.

The balancing was made in different sequence by the subjects which can be read from Table 3.1. These sequences can be seen in the figure because in general, the balancing of the same valued imaginary length with opposite sign was more successful if the subject balanced it later. This effect can be explained with learning. The subjects practiced more in the case of the later value so they learned better how to balance it. It is also visible that for the more experienced subjects (Subject 1 and Subject 5) it was easier to balance the pendulum, so their critical time delay is a higher value. The analytical critical time delay curve is a symmetric curve for the same imaginary length values with opposite sign. For the more experienced subjects the critical delay curve approaches this symmetry more precisely.

To get a proportion that it was more or less difficult to balance the stick with added imaginary length compared to the real-valued length pendulum, the critical delays can be divided by the critical delay corresponding to the real-valued length stick. With this method it can be seen that the balancing of a given imaginary length was easier if its divided value is larger than 1 and it was more difficult if its divided value is smaller than 1. The divided values of the balancing tasks can be seen in Figure 4.9 Note that the average of these values approaches a horizontal line. This observation corresponds to the theoretical result that the critical delay without taking robustness into account is independent of the imaginary stick length as shown in (2.32.)

5 Summary

5.1 Results

In physiology, it is an often used technique to add extra time delay to investigate the dynamics of the motion control. In this experiment it was used to determine the maximum time delay that the subject is still able to balance. With this method the dynamics of the complex-valued inverted pendulum was identified.

In the presence of large imaginary length the dynamics of the stick was slower, which is similar to the real-valued length case. The real-valued length pendulum dynamics is slower when the length of the stick is larger. Therefore with large imaginary length it was easier for the subjects to learn the control of the pendulum. This effect explains the larger critical delay difference between the large positive and negative imaginary lengths and the smaller difference between the small positive and negative imaginary length. However with larger imaginary length the powerlessness was more uncertain (e.g. the stick leaned out from the screen easier).

With the change of the imaginary length the balancing strategy of the subjects had changed. With small imaginary lengths the control was continuous with small and frequent movements because the stick moved faster. This control strategy is approaching the PD controller strategy better. If the imaginary length was large, the control was less continuous with big and less frequent movements, which is often called intermittent control in the literature. Often the subjects moved the pendulum at one side of the screen waited until it falls a bit and then moved it at the other side of the screen. This control strategy approach some kind of bang-bang control strategy where the control actuation is only the positive and the negative maximum actuation.

From Figure 4.9, it can be seen that the average critical time delay approaches a horizontal line. From the theoretical part it can be seen that if the critical time delay indeed does not depend on the imaginary length. This is a surprising result because this means that the decreasing robustness of the system has a small effect in human balancing. This means that if the human neural system is modeled as a PD controller, it can select the proportional and the derivative gains very precisely.

5.2 Conclusions, outlook

As an overall conclusion, it can be said that the imaginary length has a small effect on the human balancing time, but has a larger effect on the control strategy during balancing. This experiment can be further developed in several ways. The measurement could be done with more subject to get a wider spectrum in the measurement. Also, the measurement could be done with the variation of the real length too. The tolerance of the human velocity and position sense is different, so this measurement could be done by varying the velocity and position tolerances with different values.

Összefoglaló

Az élettani tudományokban gyakran használt eljárás a mozgás szabályozás dinamikai vizsgálatában, hogy mesterséges időkésés van hozzáadva a rendszerhez. Ebben a kutatásban ez a módszer volt használva, azzal a céllal, hogy meghatározzuk azt a maximális hozzáadott időkésést, amit a mérési alany még épphogy tud egyensúlyozni és így meghatározzuk a komplex értékű rúd hossz hatását az inverz inga emberi egyensúlyozhatóságára.

