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**THE AMERICAN APPORTIONMENT SYSTEM AND ITS
ADAPTATION TO THE EUROPEAN UNION**

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1. INTRODUCTION

The system to elect the President of the United States has been controversial for a long time. Instead of directly electing the President, which is more common in presidential systems, the United States uses the Electoral College to elect its leader. The Electoral College assigns each state a certain number of electoral votes roughly proportional to its population. The number assigned to a state is the sum of the number of its senators and representatives in Congress. Every state has two senators, and the number of representatives is assigned proportionally to the state's population using a method devised by Edward V. Huntington. For this work, we will be using two articles ([Hun28], [Hun21]) in which Huntington describes his method. He also mentions alternative methods for apportionment, which I will describe in more detail.

Later in this work, we will also be adapting Huntington's method (the method of the geometric mean) and its variations (the method of the harmonic mean and the method of the arithmetic mean) to the European electoral system. After briefly describing how the current apportionment system for the European Parliament works, we will generate the apportionment for the European Parliament in two ways: with and without respecting the principle of degressive proportionality (see [Gri12, Page 2]).

For this work, I have also developed a Python program to make the apportionment easier. I made six versions: for each of the three methods, with and without degressive proportionality.

In Section 2, we will start by summarising the historical background of the Electoral College. After that, we will describe how Huntington arrived at the method of the geometric mean. A crucial question for the method will be the use of the absolute and the relative difference. When we understand how Huntington arrived at the method, we can look at the working rule, which establishes how we can use this apportionment method in practice. In this section, we will also see how we can arrive at alternative methods by taking slightly different steps compared to the ones Huntington took when developing his method. We will also be comparing the alternative methods to Huntington's method. Among other factors, we will see which method favours larger states over smaller states. At the end of this section, we will be modifying the calculations that Professor George Pólya wrote about in [Pó61] to see what the smallest fraction of the popular vote a candidate needs to win the presidential election is.

The next section will be about adapting Huntington's method to elect the President of the European Commission. First, we will look at how the current system works using information from [Gri12]. After that, we will be adapting Huntington's method for apportioning the European Parliament with and without the principle of degressive proportionality. Using this data, we can assign these fictional electoral votes to every member state and calculate (using the method previously discussed) the smallest fraction of the popular vote a candidate needs to become the President of the European Commission.

We will look at the Python code that I developed for apportionment in Section 4, along with a line-by-line explanation. I have also modified the code to consider the principle of degressive proportionality, and I will also include the modified code along with a line-by-line explanation.

Some simulations can be found in the final section about fictional scenarios. Among others, it will show how the US House of Representatives would look if the District of Columbia or Puerto Rico were states, and how the European Parliament would look if Ukraine joined the European Union.

2. THE UNITED STATES ELECTORAL SYSTEM

2.1. THE HISTORICAL BACKGROUND OF THE ELECTORAL COLLEGE

After the foundation of the United States in 1776, the Founding Fathers needed to establish a method for choosing the officeholders in the different branches of government.

The keyword when looking at the history American electoral system is "compromise". The founders needed to resolve the conflict between the smaller and larger states. The small states wanted each state to be equally represented in the decision-making, while the large states wanted representation to be proportional to the population. The result of this compromise was a bicameral legislature. In the Senate, every state has two senators, thus giving the smaller states a lot of power in the legislative process. In the House of Representatives, each state has a certain number of representatives proportional to its population, with each state having at least one seat in the House. The US constitution requires that the seats assigned to each state be proportional to its population, but it doesn't specify any method to use for apportionment.

The method for electing the executive is derived from the method for electing the legislature. The President is not directly elected by the people, rather by the Electoral College. According to this system, each state is assigned a certain number of electors, who will vote according to the will of the voters¹. Each state is assigned the same number of electors as seats in the Congress (representatives + senators). Most states² award all their electors to the candidate who received the most votes in their state³. To win the election, a candidate needs the votes of a majority of all electors.

2.2. THE HUNTINGTON METHOD

The current method of apportionment was developed by an American mathematician, Edward V. Huntington in the early 20th century. The method was described in [Hun28], [Hun28], [Sul72] and [Sul82].

¹ In some cases, electors have voted against the will of the voters of their state. They are referred to as faithless electors.

² Currently, every state but two use this method. The states of Nebraska and Maine award two of their electoral votes to the person who won the most votes state-wide. The rest are awarded according to who won the most votes in each of the state's congressional districts.

³ The candidate doesn't need an absolute majority of votes to receive all of a state's electors; a plurality of votes is enough.

2.2.1. The requirements for the system

Huntington determined, that given two states, which have populations A and B respectively, have a and b seats in the House, the perfect system would give a result where

$$\frac{A}{a} = \frac{B}{b}.$$

As it would be basically impossible to satisfy this equation in a representative democracy, Huntington's goal was to make this inequality as small as possible.

2.2.2. The absolute difference and relative difference

A crucial question in developing the system was whether to use the absolute or the relative difference for the apportionment process. For example, if there are two districts, one with a population of 1,000 and another one with 1,001 people, the absolute difference here is 1, and the relative difference is 0.1%, but if two districts have a population of 10 and 11, the absolute difference would still be 1, but the relative difference would be 10%. Due to this reason, Huntington recommended the use of the relative difference instead of the absolute one. We can achieve the relative difference between states A and B by calculating

$$\frac{\frac{A}{a} - \frac{B}{b}}{\frac{B}{b}}.$$

2.2.3. The first test for the method of equal proportions

Huntington defined the first test for his method (known as the method of equal proportions or the method of the geometric mean, which we will be abbreviating as MGM) as follows (see [Hun28, Page 3]):

"If the relative difference between the congressional districts, $\frac{A}{a}$ and $\frac{B}{b}$, belonging to any two states can be reduced by a transfer of a representative from one state to the other, then this transfer should be made."

We can write the following "inequality"⁴ for the test:

$$\frac{\frac{A}{a+1} - \frac{B}{b}}{\frac{B}{b}} < \frac{\frac{B}{b+1} - \frac{A}{a}}{\frac{A}{a}}$$

⁴ Please note that this is not a "real inequality". We are writing this as an inequality to transform these fractions into the desired form.

We rearrange this inequality to get the multiplier that can be used for successive applications of the test:⁵

$$\begin{aligned}
\frac{\frac{A}{a+1} - \frac{B}{b}}{\frac{B}{b}} &< \frac{\frac{B}{b+1} - \frac{A}{a}}{\frac{A}{a}} \Rightarrow \\
\left(\frac{A}{a}\right)\left(\frac{A}{a+1} - \frac{B}{b}\right) &< \left(\frac{B}{b}\right)\left(\frac{B}{b+1} - \frac{A}{a}\right) \Rightarrow \\
\frac{A^2}{a(a+1)} - \frac{AB}{ab} &< \frac{B^2}{b(b+1)} - \frac{AB}{ab} \Rightarrow \\
\frac{A^2}{a(a+1)} &< \frac{B^2}{b(b+1)} \Rightarrow \\
\frac{A}{\sqrt{a(a+1)}} &< \frac{B}{\sqrt{b(b+1)}}
\end{aligned}$$

2.2.4. The second test for the method of equal proportions

This is like the first one, but instead of the ratio $\frac{A}{a}$, it uses $\frac{a}{A}$. Here, we divide the number of representatives by the population. We can think of this as the "individual share" each person living in the state gets of the representative. In perfect apportionment, the equality would look like this:

$$\frac{a}{A} = \frac{b}{B}$$

Huntington defined the second test as follows (see [Hun28, Page 4]):

"If the relative difference between the two "individual shares", $\frac{a}{A}$ and $\frac{b}{B}$, belonging to any two states, can be reduced by a transfer of a representative from one state to the other, then this transfer should be made."

We can see that the equality in the second test is the reciprocal of the one in the first test. If we perform transformations as seen in the previous subsection, we will arrive at the following inequality (which is also the reciprocal of the one we previously arrived at):

$$\frac{\sqrt{a(a+1)}}{A} < \frac{\sqrt{b(b+1)}}{B}$$

2.2.5. Working rule for the method of equal proportions

In subsection 2.5, we transformed the inequality into the following:

⁵ We know that the population of a state and the number of representatives assigned to it will be a positive integer, so the direction of the inequality will not change, and we can take the square root of these numbers on the set of real numbers.

$$\frac{A}{\sqrt{a(a+1)}} < \frac{B}{\sqrt{b(b+1)}}$$

As we are dividing the population of a state, we can interpret it as multiplying the population of state A by the following fraction:

$$\frac{1}{\sqrt{a(a+1)}}$$

Let's start by writing down the notations we will be using:

- n : the number of states
- r : the number of representatives that we need to assign
- A : the population of state A
- a : the number of representatives assigned to state A

The first thing we need to do is to write down each state, its population and the number of representatives assigned to it. At the start, each state will be assigned one representative. Now we see that we need to perform the procedure $n - r$ times.

Every time we assign a representative, we need to multiply the population of each state by

$$\frac{1}{\sqrt{a(a+1)}}$$

where a will be the number of representatives the state currently holds (after each iteration). The state that will be assigned the representative will be the state with the highest population after performing the multiplication.

