

Weak vs. strong breaking of integrability in interacting scalar quantum field theories

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1 Introduction

Integrability and its violation play an important role in the dynamics of quantum many-body systems, which are nowadays under intense investigation. Integrability imposes several constraints on the equilibrium and non-equilibrium dynamics of quantum systems, and its violation is expected to induce a transition to chaotic behaviour. A particularly active area of research is the modelling of low-dimensional quantum systems, where many consequences of integrability breaking—such as thermalisation and ergodicity—can be investigated. In this work, we study integrability breaking in $1+1$ dimensional relativistic scalar quantum field theories using a truncated Hamiltonian approach and methods from random matrix theory.

1.1 Classical integrability breaking

Before diving into the quantum realm, it is insightful to briefly elaborate on the classical case where integrability and its breaking have already been comprehensively studied. Let $H_0(p, q)$ be a classical Hamiltonian with n degrees of freedom. Then it is integrable if (and only if) there exist n functionally independent constants of motions I_k ($k = 1, \dots, n$) that are in *involution*, i.e., $\{I_k, I_l\} = 0$ [1]. In this case (for bounded motion), the trajectories are constrained to an n -dimensional torus and conveniently described by the so-called action-angle variables after an appropriate canonical transformation.

Let H_0 be an integrable Hamiltonian and

$$H = H_0 + \varepsilon H_1 \tag{1.1}$$

is such that it is non-integrable whenever the coupling is non-zero ($\varepsilon \neq 0$). In this case, we use the (somewhat hand-wavy) terminology as the H_1 perturbation *breaks* the integrability. Kolmogorov, Arnold, and Moser found that not arbitrarily small perturbation leads to *chaos*. The Kolmogorov–Arnold–Moser (KAM) theorem states that integrable dynamics is preserved in the fraction $1 - \mathcal{O}(\varepsilon^{1/2})$ of the phase space, that is, the integrable tori—although they are deformed—remain stable for small enough perturbation¹ [2–4]. So, even though the presence of integrability is a yes-no question, the change in the dynamics is not instantaneous as we turn on the perturbation. There is a typical scale in the coupling ε when the motion becomes chaotic in a considerable fraction of the phase space; this scale can be characterized by an ε_* coupling-like quantity that we call the crossover coupling.

To understand how robust integrability is against perturbations, it is desirable to understand how ε_* depends on the system’s parameters. A widespread wisdom states that ε_* goes to 0 as the number of degrees of freedom (n) is increased². In fact, for a system with infinitely many degrees of freedom, the change in the dynamics is instantaneous. In this work, we investigate the quantum counterpart of this behaviour.

1.2 Weak breaking of integrability

Quantum integrability is related to the presence of an extensive number of local conserved charges [8], resulting in characteristic restrictions concerning the equilibrium and non-

¹Except for a zero-measure set of tori corresponding to resonant frequencies. [1]

²There are exact mathematical proofs supporting this statement [5–7], still, these exact predictions are far from being of practical utility.

equilibrium dynamics of the system. For integrable quantum models, the number of (quasi)particle excitations is conserved, the set of incoming and outgoing momenta equals, and the multi-particle S -matrix factorises to $2 \rightarrow 2$ scattering processes [9].

Considering an integrable system, there is a class of exceptional deformations that preserves integrability [10, 11], while the typical deformations break conserved charges and violate integrability. Recent developments revealed that integrability breaking can be classified by which order in the perturbation strength the higher conservation laws are violated [12, 13]. First-order violation is the generic case, while higher order results in weaker breaking of integrability [14, 15].

The signatures of integrability are present in the statistical properties of the energy spectrum, which can be effectively studied by random matrix methods. For integrable theories, the level spacing statistics is Poissonian, while it is Wigner–Dyson in the chaotic case. Random matrix theory (RMT) has already been successfully applied in a handful of quantum many-body systems and quantum field theories [7, 16–19]. Similarly to the aforementioned classical case, it is also possible to define a crossover coupling in the quantum case based on the continuous *crossover* between the Poisson and Wigner–Dyson level spacing statistics in a finite system. With increasing system size, the crossover becomes sharper, and the crossover coupling tends to zero. The scaling of the crossover coupling by the system size was studied in gapless spin chains of length L : while the crossover coupling scales as L^{-3} for the generic integrability breaking perturbation [20, 21], recent studies found L^{-2} [15] and L^{-1} [22] scaling behaviours supporting the recently proposed classification of integrability breaking perturbations by strength [14, 15]. In Ref. [15], the gapped case was also studied, where the authors found an exponential relation between the crossover coupling and the volume with significantly lower decay for weak breaking of integrability compared to the strong case. It is, therefore, well-motivated to study the integrability breaking strength of perturbations via the finite size scaling of the crossover coupling.

The $1+1$ dimensional scalar field theories are paradigmatic examples of quantum field theories in low dimensions. We study the free massive boson perturbed by the ϕ^4 and ϕ^6 interactions. We adopt the RG-improved truncated Hamiltonian approach—a powerful non-perturbative numerical method—to extract the spectrum for statistical analysis. Since the $2 \rightarrow 2$ scattering processes can not change momenta in $1+1$ dimensions, it can be argued that the ϕ^4 interaction violates integrability at the second order in the coupling. In contrast, the ϕ^6 interaction is expected to violate it at the first order. Utilizing these properties by applying a Boltzmann equation approach [14], it was found that the ϕ^4 theory does not thermalize—in contrast to the ϕ^6 —which was also confirmed numerically in Ref. [23]. This observation suggests that the ϕ^4 interaction might be similar to the weak integrability-breaking perturbations observed in spin chains. The primary goal of this work is to compare the ϕ^4 and ϕ^6 interactions in terms of integrability breaking strength via the finite size scaling of the crossover coupling.

The outline of this paper is as follows. In Section 2, the relevant scalar field theories are introduced, and the fundamentals of the truncated Hamiltonian approach are explained, while the details of the renormalisation group improvement are relegated to Appendix A. In Section 3, we establish a method to quantify integrability breaking perturbations—via using random matrix theory, utilizing the characteristics of the statistical crossover, and studying the finite size scaling of the crossover coupling. It turns out that the scaling in the volume does not give a satisfactory result; instead, in Section 4, we study the

scaling in the number of particles, which yields much better and novel results. The main conclusions of this work are summarized in Section 5.

The main findings of my research have been recently published in our paper [24]; the present work, however, provides a more detailed account of the theoretical basis and the methods used.

2 The truncated Hamiltonian approach

In this section, the studied quantum field theories and the applied numerical method are introduced. First, we review the most important properties of the 1+1 dimensional Klein–Gordon field perturbed by a ϕ^n interaction. Next, the truncated Hamiltonian approach—a method providing an effective way to calculate the energy spectrum of the examined models—is introduced. Finally, some details about the numerical implementation are presented. Though it is a previous work of mine detailed in this section, it is inevitable to discuss these concepts as we build upon them later. For the sake of simplicity, we work in natural units, $c = \hbar = 1$ throughout this paper.

