

Weighted exchange distance of basis pairs

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Abstract

Two pairs of disjoint bases $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$ of a matroid M are called *equivalent* if $\mathbf{P}_1$ can be transformed into $\mathbf{P}_2$ by a series of symmetric exchanges. In 1980, White conjectured that such a sequence always exists whenever $R_1 \cup B_1 = R_2 \cup B_2$. A strengthening of the conjecture was proposed by Hamidoune, stating that the exchange distance of such basis pairs is at most the rank of the matroid.

We propose a weighted variant of Hamidoune’s conjecture, where the weight of an exchange depends on the weights of the exchanged elements. We prove the conjecture for two matroid classes: spikes and graphic matroids of wheel graphs.

Keywords: Exchange distance, Sequential symmetric basis exchange, Spikes, Wheels

1 Introduction

Given a matroid M on a ground set S , the exchange axiom implies that for any pair of bases R and B there exists a sequence of exchanges that transforms R into B , and the shortest length of such a sequence is $|R - B|$. In the light of this, it is natural to ask whether analogous results hold for basis pairs instead of single basis. More precisely, let (R, B) be an ordered pair of disjoint bases of M , and assume that there exist $e \in R \setminus B$ and $f \in B \setminus R$ such that both $R' := R - e + f$ and $B' := B + e - f$ are bases. In such a case we call the exchange **feasible** and say that the pair (R', B') is obtained from (R, B) by a **symmetric exchange**. Using this terminology, we define the **exchange distance** (or **distance** for short) of two pairs of disjoint bases $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$ to be the minimum number of symmetric exchanges needed to transform the former into the latter if such a sequence exists and $+\infty$ otherwise. We call two pairs of disjoint bases **equivalent** if their exchange distance is finite. A sequence of symmetric exchanges is called **strictly monotone** if each step decreases the difference between the first members of the pairs. In other words, a strictly monotone exchange sequence uses exactly the elements in $(R_1 \cap B_2) \cup (R_2 \cap B_1)$ and exactly once.

At this point it is not clear (A) when the distance of two pairs will be finite, and (B) if their distance is finite, whether we can give an upper bound on it. Regarding question (A), one can formulate an obvious necessary condition for the distance of $\mathbf{P}_1$ and $\mathbf{P}_2$ to be finite, namely $R_1 \cup B_1 = R_2 \cup B_2$ should certainly hold – two pairs with this property are called **compatible**. In [11], White conjectured that *two basis pairs $\mathbf{P}_1$ and $\mathbf{P}_2$ are equivalent if and only if they are compatible*. While the conjecture was verified for various matroid classes, it remains open in general.

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Much less is known about question (B), that is, the optimization version of the problem. Gabow [7] studied sequential symmetric exchanges and posed the following problem, which was later stated as a conjecture by Wiedemann [12] and by Cordovil and Moreira [4]: *for any two disjoint bases R and B of a rank- r matroid M , there is a sequence of r symmetric exchanges that transforms the pair $\mathbf{P}_1 = (R, B)$ into $\mathbf{P}_2 = (B, R)$* . The rank of the matroid is a trivial lower bound on the minimum number of exchanges needed to transform a pair (R, B) into (B, R) , and the essence of Gabow’s conjecture is that that many steps might always suffice. The relation between the conjectures of White and Gabow is immediate: the latter would imply the former for sequences of the form (R, B) and (B, R) .

In general, if M has rank r , then $r - |R_1 \cap R_2|$ is an obvious lower bound on the exchange distance of $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$. However, it might happen that more symmetric exchanges are needed even if M is a graphic matroid; see [5] or Figure 5 for a counterexample. As a common generalization of the conjectures of White and Gabow, Hamidoune [4] proposed a rather optimistic variant stating that *the exchange distance of compatible basis pairs is at most the rank of the matroid*.

Previous work. By relying on the constructive characterization of bispanning graphs, Farber, Richter, and Shank [6] proved White’s conjecture for graphic and cographic matroids, while Farber [5] settled the statement for transversal matroids. Bonin [3] verified the conjecture for sparse paving matroids. The case of strongly base orderable matroids was solved by Lasoń and Michałek [8]. McGuinness [9] extended the graphic case to frame matroids satisfying a certain linearity condition. Recently, Bérczi and Schwarcz [2] showed that Hamidoune’s conjecture hold for split matroids, a large class that contains paving matroids as well.

While studying a specific maker-beaker game on bispanning graphs, Andres, Hochstättler and Merkel [1] showed that there is an exchange sequence between any two pairs of disjoint spanning trees of a wheel of rank at least four using only so-called left unique exchanges. They also asked whether the exchange distance of compatible basis pairs of a matroid can be bounded by a polynomial of the rank – this latter problem is a weakening of Hamidoune’s conjecture.

Our results. Let $w: S \rightarrow \mathbb{R}_+$ be a weight function on the elements of the ground set S . Given a pair (R, B) of disjoint bases, we define the **weight of a symmetric exchange** $R' := R - e + f$ and $B' := B + e - f$ to be $w(e) + w(f)$, that is, the sum of the weights of the exchanged elements. Analogously to the unweighted setting, we define the **weighted exchange distance** (or **weighted distance** for short) of two pairs of disjoint bases $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$ to be the minimum total weight of symmetric exchanges needed to transform the former into the latter if such a sequence exists and $+\infty$ otherwise.

As a weighted extension of Hamidoune’s conjecture, we propose the following.

Conjecture 1. *Let $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$ be compatible basis pairs. Then the weighted exchange distance of $\mathbf{P}_1$ and $\mathbf{P}_2$ is at most $w(R_1 \cup B_1) = w(R_2 \cup B_2)$.*

By setting the weight function to be identically one, we get back Hamidoune’s conjecture. It is worth mentioning that a strictly monotone exchange sequence transforming $\mathbf{P}_1$ into $\mathbf{P}_2$ is optimal in every sense, i.e., it has both minimum length and minimum weight.

Our contribution is twofold. First we verify a further strengthening of Conjecture 1 for graphic matroids of wheels. Though wheels are structurally simple, the proof for this graph class is already non-trivial and requires a thorough understanding of feasible exchanges. As a corollary, we show that the minimum number of steps required to transform $\mathbf{P}_1$ into $\mathbf{P}_2$ can be arbitrarily large compared to the lower bound $r - |R_1 \cap R_2|$.

We also prove the conjecture for spikes, an important class of 3-connected matroids. Spikes are interesting from the viewpoint of Hamidoune’s conjecture because, as we show, one can define an arbitrarily large number of basis pairs without a strictly monotone exchange sequence between any two of them. This is in sharp contrast to the case of wheels, where for any three pairs of bases, there exist two with a strictly monotone exchange sequence between them.

The rest of the paper is organized as follows. Basic notions and definitions are given in Section 2, together with some elementary observations on wheels. We verify Conjecture 1 for wheels in Section 3, while the class of spikes is discussed in Section 4. We close the paper with further observations in Section 5.

2 Preliminaries

General notation. The set of nonnegative real numbers is denoted by $\mathbb{R}_+$. For subsets $X, Y \subseteq S$, their **symmetric difference** is defined as $X \triangle Y := (X \setminus Y) \cup (Y \setminus X)$. When Y consist of a single element y , then $X \setminus \{y\}$ and $X \cup \{y\}$ are abbreviated as $X - y$ and $X + y$, respectively. Given a weight function $w : S \rightarrow \mathbb{R}_+$ and a subset $X \subseteq S$, we use the notation $w(X) = \sum_{s \in X} w(s)$.