2.3. ALTERNATIVE METHODS FOR APPORTIONMENT

We can achieve alternative methods for apportionment by slightly modifying the tests defined by Huntington for the method of the geometric mean. By using the absolute difference instead of the relative difference, we can arrive at the method of the harmonic mean and the method of the arithmetic mean. These methods have been described in [Hun28], [Hun28] and [Sul72].

2.3.1. The method of the arithmetic mean

We can achieve the method of the arithmetic mean⁶ (which we will be abbreviating as MAM) by taking the absolute difference between the individual shares of two states⁷.

⁶ This method was devised by W. F. Willcox and was used for apportionment after the 1910 census.

⁷ It is important that every state shall have one representative at the start, because only then could the denominator of the multiplier be greater than one.

Huntington defined the test for the method of the arithmetic mean (or the method of major fractions) as follows (see [Hun28, Page 7]):

"If the absolute difference between the two "individual shares", $\frac{a}{A}$ and $\frac{b}{B}$, can be reduced by a transfer of a representative from one state to the other, then this transfer should be made (except that no state shall be left without at least one representative)."

We can write the following inequality using this test (using the same notations as before):

$$\frac{a+1}{A} - \frac{b}{B} < \frac{b+1}{B} - \frac{a}{A}$$

To achieve the working rule for the method of the arithmetic mean, we need to transform the inequality like so:

$$\begin{aligned} \frac{a+1}{A} - \frac{b}{B} &< \frac{b+1}{B} - \frac{a}{A} \Rightarrow \\ \frac{a+1}{A} + \frac{a}{A} &< \frac{b+1}{B} + \frac{b}{B} \Rightarrow \\ \frac{2a+1}{A} &< \frac{2b+1}{B} \end{aligned}$$

Taking the reciprocals, we get:

$$\begin{aligned} \frac{A}{2a+1} &> \frac{B}{2b+1} \Rightarrow \\ \frac{A}{a+\frac{1}{2}} &> \frac{B}{b+\frac{1}{2}} \end{aligned}$$

From this result, we can see that the multiplier for state A for the method of the arithmetic mean will be:

$$\frac{1}{a+\frac{1}{2}}$$

With this multiplier, we can arrive at the working rule for this method, which is like that of the method of the geometric mean. We need to perform the same procedure as we did for the method of equal proportions, with the only difference being the multiplier. Instead of using the multiplier for the method of the geometric mean, we shall use the one we arrived at in this section.

2.3.2. The method of the harmonic mean

We can achieve the method of the harmonic mean (which we will be abbreviating as MHM) by taking the absolute difference between the population of two districts. Huntington defined the test for the method of the harmonic mean as follows (see [Hun28, Page 7]):

"If the absolute difference between the two congressional districts $\frac{A}{a}$ and $\frac{B}{b}$, can be reduced by a transfer of a representative from one state to the other, then this transfer should be made."

We can write the following inequality using this test (using the same notations as before):

$$\frac{A}{a+1} - \frac{B}{b} < \frac{B}{b+1} - \frac{A}{a}$$

To achieve the working rule for the method of the arithmetic mean, we need to transform the inequality like so:

$$\begin{aligned} \frac{A}{a+1} - \frac{B}{b} &< \frac{B}{b+1} - \frac{A}{a} \Rightarrow \\ \frac{A}{a+1} + \frac{A}{a} &< \frac{B}{b+1} + \frac{B}{b} \Rightarrow \\ A \left(\frac{1}{a+1} + \frac{1}{a} \right) &< B \left(\frac{1}{b+1} + \frac{1}{b} \right) \Rightarrow \\ A \left(\frac{a}{a(a+1)} + \frac{a+1}{a(a+1)} \right) &< B \left(\frac{b}{b(b+1)} + \frac{b+1}{b(b+1)} \right) \Rightarrow \\ A \left(\frac{2a+1}{a(a+1)} \right) &< B \left(\frac{2b+1}{b(b+1)} \right) \Rightarrow \\ \frac{A}{\frac{a(a+1)}{2a+1}} &< \frac{B}{\frac{b(b+1)}{2b+1}} \Rightarrow \\ \frac{A}{2 \cdot \frac{a(a+1)}{2a+1}} &< \frac{B}{2 \cdot \frac{b(b+1)}{2b+1}} \end{aligned}$$

From this result, we can see that the multiplier for state \$A\$ for the method of the harmonic mean will be:

$$\frac{1}{2 \cdot \frac{a(a+1)}{2a+1}}$$

With this multiplier, we can arrive at the working rule for this method, which is like that of the method of the geometric mean. We need to perform the same procedure as we

did for the method of equal proportions, with the only difference being the multiplier. Instead of using the multiplier previously mentioned, we shall use the one seen above.

2.4. COMPARING APPORTIONMENT BETWEEN THE DIFFERENT METHODS

We can use the working rules for the different methods for the apportionment of the US House of Representatives. Using the code in section 5, I have calculated the apportionment using the data from the 2020 US census.

In the following table, we can see the name of the state in the first column, its population in the second column (according to data from the 2020 census which can be seen in [US 21]), the seats assigned to it using the method of the geometric mean in the third column, the seats assigned to it using the method of the arithmetic mean in the fourth column and the seats assigned to it using the method of the harmonic mean in the fifth column.

The data in the third column is also the official apportionment calculated by the U.S. Census Bureau, which will be used⁸ in the 2022 midterm elections in November. We should also note that the District of Columbia isn't considered in the apportionment process. It is represented by one non-voting delegate in the House (the number of delegates is fixed at one), but it currently has three electoral votes⁹ in the Electoral College according to the Twenty-third Amendment to the United States Constitution (see Table 1).

Table 1 Apportionment using the 2020 census data

State	Population	MGM	MAM	MHM
Alabama	5,030,053	7	7	7
Alaska	736,081	1	1	1
Arizona	7,158,923	9	9	9
Arkansas	3,013,756	4	4	4
California	39,576,757	52	52	52
Colorado	5,782,171	8	8	8
Connecticut	3,608,298	5	5	5
Delaware	990,837	1	1	1
Florida	21,570,527	28	28	28
Georgia	10,725,274	14	14	14
Hawaii	1,460,137	2	2	2
Idaho	1,841,377	2	2	3
Illinois	12,822,739	17	17	17
Indiana	6,790,280	9	9	9
Iowa	3,192,406	4	4	4
Kansas	2,194,865	4	4	4

⁸ At the time of writing, the midterm elections haven't happened yet.

⁹ According to the Twenty-third Amendment, the number of electoral votes assigned to the District of Columbia shall equal the number of electoral votes assigned to the state with the least number of votes.

Kentucky	4,509,342	6	6	6
Louisiana	4,661,468	6	6	6
Maine	1,363,582	2	2	2
Maryland	6,185,278	8	8	8
Massachusetts	7,033,469	9	9	9
Michigan	10,084,442	13	13	13
Minnesota	5,709,752	8	8	7
Mississippi	2,963,914	4	4	4
Missouri	6,160,281	8	8	8
Montana	1,085,407	2	1	2
Nebraska	1,963,333	3	3	3
Nevada	3,108,462	4	4	4
New Hampshire	1,379,089	2	2	2
New Jersey	9,294,493	12	12	12
New Mexico	2,120,220	3	3	3
New York	20,215,751	26	27	26
North Carolina	10,453,948	14	14	14
North Dakota	779,702	1	1	1
Ohio	11,808,848	15	16	15
Oklahoma	3,963,516	5	5	5
Oregon	4,241,500	6	6	6
Pennsylvania	13,011,844	17	17	17
Rhode Island	1,098,163	2	1	2
South Carolina	5,124,712	7	7	7
South Dakota	887,770	1	1	1
Tennessee	6,916,897	9	9	9
Texas	29,183,290	38	38	38
Utah	3,275,252	4	4	4
Vermont	643,503	1	1	1
Virginia	8,654,542	11	11	11
Washington	7,715,946	10	10	10
West Virginia	1,795,045	2	2	2
Wisconsin	5,897,473	8	8	8
Wyoming	577,719	1	1	1

2.4.1. Comparing the geometric mean and the arithmetic mean

With this population data, we can find differences in apportionment for four states. The method of the arithmetic mean assigns one seat each to the states of Rhode Island and Montana, 16 seats to the state of Ohio and 27 seats to the state of New York. The method of the geometric mean assigns two seats each to the states of Rhode Island and Montana, 15 seats to the state of Ohio and 26 seats to the state of New York. From this data, we can clearly see how the method of the arithmetic mean favours larger states over smaller states.

2.4.2. Comparing the geometric mean and the harmonic mean

We can see differences in the apportionment for two states for these two methods. These are the states of Idaho and Minnesota. The method of the geometric mean assigns two seats to the former and eight seats to the latter, whereas the method of the harmonic mean assigns three seats to the former and seven seats to the latter. This shows us that the method of the harmonic mean favours smaller states even more than the method of the geometric mean, but in this case at the expense of a medium-sized state.