2.1 The 1 + 1 dimensional ϕ^n theory

We consider massive scalar field theories in 1 + 1 (one space and one time) dimensions. Despite the simple formalism of such models, they carry much of the non-trivial characteristics of quantum field theories making them a fruitful subject to study. The free theory is described by the Klein–Gordon Hamiltonian

$$H_{\text{KG}} = \frac{1}{2} \int dx : ((\partial_t \phi)^2 + (\partial_x \phi)^2 + m^2 \phi^2) :, \quad (2.1)$$

where $: \dots :$ denotes normal ordering with respect to the free modes. Perturbations of the form

$$V_n = \int dx : \phi^n(x) : \quad (2.2)$$

are considered (with n being an even number). Particularly, we focus on the cases $n = 4$ and 6, corresponding to the so-called ϕ^4 and ϕ^6 theories described by the

$$H_{\phi^4} = H_{\text{KG}} + g_4 V_4, \quad (2.3)$$

$$H_{\phi^6} = H_{\text{KG}} + g_6 V_6 \quad (2.4)$$

Hamiltonians with some g_4 and g_6 couplings ($g_n \geq 0$).

We remark that the field $\phi(x)$ is dimensionless in 1 + 1 dimensions, therefore the coupling g_n is of dimension m^2 for every n . In the perturbative description, a coupling of positive powers in the mass implies that the convergence of Feynman diagrams gets better when increasing the order—which property is referred to as the theory being superrenormalisable. Particularly, in the 1 + 1 dimensional ϕ^n theories, only the one loop levels give divergent contributions that we already eliminated by prescribing normal ordering in Eq. (2.2). So we expect that the calculations converge in the UV regime implying that the ϕ^n interaction is relevant in the renormalization group (RG) sense for every (positive) n in 1 + 1 dimensions.

We consider the models in finite volume $0 \leq x \leq L$ with periodic boundary conditions $\phi(0) = \phi(L)$. Compared to the infinite volume case, the finite volume restriction implies some discrepancies in the physical parameters (vacuum energy density, mass gap, interaction strength). To retain meaningful results, we aim to keep these physical quantities constant when comparing theories in different volumes. The discrepancies can be compensated by adding some correction terms to the ‘naive’ Hamiltonian [25, 26]. However,

these finite volume corrections decrease exponentially with the volume and turn out to be negligible in the examined cases, so we simply omit them.

Finally, we mention that the models considered above have three ‘trivial’ symmetries:

- translation invariance resulting in conservation of the total momentum (P),
- spatial parity ($\mathbb{P}$), that is, the theory is invariant under the $x \rightarrow -x$ transformation, and
- the $\mathbb{Z}_2$ symmetry coming from the fact that field $\phi(x)$ appears only on even powers in the Hamiltonian, so the theory is invariant under the $\phi(x) \rightarrow -\phi(x)$ transformation (i.e. states containing even and odd number of particles (free modes) are decoupled).

To avoid trivial degeneracies (and make the numerical calculations efficient), we restrict our scope to an individual symmetry sector. Specifically, we consider states of 0 total momentum, which are also even under both parity and $\mathbb{Z}_2$ symmetry, that is, states of the symmetry sector containing the vacuum.

2.2 The RG-improved truncated Hamiltonian approach

Having the aforementioned quantum field theory in mind, we are aiming to develop a method that calculates the spectrum in a given energy window $[E_1, E_2]$. Our approach follows the idea invented by Yurov and Zamolodchikov [27, 28] and later works that extended the method to the ϕ^4 model [25, 29–32].

The first step is to solve the free theory. The eigenvectors of the non-interacting Hamiltonian, $H_{\text{KG}}|\psi_k^{(0)}\rangle = E_k^{(0)}|\psi_k^{(0)}\rangle$ can be constructed as a multi-particle state built up from free modes. In the finite volume setting, the allowed momenta of the individual modes are discrete and so are the $|\psi_k^{(0)}\rangle$ states. We aim to retain a finite set of these eigenstates, therefore we introduce a so-called cutoff-energy, Λ , and keep only those $|\psi_k^{(0)}\rangle$ eigenstates whose corresponding eigenvalue is smaller than the cutoff-energy, $E_k^{(0)} < \Lambda$. This procedure—neglecting a large part of the Hilbert space resulting in a finite basis—is called the truncation. In this finite basis, we extract the full (i.e. interacting) Hamiltonian as a finite matrix. Finally, we numerically diagonalize the matrix to get the spectrum of the truncated Hamiltonian.

The spectrum of the truncated Hamiltonian is an approximation of the exact spectrum, with truncation errors dependent on the cutoff Λ . Since the V_n operators are relevant (in the RG sense), the individual energy levels converge to a finite value—the energy eigenvalue corresponding to the exact Hamiltonian—as we push the cutoff to infinity. Moreover, the convergence can even be improved by adding some Λ -dependent correction terms to the truncated Hamiltonian. Following RG methods developed in [31], we can eliminate the leading Λ -dependence for the Hamiltonian

$$H = H_{\text{KG}} + \sum_n g_n V_n \quad (2.5)$$

by adding the following correction term

$$\Delta H = \sum_n \kappa_n V_n, \quad \text{where} \quad \kappa_k = - \sum_{n \geq m} g_n g_m \int_{\Lambda}^{\infty} dE \frac{\mu_{nmk}(E)}{E - \mathcal{E}}. \quad (2.6)$$

The $\mu_{nmk}(E)$ functions describe the running of the couplings with the cutoff-energy and are summarized in Eq.(A.20). The $\mathcal{E}$ parameter is a reference energy that should be set in the range of the energy levels we are interested in (in our finite energy window setting, $\mathcal{E} = (E_1 + E_2)/2$ is a reasonable choice). For more details on the RG improvement, the reader is referred to Appendix A. While the $\mu_{nmk}(E)$ functions concerning the ϕ^4 theory are already determined in [31], the results for the ϕ^6 interaction is my own contribution.

2.3 Numerical implementation

The numerical task is uniquely specified by the $\{mL, \tilde{g}_n = g_n/m^2, \Lambda/m, E_1/m, E_2/m\}$ dimensionless parameters. The implementation breaks down into three main steps:

1. generate the basis,
2. initialize the matrix elements, and
3. diagonalize the Hamiltonian matrix.

First, generating the THA basis at a given volume (L) and cutoff energy (Λ) is a combinatorial exercise. Table 2.1. shows the cardinality of the basis at some volumes and cutoff energies; we observe that the cardinality increases rapidly (roughly exponentially) as a function of both parameters. We store the basis vectors as positive integer strings, $|n_{-k_{\max}}, \dots, n_{-1}, n_0, n_1, \dots, n_{k_{\max}}\rangle$ that represents the occupation numbers of the free modes and is easy to handle numerically.

$mL \setminus \Lambda/m$	6	9	12	16	20
8	10	40	152	792	3 774
12	19	112	617	5 267	38 617
16	30	263	2 031	25 484	259 434
24	69	1 076	13 846	314 359	5 406 267
30	116	2 550	44 817	1 449 230	34 289 883

Table 2.1: Cardinality of the THA basis at different cutoffs and volumes.