A rendszer stabilitását a karakterisztikus egyenlet gyökeinek, valós részének vizsgálatával lehet elvégezni. Mivel a karakterisztikus egyenlet együtthatói komplex számok, a karakterisztikus gyökök nem konjugált párok. Ennek megfelelően a stabilitási határok negatív és pozitív frekvenciákra különbözőek lesznek. Az időkésés növelésével a stabilitási tartomány csökken, és egy kritikus időkésés esetén eltűnik a stabil tartomány. A kritikusnál nagyobb időkésés esetén nem lehet stabilizálni az ingát. A képzetes rúd hossz növelésével szintén csökken a stabil tartomány mérete, de a kritikus időkésés nem függ a képzetes rúdhosztól.

Nagy értékű képzetes rúd hossz esetén, az inga dinamikája lassabb, hasonlóan a valós értékű inga hossz esetéhez, ahol szintén lassúbbá válik a dinamika, ha növeljük a rúd hosszát. Emiatt nagy értékű képzetes rúd hossz esetén a mérési alanyoknak könnyebb volt megtanulni az inga szabályozását, hiszen az lassabban mozgott. Ez a hatás magyarázza, hogy miért voltak különböző értékűek és nem szimmetrikusak a kritikus időkésések ugyanakkora értékű pozitív és negatív képzetes rúd hossz esetén, holott az elmélet szerint ez szimmetrikus. Ezt az is alátámasztja, hogy a tapasztaltabb alanyoknak jobban közelítettek a mérési eredményei a szimmetrikus görbéhez. Ugyanakkor a képzetes hossz növelésével nőtt a rendszer tehetetlenségének a bizonytalansága és kiszámíthatatlansága (például az inga könnyebben kidőlt a képernyőből, még akkor is, ha mélyebb helyzetből is indult, mint a csak valós rúd hosszú inga).

A képzetes rúd hossz változtatásával változott az alanyok egyensúlyozási stratégiája. Kis képzetes rúd hossz esetén a szabályozás folyamatosabb volt, kicsi és gyakori mozdulatokkal, hiszen az inga gyorsabban mozgott. Ez a szabályozási stratégia jobban közelíti a PD szabályozást. Ha a képzetes rúd hossz nagy volt, akkor a szabályozás kevésbé volt folyamatos és az alanyok nagy és kevésbé gyakori mozdulatokkal irányították az inga mozgását. Ezt a szabályozást a szakirodalomban gyakran megszakított szabályozásnak

nevezik. Nagyobb képzetes hosszok jelenlétében, a mérési alanyok gyakran a képernyő egyik végébe mozgatták az ingát, hagyták, hogy kicsit eldőljön, majd a képernyő másik végébe mozgatták. Ez a szabályozási stratégia egyfajta bang-bang szabályozáshoz közelít, ahol a szabályozási erő csak egy meghatározott minimum és maximum erő és a szabályozó ezeket váltogatja. A szabályozási stratégia változásából látszik, hogy az emberi idegrendszer jól tud alkalmazkodni eltérő szabályozási helyzetekhez.

A 4.9 Ábrán látszik, hogy az átlagos kritikus időkésés egy vízszintes vonalhoz közelít. A kutatás elméleti részéből látható, hogy a kritikus időkésés valóban nem függ a képzetes rúd hosszától. Ez egy meglepő eredmény, hiszen ez azt jelenti, hogy a rendszer csökkenő robosztussága nem befolyásolja az emberi egyensúlyozást. Ez azt jelenti, hogy ha az emberi idegrendszer PD szabályozóként van modellezve, akkor nagyon pontosan tudja kiválasztani a megfelelő arányos és derivált tagokat.

Összefoglalásként kijelenthető, hogy a képzetes rúd hossz kicsi hatással van az emberi egyensúlyozásra, azonban nagyobb hatása van a szabályozási stratégiára egyensúlyozás közben. Ez a kutatás sok féleképpen tovább fejleszthető. A mérést több alany részvételével is el lehetne végezni, különböző valós hosszokra, hogy egy szélesebb spektrumon vizsgáljuk a jelenséget. A valóságban az emberi sebesség és pozíció érzékelés bizonytalansága különböző, szóval a mérést úgy is el lehetne végezni, hogy külön változtatjuk a sebesség és a pozíció bizonytalanságát.

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