2.4.3. Comparing the arithmetic mean and the harmonic mean

This comparison shows us more clearly that the method of the arithmetic mean favours the larger states, and the method of the harmonic mean favours smaller states. In this comparison, we can see elements of both previous comparisons. For the states of Montana, Rhode Island and Idaho, the method of the arithmetic mean assigns one seat fewer than the method of the harmonic mean, and for the states of Minnesota, Ohio and New York, the method of the arithmetic mean assigns one seat more than the method of the harmonic mean.

2.4.4. Summary of the comparison

In summary, we can see that the method of the arithmetic mean favours larger states, the method of the harmonic mean favours smaller states, and apportionment using the method of the geometric mean can lean either way, but is usually fairer than both methods.

2.5. WHAT'S THE SMALLEST FRACTION OF VOTES SOMEONE NEEDS TO WIN THE PRESIDENTIAL ELECTION?

Professor George Pólya calculated the minimum required fraction of the popular vote to win a presidential election in [Pó61] using the apportionment data available in 1961. We can use the method devised by Pólya to calculate the minimum fraction of the popular vote required to win the 2024 presidential election with apportionment using any of the methods described earlier. At the time of the writing of Pólya's article, the Twenty-third Amendment hadn't been used in a presidential election yet, so the District of Columbia wasn't allowed to participate in a presidential election until 1964. The Twenty-third Amendment states that Washington DC shall have the same number of electoral votes as the state with the least number of electoral votes. In practice, this means that DC will have 3 electoral votes. To make our calculations simpler, we will be assuming that the District of Columbia is a state with one representative in the House, giving it 3 electoral votes.

2.5.1. Notations used

We will be using the following notations for this section:

- T : total of number of votes cast in the election
- W : number of votes received by the winner
- r : number of representatives a certain state has
- m : number of states that have r representatives
- s : number of states a candidate carried

2.5.2. Composition of the House

Using the data previously generated, we have arrived at the composition for the US House of Representatives, which can be found in Table 2 (Please note that the r values are the number of representatives a state has. The number of electoral votes is the number of representatives plus two.).

Table 2 Composition of the House

r	m (MGM)	mr (MGM)	m (MAM)	mr (MAM)	m (MHM)	mr (MHM)
1	7	7	9	9	7	7
2	7	14	5	10	7	14
3	2	6	2	6	2	6
4	5	20	5	20	5	20
5	2	10	2	10	2	10
6	3	18	3	18	3	18
7	3	21	3	21	3	21
8	4	32	4	32	4	32
9	3	27	3	27	3	27
10	2	20	2	20	2	20
11	1	11	1	11	1	11
12	1	12	1	12	1	12
13	1	13	1	13	1	13
14	2	28	2	28	2	28
15	1	15	1	15	1	15
17	2	34	2	34	2	34
26	1	26	1	26	1	26
28	1	28	0	0	1	28
29	0	0	1	29	0	0
39	1	39	1	39	1	39
51	1	51	0	0	1	51
52	0	0	1	52	0	0
Σ	51	436	51	436	51	436

2.5.3. Constants

As we are using this method for the 2024 election, we will have the following constants during our calculations:

$\Sigma m = 51$: the number of states

$\Sigma(mr) = 436$: the total number of congressional districts

$\Sigma(mr) + \Sigma m \cdot 2 = 538$: the total number of electoral votes (every state has $r + 2$ electoral votes)

$N = 759424$: population per congressional district (using the 2020 census data; see the Assumptions for details)¹⁰

2.5.4. Assumptions

We need to make some assumptions to arrive at a problem that is possible to solve.

1. The number of representatives assigned to a state is exactly proportional to its population. This leads to the following equation (assumption of proportionality):

$$T = \Sigma(mr)N$$

2. There are only two candidates competing in the presidential election. This means that the winner has a majority of the votes, and we can state that the loser has $T - W$ votes.¹¹
3. Each state awards every one of its electoral votes to the candidate who receives the most votes.

2.5.5. Solution

Our goal with this problem is to arrive at the least possible $\frac{W}{T}$.

For a candidate to win the presidential election, they need to receive more than half of the electoral votes (which is at least 270 with the current data) in the Electoral College. If the winner carries s states, with $r_1, r_2, \dots, r_s$ representatives, we arrive at the following inequality:

$$(r_1 + 2) + (r_2 + 2) + \dots + (r_s + 2) \geq 270 \Rightarrow \\ r_1 + r_2 + \dots + r_s \geq 270 - 2s$$

¹⁰ We have arrived at this result by taking the sum of the population of each state and the District of Columbia, dividing it by $\Sigma(mr)$, and rounding it up to arrive at an even number, as we will be assuming later that N is an even number (rounding down would give us an odd number in this case).

¹¹ Letting more candidates participate in the election and awarding the electoral votes to the candidate who wins a plurality of votes could lead to more disproportional results.

Using our assumption that there are only two candidates running in the election, to carry a state with $r + 2$ electoral votes, a candidate needs $\frac{rN}{2} + 1$ votes. (We are assuming that N is even.). We can write an inequality to arrive at W , which is the total of votes the candidate needs from the s states carried.

$$W \geq \left(\frac{1}{2}r_1N + 1\right) + \left(\frac{1}{2}r_2N + 1\right) + \cdots + \left(\frac{1}{2}r_sN + 1\right) \Rightarrow$$

$$W \geq \frac{N}{2}(r_1 + r_2 + \cdots + r_s) + s$$

We can divide both sides of the inequality by $T \in \mathbb{Z}^+$, and substitute the assumption of proportionality in the right side to arrive at the following inequality:

$$\frac{W}{T} \geq \frac{r_1 + r_2 + \cdots + r_s}{872} + \frac{s}{436N}$$

Using the inequality on the previous page, we can substitute on the right side:

$$\frac{W}{T} \geq \frac{270 - 2s}{872} + \frac{s}{436N}$$

A case of equality here only exists if, and only if a case of equality also exists in the inequality on the previous page and in the first inequality on this page, which we used to arrive at this result. Equality in the inequality on the previous page only exists, if the winner wins exactly 270 electoral votes. To arrive at an equality in the first inequality on this page, the winner needs to win with the smallest possible majority in s states and win zero votes every other state. This is compatible with the assumptions, so we will arrive at our solution if an equality exists, so our solution looks like this:

$$\frac{W}{T} = \frac{270 - 2s}{872} + \frac{s}{436N}$$

We can see that we will also need the greatest possible number of states which jointly have 270 electoral votes to achieve the smallest fraction of the popular vote. We can start by counting the states in an increasing order of population until we arrive at 270 electoral votes.

Using the apportionment data for the method of the geometric mean, we need to include every state with no fewer than 13 electoral votes (11 representatives), except for a state with 3 electoral votes (1 representative) and including one state with 16 electoral votes (14 representatives). This means that $s = 40$. The number of states that fit these conditions is 38. Substituting into the solution, we get the following result:

$$\frac{W}{T} = \frac{270 - 2 \cdot 40}{872} + \frac{40}{436 \cdot 759424} \approx 0.217890029$$

We have arrived at the result that, in an extreme and unlikely scenario, the minimum fraction of the popular vote a candidate needs to win the 2024 presidential election is about 21.79%.

If wish to calculate the same fraction for the method of the arithmetic mean, we only need to make a slight modification to the substitution in the solution. As this method produces a different result than the method of the geometric mean, we need a different combination of states to reach exactly 270 electoral votes. This combination will be every state with no fewer than 12 electoral votes (10 representatives), and a state with 15 electoral votes (13 representatives). From this, we know that $s = 40$. Substituting this into the solution, we get the following result:

$$\frac{W}{T} = \frac{270 - 2 \cdot 40}{872} + \frac{40}{436 \cdot 759424} \approx 0.217890029$$

We have arrived at the result that, in an extreme and unlikely scenario, the minimum fraction of the popular vote a candidate needs to win the 2024 presidential election, if apportionment used the method of the arithmetic mean is about 21.79%.

We can also use the same process for the method of the harmonic mean. In this case, the winner needs to carry every state with no fewer than 13 electoral votes (11 representatives). This leads us to $s = 40$, so substituting into the solution:

$$\frac{W}{T} = \frac{270 - 2 \cdot 40}{872} + \frac{40}{436 \cdot 759424} \approx 0.217890029$$

We have arrived at the result that, in an extreme and unlikely scenario, the minimum fraction of the popular vote a candidate needs to win the 2024 presidential election, if apportionment used the method of the harmonic mean is about 21.79%.

In the solution, we can see that only two variables influence the value of the minimum fraction. The first is N , which in practice means that the population of the country influences the result. The second is s , which is the number of states carried by the winner. Using the 2020 census data, we have generated the apportionment for the three methods, and in all three cases, the number of states the winner needed to carry was 39, so we have arrived at the same result.

3. ADAPTING THE HUNTINGTON METHOD TO CHOOSE THE PRESIDENT OF THE EUROPEAN COMMISSION

3.1. INTRODUCTION

The President of the European Commission is the most powerful official in the European Union. This position is comparable to prime ministers in parliamentary democracies or presidents in presidential republics.