Given the THA basis, the next step is to calculate the matrix elements of the $H = H_{\text{KG}} + \sum_n (g_n + \kappa_n(\Lambda)) V_n$ RG-improved truncated Hamiltonian. The H_{KG} is diagonal in the THA basis, and so it is straightforward to initialize. To efficiently calculate the V_n matrix elements, we first determine the matrix elements of the operator

$$\hat{V}_n = L \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{l} \sum_{k_1 \leq \dots \leq k_l} \sum_{k_{l+1} \leq \dots \leq k_n} 2^{-\delta_{l,n/2}} \delta_{0, \sum k} \frac{f(k_1, \dots, k_l) f(k_{l+1}, \dots, k_n)}{\prod_i \sqrt{2L\omega_{p_{k_i}}}} \prod_{j'=1}^l a_{-k_{j'}}^\dagger \prod_{j=l+1}^n a_{k_j}, \quad (2.7)$$

where $a_k^\dagger$ and a_k are the usual bosonic creation and annihilation operators, and $f(k_1, \dots, k_q)$ is the number of all the different permutations of the $k_1, \dots, k_q$ indices. Then it is easy to initialize the V_n matrices as

$$V_n = \hat{V}_n + \hat{V}_n^\dagger. \quad (2.8)$$

I wrote a code in `cpp` to generate the basis and initialize the matrix elements. We remark that these matrices are *sparse*, that is, most of its elements are zero.

Finally, I used `Matlab` to diagonalize the H matrix. We typically do not determine all the eigenvalues, just the ones corresponding to a given energy window $[E_1, E_2]$. We remark that the dominant part of the runtime is the diagonalization; generating the basis and initializing the matrices require significantly fewer resources.

3 Quantifying integrability breaking

In this section, a method to quantify integrability breaking in quantum field theories is established. First, we review the consequences of integrability and chaos concerning the statistical properties of the energy spectrum and introduce the Poisson and GOE statistics. Next, measures characterising temporary—between the Poisson and GOE—level spacing statistics are introduced. After that, we quantify the characteristic coupling where the integrability breaking *crossover* happens. Finally, we investigate the finite volume scaling of the crossover coupling in the ϕ^4 and ϕ^6 theories.

3.1 Signatures of integrability

The fingerprints of integrability are present in the equilibrium dynamics, specifically in the statistical properties of the energy spectrum. Suppose that the spectrum is discrete and free of degeneracies and consider a small energy window where the level density is approximately constant. Following the idea of Berry and Tabor [16, 33], the different levels (statistically) decouple if there is an extensive number of conserved charges. So, in the integrable case, the level spacing is described by Poisson statistics, and the gap between the neighbouring levels follows an exponential distribution

$$P_{\text{Poi}}(\Delta E_n) \propto e^{\omega(E_n)\Delta E_n}, \quad (3.1)$$

where E_n is the n -th energy level, $\Delta E_n = E_{n+1} - E_n$, and $\omega(E_n)$ is the smoothed level density around E_n .

In the chaotic case, we follow a different approach. Wigner and Dyson found that the spectrum of (appropriately chosen) random matrices have similar statistical properties as the energy levels in heavy nuclei [16]. Moreover, the comprehensive study of quantum models that are the counterpart of classically chaotic systems showed that the spectrum statistics follows the predictions of random matrix theory (RMT) in the generic case [16]. It is well-motivated to say a quantum system is chaotic if RMT describes the statistical properties of its spectrum.

For unitary (quantum) models that are invariant under time reversal, the Hamiltonian matrix (H) is real and symmetric in an appropriately chosen basis. Then, it can be modelled as a random matrix where the joint distribution of the elements follows the Gaussian orthogonal ensemble (GOE)

$$P(H) \propto \exp \left[-\frac{1}{2a^2} \text{Tr} (H^2) \right], \quad (3.2)$$

where a is a constant parameter describing the scale of the spectrum. Then, the joint distribution of the spectrum is

$$P(\{E_j\}) \propto \prod_{i < j} |E_i - E_j| \prod_k e^{-E_k^2/2}, \quad (3.3)$$

while the approximate distribution for the gap between consecutive levels is

$$P_{\text{GOE}}(s) = \frac{\pi}{2} s \exp \left(-\frac{\pi}{4} s^2 \right), \quad (3.4)$$

where $s_n = \omega(E_n)\Delta E_n$. We will refer to these formulas as the spectrum follows Wigner–Dyson statistics.

The quantum field theories introduced in Section 2.1 are unitary and invariant under time reversal and integrable only if $g_n = 0$. There are two subtleties to be taken care of before we apply the statistical analysis. Firstly, the level density $\omega(E)$ increases roughly exponentially. It is numerically infeasible to take an energy window with enough levels (for statistical analysis). At the same time, $\omega(E)$ does not change much, and the examined states are well below the cutoff. Therefore, we must choose the energy window somewhat wider and take care of the change in the level density. Calculating with the “normalized” gaps $s = \omega(E)\Delta E$ instead of the naive ΔE solves this problem.

The other subtlety is coming from the structure of the interaction. Since the models considered earlier are relativistic quantum field theories, the interaction should be *local*. It is not obvious that random matrix theory applies to such models: the lowest excitations of a massive local theory are always in the form of quasi-particles, and their structure deviates from the RMT predictions. A similar problem was studied in quantum many-body systems, and it was found that RMT is still applicable in the *middle* of the spectrum [16]. Analogously to their findings, we can still use RMT if we stay sufficiently away from the low-energy states (that is, we are well above the mass scale). On the other hand, we also need to stay well below the truncation scale (Λ) to reduce the truncation error. Therefore, we aim to determine the energy spectrum in some energy window $[E_1, E_2]$, which is ideally chosen as $m \ll E_1 < E_2 \ll \Lambda$.

3.2 Quantifying level spacing statistics

Considering the models introduced in Section 2.1, the integrability of the free field theory breaks at any non-zero coupling. However—analogously to the classical case—the change in the equilibrium dynamics within a finite energy window is not instantaneous. In the finite volume setting, there are a finite number of levels in a given energy window $[E_1, E_2]$. Since these levels are continuous functions of the coupling g_n , the spectrum statistics can not change instantaneously. Therefore, we expect a continuous *crossover* between the Poisson and the Wigner–Dyson statistics as we turn on the interaction.

The presence of the statistical crossover necessitates the characterisation of level spacing statistics intermediate between the fully integrable and chaotic cases. Therefore, we define statistical measures that assign a real number to a level spacing such that 0 corresponds to Poisson and 1 corresponds to Wigner–Dyson statistics. Since such a measure is not unique, we introduce two different ones to test the independence of our conclusions on its choice.

3.2.1 Consecutive level spacing ratios

The consecutive level spacing ratios are defined as [17, 34]

$$r_n = \frac{s_n}{s_{n-1}}, \quad \tilde{r}_n = \min(r_n, 1/r_n). \quad (3.5)$$

The average value of this variable is $\langle \tilde{r}_{\text{Poi}} \rangle = 2 \log 2 - 1 \approx 0.386$ for Poissonian, and $\langle \tilde{r}_{\text{GOE}} \rangle = 4 - 2\sqrt{3} \approx 0.534$ for GOE statistics, respectively. We consider an affine map