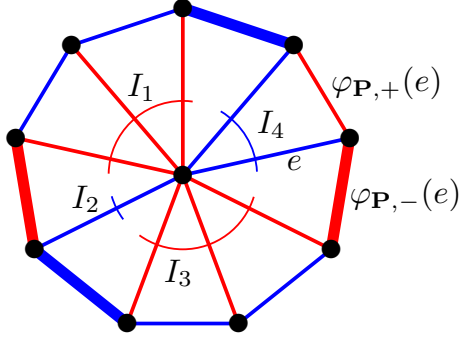
Matroids. For basic definitions on matroids, we refer the reader to [10]. If M is a rank- r matroid over a ground set S of size $2r$ such that S decomposes into two disjoint bases R and B of M , then such a decomposition is called a **coloring** of M ¹. A **feasible exchange** of elements $r \in R$ and $b \in B$ is denoted by (r, b) . We extend this notation to a **sequence of symmetric exchanges** as well by writing $(r_1, b_1), \dots, (r_k, b_k)$, meaning that $(R \setminus \{r_1, \dots, r_i\}) \cup \{b_1, \dots, b_i\}$ and $(B \cup \{r_1, \dots, r_i\}) \setminus \{b_1, \dots, b_i\}$ are bases for $i = 1, \dots, k$.

For a graph $G = (V, E)$ on n vertices, the **graphic matroid** $M = (E, \mathcal{I})$ of G is defined on the edge set by considering a subset $F \subseteq E$ to be independent if it is a forest, that is, $\mathcal{I} = \{F \subseteq E : F \text{ does not contain a cycle}\}$. If the graph is connected, then the bases of the graphic matroid are exactly the spanning trees of G and the rank of the matroid is $n - 1$.

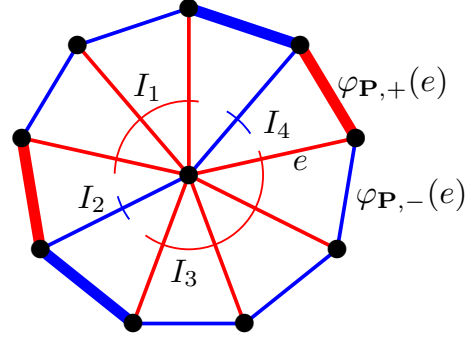
Let $S = \{t, x_1, y_1, \dots, x_r, y_r\}$ be a ground set of size $2r + 1$ for some $r \geq 3$, and let $\mathcal{C}_1 = \{\{t, x_i, y_i\} \mid 1 \leq i \leq r\}$ and $\mathcal{C}_2 = \{\{x_i, y_i, x_j, y_j\} \mid 1 \leq i < j \leq r\}$. Furthermore, let $\mathcal{C}_3 \subseteq \{Z \subseteq S \mid |Z| = r, |Z \cap \{x_i, y_i\}| = 1 \text{ for } 1 \leq i \leq r\}$ be such that the intersection of any two members of $\mathcal{C}_3$ has size at most $r - 2$. Note that $\mathcal{C}_3$ might be empty. Finally, define $\mathcal{C}_4 = \{C \subseteq S \mid |C| = r + 1, C' \not\subseteq C \text{ for } C' \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3\}$. Then the family $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$ satisfies the circuit axioms, hence $M = (S, \mathcal{C})$ is a rank- r matroid with circuit family $\mathcal{C}$. Matroids arising this way are called **spikes**, where t and the pairs $\{x_i, y_i\}$ are called the **tip** and the **legs** of the spike, respectively. It is not difficult to check that by restricting M to any $2r$ of its elements (or in other words, deleting any of its elements) results in a matroid whose ground set decomposes into two disjoint bases.

Wheels. A graph $G = (V, E)$ is called a **wheel graph** (or **wheel** for short) if it is obtained by connecting a vertex, called the **center of the wheel**, to all the vertices of a cycle of length at least three, called the **outer cycle** of the wheel. In particular, wheels have at least four vertices. Edges connecting the center vertex with the vertices of the outer cycle are called **spokes**, while the edges of the outer cycle are called **rim edges**. Wheels are clearly planar, and so the order of the vertices on the outer cycle implies a natural cyclic ordering of the spokes and the rim edges as well.

¹Throughout the paper, we will refer to the elements of R and B as ‘red’ and ‘blue’, respectively.



(a) The red and blue intervals are denoted by I_1, I_3 and I_2, I_4 , respectively.



(b) The coloring obtained by symmetrically exchanging e and its pair $\varphi_{\mathbf{P},-}(e)$.

Figure 1: Colorings containing four intervals. Thick rim edges correspond to boundary edges. Note that both colorings have positive orientation.

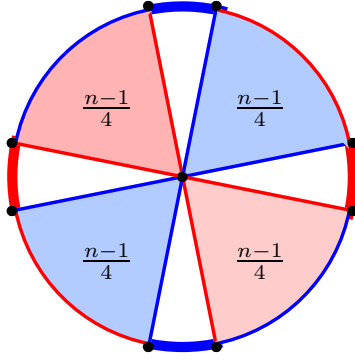
It is not difficult to check that any wheel is the disjoint union of two spanning trees. Therefore, a coloring of the graphic matroid of a wheel is basically a partition of its edge set into two colors R and B such that both color classes form a spanning tree. A nice property of wheels is that we have a fairly good understanding of the structure of their colorings. Indeed, in order to decompose a wheel into two spanning trees, we first need to split the spokes into two nonempty sets. Then, if a rim edge goes between the endpoints of two spokes having the same color, then it automatically has to be colored with the other color to obtain a basis. Hence it only remains to decide the color of the rim edges going between the endpoints of spokes having different colors. However, once we fix the color of any of those edges, it determines the color of all the remaining ones.

We call a maximal set of consecutive spokes of the same color an **interval**. By the **length** and **color of the interval** we mean the number and color of the spokes in it, respectively. Rim edges going between two intervals are called **boundary edges**. By the above, for $X \in \{R, B\}$, either every interval of color X is followed by a boundary edge of color X in a counterclockwise direction, or every interval of color X is followed by a boundary edge of color X in a clockwise direction. This property is referred to as the **orientation** of the coloring, and the orientation is called **positive** in the former and **negative** in the latter case; see Figure 1a for an example. Orientations will play a crucial role in whether one can go from one coloring to another using a small number of exchanges or not.

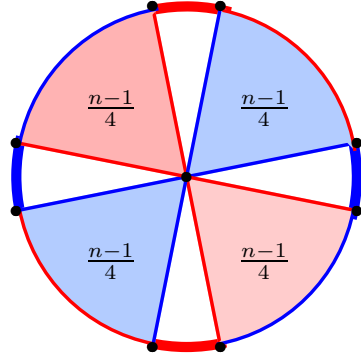
Given a coloring $\mathbf{P} = (R, B)$ of the wheel graph, each spoke e is incident to two rim edges. The rim edge sharing a vertex with e in the direction opposite of the orientation of the coloring can always be exchanged with e . We denote this rim edge by $\varphi_{\mathbf{P},-}(e)$, while the other rim edge incident to e is denoted by $\varphi_{\mathbf{P},+}(e)$; see Figure 1b for an example. Both $\varphi_{\mathbf{P},-}$ and $\varphi_{\mathbf{P},+}$ provide bijections between spokes and rim edges. Note that these bijections are already determined by the orientation of $\mathbf{P}$. If there are at least four intervals in the coloring, then the feasible exchanges are exactly the ones exchanging e and $\varphi_{\mathbf{P},-}(e)$ for some spoke e . When there are only two intervals in the coloring, then there are other pairs that can be symmetrically exchanged.

3 Wheels

Our first main result is a proof of Conjecture 1 for the graphic matroid of wheels. In fact, we prove a much stronger statement: we verify that for any pair $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$ of colorings of a wheel $G = (V, E)$, there exists a sequence of exchanges of length at most r and



(a) A coloring $\mathbf{P}_1 = (R_1, B_1)$ with positive orientation. Thick edges denote the boundary edges.



(b) A coloring $\mathbf{P}_2 = (R_2, B_2)$ with negative orientation. Thick edges denote the boundary edges.

Figure 2: A pair of $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$ with $(n-1) - |R_1 \cap R_2| = 2$. The colorings have opposite orientations, hence any sequence of exchanges that transforms $\mathbf{P}_1$ into $\mathbf{P}_2$ goes through a coloring having two intervals, implying that their exchange distance is at least $\frac{n-1}{4}$.

total weight at most $w(E)$ that transforms $\mathbf{P}_1$ into $\mathbf{P}_2$ and uses each edge at most twice.