The election of the President is a contentious issue after every European election. To resolve this issue, the Spitzenkandidaten system was developed. According to this system, the person nominated by the party with the most seats in the European Parliament will become the President of the European Commission. This system was last used in 2014, when Jean-Claude Juncker of the European People's Party was elected President, after the centre-right party won a plurality of seats in the European Parliament.

3.2. THE CURRENT METHOD OF APPORTIONMENT FOR THE EUROPEAN PARLIAMENT

The equivalent document of the US Constitution for the European Union which establishes the branches of government is the Treaty of Lisbon. It establishes the European Parliament as the primary legislative institution of the EU. Like the US Constitution, the Treaty does not specify the apportionment method, but it does set some guidelines. The document establishes the 705-seats¹² of the European Parliament. It gives every member state at least six, and at most 96 seats in the Parliament.

The Treaty also requires any apportionment method to adhere to the principle of degressive proportionality. The guidelines for the current method of apportionment were described in the Cambridge compromise, which was finalised at the Cambridge Apportionment Meeting (CAM). (See [Gri12, Page 1], [Gri17])

3.2.1. The principle of degressive proportionality

Degressive proportionality means "the ratio between the population and the number of seats of each Member State must vary in relation to their respective populations in such a way that each Member from a more populous Member State represents more citizens than each Member from a less populous Member State and conversely, but also that no less

¹² Originally, the Treaty of Lisbon created 751 seats for the European Parliament, but after the United Kingdom left the European Union, this number was reduced to 705. The United Kingdom had 73 seats. 46 of these seats have been abolished, and the remaining 27 were reapportioned to the remaining member states.

populous Member State has more seats than a more populous Member State." (This is known as the Lamassoure-Severin definition, see [Gri12, Page 2].)

The following requirements were devised at the Cambridge Apportionment Meeting:

1. "no smaller State shall receive more seats than a larger State,"
2. "the ratio population/seats shall increase as population increases."

At the time, it seemed impossible to design a system that satisfies both requirements under the hypotheticals tested. This resulted in the modification of the Lamassoure-Severin definition. After the amendment, it defined degressive proportionality as follows: "the ratio between the population and the number of seats of each Member State before rounding to whole numbers must vary in relation to their respective populations in such a way that each Member from a more populous Member State represents more citizens than each Member from a less populous Member State and conversely, but also that no less populous Member State has more seats than a more populous Member State."

After the definition was amended, the Cambridge Apportionment Meeting developed the base + prop method for apportionment.

3.2.2. Base + prop method

The base + prop method for apportionment consists of two stages, as written in [Gri12, Page 2-3]:

1. The allocation of a fixed (base) number of states to each member state.
2. The allocation of the remaining states in proportion to the member state's population (subject to rounding and capping at the maximum).

To achieve the desired size for the Parliament, a house-size divisor is introduced. Using the given base b , maximum M and divisor d , we can define the following allocation function $A_d: [0, \infty) \rightarrow [0, \infty)$:

$$A_d(p) = \min \left\{ b + \frac{p}{d}, M \right\}$$

We shall assign the seat share $A_d(p)$ to the member state with population p . After that, we need to round this to an integer, and adjust the divisor d , so that the sum of the seat numbers equals the desired Parliament size.

The Cambridge Apportionment Meeting recommended the use of base $b = 5$, and the use of upwards rounding.

3.2.3. The current apportionment for the European Parliament

The apportionment for the European Parliament since the United Kingdom's departure from the European Union can be seen on Table 3.

Table 3 Apportionment in the European Parliament

Country	MEPs
Austria	19
Belgium	21
Bulgaria	17
Croatia	12
Cyprus	6
Czech Republic	21
Denmark	14
Estonia	7
Finland	14
France	79
Germany	96
Greece	21
Hungary	21
Ireland	13
Italy	76
Latvia	8
Lithuania	11
Luxembourg	6
Malta	6
Netherlands	29
Poland	52
Portugal	21
Romania	33
Slovakia	14
Slovenia	8
Spain	59
Sweden	21
Σ	705

3.3. ADAPTING HUNTINGTON'S METHOD

There are two approaches we can take when adapting Huntington's method.

The first is to completely ignore the principle of degressive proportionality and give every state near equal representation in the European Parliament. Taking this approach would lead us to a Parliament which looks more like the US House of Representatives.

The second approach is to keep the principle of degressive proportionality but grant every member state an m minimum number of seats,¹³ and allocate the remaining seats using Huntington's method subject to a maximum of 96 seats. After a member state reaches 96 seats, it shall be assigned no more seats. These guidelines were established [Gri12, Page 3].

¹³ According to the current process for apportionment, this minimum is $m = 6$.

3.3.1. Ignoring degressive proportionality

If we wish to ignore the principle of degressive proportionality, we will get a result, where every member state has roughly equal representation. This will give us a result that gives the more populous member states (such as Germany and France) a lot more power when compared to the current system. The results, if we applied the different methods using the Python program, can be found in Table 4.

Table 4 Apportionment in the European Parliament without degressive proportionality

Country	Population	Current	MGM	MAM	MHM
Austria	8,401,940	19	13	13	13
Belgium	11,000,638	21	18	18	18
Bulgaria	7,364,570	17	12	12	12
Croatia	4,284,889	12	7	7	7
Cyprus	840,407	6	2	1	2
Czech Republic	10,436,560	21	17	17	17
Denmark	5,560,628	14	9	9	9
Estonia	1,294,455	7	2	2	2
Finland	5,375,276	14	9	9	9
France	64,933,400	79	104	104	104
Germany	80,219,695	96	128	129	128
Greece	10,816,286	21	17	17	17
Hungary	9,937,628	21	16	16	16
Ireland	4,574,888	13	7	7	7
Italy	59,433,744	76	95	95	95
Latvia	2,070,371	8	3	3	3
Lithuania	3,043,429	11	5	5	5
Luxembourg	512,353	6	1	1	1
Malta	417,432	6	1	1	1
Netherlands	16,655,799	29	27	27	27
Poland	38,044,565	52	61	61	61
Portugal	10,562,178	21	17	17	17
Romania	20,121,641	33	32	32	32
Slovakia	5,397,036	14	9	9	9
Slovenia	2,050,189	8	3	3	3
Spain	46,815,910	59	75	75	75
Sweden	9,482,855	21	15	15	15

When comparing the apportionment data between the three methods, we can only see a difference in the case of two states. The method of the arithmetic mean assigns one seat fewer to Cyprus, and one more to Germany than the other two methods. Cyprus is one of the least populous states in the European Union, and Germany is the most populous. This also highlights that the method of the arithmetic mean favours larger states over smaller states.

As we no longer use the principle of degressive proportionality, we can see the effects of the removal of the cap on the number of representatives. Germany and France have crossed the 96 seat-threshold currently used during apportionment, with Italy coming close at 95 seats.

The minimum number of seats assigned to a state has also been lowered to one. This also causes significant differences and gives less power to the smaller states.

From these results, it is clear why the principle of degressive proportionality is crucial in the balance between the power of a member state, and the power of the people living in the European Union. Using this system, the four most populous states alone would have a sizeable 402-seat (or 403-seat¹⁴) majority.

3.3.2. Keeping degressive proportionality

We will use a version of the Python code modified according to the guidelines in [Gri12, Page 3]. The code along with a full, line-by-line explanation can be found in section 4.2. Using census data from 2011 from across the European Union, we have arrived at the following composition using the different methods (see Table 5).

Using this data, we can see that all three methods generated the same results. When analysing the difference between our apportionment and the official apportionment, we can notice a few major differences.

As this simulation respects degressive proportionality, we can say that the least populous member states benefited greatly from this apportionment. We also see that there are more states with the minimum number of seats than under the current system.

The only member state that reached the maximum of 96 seats under the current system is Germany. Germany reached this threshold under our system as well, making Germany less represented than the smaller states.

We can also see that a larger number of states reached 96 seats. These states were already populous, so we can say that this system benefits the larger states as well.

Another significant statement that we can make after looking the data is that the medium-sized states were the biggest losers of our system. They lost seats at the expense of larger states.

¹⁴ If we use the method of the arithmetic mean.