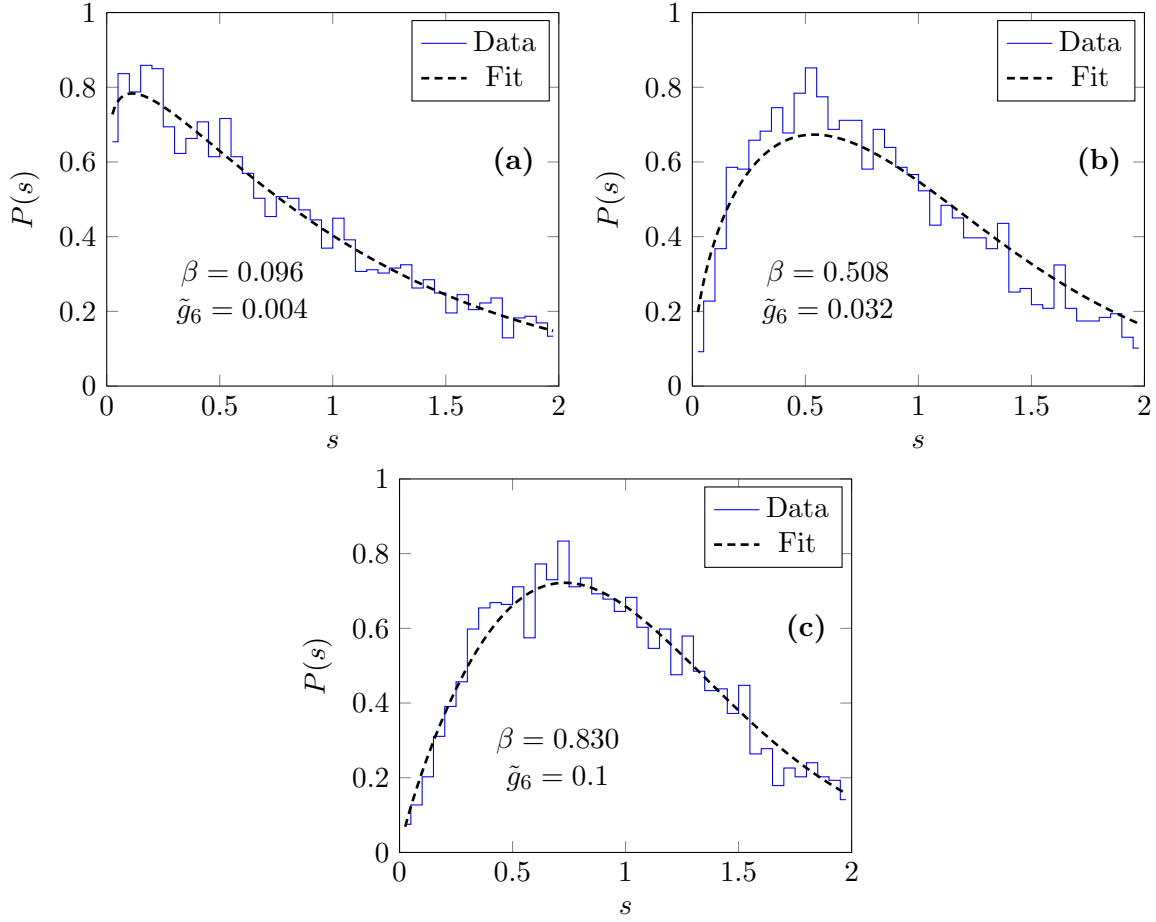


Figure 3.1: Level spacing histograms (*Data*) and the corresponding Brody distributions (*Fit*) in ϕ^6 theory at different values of the coupling, with the extracted value of β shown as part of the legend. The parameters of the spectral window are $mL = 12$, $[E_1/m, E_2/m] = [12, 16]$, $\Lambda/m = 20$.

from the interval $[\langle \tilde{r}_{\text{Poi}} \rangle, \langle \tilde{r}_{\text{GOE}} \rangle]$ to $[0, 1]$:

$$\langle \tilde{r}' \rangle = \frac{\tilde{r} - \langle \tilde{r}_{\text{Poi}} \rangle}{\langle \tilde{r}_{\text{GOE}} \rangle - \langle \tilde{r}_{\text{Poi}} \rangle}, \quad (3.6)$$

which takes value 0 for Poissonian and 1 for GOE statistics. Since this is a statistical measure, its uncertainty can be estimated using the central limit theorem and the known variances $\delta(\tilde{r}_{\text{Poi}}) \approx 0.280$ and $\delta(\tilde{r}_{\text{GOE}}) \approx 0.254$ in the two ensembles, leading to an upper estimate $\delta\langle \tilde{r}' \rangle \lesssim 1.89/\sqrt{N}$ where N is the number of levels included in the statistics.

3.2.2 Brody distribution

The Brody distribution [20]

$$P_\beta(s) = (\beta + 1)bs^\beta \exp(-bs^{\beta+1}) \quad (3.7)$$

with

$$b = \Gamma\left(\frac{\beta + 2}{\beta + 1}\right)^{\beta+1} \quad (3.8)$$

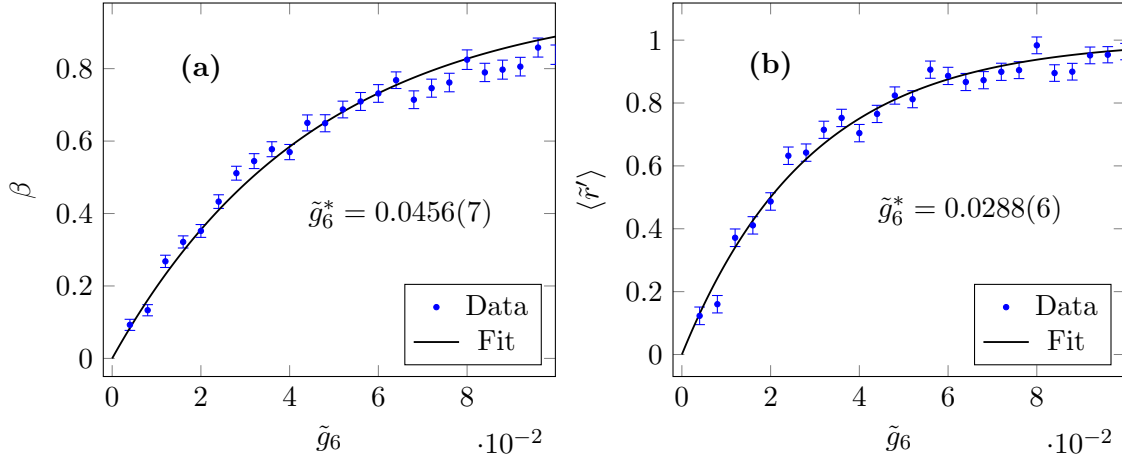


Figure 3.2: The β and $\langle \tilde{r}' \rangle$ measures as a function of the g_6 coupling in the ϕ^6 theory. We also present the fitted curve defined in Eq. (3.9), determining the coupling constant. The error bars ($\pm 1\sigma$) and the error of the crossover coupling are calculated from the statistical uncertainty. We note that $\langle \tilde{r}' \rangle(g_6 = 0) = -1.35$ and $\beta(g_6 = 0) = -2.00$ due to additional symmetries in the free theory. Parameters: $mL = 12$, $[E_1/m, E_2/m] = [12, 16]$, $\Lambda/m = 20$.

can be used to interpolate between the Poissonian and GOE cases, with $\beta = 0$ corresponding to Poisson, while $\beta = 1$ corresponds to GOE statistics. Given a set of energy levels spectrum, it can be evaluated by making a histogram from the s_n spacings with some appropriate resolution and then fitting $P_\beta(s)$ to extract β , as illustrated in Figure 3.1. Statistical fluctuations can be estimated by assigning to every histogram value its square root as an estimator for its variance and then computing the variance $\delta\beta$ of the fitted value of β using the standard least squares method.

3.3 Crossover coupling

Having the measures defined, we aim to characterise their behaviour as a function of the coupling. Figure 3.2 shows the variation of the measures $\langle \tilde{r}' \rangle$ and β in the ϕ^6 theory with increasing coupling g_6 while keeping all other parameters such as the volume, the cutoff, and the energy window fixed. At first, we observe that the statistical crossover does happen: both measures start from about 0 (Poisson) and approach 1 (Wigner–Dyson) as we turn on the interaction.