Throughout the section, we assume that $\mathbf{P}_1$ has positive orientation. For ease of notation, we introduce $\varphi_{\oplus} := \varphi_{\mathbf{P}_1, +}$ and $\varphi_{\ominus} := \varphi_{\mathbf{P}_1, -}$. Recall that both $\varphi_{\ominus}$ and $\varphi_{\oplus}$ provide a bijection between spokes and rim edges.

First we settle the case when the two colorings have the same orientation.

Lemma 1. *Let $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$ be colorings of a wheel $G = (V, E)$ with the same orientation. Then there exists a strictly monotone sequence of exchanges that transforms $\mathbf{P}_1$ into $\mathbf{P}_2$.*

Proof. Exchange each spoke e that has different color in $\mathbf{P}_1$ than in $\mathbf{P}_2$ with its pair $\varphi_{\ominus}(e)$ in an arbitrary order, only paying attention to always have at least one spoke in both color classes. Once the spokes have the right colors, that is, they are colored as in $\mathbf{P}_2$, the rim edges are also colored as required. Indeed, the orientation was not changed during the procedure and the coloring of the spokes together with the orientation of the coloring uniquely determines the colors of the rim edges. $\square$

Remark 2. Exchanging a spoke e with its pair $\varphi_{\ominus}(e)$ does not reverse the orientation of the coloring. Therefore, if the initial coloring $\mathbf{P}_1 = (R_1, B_1)$ and the target coloring $\mathbf{P}_2 = (R_2, B_2)$ have different orientations, then any sequence of feasible exchanges must go through a coloring of the graph with only two intervals. This results in reversing the orientation of a coloring to be costly, as one needs to make exchange steps that, at least seemingly, bring the colorings further away from each other in the sense that they increase the symmetric difference of R_1 and R_2 . The class of wheels is particularly interesting as they admit colorings $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$ for which the minimum number of steps required to transform $\mathbf{P}_1$ into $\mathbf{P}_2$ is arbitrarily large compared to the trivial lower bound $(n-1) - |R_1 \cap R_2|$; see Figure 2 for an example.

Next we consider colorings with different orientations and a bounded number of intervals in one of the color classes.

Lemma 3. *Let $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$ be colorings of a wheel $G = (V, E)$ with different orientations where $\mathbf{P}_1$ has at most four intervals, and $w : E \rightarrow \mathbb{R}_+$ be a weight*

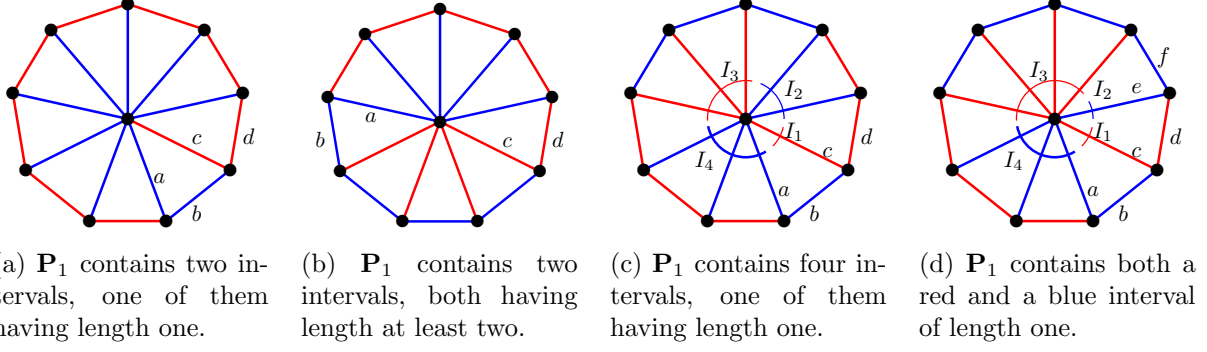


Figure 3: Illustration of the cases in the proof of Lemma 3.

function. Then there exists a sequence of exchanges of length at most $n - 1$ and total weight at most $w(E)$ that transforms $\mathbf{P}_1$ into $\mathbf{P}_2$ and uses each edge at most twice.

Proof. We will distinguish two main cases based on the number of intervals in $\mathbf{P}_1$. Recall that $\mathbf{P}_1$ is assumed to have positive orientation throughout.

Case 1. $\mathbf{P}_1$ has two intervals.

Case 1.1. $\mathbf{P}_1$ has an interval of length one.

We may assume that there exists a red interval of length one, and let c denote the unique spoke in it. We denote by a the spoke following c in negative direction, and further define $b := \varphi_{\oplus}(a) = \varphi_{\ominus}(c)$ and $d := \varphi_{\oplus}(c)$. Note that a and b are blue, while d is a red edge; see Figure 3a. As $\mathbf{P}_2$ has negative orientation, a and $\varphi_{\oplus}(a) = b$ have different colors in $\mathbf{P}_2$, and the same holds for c and $\varphi_{\oplus}(c) = d$. Hence the set of edges among a, b, c and d that have different colors in $\mathbf{P}_1$ and $\mathbf{P}_2$ is either $\{a, c\}$, $\{a, d\}$, $\{b, d\}$ or $\{b, c\}$. In the first three cases, changing the color of the two edges is a feasible exchange which reverses the orientation. Once the orientation of the coloring is reversed, there exists a strictly monotone exchange sequence to $\mathbf{P}_2$ by Lemma 1, altogether resulting in a strictly monotone exchange sequence from $\mathbf{P}_1$ to $\mathbf{P}_2$.

The only remaining case is when the set of edges among a, b, c and d that need to change color is $\{b, c\}$. In this case, the difficulty comes from the fact that these edges do not define a feasible exchange between the two color classes. In order to overcome this, extra steps are needed to reverse the orientation. Let s be an arbitrary red spoke of $\mathbf{P}_2$. Note that $s \notin \{a, c\}$ as we are in the case when a and c are blue in $\mathbf{P}_2$. As $\mathbf{P}_1$ has a unique red spoke, namely c , we get that s is blue in $\mathbf{P}_1$, and $\varphi_{\oplus}(s)$ is red in $\mathbf{P}_1$ and blue in $\mathbf{P}_2$ since $\mathbf{P}_2$ has negative orientation. Consider the two exchange sequences of length three $(b, d), (s, \varphi_{\oplus}(s)), (c, d)$ and $(a, c), (s, \varphi_{\oplus}(s)), (a, b)$. Both of these sequences reverse the orientation of the coloring and fix the colors of the edges a, b, c and d . Therefore, after applying any of them, there exists a strictly monotone exchange sequence to $\mathbf{P}_2$ by Lemma 1 that uses all the remaining edges in $E \setminus \{a, b, c, d\}$ at most once. Thus in overall, we get an exchange sequence that uses each edge in $E \setminus \{a, d\}$ at most once, does not use one of a and d and uses the other twice. Hence the length of the sequence is at most half of the number of edges, that is, $n - 1$. If $w(a) \geq w(d)$, then starting the sequence with $(b, d), (s, \varphi_{\oplus}(s)), (c, d)$, while if $w(a) < w(d)$, then starting the sequence with $(a, c), (s, \varphi_{\oplus}(s)), (a, b)$ ensures that total weight of the exchange sequence is at most $w(E)$, concluding the proof of the case.

Case 1.2. Both intervals of $\mathbf{P}_1$ have length at least two.

Let c denote the last spoke of the red interval in positive direction and let $d := \varphi_{\oplus}(c)$. Furthermore, let a be the last spoke of the blue interval in positive direction and let $b := \varphi_{\oplus}(a)$,

see Figure 3b. Similarly to Case 1.1, the set of edges among a, b, c and d that have different colors in $\mathbf{P}_1$ and $\mathbf{P}_2$ is either $\{a, c\}$, $\{a, d\}$, $\{b, d\}$ or $\{b, c\}$. However, now fixing the orientation is even simpler than before as any of the exchanges (a, c) , (a, d) , (b, c) and (b, d) is feasible. After reversing the orientation using one of these exchanges, there exists a strictly monotone exchange sequence to $\mathbf{P}_2$ by Lemma 1, altogether resulting in a strictly monotone exchange sequence from $\mathbf{P}_1$ to $\mathbf{P}_2$.