Table 5 Apportionment in the European Parliament with degressive proportionality

Country	Population	Current	Hypothetical
Austria	8,401,940	19	14
Belgium	11,000,638	21	18
Bulgaria	7,364,570	17	12
Croatia	4,284,889	12	7
Cyprus	840,407	6	6
Czech Republic	10,436,560	21	17
Denmark	5,560,628	14	9
Estonia	1,294,455	7	6
Finland	5,375,276	14	9
France	64,933,400	79	96
Germany	80,219,695	96	96
Greece	10,816,286	21	18
Hungary	9,937,628	21	17
Ireland	4,574,888	13	8
Italy	59,433,744	76	96
Latvia	2,070,371	8	6
Lithuania	3,043,429	11	6
Luxembourg	512,353	6	6
Malta	417,432	6	6
Netherlands	16,655,799	29	28
Poland	38,044,565	52	63
Portugal	10,562,178	21	18
Romania	20,121,641	33	34
Slovakia	5,397,036	14	9
Slovenia	2,050,189	8	6
Spain	46,815,910	59	78
Sweden	9,482,855	21	16

3.4. WHAT'S THE SMALLEST FRACTION OF VOTES SOMEONE NEEDS TO BECOME THE PRESIDENT OF THE EUROPEAN COMMISSION?

The European Union does not use an electoral college to elect the President of the European Commission, but using our data, we can establish a hypothetical scenario, where each member state received electoral votes equal to its representation in the European Parliament plus two votes. We will be using the same notations and methods in this subsection as we have in the subsection 3.10 solution. This would give us the following constants:

- $\Sigma m = 27$: the number of states
- $\Sigma(mr) = 705$: the size of the European Parliament
- $\Sigma(mr) + \Sigma m \cdot 2 = 759$: the total number of electoral votes (every state has $r + 2$ electoral votes)

- $N = 623616$: population per seat in the Parliament (using census data from 2011; upwards rounding to get an even result)
- $\left\lceil \frac{\Sigma(mr) + \Sigma m \cdot 2}{2} \right\rceil = 380$: the minimum number of electoral votes needed to win

Modifying the solution to fit the new variables for the European Parliament, we will get the following result:

$$\frac{W}{T} = \frac{380 - 2s}{1410} + \frac{s}{705N}$$

Under the current system, the composition of the European Parliament can be found on Table 6. (Please note that the r values in Tables 6-8 are the number of representatives. The number of electoral votes of a state is the number of its representatives plus two.)

Table 6 Composition of the European Parliament using the current system

r	m	mr
6	3	18
7	1	7
8	2	16
11	1	11
12	1	12
13	1	13
14	3	42
17	1	17
19	1	19
21	6	126
29	1	29
33	1	33
52	1	52
59	1	59
76	1	76
79	1	79
96	1	96
Σ	27	705

For this apportionment, it is not possible to reach exactly 380 electoral votes. The closest we can get is by using every state with no fewer than 31 electoral votes (29 representatives) and including one with 61 electoral votes (59 representatives). Using this result, we will have 382 electoral votes. The number of states to achieve this result is 22. Substituting into the modified solution, we get the following result:

$$\frac{W}{T} = \frac{380 - 2 \cdot 22}{1410} + \frac{22}{705 \cdot 623616} \approx 0.238297922$$

From this result, we can see that if the President of the European Commission were elected using a system like the one used in the United States, they would only need about 23.83% of the popular vote to win the election in an extreme and unlikely scenario.

3.4.1. Without degressive proportionality

If we don't adhere to the principle of degressive proportionality, the composition of the European Parliament can be seen on Table 7.

Table 7 Composition of the European Parliament without degressive proportionality

r	m (MGM/ MHM)	mr (MGM/ MHM)	m (MAM)	mr (MAM)
1	2	2	3	3
2	2	4	1	2
3	2	6	2	6
5	1	5	1	5
7	2	14	2	14
9	3	27	3	27
12	1	12	1	12
13	1	13	1	13
15	1	15	1	15
16	1	16	1	16
17	3	51	3	51
18	1	18	1	18
27	1	27	1	27
32	1	32	1	32
61	1	61	1	61
75	1	75	1	75
95	1	95	1	95
104	1	104	1	104
128	1	128	0	0
129	0	0	1	129
Σ	27	705	27	705

Using the apportionment for the method of the geometric mean and the method of the harmonic mean, we can achieve exactly 380 electoral votes. This will happen if we include every state with no fewer than 77 electoral votes (75 representatives) and exclude a state with 29 electoral votes (27 representatives) and one with 17 electoral votes (15 representatives). The number of these states is 22.

Using the apportionment for the method of the arithmetic mean, we will only see slight differences from the result using the two other methods. As this method takes away one seat from Cyprus and gives it to Germany, instead of excluding the states with 29 and 17 electoral votes, we shall exclude ones with 34 and 11 electoral votes. The number of states

will remain the same. We can use the same substitution into the modified solution for the three methods:

$$\frac{W}{T} = \frac{380 - 2 \cdot 22}{1410} + \frac{22}{705 \cdot 623616} \approx 0.238297922$$

From this result, we can see that if the President of the European Commission were elected using apportionment by the method of the geometric, arithmetic or harmonic mean without keeping degressive proportionality, they would only need about 23.83% of the popular vote to win the election in an extreme and unlikely scenario.

3.4.2. With degressive proportionality

If we adhere to the principle of degressive proportionality, the composition of the European Parliament can be seen on Table 8.

Table 8 Composition of the European Parliament with degressive proportionality

r	m	mr
6	7	42
7	1	7
8	1	8
9	3	27
12	1	12
16	1	16
17	2	34
18	3	54
28	1	28
34	1	34
63	1	63
78	1	78
96	3	288
Σ	27	705

Using this apportionment, we can't achieve exactly 380 electoral votes. The closest we can get to it is 384. We can achieve this by including every state with no fewer than 80 electoral votes (78 representatives), except for a state with 65 electoral votes (63 representatives). The number of these states is 22. Substituting into the modified solution, we arrive at the following result:

$$\frac{W}{T} = \frac{380 - 2 \cdot 22}{1410} + \frac{22}{705 \cdot 623616} \approx 0.238297922$$

From this result, we can see that if the President of the European Commission were elected using apportionment by the method of the geometric, arithmetic or harmonic mean while preserving degressive proportionality, they would only need about 23.83% of the popular vote to win the election in an extreme and unlikely scenario.

4. PYTHON CODE FOR APPORTIONMENT

I have developed the following Python code, which can be used to assign seats using the method of equal proportions. The source code can also be found at <https://github.com/zsoltlszabo/tdk> with the same comments. Please note that I will be using line breaks in this document that will render the code inexecutable if copied from here. For an executable version, please refer to the GitHub repository.

```
from math import sqrt
def assign(list_of_states: tuple, list_of_populations: tuple,
           number_of_reps: int):
    NUMBER_OF_STATES = len(list_of_states)
    STATE_KEYS = {i: list_of_states[i] for i in range(NUMBER_OF_STATES)}
    states = [[list_of_populations[i], 1] for i in range(NUMBER_OF_STATES)]
    for i in range(number_of_reps - NUMBER_OF_STATES):
        current_multiplied_values = []
        for k in states:
            current_multiplied_values.append(k[0] *
                                              (1 / sqrt(k[1] * (k[1] + 1))))
        current_max = current_multiplied_values.index(
            max(current_multiplied_values))
        states[current_max][1] += 1
    solution = {}
    for i in range(NUMBER_OF_STATES):
        solution.update({STATE_KEYS.get(i): states[i][1]})
    return solution
```

4.1. LINE-BY-LINE EXPLANATION

First, we need to import the sqrt function from the math module.

```
from math import sqrt
```

Let's start by defining the function. We will request the list of states, their respective populations as tuples, and the number of representatives to assign as an integer.

```
def assign(list_of_states: tuple, list_of_populations: tuple,
           number_of_reps: int):
```

We will store the number of states in a variable.

```
    NUMBER_OF_STATES = len(list_of_states)
```

We will be working with indexes of states instead of the names. To do this, we will create a dictionary that will map the indexes to the names.

```
    STATE_KEYS = {i: list_of_states[i] for i in range(NUMBER_OF_STATES)}
```

We will be working with the states list. It contains the following data for each state: [0] population, [1] assigned representatives (every state starts with one)

```
states = [[list_of_populations[i], 1] for i in range(NUMBER_OF_STATES)]
```

We will create a for loop which will be executed until all the representatives are assigned. As every state is assigned one representative at the start, we will only need to assign the remaining ones.

```
for i in range(number_of_reps - NUMBER_OF_STATES):
```

We will create a list which will contain the populations multiplied with the correct values for the current iteration.

```
current_multiplied_values = []
```

Next, we will create a for loop that multiplies every state with the correct multiplier.

```
for k in states:
    current_multiplied_values.append(k[0] *
                                     (1 / sqrt(k[1] * (k[1] + 1))))
```

We will store the index of the state with the largest value after multiplication in a variable.

```
current_max = current_multiplied_values.index(
    max(current_multiplied_values))
```

We will increase the number of representatives of the state with the largest value after multiplication by one.

```
states[current_max][1] += 1
```

We will create a dictionary, where we will store the solution.

```
solution = {}
```

We will create a for loop that will be executed for every state.

```
for i in range(NUMBER_OF_STATES):
```

We will add the state to the solution dictionary, with the number of representatives as the value, and we will return this dictionary as our solution.

```
solution.update({STATE_KEYS.get(i): states[i][1]})
```

4.2. MODIFYING THIS CODE FOR THE METHOD OF THE ARITHMETIC MEAN

We can modify the following line in the code to use it for calculating apportionment using the method of the arithmetic mean:

```
current_multiplied_values.append(k[0] *
                                 (1 / sqrt(k[1] * (k[1] + 1))))
```

We need to replace this line with the following code: \\

```
current_multiplied_values.append(k[0] *
                                 (1 / sqrt(k[1] * (k[1] + 1))))
```

4.3. MODIFYING THIS CODE FOR THE METHOD OF THE HARMONIC MEAN

We can modify the following line in the code to use it for calculating apportionment using the method of the harmonic mean:

```
current_multiplied_values.append(k[0] *  
    (1 / sqrt(k[1] * (k[1] + 1))))
```