Motivated by previous studies performed with spin chains [21] we fit the data with an exponential relaxation

$$f_{g^*}(g) = 1 - \exp(-g/g^*) \quad (3.9)$$

to characterise the crossover. After investigating several different parameter settings, we found that this function works surprisingly well for the general case. The only parameter to be fitted is the g^* , which is a coupling value characteristic to the scale of the crossover, and we call it the *crossover coupling*. Note that the numerical value of the crossover coupling—obviously—depends on the definition of the measure (c.f. Fig. 3.2).

We remark that the spectrum statistics in the free theory (i.e., $g_n = 0$) is not eventually Poissonian due to a higher number of degeneracies in free field theory than the one expected from integrability alone. However, turning on the interaction, these degeneracies

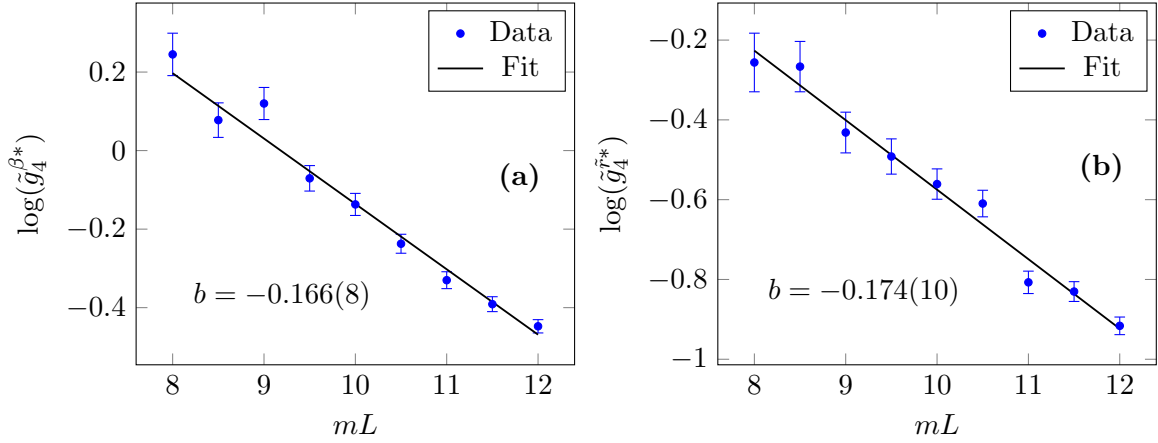


Figure 3.3: The $\log(\tilde{g}_4^*)$ vs. mL graphs in the ϕ^4 theory using the (a) β and the (b) $\langle \tilde{r}' \rangle$ measures. We also depicted the fitted line according to Eq. (3.10). The errors are calculated from the statistical uncertainty. Parameters: $[E_1/m, E_2/m] = [12, 16]$, $\Lambda/m = 20$.

are instantly resolved, and the spectrum quickly becomes Poissonian, much more rapidly than the eventual crossover to GOE³, therefore Poissonian statistics is observed for very small but non-zero couplings (i.e., $0 \neq g_n \ll g_n^*$, c.f. Fig. 3.2).

3.4 Finite size scaling

Analogously to the classical case and spin chain studies, we expect the crossover to get sharper and the crossover coupling to approach 0 as we increase the *system size*. Our aim is to characterize this process and use these characteristics to compare the integrability breaking strength of the ϕ^4 and ϕ^6 interactions.

Figure 3.3 illustrates the crossover coupling as a function of the volume L in the ϕ^4 theory while keeping the energy window and the cutoff fixed. We observe a decreasing tendency in accordance with our expectations and also see that the statistical errors decrease considerably as we increase the volume and, therefore, the number of examined levels.

Based on the exponential behaviour found for the massive regime of the XXZ spin chain [15], we fit the volume dependence $g^*(mL)$ as

$$\log(g/m^2) = a + b(mL), \quad (3.10)$$

which is also illustrated in Figure 3.3. Note that redefining the normalisation of the coupling constant (e.g. dividing g_n by $n!$) affects only a . Therefore, the eventual quantity of interest is the coefficient b , which can be taken to characterize the strength of integrability breaking of the given interaction (V_n) [15].

Unfortunately, the choice of parameters is quite constrained: on the one hand, the number of states increases rapidly (roughly exponentially) with the volume and the cutoff (c.f. Table 2.1), while on the other hand, the energy window should be kept wide enough to contain enough states for the statistical analysis. As a result, it is necessary to vary the energy window setting to access different volume ranges.

³Except for very small volumes $mL \lesssim 2$.

$[E_1/m, E_2/m]$	Λ/m	$b_{\phi^4}^\beta$	$b_{\phi^4}^{\tilde{r}}$	$b_{\phi^6}^\beta$	$b_{\phi^6}^{\tilde{r}}$
[12.0, 16.0]	20.0	-0.166(8)	-0.174(10)	-0.215(6)	-0.225(8)
[8.0, 10.7]	14.0	-0.058(4)	-0.059(6)	-0.076(3)	-0.080(5)
[8.0, 10.7]	13.0	-0.071(4)	-0.069(6)	-0.077(3)	-0.079(5)
[6.5, 9.1]	11.5	-0.053(4)	-0.048(5)	-0.046(3)	-0.053(4)
[6.5, 9.1]	10.5	-0.053(4)	-0.070(5)	-0.049(3)	-0.051(4)

Table 3.1: Finite size scaling of the crossover coupling in the ϕ^4 and ϕ^6 theories using β and $\langle \tilde{r}' \rangle$ measures at different energy windows and cutoff energies. The error in the brackets is calculated from the statistical uncertainty. The examined volume regimes at the corresponding energy windows: [12.0, 16.0] : $mL = 8, 8.5, \dots, 12$; [8.0, 10.7] : $mL = 16, 17, \dots, 23$; [6.5, 9.1] : $mL = 24, 24.5, \dots, 30$.

The calculated b values corresponding to different volume regions at different energy windows and cutoffs are summarized in Table 3.1. Considering that the definition of the measures was quite different, the agreement between the corresponding b^β and $b^{\tilde{r}}$ values are quite notable. This agreement indicates that the exponent is independent of the definition of the measure used to define the crossover couplings and, therefore, physically meaningful. However, b substantially depends on the choice of the energy window. Moreover, the b values are remarkably small, and the crossover coupling covers only a narrow range on the log scale (c.f. Fig. 3.3), in stark contrast to the results obtained for spin chain results [15]. The latter observations call into question the physical significance of the b values. We conclude that studying the finite size dependence of the crossover in terms of the system volume is not quite satisfactory, as it does not capture the integrability-breaking crossover clearly. In the next section, we demonstrate that parameterising the system size differently leads to very robust results.

4 Scaling by the number of particles

An alternative notion of system size is the number of degrees of freedom that can be associated with the number of (quasi)particle excitations. For spin chains, their length L is, in fact, the same parameter since the Hilbert space is dominated by states where the number of quasi-particle excitations is of order L . Analogously to the former analysis, we investigate the dependence of the crossover coupling on the total number of particles. However, it is not immediately apparent how to interpret the particle number for non-integrable interacting quantum field theories, so we should first establish a meaningful approach before getting to the results.