We denote the number of spokes in R_1, R_2, B_1 and B_2 by r_1, r_2, b_1 and b_2 , respectively. Let $2q$ denote the number of intervals in $\mathbf{P}_1$, and let $I_1, \dots, I_{2q}$ denote the intervals in a positive direction, where intervals with odd indices have color red and intervals with even indices have color blue. Furthermore, for $1 \leq i \leq 2q$, we define

$$\begin{aligned} x_i &:= \sum [w(e) \mid e \in I_i \cup \varphi_{\ominus}(I_i), e \text{ has the same color in } \mathbf{P}_1 \text{ and } \mathbf{P}_2], \\ y_i &:= \sum [w(e) \mid e \in I_i \cup \varphi_{\ominus}(I_i), e \text{ has different colors in } \mathbf{P}_1 \text{ and } \mathbf{P}_2]. \end{aligned}$$

By the above definitions, we have $w(I_i \cup \varphi_{\ominus}(I_i)) = x_i + y_i$ for $1 \leq i \leq 2q$, and $\sum_{i=1}^{2q} (x_i + y_i) = w(E)$.

Case 2. $\mathbf{P}_1$ has four intervals.

Since $r_1 + r_2 + b_1 + b_2 = 2(n - 1)$, we have $\min\{r_1 + r_2, b_1 + b_2\} \leq n - 1$. We may assume that $r_1 + r_2 \leq n - 1$. We distinguish two cases based on the structure of the red intervals in $\mathbf{P}_1$.

Case 2.1. $\mathbf{P}_1$ has no red interval of length one.

Take one of the intervals $I_j \in \{I_1, I_3\}$ and exchange each spoke p in I_j with its pair $\varphi_{\ominus}(p)$. The coloring thus obtained has only two intervals from which the red has length more than one, hence there exists a strictly monotone exchange sequence to $\mathbf{P}_2$ as in Case 1.2. Note that each edge is used at most twice during this process, and the edges used twice are exactly the edges of I_j having the same color in $\mathbf{P}_1$ and $\mathbf{P}_2$. The length of the exchange sequence thus obtained is

$$\begin{aligned} |I_j| + |(I_j \cup I_2 \cup I_4) \cap R_2| + |I_{4-j} \cap B_2| &\leq |I_j| + |(I_1 \cup I_2 \cup I_3 \cup I_4) \cap R_2| + |I_{4-j}| \\ &= r_1 + r_2 \\ &\leq n - 1. \end{aligned}$$

In order to bound the total weight of the exchanges, note that we used edges twice only from $I_j \cup \varphi_{\ominus}(I_j)$, and all the other edges were used only if their color had to change. Thus the total weight of the exchange sequence is $2x_j + y_1 + y_2 + y_3 + y_4$. As $2 \min\{x_1, x_3\} \leq x_1 + \dots + x_4$, we get $2 \min\{x_1, x_3\} + y_1 + y_2 + y_3 + y_4 \leq w(E)$. Hence choosing $j \in \{1, 3\}$ such that $x_j = \min\{x_1, x_3\}$ leads to an exchange sequence with the required properties.

Case 2.2. $\mathbf{P}_1$ has a red interval of length one.

Our approach is similar to that of Case 2.1 with slight modifications. By applying the same algorithm, the analysis goes through if the the red interval whose edges are not used before reversing the orientation has length at least two. Indeed, the bounds on the length and weight of the exchange sequence remain valid since we did not use the length of the intervals in the proof. However, a problem occurs if at the orientation reversal step we have an interval of length one, that is, when the algorithm starts with a red interval, say I_3 due to $x_3 \leq x_1$, and the other red interval I_1 has length one.

Let $\mathbf{P}'_1$ denote the coloring obtained by the exchanges so far, that is, by exchanging each spoke p in I_3 with its pair $\varphi_{\ominus}(p)$. At this point, our goal is to reverse the orientation and then find a strictly monotone sequence to $\mathbf{P}_2$. Note that this is exactly the same scenario we had in Case 1.1, hence we use an analogous notation that we had there. That is, let c denote the

unique red spoke of $\mathbf{P}'_1$ and a be the spoke following c in a negative direction. Furthermore, define $b := \varphi_{\oplus}(a) = \varphi_{\ominus}(c)$ and $d := \varphi_{\oplus}(c)$, see Figure 3c. Similarly to Case 1.1, the set of edges among a, b, c and d that have different colors in $\mathbf{P}'_1$ and $\mathbf{P}_2$ is either $\{a, c\}$, $\{a, d\}$, $\{b, d\}$ or $\{b, c\}$. In the first three cases, changing the color of the two edges is a feasible exchange which reverses the orientation and fixes the color of all four edges a, b, c and d . Once the coloring is reversed, there exists a strictly monotone exchange sequence to $\mathbf{P}_2$ by Lemma 1. The length and weight of the exchange sequence can be bounded analogously to Case 2.1.

The only remaining case is when the set of edges among a, b, c and d that need to change color is $\{b, c\}$. Let s be an arbitrary red spoke of $\mathbf{P}_2$. Note that $s \notin \{a, c\}$ as we are in the case when a and c are blue in $\mathbf{P}_2$. As $\mathbf{P}'_1$ has a unique red spoke, namely c , we get that s is blue in $\mathbf{P}'_1$, and $\varphi_{\oplus}(s)$ is red in $\mathbf{P}'_1$ and blue in $\mathbf{P}_2$ since $\mathbf{P}_2$ has negative orientation. Consider the two exchange sequences of length three $(b, d), (s, \varphi_{\oplus}(s)), (c, d)$ and $(a, c), (s, \varphi_{\oplus}(s)), (a, b)$. Both of these sequences change the orientation of the coloring and fix the colors of the edges a, b, c and d . Therefore, after applying any of them, there exists a strictly monotone exchange sequence to $\mathbf{P}_2$ by Lemma 1. If $w(a) \geq w(d)$, then use the sequence $(b, d), (s, \varphi_{\oplus}(s)), (c, d)$ for reversing the orientation, and use $(a, c), (s, \varphi_{\oplus}(s)), (a, b)$ otherwise. In what follows, we bound the length and weight of the sequence obtained.

Suppose first that $r_1 + r_2 < n - 1$. The length of the exchange sequence is

$$|I_3| + 3 + |(I_2 \cup I_3 \cup I_4 - s) \cap R_2| = (r_1 - 1) + 3 + (r_2 - 1) = r_1 + r_2 + 1 \leq n - 1.$$

We used edges twice only from $I_3 \cup \varphi_{\ominus}(I_3) \cup \{d\}$ if $w(a) \geq w(d)$, and only from $I_3 \cup \varphi_{\ominus}(I_3) \cup \{a\}$ otherwise. Thus the total weight of the exchange sequence is at most

$$2x_3 + 2 \min\{w(a), w(d)\} + y_1 + y_2 + y_3 + y_4 \leq 2x_3 + x_2 + x_4 + y_1 + y_2 + y_3 + y_4 \leq w(E),$$

where we used that $2 \min\{w(a), w(d)\} \leq 2 \min\{x_4, x_2\} \leq x_2 + x_4$ and $x_3 \leq x_1$.