We need to replace this line with the following code:

```
current_multiplied_values.append(k[0] / ((2 *  
    (k[1] * (k[1] + 1))) / (k[1] + k[1] + 1)))
```

4.4. MODIFYING THIS CODE FOR THE PRINCIPLE OF DEGRESSIVE PROPORTIONALITY

We can modify this code to honour the principle of degressive proportionality, which is used during the apportionment process for the European Parliament. After the modifications, the code looks like this:

```
from math import sqrt  
def assign(list_of_states: tuple, list_of_populations: tuple,  
          number_of_reps: int):  
    NUMBER_OF_STATES = len(list_of_states)  
    STATE_KEYS = {i: list_of_states[i] for i in range(NUMBER_OF_STATES)}  
    states = [[list_of_populations[i], 6] for i in range(NUMBER_OF_STATES)]  
    for i in range(number_of_reps - NUMBER_OF_STATES * 6):  
        current_multiplied_values = []  
        for k in states:  
            current_multiplied_values.append(k[0] *  
                (1 / sqrt(k[1] * (k[1] + 1))))  
        while True:  
            current_max = current_multiplied_values.index(  
                max(current_multiplied_values))  
            if states[current_max][1] == 96:  
                current_multiplied_values[current_max] = 0  
            else:  
                states[current_max][1] += 1  
                break  
    solution = {}  
    for i in range(NUMBER_OF_STATES):  
        solution.update({STATE_KEYS.get(i): states[i][1]})  
    return solution
```

4.4.1. Line-by-line explanation

First, we need to import the sqrt function from the math module.

```
from math import sqrt
```

Let's start by defining the function. We will request the list of states, their respective populations as tuples, and the number of representatives to assign as an integer.

```
def assign(list_of_states: tuple, list_of_populations: tuple,
```

```
number_of_reps: int):
```

We will store the number of states in a variable.

```
NUMBER_OF_STATES = len(list_of_states)
```

We will be working with indexes of states instead of the names. To do this, we will create a dictionary that will map the indexes to the names.

```
STATE_KEYS = {i: list_of_states[i] for i in range(NUMBER_OF_STATES)}
```

We will be working with the states list. It contains the following data for each state: [0] population, [1] assigned representatives (every state starts with six)

```
states = [[list_of_populations[i], 6] for i in range(NUMBER_OF_STATES)]
```

We will create a for loop which will be executed until all the representatives are assigned. As every state is assigned six representatives at the start, we will only need to assign the remaining ones.

```
for i in range(number_of_reps - NUMBER_OF_STATES * 6):
```

We will create a list which will contain the populations multiplied with the correct values for the current iteration.

```
current_multiplied_values = []
```

Next, we will create a for loop that multiplies every state with the correct multiplier.

```
for k in states:
    current_multiplied_values.append(k[0] *
                                      (1 / sqrt(k[1] * (k[1] + 1))))
```

This while loop will check if a certain member state has reached the maximum amount of seats.

```
while True:
```

We will store the index of the state with the largest value after multiplication in a variable.

```
current_max = current_multiplied_values.index(
    max(current_multiplied_values))
```

If the state with the largest value already has 96 representatives, the value will be decreased to zero, thus putting the state at the end of the list.

```
if states[current_max][1] == 96:
    current_multiplied_values[current_max] = 0
```

If the state has fewer than 96 representatives, we will increase the number of representatives by one.

```
else:
    states[current_max][1] += 1
    break
```

We will create a dictionary, where we will store the solution.

```
solution = {}
```

We will create a for loop that will be executed for every state.

```
for i in range(NUMBER_OF_STATES):
```

We will add the state to the solution dictionary, with the number of representatives as the value, and we will return this dictionary as our solution.

```
    solution.update({STATE_KEYS.get(i): states[i][1]})  
return solution
```

This code can also be modified the same way as previously described to be used for the methods of the arithmetic and harmonic mean.

4.5. USING THE PROGRAM TO SIMULATE DIFFERENT SCENARIOS

I have run some simulations for fictional scenarios using this code, which can be found in the final section.

5. HYPOTHETICAL SCENARIOS

Using the Python code, we can run apportionment for different fictional scenarios. A few cases can be found here for the US House of Representatives and the European Parliament. As we are working with many states here, the tables with the generated data can be found at the end of this section to make reading easier.

5.1. SIMULATIONS FOR THEORETICAL SCENARIOS FOR THE UNITED STATES CONGRESS

5.1.1. Apportionment if the District of Columbia were a state

The question of statehood for the District of Columbia has been a popular issue for decades. Since the 1964 presidential election, the District has had a vote in presidential elections, but not in Congress. Instead, one non-voting delegate represents the residents of the capital in Congress. As DC has more people than two states, there have been calls to give the city statehood. We will be running a simulation for the three methods. The size of the House will stay at 435. The population data for the US states is from [US 21], while the population data for DC is in [Ame21a]. See Table 9 for the results.

As we can see, the District of Columbia would be one of the smallest states. All three methods give it one representative. We should look at which state's seat DC gets. For the method of the geometric mean and the method of the arithmetic mean, it is Minnesota. When we compared the three methods in subsection 2.4, we saw that the method of the harmonic mean assigned the state one seat fewer when compared to the other methods. For the method of the harmonic mean, it is Idaho. We also saw that the method originally assigned one more seat when compared to the other methods.

5.1.2. Apportionment if Puerto Rico were a state

There have also been debates (and several referenda) about the status of Puerto Rico. It is currently a commonwealth of the United States. It is represented by a non-voting delegate in the House, but has special, more beneficial tax regulations when compared to DC. With a population of over 3 million, Puerto Rico would be a significantly sized state in the Union. We will be running the simulation using the three methods, with the size of the House staying at 435. The population data for the US states is from [US 21], while the population data for Puerto Rico is in [Ame21b]. See Table 10 for the results.

We can see that all three methods assigned 4 seats to Puerto Rico. In this case, it's also important to look at which states lost by Puerto Rico getting these seats. For the method of the geometric mean, California, Colorado, Minnesota and Montana each lost one seat when

compared to the original results. In the case of the arithmetic mean, California, Minnesota, New York and Ohio lost one seat each. When using the method of the harmonic mean, we can see that California, Colorado, Idaho and North Carolina lost one seat each.

5.1.3. Apportionment if both the District of Columbia and Puerto Rico were states

We can also run a simulation where both the District of Columbia and Puerto Rico were states. We will be using the same three methods with the same House size of 435. The population data for the US states is from [US 21], the population data for Puerto Rico is in [Ame21b] and the population data for DC is in [Ame21a]. See Table 11 for the results.

We can see that similarly to the previous two simulations, in this case, all three methods give one and four seats to the District of Columbia and Puerto Rico respectively.

Let's start by comparing the original and the simulation for the method of the geometric mean. Adding DC and Puerto Rico resulted in five states losing one seat each. These states were California, Colorado, Minnesota, Montana and Oregon.

In the case of the arithmetic mean, DC and Puerto Rico were assigned a total of five seats at the expense of five states, with each state losing one representative. These states were California, Colorado, Minnesota, New York and Ohio.

When we look at the result generated by the method of the harmonic mean, we also see five states losing one seat each. These states are California, Colorado, Idaho, North Carolina and Oregon.

5.2. SIMULATIONS FOR THEORETICAL SCENARIOS FOR THE EUROPEAN PARLIAMENT

5.2.1. Apportionment if all EEA members were included

The countries that are members of the European single market comprise the European Economic Area. In addition to EU member states, this organisation also includes Norway, Iceland and Liechtenstein. These three countries have a looser relationship with the European Union than members of the Union, thus they are not currently represented in the European Parliament. We can use the Python code to create an apportionment for a parliament that would include Norway, Iceland and Liechtenstein. The population data is from [Eur].

First, we will look at the results that don't consider degressive proportionality (see Table 12).

Using this data, the three methods only produce slight differences. We get the same results from the method of the arithmetic mean and the method of the geometric mean. When comparing these two with the method of the harmonic mean, we only see a difference

when looking at two states. The method of the harmonic mean assigns one seat more to Finland, and one seat fewer to France. This also shows us that among the three methods, the method of the harmonic mean favours smaller states the most.

The three methods assign one, one and eight seats to Iceland, Liechtenstein and Norway respectively, thus requiring ten seats to be taken away from other states.

For the method of the methods of the arithmetic and geometric mean, the difference from the original result is the same. It takes away two seats from Germany, and one seat each from Belgium, the Czech Republic, Finland, France, Italy, the Netherlands, Poland and Spain.

The method of the harmonic mean takes away two seats from France, and one seat each from Belgium, Cyprus, the Czech Republic, Germany, Italy, the Netherlands, Poland and Spain.

We can also run the same scenario using the program that keeps degressive proportionality (see Table 13).

In this case, all three methods produce the exact same result.

Respecting the principle of degressive proportionality has a larger impact on the apportionment. Iceland, Liechtenstein and Norway have six, six and eight seats respectively, thus requiring twenty seats to be taken away from other states.