4.1 Integrability breaking via scattering theory

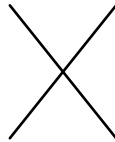
Scattering theory provides a useful perspective to investigate integrability. In a massive theory (with a gapped spectrum), the Hilbert space can be built up from asymptotic states (which statement is known as asymptotic completeness). In the asymptotic sense, the notion of particles is well-defined, and an asymptotic state can be represented as

$$|A_{a_1}(p_1), \dots, A_{a_n}(p_n)\rangle, \quad (4.1)$$

where p_k is the momentum of the k -th particle and a_k is some additional freedom (if there are any, e.g. particle type or spin). The $1 + 1$ dimensional structure implies that $p_1 > \dots > p_n$ for an *in*-state (i.e., $t \rightarrow -\infty$ asymptotic state), and $p_1 < \dots < p_n$ for an *out*-state (i.e., $t \rightarrow +\infty$ asymptotic state). The transformation between the *in* and *out* basis is described by the so-called S -matrix.

If the system is integrable, the presence of an extensive number of local conserved charges entails particular restrictions to the S -matrix, specifically, the multi-particle S -matrix factorises to $2 \rightarrow 2$ scattering processes [9]. This implies that the number of particles can not change. Therefore, the notion of the “total number of particles” is well-defined (i.e., there exists a corresponding Hermitian operator, $\hat{N}$). Moreover, in the absence of internal degrees of freedom, the $2 \rightarrow 2$ processes preserve the momenta due to energy-momentum conservation (which is an exceptional property of the $1 + 1$ dimensions), and thus the sets of momenta corresponding to the multi-particle states are conserved. Therefore, to break integrability, scattering processes producing non-trivial momentum transfer should be present to lift the level crossings.

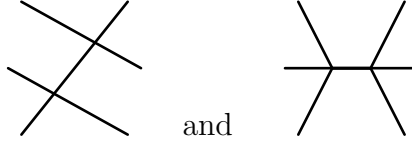
Let us consider the weak coupling limit ($g_n \rightarrow 0$) in the ϕ^4 and ϕ^6 theories. The leading order scattering in the ϕ^4 model is the $2 \rightarrow 2$ process described by the



tree-level diagram. The outgoing momenta are equal to the incoming ones due to energy-momentum conservation, so — up to a phase factor — this diagram is trivial and thus irrelevant to integrability breaking. Since the $1 \rightarrow 3$ process (and its crossing counterpart) is forbidden by energy-momentum conservation, we need at least 6 particles to produce non-trivial momenta⁴. Therefore, the lowest order processes that change the set

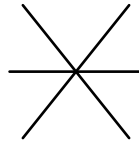
⁴The S -matrix elements of asymptotic states including even and odd numbers of particles (e.g., the $2 \rightarrow 3$ process) are always 0 due to $\mathbb{Z}_2$ symmetry.

of momenta—and hence relevant in terms of integrability breaking—are the $2 \rightarrow 4$, the $3 \rightarrow 3$ and the $4 \rightarrow 2$ scatterings. These processes are of order 2 in the g_4 coupling as the tree-level graphs already include two vertices; for example, the leading order graphs corresponding to the $3 \rightarrow 3$ process are



with the appropriate permutations of the external legs.

In contrast to the ϕ^4 theory, the processes involving 6 particles are still first order in the coupling in the ϕ^6 model, e.g. the tree-level diagram of the $3 \rightarrow 3$ scattering is as follows.



We thus see that the integrability breaking processes are second order in the coupling for the ϕ^4 but first order for the ϕ^6 (and—in fact—for every ϕ^n , $n > 4$ even) interaction. This observation has remarkable consequences concerning the dynamics of the system. In Ref. [14], the authors found that—unlike the ϕ^6 —the ϕ^4 theory does not thermalize in the Boltzmann regime (which was numerically confirmed in Ref. [23]). In principle, the ϕ^4 perturbation breaks the integrability as well; however, its effect on the non-equilibrium dynamics is considerably “weaker” than that of the ϕ^6 (and the generic integrability breaking perturbation).

This raises the interesting question of whether we can pinpoint a similar difference between the ϕ^4 and ϕ^6 interactions via studying the equilibrium dynamics—specifically, the scaling of the crossover coupling by the number of particles.

4.2 Numerical results

To apply the THA and the aforementioned statistical analysis, we must specify how to associate a crossover coupling to a given number of particles N . First, note that the crossover coupling is expected to go to zero with increasing system size. Therefore, it is reasonable to consider integrability breaking in the weak coupling limit. For the theories considered here, it is natural to classify the states of the computational basis by the eigenvalue of the free boson particle number operator $\hat{N}$, which classification retains much of its relevance even in the weakly interacting case (when the renormalisation of the physical parameters are negligible).

As it was discussed in Section 4.1, considering the ϕ^4 and ϕ^6 , the lowest order scattering processes that change the set of momenta involves 6 particles: the $2 \rightarrow 4$, the $3 \rightarrow 3$ and the $4 \rightarrow 2$ scatterings. Now, the claim is as follows: out of these three processes, the dominant (in the integrability-breaking sense) is the $3 \rightarrow 3$ scattering while the particle number changing processes are suppressed. On one hand, the $2 \rightarrow 4$ process is kinematically constrained as it requires a threshold energy. On the other hand, the $4 \rightarrow 2$ process is suppressed compared to the $3 \rightarrow 3$ as it requires 4 particles to meet at once (within the finite range of interaction imposed by the mass gap). As a result, one expects

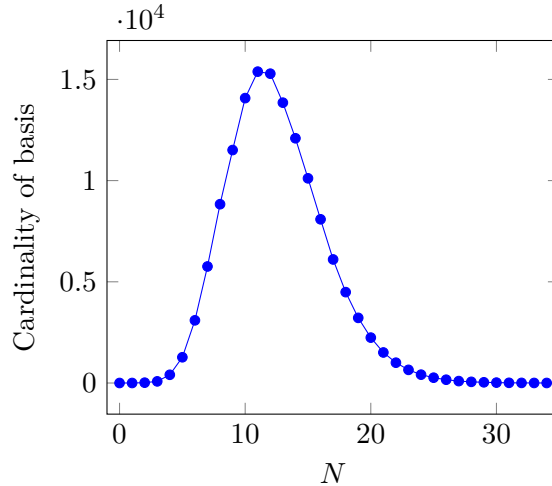


Figure 4.1: Cardinality of the THA basis as a function of the particle number. Parameters: $mL = 6$, $\Lambda/m = 35$.

that the level spacing statistics restricted to subspaces with a fixed number of particles show a much faster crossover than the statistics for the overall spectrum. While these considerations are quite heuristic, the results obtained below are consistent with them.

Therefore, we consider the interacting Hamiltonians (2.3) and (2.4) within individual subspaces of a given number N of particles; that is, we neglect the interaction between the different N -sectors and consider them individually. To get an idea of whether the number of states with a given number of particles is sufficient for statistical analysis, one can determine the cardinality of the computational basis for given values of the volume and the cutoff as a function of N , which is illustrated in Figure 4.1. We then perform the aforementioned statistical analysis for the spectra corresponding to an individual N -sector as we switch on the interaction, from which we determine the $g^*(N)$ crossover coupling.