We are left with the case when $r_1 + r_2 = n - 1$, there exists a red interval consisting of a single spoke c , and the edges c and $\varphi_{\ominus}(c)$ have different colors in $\mathbf{P}'_1$ and $\mathbf{P}_2$. Then, by $r_1 + r_2 + b_1 + b_2 = 2n - 2$, we get that $b_1 + b_2 = n - 1$ also holds. Up to this point the two colors played different roles in the proof of Case 2 as we used the fact that $r_1 + r_2 \leq n - 1$. However, now the same inequality holds for the number of blue spokes as well, hence we can switch the roles of the two colors. In particular, if none of the blue intervals have length one, then we are done. Thus suppose that the red interval I_1 consisting of the single spoke c is followed in positive direction by a blue interval I_2 consisting of the single spoke e ; the case when a blue interval of length one is followed in positive direction by a red interval of length one can be proved analogously. We denote by a the spoke before c in negative direction, and further define $b := \varphi_{\oplus}(a) = \varphi_{\ominus}(c)$, $d := \varphi_{\oplus}(c) = \varphi_{\ominus}(e)$ and $f := \varphi_{\oplus}(e)$, see Figure 3d. Recall that we are in the case when the set of edges among a, b, c and d that need to change color is $\{b, c\}$. However, this means that among the edges c, d, e and f , the set of edges that need to change color cannot be $\{d, e\}$. That is, if we start by exchanging each spoke p in I_4 with its pair $\varphi_{\ominus}(p)$, then the resulting coloring $\mathbf{P}''_1$ can be transformed into $\mathbf{P}_2$ by a strictly monotone sequence of exchanges, and the analysis of the second paragraph of Case 2.2 applies. $\square$

Our last technical lemma shows that when one of the colorings have at least six intervals, then there exists a sequence of exchanges that has low weight with respect to two arbitrary weight functions simultaneously.

Lemma 4. *Let $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$ be colorings of a wheel $G = (V, E)$ with different orientations such that $\mathbf{P}_1$ has at least six intervals, and $w_1, w_2 : E \rightarrow \mathbb{R}_+$ be weight functions. Then there exists a sequence of exchanges of total w_i -weight at most $w_i(E)$ for $i = 1, 2$ that transforms $\mathbf{P}_1$ into $\mathbf{P}_2$ and uses each edge at most twice.*

Proof. We distinguish two cases based on the remainder of the number of intervals modulo four.

Case 1. $q = 2k + 1$ for some integer $k \geq 1$.

For an index $1 \leq j \leq 4k + 2$, exchange each spoke $e \in \bigcup_{i=1}^k I_{j+2i-1}$ with its pair $\varphi_{\ominus}(e)$, and do the same for each spoke $e \in \bigcup_{i=k+1}^{2k} I_{j+2i}$. After these exchanges, the resulting coloring $\mathbf{P}'_1$ has two intervals: $I_j \cup I_{j+1} \cup \dots \cup I_{j+2k}$ has the same color in $\mathbf{P}'_1$ as I_j in $\mathbf{P}_1$, and $I_{j+2k+1} \cup I_{j+2k+2} \cup \dots \cup I_{j+4k+1}$ has the other color. Note that none of these two intervals has length one as $k \geq 1$. Therefore, there exists a strictly monotone exchange sequence from $\mathbf{P}'_1$ to $\mathbf{P}_2$ by Case 1.2 of Lemma 3. Let $w \in \{w_1, w_2\}$, and let us define I_i , x_i and y_i for $1 \leq i \leq 2q$ as in the proof of Lemma 3, where the x_i and y_i values are computed with respect to w . Our goal is to bound the w -weight of the above defined sequence of exchanges.

Exchanging each spoke e in $\bigcup_{i=1}^k I_{j+2i-1} \cup \bigcup_{i=k+1}^{2k} I_{j+2i}$ with its pair $\varphi_{\ominus}(e)$ has weight

$$\sum_{i=1}^k (x_{j+2i-1} + y_{j+2i-1}) + \sum_{i=k+1}^{2k} (x_{j+2i} + y_{j+2i}).$$

Then the strictly monotone sequence to $\mathbf{P}_2$ has weight

$$\sum_{i=0}^k y_{j+2i} + \sum_{i=1}^k x_{j+2i-1} + \sum_{i=k}^{2k} y_{j+2i+1} + \sum_{i=k+1}^{2k} x_{j+2i}.$$

The total weight is then

$$2 \cdot \left(\sum_{i=1}^k x_{j+2i-1} + \sum_{i=k+1}^{2k} x_{j+2i} \right) + \sum_{i=1}^{4k+2} y_i.$$

Therefore the total w -weight of the exchange sequence is at most $w(E) = \sum_{i=1}^{4k+2} (x_i + y_i)$ if and only if

$$(A_w(j)) \quad \sum_{i=1}^k x_{j+2i-1} + \sum_{i=k+1}^{2k} x_{j+2i} \leq \sum_{i=0}^k x_{j+2i} + \sum_{i=k}^{2k} x_{j+2i+1}.$$

Consider inequalities $A_w(j)$ and $A_w(j+1)$. The sum of these two inequalities gives

$$\left(\sum_{i=1}^{4k+2} x_i \right) - (x_j + x_{j+2k+1}) \leq \left(\sum_{i=1}^{4k+2} x_i \right) + (x_j + x_{j+2k+1}).$$

As this inequality clearly holds, at least one of $A_w(j)$ and $A_w(j+1)$ must hold as well. Furthermore, $A_w(j)$ is identical to $A_w(j+2k+1)$. These together imply that $A_w(j)$ holds for at least $k+1$ choices of j from $\{1, \dots, 2k+1\}$ for $w \in \{w_1, w_2\}$. Therefore, there exists an index j for which both $A_{w_1}(j)$ and $A_{w_2}(j)$ are satisfied. As each edge is used at most twice, the statement follows.

Case 2. $q = 2k$ for some integer $k \geq 2$.

Our approach is similar to that of Case 1. For an index $1 \leq j \leq 4k$, exchange each spoke $e \in \bigcup_{i=1}^{k-1} I_{j+2i-1}$ with its pair $\varphi_{\ominus}(e)$, and do the same for each spoke $e \in \bigcup_{i=k}^{2k-1} I_{j+2i}$. After these exchanges, the resulting coloring $\mathbf{P}'_1$ has two intervals: $I_j \cup I_{j+1} \cup \dots \cup I_{j+2k-1}$ has the same color in $\mathbf{P}'_1$ as I_j in $\mathbf{P}_1$, and $I_{j+2k} \cup I_{j+2k+1} \cup \dots \cup I_{j+4k}$ has the other color. Note that none of these two intervals has length one as $k \geq 2$. Therefore, there exists a strictly monotone exchange sequence from $\mathbf{P}'_1$ to $\mathbf{P}_2$ by Case 1.2 of Lemma 3. Let $w \in \{w_1, w_2\}$, and let us

define I_i , x_i and y_i for $1 \leq i \leq 2q$ as in the proof of Lemma 3, where the x_i and y_i values are computed with respect to w . Our goal is to bound the w -weight of the above defined sequence of exchanges.

Exchanging each spoke e in $\bigcup_{i=1}^{k-1} I_{j+2i-1} \cup \bigcup_{i=k}^{2k-1} I_{j+2i}$ with its pair $\varphi_{\ominus}(e)$ has weight

$$\sum_{i=1}^{k-1} (x_{j+2i-1} + y_{j+2i-1}) + \sum_{i=k}^{2k-1} (x_{j+2i} + y_{j+2i}).$$

Then the strictly monotone sequence to $\mathbf{P}_2$ has weight

$$\sum_{i=0}^{k-1} y_{j+2i} + \sum_{i=1}^{k-1} x_{j+2i-1} + \sum_{i=k}^{2k} y_{j+2i-1} + \sum_{i=k}^{2k-1} x_{j+2i}.$$

The total weight is then

$$2 \cdot \left(\sum_{i=1}^{k-1} x_{j+2i-1} + \sum_{i=k}^{2k-1} x_{j+2i} \right) + \sum_{i=1}^{4k} y_i.$$

Therefore the total w -weight of the exchange sequence is at most $w(E) = \sum_{i=1}^{4k} (x_i + y_i)$ if and only if

$$(B_w(j)) \quad \sum_{i=1}^{k-1} x_{j+2i-1} + \sum_{i=k}^{2k-1} x_{j+2i} \leq \sum_{i=0}^{k-1} x_{j+2i} + \sum_{i=k-1}^{2k-1} x_{j+2i+1}.$$

Consider inequalities $B_w(j)$ and $B_w(j+1)$. The sum of these two inequalities gives

$$\left(\sum_{i=1}^{4k} x_i \right) - (x_j + x_{j+2k-1}) \leq \left(\sum_{i=1}^{4k} x_i \right) + (x_j + x_{j+2k-1}).$$

As this inequality clearly holds, at least one of $B_w(j)$ and $B_w(j+1)$ must hold as well. This implies that $B_w(j)$ holds for at least $2k$ choices of j from $\{1, \dots, 4k\}$. Note that if the number of such choices is exactly $2k$, then $B_w(j)$ holds either for all odd or for all even indices.