In this case, as both the EU27 and EEA scenarios gave us results where all three methods produced the same result, we can discuss these together. After including the three countries, Spain loses four seats, Poland loses three, Italy, Romania and the Netherlands lose two each, and Austria, Belgium, Greece, Hungary, Ireland, Portugal and Sweden lose one seat each.

5.2.2. Apportionment if all EEA members and Switzerland were included

Switzerland is not a member of the European Economic Area, but it cooperates with the European Union in other areas (such as the Schengen area). In this simulation, we will be generating the apportionment for every EEA member and Switzerland (see Table 14). The population data is from [Eur].

When we include Switzerland, we will get the same results from all three methods. In this case, we will be comparing the results to the simulation that includes the EU27 and Liechtenstein, Iceland and Norway.

Comparing these results to the results we got in the previous scenario using the methods of the geometric and arithmetic mean, we will notice differences, since we need to

take 12 seats away from other states. France and Germany will lose two seats each, while Bulgaria, Hungary, Italy, Poland, Portugal, Romania, Slovakia and Spain will only lose one seat each.

If we compare this data with the results that we previously generated using the method of the harmonic mean, we can see a slightly different apportionment. In this case, 12 seats need to be taken away as well. As a result, Germany will lose two seats, Bulgaria, Finland, France, Hungary, Italy, Poland, Portugal, Romania, Slovakia and Spain will lose one seat each.

We can also run a simulation where we adhere to the principle of degressive proportionality (see Table 15).

As both the previous and current scenarios generated data, where the three methods gave us the same results, we can treat them as one case. As we included Switzerland with 12 seats, Italy and Spain lost two seats each, while Bulgaria, the Czech Republic, Finland, Hungary, Poland, Portugal, Romania, Slovakia and Spain only lost one seat each.

5.2.3. Apportionment if Ukraine were included

Since the Russian invasion of Ukraine in February 2022, the question of Ukrainian EU membership has become a crucial question in European politics. After the invasion, Ukraine officially applied to join the European Union. The leaders of the EU member states gave mostly positive reactions to the announcement.

Ukraine's population in 2022 is estimated to be around 43.5 million according to [Cen22]¹⁵, which would make it the fifth largest member state ranking behind Spain, but ahead of Poland. This would have a large impact on the composition of the European Parliament. Assuming the number of seats in the European Parliament would not be raised (and thus stay at 705), the apportionment using the different methods without degressive proportionality can be found in Table 16.

Using this data, the methods of the geometric and harmonic mean gave us the same result, with the method of the arithmetic mean giving us a slightly different result: it takes away one seat from Hungary and gives it to Ukraine, which makes sense, as we have already established that this method favours larger states.

¹⁵ We will be using the population data from the CIA World Factbook for this simulation.

As Ukraine has a significantly larger population than any of the countries included in previous simulations, including it has a large impact on apportionment. It will clearly show that the larger states will lose more as a result of the inclusion of Ukraine.

As a result of Ukraine being assigned 63 (or 64) seats, Germany would lose 12 seats, France would lose 9, Italy would lose 8, Spain would lose 7, Poland would lose 5, Romania and the Netherlands would lose 2 each, Belgium, the Czech Republic and Portugal would lose 2, Austria, Bulgaria, Croatia, Denmark, Finland, Greece, Hungary, Lithuania, Slovakia and Sweden would lose one seat each.

Finally, we can also generate this data while preserving the principle of degressive proportionality (see Table 17).

The three methods gave us the same results in this case as well. As the original apportionment with degressive proportionality also gave us one result for the three methods, we can discuss them together here.

If we used degressive proportionality, Spain would lose 11 seats, Italy would lose 10, Poland would lose 8, Romania would lose 5, the Netherlands would lose 4 each, France, Hungary and Portugal would lose 3 each, Austria, Belgium, the Czech Republic, Greece and Sweden would lose 2 each, and Bulgaria, Croatia, Denmark, Finland, Ireland and Slovakia would lose 1 seat each.

5.3. APPORTIONMENT DATA

For typographic reasons, the data generated by the Python program for the different scenarios can be found here. The tables containing the apportionment are displayed here in order of appearance in the text, along with captions indicating which scenario they belong to.

There are some cases, where two or more of the methods produced the same result. If two methods produced the same result, while the other one gave us a different one, the data is displayed in three columns. If all three methods arrived at the same apportionment, the data can be found in one column.

Table 9 Apportionment using the 2020 census data with DC statehood

State	Population	MGM	MAM	MHM
Alabama	5,030,053	7	7	7
Alaska	736,081	1	1	1
Arizona	7,158,923	9	9	9
Arkansas	3,013,756	4	4	4
California	39,576,757	52	52	52
Colorado	5,782,171	8	8	8
Connecticut	3,608,298	5	5	5
Delaware	990,837	1	1	1
District of Columbia	689,545	1	1	1
Florida	21,570,527	28	28	28
Georgia	10,725,274	14	14	14
Hawaii	1,460,137	2	2	2
Idaho	1,841,377	2	2	2
Illinois	12,822,739	17	17	17
Indiana	6,790,280	9	9	9
Iowa	3,192,406	4	4	4
Kansas	2,194,865	4	4	4
Kentucky	4,509,342	6	6	6
Louisiana	4,661,468	6	6	6
Maine	1,363,582	2	2	2
Maryland	6,185,278	8	8	8
Massachusetts	7,033,469	9	9	9
Michigan	10,084,442	13	13	13
Minnesota	5,709,752	7	7	7
Mississippi	2,963,914	4	4	4
Missouri	6,160,281	8	8	8
Montana	1,085,407	2	1	2
Nebraska	1,963,333	3	3	3
Nevada	3,108,462	4	4	4
New Hampshire	1,379,089	2	2	2
New Jersey	9,294,493	12	12	12
New Mexico	2,120,220	3	3	3
New York	20,215,751	26	27	26
North Carolina	10,453,948	14	14	14
North Dakota	779,702	1	1	1
Ohio	11,808,848	15	16	15
Oklahoma	3,963,516	5	5	5
Oregon	4,241,500	6	6	6
Pennsylvania	13,011,844	17	17	17
Rhode Island	1,098,163	2	1	2
South Carolina	5,124,712	7	7	7
South Dakota	887,770	1	1	1
Tennessee	6,916,897	9	9	9
Texas	29,183,290	38	38	38
Utah	3,275,252	4	4	4
Vermont	643,503	1	1	1
Virginia	8,654,542	11	11	11
Washington	7,715,946	10	10	10
West Virginia	1,795,045	2	2	2
Wisconsin	5,897,473	8	8	8
Wyoming	577,719	1	1	1

Table 10 Apportionment using the 2020 census data with Puerto Rico statehood

State	Population	MGM	MAM	MHM
Alabama	5,030,053	7	7	7
Alaska	736,081	1	1	1
Arizona	7,158,923	9	9	9
Arkansas	3,013,756	4	4	4
California	39,576,757	51	51	51
Colorado	5,782,171	7	8	7
Connecticut	3,608,298	5	5	5
Delaware	990,837	1	1	1
Florida	21,570,527	28	28	28
Georgia	10,725,274	14	14	14
Hawaii	1,460,137	2	2	2
Idaho	1,841,377	2	2	2
Illinois	12,822,739	17	17	17
Indiana	6,790,280	9	9	9
Iowa	3,192,406	4	4	4
Kansas	2,194,865	4	4	4
Kentucky	4,509,342	6	6	6
Louisiana	4,661,468	6	6	6
Maine	1,363,582	2	2	2
Maryland	6,185,278	8	8	8
Massachusetts	7,033,469	9	9	9
Michigan	10,084,442	13	13	13
Minnesota	5,709,752	7	7	7
Mississippi	2,963,914	4	4	4
Missouri	6,160,281	8	8	8
Montana	1,085,407	1	1	2
Nebraska	1,963,333	3	3	3
Nevada	3,108,462	4	4	4
New Hampshire	1,379,089	2	2	2
New Jersey	9,294,493	12	12	12
New Mexico	2,120,220	3	3	3
New York	20,215,751	26	26	26
North Carolina	10,453,948	14	14	13
North Dakota	779,702	1	1	1
Ohio	11,808,848	15	15	15
Oklahoma	3,963,516	5	5	5
Oregon	4,241,500	6	6	6
Pennsylvania	13,011,844	17	17	17
Puerto Rico	3,285,874	4	4	4
Rhode Island	1,098,163	2	1	2
South Carolina	5,124,712	7	7	7
South Dakota	887,770	1	1	1
Tennessee	6,916,897	9	9	9
Texas	29,183,290	38	38	38
Utah	3,275,252	4	4	4
Vermont	643,503	1	1	1
Virginia	8,654,542	11	11	11
Washington	7,715,946	10	10	10
West Virginia	1,795,045	2	2	2
Wisconsin	5,897,473	8	8	8
Wyoming	577,719	1	1	1

Table 11 Apportionment using the 2020 census data with DC and Puerto Rico statehood