Figure 4.2 illustrates the crossover coupling as a function of the particle number in the ϕ^4 and ϕ^6 theory at $mL = 6$ volume. We first observe that the crossover coupling covers a remarkably wider range on the log scale than in the volume scaling case illustrated in Figure 3.3. We also note that the couplings become especially small as we increase N , which shows our assumptions are self-consistent. For comparison, we also determined the crossover coupling using the complete spectrum at $mL = 6$ as in Section 3.4, including particle-changing interaction. The crossover couplings corresponding to the full spectra (at the same volume, energy window, and cutoff as in Fig. 4.2) are found to be

$$\log \tilde{g}_4^* = -0.72(3), \quad \text{and} \quad \log \tilde{g}_6^* = -2.86(2). \quad (4.2)$$

Comparing to Figure 4.2 we find that the typical value of $\tilde{g}^*(N)$ is significantly smaller than the crossover coupling corresponding to the full spectrum, supporting the argument that the level crossings within the N -sectors are lifted much faster than those occurring between states corresponding to different N .

Although we have no particular argument for a power scaling of $\tilde{g}^*(N)$ vs N , Figure 4.2 strongly suggests applying a linear fit to the $\log g^*$ vs. $\log N$ data. We define the exponent γ by assuming the dependence

$$\log g^* = \gamma \log N + \alpha, \quad (4.3)$$

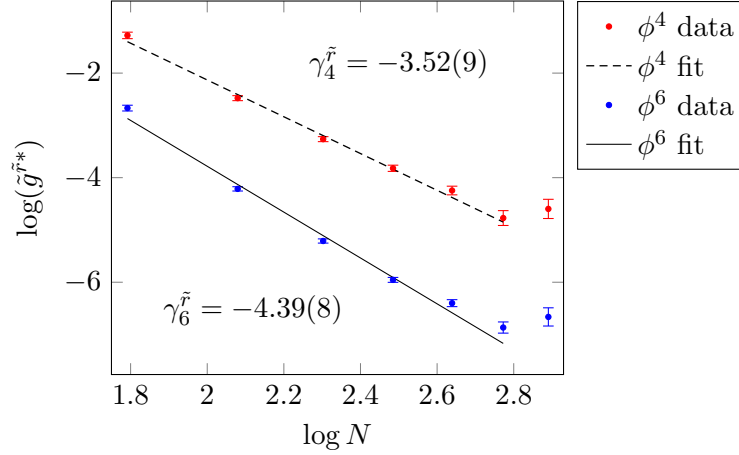


Figure 4.2: Crossover coupling vs. particle number ($N = 6, 8, \dots, 18$) in the ϕ^4 and ϕ^6 theories, together with linear fits according to Eq. (4.3) (the $N = 18$ points were ignored). The errors are calculated from statistical uncertainty. Parameters: $mL = 6$, $[E_1/m, E_2/m] = [20, 25]$, and $\Lambda/m = 35$.

mL	$[E_1/m, E_2/m]$	Λ/m	γ_4^β	$\gamma_4^{\tilde{r}}$	γ_6^β	$\gamma_6^{\tilde{r}}$
4	[30, 40]	50	-3.15(3)	-3.00(4)	-4.50(3)	-4.21(4)
5	[29, 35]	45	-3.14(3)	-3.19(4)	-4.44(3)	-4.45(3)
6	[20, 25]	35	-3.35(6)	-3.52(9)	-4.34(6)	-4.39(8)
7	[20, 25]	35	-3.45(4)	-3.57(6)	-4.76(4)	-4.84(5)
8	[17, 22]	28	-3.62(5)	-3.70(8)	-4.80(4)	-4.76(6)

Table 4.1: Scaling parameter characterizing the crossover coupling vs. particle number dependence in the ϕ^4 and ϕ^6 theories using β and $\langle \tilde{r}' \rangle$ measures at different volumes, energy windows, and cutoff energies. The uncertainty is calculated from the statistical fluctuations.

i.e., $g^* \sim N^\gamma$. Similarly to the finite volume scaling parameter b defined in (3.10), the exponent γ is independent of the normalisation of the coupling. The fitted lines are illustrated in Figure 4.2; note that the $N = 18$ points were ignored as the relevant energy levels are too close to the edge of the energy window.

We repeated the computations of the exponent γ at different volumes, with the results summarized in Table 4.1. Now the choice of the parameters is less constrained than for the analysis of the full spectrum, as the number of states in an N -sector only increases polynomially with the cutoff. Nevertheless, it is still necessary to change the energy window and the cutoff as the volume is increased.

The results demonstrate some key features:

1. Similarly to the b values obtained from finite volume scaling, the γ values coming from the Brody distribution and the consecutive level ratios match pretty well, indicating that the exponent does capture universal features that only depend on the interaction.
2. γ values only change by about 10-20% as the volume is doubled, which makes their value much more consistent than the b values for finite volume scaling (c.f. Table 3.1).

3. Comparing the γ values for ϕ^4 and ϕ^6 perturbations, the γ_4 are significantly smaller than the γ_6 , which can be interpreted as a clear signal that the ϕ^6 interaction violates integrability stronger than the ϕ^4 .

This leads us to the conclusion that the relevant system size parameter for the crossover from integrable to non-integrable level spacing statistics is the particle number rather than the volume. As noted before, this eventually does not contradict previous results obtained in spin chains since, in those systems, the two parameters are closely related when considering the middle of the spectrum used in the analysis of level spacing statistics.

To interpret the γ values numerically, we invoke the literature. In Ref. [7], using the concept of the so-called Fock space delocalisation transition, the authors found that the crossover coupling behaves as

$$g^*(N) \sim \frac{1}{N^{(n_h+1)/2} \log N}, \quad (4.4)$$

for a quantum dot model, where N is the number of fermionic levels, and n_h is the number of states connected by an elementary interaction (e.g., it is $n_h = n$ for ϕ^n). Considering that their result might not fully apply to a massive scalar boson field, it is still insightful to associate their predictions with our results. Neglecting the logarithm, we get that

$$\gamma_n = -\frac{n+1}{2}. \quad (4.5)$$

Though this formula does not recover the exact values presented in Table 4.1, it suggests that $|\gamma_6 - \gamma_4| = 1$, which seems to agree with our results. In summary, studying the Fock space delocalisation provides a promising approach to interpreting these results. However, its precise elaboration for the massive scalar boson is left for future research.

5 Summary and outlook

In this work, we investigated integrability breaking in scalar quantum field theories in $1 + 1$ dimensions. First, we reviewed the most important properties of the ϕ^4 and ϕ^6 models and worked out the truncated Hamiltonian approach. Next, we elaborated on the characterisation of integrability breaking by analyzing the statistical properties of the energy spectrum and introducing the crossover coupling. After that—analogously to previous studies—we studied the finite size scaling behaviour of the crossover coupling to compare the ϕ^4 and ϕ^6 interactions regarding the strength of integrability breaking.

Our first main result is that the relevant system size parameter regarding integrability breaking is not the volume but the particle number. The scaling of the crossover coupling as a function of the volume turned out to be particularly weak, and thus the results were inconclusive. On the contrary, we found a more robust scaling as a function of the particle number N . In both cases, we found that the scaling exponent showed good agreement between two different determinations of the crossover coupling, which is an important consistency check for the approach.

We also found that the dominant processes—in terms of integrability breaking—are those preserving the number of particles. We first established this statement using a somewhat heuristic argument to support the idea of neglecting particle-changing interaction. Later, we found that the crossover coupling corresponding to the individual N -sectors is typically well below the one corresponding to the full spectrum. This means that our argument is indeed self-consistent and numerically confirmed.