Now consider inequalities $B_w(j)$ and $B_w(j+2k)$. The sum of these two inequalities gives

$$\left(\sum_{i=1}^{4k} x_i \right) - (x_{j+2k-1} + x_{j+4k-1}) \leq \left(\sum_{i=1}^{4k} x_i \right) + (x_{j+2k-1} + x_{j+4k-1}).$$

As this inequality clearly holds, at least one of $B_w(j)$ and $B_w(j+2k)$ must hold as well. As the parities of j and $j+2k$ are the same, this, together with the above observation, implies that $B_w(j)$ holds for at least $2k+1$ choices of j from $\{1, \dots, 4k\}$ for $w \in \{w_1, w_2\}$. Therefore, there exists an index j for which both $B_{w_1}(j)$ and $B_{w_2}(j)$ are satisfied. As each edge is used at most twice, the statement follows. $\square$

With the help of Lemmas 1, 3 and 4, we are ready to prove the main result of the paper.

Theorem 5. *Let $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$ be colorings of a wheel $G = (V, E)$, and $w : E \rightarrow \mathbb{R}_+$ be a weight function. Then there exists a sequence of exchanges of length at most $n-1$ and total weight at most $w(E)$ that transforms $\mathbf{P}_1$ into $\mathbf{P}_2$ and uses each edge at most twice.*

Proof. If the colorings have identical orientation, then the theorem follows by Lemma 1. Indeed, the length and the weight of any strictly monotone sequence of exchanges that transforms $\mathbf{P}_1$ into $\mathbf{P}_2$, achieves the natural lower bounds $(n-1) - |R_1 \cap R_2| \leq n-1$ and $w(R_1 \triangle R_2) \leq w(E)$, respectively, and uses each edge at most once.

Hence assume that the colorings have different orientations. If $\mathbf{P}_1$ has at most four intervals, then the theorem immediately follows by Lemma 3. Otherwise, Lemma 4 with the choice $w_1 := w$ and $w_2 \equiv 1$ ensures the existence of a sequence of exchanges of total weight at most $w_1(E) = w(E)$ and length at most $w_2(E)/2 = |E|/2 = n-1$ that uses each edge at most twice, concluding the proof. $\square$

4 Spikes

In this section, we prove that a strengthening of Conjecture 1 analogous to Theorem 5 holds for spikes as well. That is, consider a rank- r spike M over ground set S , and let $w : S \rightarrow \mathbb{R}_+$ be a weight function. We show that for any two compatible basis pairs $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$, there exists a sequence of exchanges of length at most r and total weight at most $w(S)$ that transforms $\mathbf{P}_1$ into $\mathbf{P}_2$ and uses each element at most twice.

Recall that $S = \{t, x_1, y_1, \dots, x_r, y_r\}$, where t is the tip and $\{x_i, y_i\}$ for $1 \leq i \leq r$ are the legs of the spike. Hence $R_1 \cup B_1 = R_2 \cup B_2$ does not contain exactly one element s of S ; for short, we say that the pairs $\mathbf{P}_1$ and $\mathbf{P}_2$ **miss** the element s . We distinguish two cases depending on whether this element is the tip of M or not.

Lemma 6. *Let $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$ be compatible pairs of disjoint bases of a rank- r spike M missing the tip t , and let $w : S - t \rightarrow \mathbb{R}_+$ be a weight function. Then there exists a sequence of exchanges of length at most r and total weight at most $w(S - t)$ that transforms $\mathbf{P}_1$ into $\mathbf{P}_2$ and uses each element at most twice.*

Proof. Let $Z \subseteq S - t$ be a subset of size r . Then Z is a basis of M if and only if one of the followings hold.

1. There exist indices k and ℓ such that $\{x_k, y_k\} \subseteq Z$, $\{x_\ell, y_\ell\} \cap Z = \emptyset$ and $|\{x_i, y_i\} \cap Z| = 1$ for all $1 \leq i \leq r$, $i \notin \{k, \ell\}$.
2. $|Z \cap \{x_i, y_i\}| = 1$ for all $1 \leq i \leq r$ and Z is not a circuit of M .

Bases of the latter type are called **transversal bases** as they intersect every leg of M , while bases of the former type are called **non-transversal bases**.

Case 1. Both R_1 and R_2 are non-transversal.

We may assume that both x_1 and y_1 are red and both x_r and y_r are blue in $\mathbf{P}_1$. Note that in this case exchanging the elements x_i and y_i is feasible for $2 \leq i \leq r-1$. As R_2 is non-transversal, there exists indices $1 \leq k, \ell \leq r$ such that x_k and y_k are both red and x_ℓ and y_ℓ are both blue in $\mathbf{P}_2$. We distinguish cases based on the values of k and ℓ .

Case 1.1. $k = 1$.

Exchange x_i and y_i for all $2 \leq i \leq r-1$ for which $i \neq \ell$ and the color of x_i differs in $\mathbf{P}_1$ and $\mathbf{P}_2$. If $\ell \neq r$, then exchange the unique element of $\{x_\ell, y_\ell\} \cap R_1$ with the unique element of $\{x_r, y_r\} \cap R_2$.

Case 1.2. $2 \leq k \leq r-1$.

We may assume that x_1 and x_r are blue in $\mathbf{P}_2$ and x_k is red in $\mathbf{P}_1$. Exchange x_1 with y_k , then exchange x_i and y_i for all $2 \leq i \leq r-1$ for which $i \notin \{k, \ell\}$ and the color of x_i differs in $\mathbf{P}_1$ and $\mathbf{P}_2$. Moreover, if $\ell \neq r$ then exchange the unique element of $\{x_\ell, y_\ell\} \cap R_1$ with y_r .

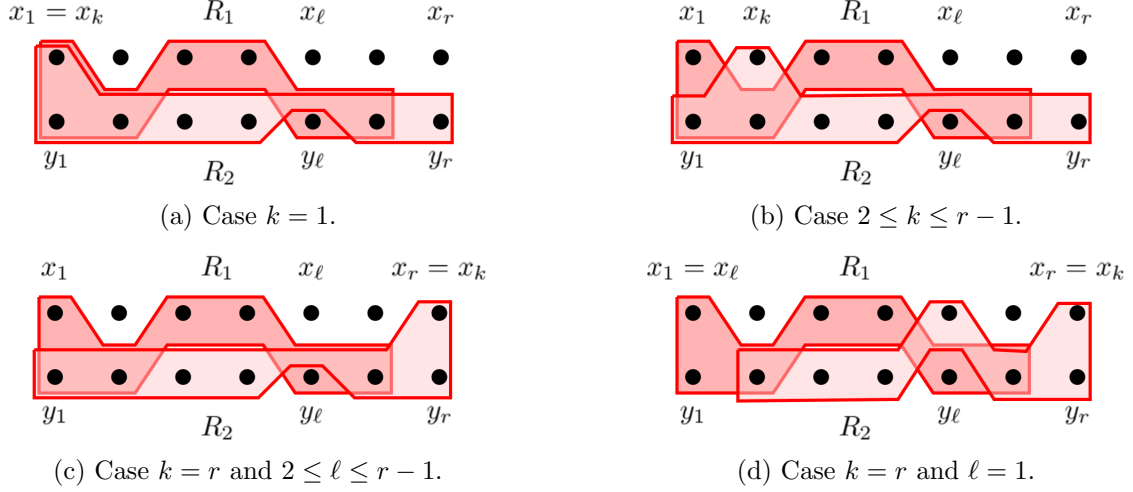


Figure 4: Illustration of Case 1 of Lemma 6.

Case 1.3. $k = r$ and $2 \leq \ell \leq r - 1$.