State	Population	MGM	MAM	MHM
Alabama	5,030,053	7	7	7
Alaska	736,081	1	1	1
Arizona	7,158,923	9	9	9
Arkansas	3,013,756	4	4	4
California	39,576,757	51	51	51
Colorado	5,782,171	7	7	7
Connecticut	3,608,298	5	5	5
Delaware	990,837	1	1	1
District of Columbia	689,545	1	1	1
Florida	21,570,527	28	28	28
Georgia	10,725,274	14	14	14
Hawaii	1,460,137	2	2	2
Idaho	1,841,377	2	2	2
Illinois	12,822,739	17	17	17
Indiana	6,790,280	9	9	9
Iowa	3,192,406	4	4	4
Kansas	2,194,865	4	4	4
Kentucky	4,509,342	6	6	6
Louisiana	4,661,468	6	6	6
Maine	1,363,582	2	2	2
Maryland	6,185,278	8	8	8
Massachusetts	7,033,469	9	9	9
Michigan	10,084,442	13	13	13
Minnesota	5,709,752	7	7	7
Mississippi	2,963,914	4	4	4
Missouri	6,160,281	8	8	8
Montana	1,085,407	1	1	2
Nebraska	1,963,333	3	3	3
Nevada	3,108,462	4	4	4
New Hampshire	1,379,089	2	2	2
New Jersey	9,294,493	12	12	12
New Mexico	2,120,220	3	3	3
New York	20,215,751	26	26	26
North Carolina	10,453,948	14	14	13
North Dakota	779,702	1	1	1
Ohio	11,808,848	15	15	15
Oklahoma	3,963,516	5	5	5
Oregon	4,241,500	5	6	5
Pennsylvania	13,011,844	17	17	17
Puerto Rico	3,285,874	4	4	4
Rhode Island	1,098,163	2	1	2
South Carolina	5,124,712	7	7	7
South Dakota	887,770	1	1	1
Tennessee	6,916,897	9	9	9
Texas	29,183,290	38	38	38
Utah	3,275,252	4	4	4
Vermont	643,503	1	1	1
Virginia	8,654,542	11	11	11
Washington	7,715,946	10	10	10
West Virginia	1,795,045	2	2	2
Wisconsin	5,897,473	8	8	8
Wyoming	577,719	1	1	1

*Table 12 Apportionment in the European Parliament without degressive proportionality
(EEA member states)*

Country	Population	MGM	MAM	MHM
Austria	8,401,940	13	13	13
Belgium	11,000,638	17	17	17
Bulgaria	7,364,570	12	12	12
Croatia	4,284,889	7	7	7
Cyprus	840,407	1	1	1
Czech Republic	10,436,560	16	16	16
Denmark	5,560,628	9	9	9
Estonia	1,294,455	2	2	2
Finland	5,375,276	8	8	9
France	64,933,400	103	103	102
Germany	80,219,695	127	127	127
Greece	10,816,286	17	17	17
Hungary	9,937,628	16	16	16
Iceland	315,556	1	1	1
Ireland	4,574,888	7	7	7
Italy	59,433,744	94	94	94
Latvia	2,070,371	3	3	3
Liechtenstein	36,149	1	1	1
Lithuania	3,043,429	5	5	5
Luxembourg	512,353	1	1	1
Malta	417,432	1	1	1
Netherlands	16,655,799	26	26	26
Norway	4,979,954	8	8	8
Poland	38,044,565	60	60	60
Portugal	10,562,178	17	17	17
Romania	20,121,641	32	32	32
Slovakia	5,397,036	9	9	9
Slovenia	2,050,189	3	3	3
Spain	46,815,910	75	75	75
Sweden	9,482,855	15	15	15

*Table 13 Apportionment in the European Parliament with degressive proportionality
(EEA member states)*

Country	Population	Seats
Austria	8,401,940	13
Belgium	11,000,638	17
Bulgaria	7,364,570	12
Croatia	4,284,889	7
Cyprus	840,407	6
Czech Republic	10,436,560	17
Denmark	5,560,628	9
Estonia	1,294,455	6
Finland	5,375,276	9
France	64,933,400	96
Germany	80,219,695	96
Greece	10,816,286	17
Hungary	9,937,628	16
Iceland	315,556	6
Ireland	4,574,888	7
Italy	59,433,744	96
Latvia	2,070,371	6
Liechtenstein	36,149	6
Lithuania	3,043,429	6
Luxembourg	512,353	6
Malta	417,432	6
Netherlands	16,655,799	26
Norway	4,979,954	8
Poland	38,044,565	60
Portugal	10,562,178	17
Romania	20,121,641	32
Slovakia	5,397,036	9
Slovenia	2,050,189	6
Spain	46,815,910	74
Sweden	9,482,855	15

*Table 14 Apportionment in the European Parliament without degressive proportionality
(EEA member states and Switzerland)*

Country	Population	Seats
Austria	8,401,940	13
Belgium	11,000,638	17
Bulgaria	7,364,570	11
Croatia	4,284,889	7
Cyprus	840,407	1
Czech Republic	10,436,560	16
Denmark	5,560,628	9
Estonia	1,294,455	2
Finland	5,375,276	8
France	64,933,400	101
Germany	80,219,695	125
Greece	10,816,286	17
Hungary	9,937,628	15
Iceland	315,556	1
Ireland	4,574,888	7
Italy	59,433,744	93
Latvia	2,070,371	3
Liechtenstein	36,149	1
Lithuania	3,043,429	5
Luxembourg	512,353	1
Malta	417,432	1
Netherlands	16,655,799	26
Norway	4,979,954	8
Poland	38,044,565	59
Portugal	10,562,178	16
Romania	20,121,641	31
Slovakia	5,397,036	8
Slovenia	2,050,189	3
Spain	46,815,910	73
Sweden	9,482,855	15
Switzerland	7,954,662	12

*Table 15 Apportionment in the European Parliament with degressive proportionality
(EEA member states and Switzerland)*

Country	Population	Seats
Austria	8,401,940	13
Belgium	11,000,638	17
Bulgaria	7,364,570	11
Croatia	4,284,889	7
Cyprus	840,407	6
Czech Republic	10,436,560	16
Denmark	5,560,628	9
Estonia	1,294,455	6
Finland	5,375,276	8
France	64,933,400	96
Germany	80,219,695	96
Greece	10,816,286	17
Hungary	9,937,628	15
Iceland	315,556	6
Ireland	4,574,888	7
Italy	59,433,744	92
Latvia	2,070,371	6
Liechtenstein	36,149	6
Lithuania	3,043,429	6
Luxembourg	512,353	6
Malta	417,432	6
Netherlands	16,655,799	26
Norway	4,979,954	8
Poland	38,044,565	59
Portugal	10,562,178	16
Romania	20,121,641	31
Slovakia	5,397,036	8
Slovenia	2,050,189	6
Spain	46,815,910	72
Sweden	9,482,855	15
Switzerland	7,954,662	12

Table 16 Apportionment in the European Parliament without degressive proportionality (EU27 and Ukraine)

Country	Population	MGM	MAM	MHM
Austria	8,401,940	12	12	12
Belgium	11,000,638	16	16	16
Bulgaria	7,364,570	11	11	11
Croatia	4,284,889	6	6	6
Cyprus	840,407	1	1	1
Czech Republic	10,436,560	15	15	15
Denmark	5,560,628	8	8	8
Estonia	1,294,455	2	2	2
Finland	5,375,276	8	8	8
France	64,933,400	95	95	95
Germany	80,219,695	117	117	117
Greece	10,816,286	16	16	16
Hungary	9,937,628	15	14	15
Ireland	4,574,888	7	7	7
Italy	59,433,744	87	87	87
Latvia	2,070,371	3	3	3
Lithuania	3,043,429	4	4	4
Luxembourg	512,353	1	1	1
Malta	417,432	1	1	1
Netherlands	16,655,799	24	24	24
Poland	38,044,565	56	56	56
Portugal	10,562,178	15	15	15
Romania	20,121,641	29	29	29
Slovakia	5,397,036	8	8	8
Slovenia	2,050,189	3	3	3
Spain	46,815,910	68	68	68
Sweden	9,482,855	14	14	14
Ukraine	43,528,136	63	64	63

Table 17 Apportionment in the European Parliament with degressive proportionality (EU27 and Ukraine)

Country	Population	Seats
Austria	8,401,940	12
Belgium	11,000,638	16
Bulgaria	7,364,570	11
Croatia	4,284,889	6
Cyprus	840,407	6
Czech Republic	10,436,560	15
Denmark	5,560,628	8
Estonia	1,294,455	6
Finland	5,375,276	8
France	64,933,400	93
Germany	80,219,695	96
Greece	10,816,286	16
Hungary	9,937,628	14
Ireland	4,574,888	7
Italy	59,433,744	86
Latvia	2,070,371	6
Lithuania	3,043,429	6
Luxembourg	512,353	6
Malta	417,432	6
Netherlands	16,655,799	24
Poland	38,044,565	55
Portugal	10,562,178	15
Romania	20,121,641	29
Slovakia	5,397,036	8
Slovenia	2,050,189	6
Spain	46,815,910	67
Sweden	9,482,855	14
Ukraine	43,528,136	63

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