Exchange the unique element of $\{x_\ell, y_\ell\} \cap R_1$ with x_r . Then exchange x_i and y_i for all $2 \leq i \leq r - 1$ for which $i \neq \ell$ and the color of x_i differs in $\mathbf{P}_1$ and $\mathbf{P}_2$. Finally, exchange the unique element of $\{x_1, y_1\} \cap B_2$ with y_r .

Case 1.4. $k = r$ and $\ell = 1$.

Suppose first that there exists an index $2 \leq j \leq r - 1$ such that x_j has different colors in $\mathbf{P}_1$ and $\mathbf{P}_2$. We may assume that x_j is red in $\mathbf{P}_1$. Exchange x_1 with y_j , then exchange x_i and y_i for all $2 \leq i \leq r - 1$ for which $i \neq j$ and x_i has different colors in $\mathbf{P}_1$ and $\mathbf{P}_2$. Finally, exchange y_1 and x_r , and exchange x_j with y_r . This results in a strictly monotone exchange sequence that transforms $\mathbf{P}_1$ into $\mathbf{P}_2$.

The only remaining case is when x_i has the same color in $\mathbf{P}_1$ and $\mathbf{P}_2$ for all $2 \leq i \leq r - 1$, we may assume that they are all red. If $r = 2$ then at least one of the exchanges (x_1, x_2) , (x_1, y_2) is feasible, exchanging them and exchanging the other two elements afterwards gives a strictly monotone exchange sequence. If $r \geq 3$ then we may assume that $w(y_2) \leq w(x_2)$. Consider the exchange sequence (x_1, y_2) , (y_1, x_r) , (y_2, y_r) of length $3 \leq r$. The only element that is used twice is y_2 , while x_2 was not used at all, hence the sequence has total weight at most $w(E)$ by $w(y_2) \leq w(x_2)$.

Case 2. Both R_1 and R_2 are transversal bases.

If R_1 and R_2 differ in only one leg, say $R_1 \cap \{x_i, y_i\} \neq R_2 \cap \{x_i, y_i\}$, then exchanging x_i and y_i is feasible. Otherwise we may assume that $x_1, x_2 \in R_1$ and $y_1, y_2 \in R_2$. Exchange x_1 and y_2 , then exchange x_i and y_i for all $3 \leq i \leq r$ for which the color of x_i differs in $\mathbf{P}_1$ and $\mathbf{P}_2$. Finally, exchange x_2 and y_1 .

Case 3. Exactly one of R_1 and R_2 is a transversal basis.

We may assume that R_1 is non-transversal such that both x_1 and y_1 are red and both x_r and y_r are blue in $\mathbf{P}_1$. Exchange x_i and y_i for all $2 \leq i \leq r - 1$ for which the color of x_i differs in $\mathbf{P}_1$ and $\mathbf{P}_2$. Then exchange the unique element of $\{x_1, y_1\} \cap B_2$ with the unique element of $\{x_r, y_r\} \cap R_2$.

Concluding the above, in all cases except Case 1.4, the resulting sequence of exchanges is strictly monotone and uses each element at most twice, hence the bounds on the length and the total weight follow. In Case 1.4, we showed that the bounds hold, concluding the proof of the lemma. $\square$

Now we consider the case when $\mathbf{P}_1$ and $\mathbf{P}_2$ miss a non-tip element, say, x_1 .

Lemma 7. *Let $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$ be compatible pairs of disjoint bases of a rank- r spike M missing the non-tip element x_1 , and let $w : S - x_1 \rightarrow \mathbb{R}_+$ be a weight function. Then there exists a sequence of exchanges of length at most r and total weight at most $w(S - x_1)$ that transforms $\mathbf{P}_1$ into $\mathbf{P}_2$ and uses each element at most twice.*

Proof. Let $Z \subseteq S - x_1$ be a subset of size r . Then Z is a basis of M if and only if one of the followings hold.

1. $t \in Z$, $y_1 \in Z$, $\{x_\ell, y_\ell\} \cap Z = \emptyset$ for an index $2 \leq \ell \leq r$, $|\{x_i, y_i\} \cap Z| = 1$ for all $i \in \{2, \dots, r\}$, $i \neq \ell$.
2. $t \in Z$, $y_1 \notin Z$, $|\{x_i, y_i\} \cap Z| = 1$ for all $2 \leq i \leq r$.
3. $t \notin Z$, $y_1 \in Z$, $|\{x_i, y_i\} \cap Z| = 1$ for all $2 \leq i \leq r$, Z is not a circuit.
4. $t \notin Z$, $y_1 \notin Z$, $\{x_\ell, y_\ell\} \subseteq Z$ for an index $2 \leq \ell \leq r$, $|Z \cap \{x_i, y_i\}| = 1$ for all $i \in \{2, \dots, r\}$, $i \neq \ell$.

We will distinguish two cases based on the coloring of t and y_1 .

Case 1. t and y_1 have the same color in at least one of $\mathbf{P}_1$ and $\mathbf{P}_2$.

We may assume that t and y_1 are both red, and x_r and y_r are both blue in $\mathbf{P}_1$. Exchange x_i and y_i for all $2 \leq i \leq r-1$ for which $|R_2 \cap \{x_i, y_i\}| = |B_2 \cap \{x_i, y_i\}| = 1$ and x_i have different colors in $\mathbf{P}_1$ and $\mathbf{P}_2$. Let $\mathbf{P}'_1 = (R'_1, B'_1)$ denote the coloring obtained this way. Note that all elements have the same color in $\mathbf{P}'_1$ and $\mathbf{P}_2$, except possibly t, y_1, x_r, y_r and at most one leg $\{x_\ell, y_\ell\}$. We distinguish further cases based on the colors of t and y_1 in $\mathbf{P}_2$.

Case 1.1. t and y_1 are both red in $\mathbf{P}_2$.

Let ℓ denote the unique index for which both x_ℓ and y_ℓ are blue in $\mathbf{P}_2$. If $\ell = r$, we need no further exchanges. If $\ell \neq r$, exchange the element of the leg $\{x_\ell, y_\ell\}$ which is red in $\mathbf{P}_1$ with the element of leg $\{x_r, y_r\}$ which is red in $\mathbf{P}_2$.

Case 1.2. Exactly one of t and y_1 is red in $\mathbf{P}_2$. In this case exactly one of t and y_1 and exactly one of x_r and y_r need to change colors and their exchange is feasible.

Case 1.3 Both t and y_1 are blue in $\mathbf{P}_2$.

Let ℓ denote the unique index for which x_ℓ and y_ℓ are both red in $\mathbf{P}_2$. If $\ell = r$, then consider the exchange sequences (y_1, x_r) , (t, y_r) and (y_1, y_r) , (t, x_r) of length two. At least one of them is feasible since at most one of $B'_1 - x_r + y_1$ and $B'_1 - y_r + y_1$ forms a circuit by $|(B'_1 - x_r + y_1) \cap (B'_1 - y_r + y_1)| = r - 1$. Adding this to the exchange sequence that transformed $\mathbf{P}_1$ into $\mathbf{P}'_1$ we get a strictly monotone exchange sequence from $\mathbf{P}_1$ to $\mathbf{P}_2$.

If $2 \leq \ell \leq r-1$, then we may assume that x_ℓ is red and y_ℓ is blue in $\mathbf{P}_1$, while x_r is red and y_r is blue in $\mathbf{P}_2$. If the exchange (y_1, x_r) is feasible, then adding (y_1, x_r) , (t, y_ℓ) to the exchange sequence that transformed $\mathbf{P}_1$ into $\mathbf{P}'_1$ results in a strictly monotone exchange sequence. From now on we assume that (y_1, x_r) is not feasible which means that $B'_1 - x_r + y_1$ forms a circuit. As the intersection of two circuits cannot have size $r - 1$, this implies that both $B'_1 - y_r + y_1$ and $(B'_1 \setminus \{x_r, y_\ell\}) \cup \{x_\ell, y_1\}$ are bases.

If $w(x_\ell) \geq w(y_r)$, then extend the exchange sequence from $\mathbf{P}_1$ to $\mathbf{P}'_1$ by the exchanges (y_1, y_r) , (y_ℓ, t) , (x_r, y_r) . The feasibility of these exchanges follows from $B'_1 - y_r + y_1$ being a basis. The total length of the exchange sequence is at most $(r - 3) + 3 = r$. The sequence does not use x_ℓ , it uses y_r twice, and all the other elements at most once, so it has weight at most $w(E)$ by the assumption $w(x_\ell) \geq w(y_r)$.

If $w(x_\ell) < w(y_r)$, then extend the exchange sequence from $\mathbf{P}_1$ to $\mathbf{P}'_1$ by the exchanges (x_ℓ, x_r) , (y_1, y_ℓ) , (x_ℓ, t) . The feasibility of these exchanges follows from $(B'_1 \setminus \{x_r, y_\ell\}) \cup \{x_\ell, y_1\}$

being a basis. The total length of the exchange sequence is at most $(r-3)+3 = r$. The sequence does not use y_r , it uses x_ℓ twice, and all the other elements at most once, so it has weight at most $w(E)$ by the assumption $w(x_\ell) < w(y_r)$.

Case 2. t and y_1 have different colors in both $\mathbf{P}_1$ and $\mathbf{P}_2$.

We may assume that y_1 is red and t is blue in $\mathbf{P}_1$. Observe that each leg contains one red and one blue element in any of the colorings $\mathbf{P}_1$ and $\mathbf{P}_2$. If every leg is colored the same way in $\mathbf{P}_1$ and $\mathbf{P}_2$, then the exchange distance of $\mathbf{P}_1$ and $\mathbf{P}_2$ is at most one. Hence we may assume that $x_2 \in B_1 \cap R_2$ and $y_2 \in B_2 \cap R_1$.

Case 2.1. y_1 is blue and t is red in $\mathbf{P}_2$.

Exchange x_2 with y_1 . After this step, both y_1 and t is blue, hence we can exchange x_i and y_i for all $3 \leq i \leq r$ for which x_i has different colors in $\mathbf{P}_1$ and $\mathbf{P}_2$. Finally, exchange y_2 with t .

Case 2.2. y_1 is red and t is blue in $\mathbf{P}_2$.

If $w(y_1) \leq w(t)$, exchange y_1 with x_2 . After this step, exchange x_i and y_i for all $3 \leq i \leq r$ for which x_i has different colors in $\mathbf{P}_1$ and $\mathbf{P}_2$. Finally, exchange y_1 with y_2 . This exchange sequence does not use t , uses y_1 twice and all the other elements at most once. These imply that its length is at most r and total weight is at most $w(E)$ by $w(y_1) \leq w(t)$.

If $w(y_1) > w(t)$, exchange t with y_2 . After this step, exchange x_i and y_i for all $3 \leq i \leq r$ for which x_i has different colors in $\mathbf{P}_1$ and $\mathbf{P}_2$. Finally, exchange t with x_2 . This exchange sequence does not use y_1 , uses t twice and all the other elements at most once. These imply that its length is at most r and total weight is at most $w(E)$ by $w(y_1) > w(t)$.

Concluding the above, in all cases except Cases 1.3 and 2.2, the resulting sequence of exchanges is strictly monotone and uses each element at most twice, hence the bounds on the length and the total weight follow. In Cases 1.3 and 2.2, we showed that the bounds hold, concluding the proof of the lemma. $\square$

Theorem 8. *Let $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$ be compatible pairs of disjoint bases of a rank- r spike M , and $w : S \rightarrow \mathbb{R}_+$ be a weight function. Then there exists a sequence of exchanges of length at most r and total weight at most $w(S)$ that transforms $\mathbf{P}_1$ into $\mathbf{P}_2$ and uses each element at most twice.*

Proof. The theorem follows by combining Lemmas 6 and 7. $\square$

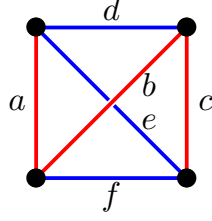
Remark 9. The orientation of colorings of wheels played a crucial role in the existence of strictly monotone exchange sequences. Spikes are interesting because one can define an arbitrarily large number of colorings without a strictly monotone exchange sequence between any two of them.

Consider the spike² M defined by

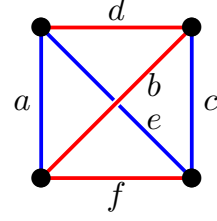
$$\mathcal{C}_3 := \{C \subseteq S : |C| = r, |C \cap \{x_i, y_i\}| = 1 \text{ for } 1 \leq i \leq r, |C \cap \{x_1, \dots, x_r\}| \text{ is odd}\}.$$

Consider all pairs (R, B) of disjoint sets of size r for which $y_1 \in R$, $t \in B$, $|R \cap \{x_i, y_i\}| = |B \cap \{x_i, y_i\}| = 1$ for all $2 \leq i \leq r$, and $|R \cap \{x_2, \dots, x_r\}|$ is even. There are 2^{r-2} such pairs. Note that all such pairs (R, B) are colorings of the matroid obtained from M by deleting x_1 . Further observe that any feasible exchange uses at least one of y_1 and t . Indeed, exchanging x_i with y_j for some $2 \leq i, j \leq r$ results in an odd number of red elements in $\{x_2, \dots, x_r\}$, meaning that the red elements form a circuit, showing that the exchange is non-feasible. Exchanging x_i with x_j or y_i with y_j for some $2 \leq i, j \leq r$, $i \neq j$ results in either $\{t, x_i, y_i\}$ or $\{t, x_j, y_j\}$ forming a blue circuit, hence the exchange is non-feasible. This implies that there exists no strictly monotone exchange sequence between any two distinct colorings of this form.

²This matroid is interesting on its own as it is isomorphic to the unique rank- r binary spike, see [10] for details.



(a) Coloring $\mathbf{P}_1 = (R_1, B_1)$ of K_4 .



(b) Coloring $\mathbf{P}_2 = (R_2, B_2)$ of K_4 .

Figure 5: Example showing that the statement of Lemma 4 is not true in general. Any exchange sequence that transforms (R_1, B_1) into (R_2, B_2) uses one of b and e twice. If $w_1(b) = 1$ and 0 otherwise, and $w_2(e) = 1$ and 0 otherwise, then any sequence violates the weight constraint for at least one the weight functions.

5 Conclusions

In this paper, we proposed a weighted generalization of Hamidoune’s conjecture on the exchange distance of compatible basis pairs. We verified the conjecture for two classes of matroids: graphic matroids of wheels and spikes. In both cases, we proved a stronger result stating that there exists an exchange sequence that has small length and weight at the same time.

Motivated by Lemma 4, it would be tempting to formulate a conjecture stating that there always exists an exchange sequence that has small weight with respect to two weight functions w_1 and w_2 simultaneously. However, this is not true in general. Let $G = (V, E)$ be a complete graph on four vertices with edge set $E = \{a, b, c, d, e, f\}$, and let M be the graphic matroid of G , see Figure 5. Interestingly, G is a wheel, but M can be also obtained by deleting the tip of a spike. Consider two colorings $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$ of G where $R_1 = \{a, b, c\}$, $B_1 = \{d, e, f\}$, $R_2 = \{b, d, f\}$ and $B_2 = \{a, c, e\}$. It is not difficult to check that any sequence of exchanges that transforms $\mathbf{P}_1$ into $\mathbf{P}_2$ uses at least one of the edges b and e twice. This implies that if we set $w_1(b) := 1$ and 0 otherwise, and we set $w_2(e) := 1$ and 0 otherwise, then any exchange sequence has w_1 -weight or w_2 -weight at least $2 = 2w_1(E) = 2w_2(E)$.

On the other hand, no counterexample is known for the case when $w_2 \equiv 1$, that is, when one would like to find an exchange sequence that has small length and weight simultaneously. Even more, both Theorems 5 and 8 ensured the existence of an exchange sequence that uses each element at most twice. A rather optimistic conjecture would state the existence of a sequence of exchanges that, besides having small length and weight, also has this property.

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