These results are in good agreement with the classical expectations as the crossover coupling goes to zero while increasing the number of degrees of freedom. Classically, the number of degrees of freedom is associated with the number of (classical) particles, not the volume, which agrees with our findings for the quantum case. Following spin chain studies, the naive intuition suggested investigating the volume dependence of the crossover coupling as the particle number seemed less fundamental for an interacting quantum field theory. However, on the one hand, particle number can always be interpreted in the integrable case⁵. Therefore, it is meaningful to consider it when studying weakly perturbed integrable systems. On the other hand, our observation does not in any way contradict the scaling of the crossover with the length of the system observed in spin chains since, for the relevant states (those in the middle of the spectrum), the length of the chain and the number of quasi-particle excitations are essentially the same parameters.

The second main conclusion of our research is that the scaling of the crossover coupling, considered as a function of particle number supports the distinction proposed in [14] based on non-equilibrium evolution modelled using a Boltzmann kinetic equation approach, according to which the ϕ^4 induces weak integrability breaking as opposed to the strong one induced by ϕ^6 . To interpret the numerical values of the scaling exponents, we adopted the approach outlined in Ref. [7]. Although the predictions partially supported our findings, a comprehensive explanation of the exponents necessitates further elaboration on the Hilbert space connectivity theory, which remains an avenue for future research. An other interesting open issue is to compare the full quantum evolution in the ϕ^4 and ϕ^6 theories, e.g. following [23]; we performed some preliminary studies which indicated that the THA must be improved further to arrive at reliable results, which we leave as an open problem for the future.

⁵Note that the particle picture is not unique for massless theories.

A Leading order renormalization group improvement for the ϕ^4 and ϕ^6 THA

In this appendix, I summarize the derivation of the leading order RG improvement following the approach introduced in Ref. [31].

Let H denote the full Hamiltonian, $\mathcal{E}$ one of its eigenvalues, and $\psi_{\mathcal{E}}$ the corresponding eigenstate:

$$H\psi_{\mathcal{E}} = \mathcal{E}\psi_{\mathcal{E}}. \quad (\text{A.1})$$

The truncation splits the Hilbert space into the low-energy subspace ($\mathcal{H}_l$) retained for the numerical calculations and the high-energy ($\mathcal{H}_h$) subspace which is discarded. Then the full Hilbert space is $\mathcal{H} = \mathcal{H}_l \oplus \mathcal{H}_h$ and the eigenvalue equation can be decomposed as

$$\begin{pmatrix} H_{ll} & H_{lh} \\ H_{hl} & H_{hh} \end{pmatrix} \begin{pmatrix} \psi_{\mathcal{E},l} \\ \psi_{\mathcal{E},h} \end{pmatrix} = \mathcal{E} \begin{pmatrix} \psi_{\mathcal{E},l} \\ \psi_{\mathcal{E},h} \end{pmatrix}, \quad (\text{A.2})$$

where H_{ll} is the naive truncated Hamiltonian. Assuming that the Hamiltonian has the form $H = H_0 + V$ with a diagonal part H_0 and a perturbation V yields

$$(H_{ll} + \Delta H)\psi_{\mathcal{E},l} = \mathcal{E}\psi_{\mathcal{E},l} \quad (\text{A.3})$$

where

$$\Delta H = -V_{lh}(H_0 + V_{hh} - \mathcal{E})^{-1}V_{hl} = -V_{lh}(H_0 - \mathcal{E})^{-1}V_{hl} + \mathcal{O}(V^3) \quad (\text{A.4})$$

are counter terms that eliminate the cut-off dependence of the energy level. We introduce a computational basis $|k\rangle$ composed of the eigenstates of H_0 :

$$H_0 |k\rangle = E_k |k\rangle \quad (\text{A.5})$$

and approximate the counter terms by keeping only the leading order term, which has the matrix elements

$$(\Delta H)_{ij} = - \sum_{k: E_k > \Lambda} \frac{V_{ik}V_{kj}}{E_k - \mathcal{E}} = - \int_{\Lambda}^{\infty} dE \frac{M(E)_{ij}}{E - \mathcal{E}}, \quad (\text{A.6})$$

where

$$M(E)_{ij}dE = \sum_{k: E < E_k < E+dE} V_{ik}V_{kj}. \quad (\text{A.7})$$

Since the relevant contribution comes from the high-energy asymptotics of $M(E)$, it is reasonable to approximate $M(E)$ as a continuous distribution due to the high density of energy levels in $\mathcal{H}_h$. The matrix elements $M(E)_{ij}$ can be evaluated explicitly by introducing the quantity

$$C_{ij}(\tau) = \langle i | V(\tau/2) V(-\tau/2) | j \rangle = \int_0^{\infty} dE e^{-(E - (E_i + E_j)/2)\tau} M(E)_{ij}, \quad (\text{A.8})$$

where

$$V(\tau) = e^{H_0\tau} V e^{-H_0\tau} \quad (\text{A.9})$$

is the perturbation in the (Euclidean) interaction picture. The high-energy behaviour can be determined from the behaviour for small τ , which can be obtained using Laplace-transformation. Assuming that the Hamiltonian has the form

$$\begin{aligned} H_0 &= \frac{1}{2} \int dx : ((\partial_t \phi)^2 + (\partial_x \phi)^2 m^2 \phi^2) : , \\ V &= \sum_n g_n V_n , \quad V_n = \int dx : \phi^n(x) : , \end{aligned} \quad (\text{A.10})$$

and applying Wick's theorem yields

$$\frac{d}{d\tau} C_{ij}(\tau) = \sum_{n,m} g_n g_m \sum_{0 \leq k \leq \min(n,m)} k! \binom{n}{k} \binom{m}{k} I'_k(\tau) \langle i | V_{n+m-2k} | j \rangle \quad (\text{A.11})$$

$$= \int_0^\infty dE e^{-(E-(E_i+E_j)/2)\tau} [-(E-(E_i+E_j)/2)M(E)] , \quad (\text{A.12})$$

where

$$I_k(\tau) = \int_{-L/2}^{L/2} G_L(\tau, z)^k , \quad (\text{A.13})$$

and $G_L(\tau, z)$ is the Euclidean propagator

$$G_L(\rho) = \frac{1}{2\pi} K_0(m\rho) \approx -\frac{1}{2\pi} \log \left(\frac{e^\gamma}{2} m\rho \right) [1 + \mathcal{O}(m^2 \rho^2)] , \quad \rho m \ll 1 . \quad (\text{A.14})$$

Here $\rho = \sqrt{\tau^2 + z^2}$ is the Euclidean distance, and K_0 is the modified Bessel function of the second kind. The derivative with respect to τ was introduced to eliminate some spurious IR divergences [31].

We only keep the leading non-analytic behaviour of $I'_k(\tau)$ for $\tau \rightarrow 0$, which corresponds to the leading order in the cutoff dependence, expressed as a Laplace transform of some function $\mu_k(E)$

$$I'_k(\tau) = \int_\varepsilon^\infty dE e^{-E\tau} \mu_k(E) + \text{subleading contributions} . \quad (\text{A.15})$$

After substituting back and reordering the sum, the final expression for the correction term is

$$(\Delta H)_{ij} = - \sum_{k'} (V_{k'})_{ij} \sum_{n \geq m} g_n g_m \int_{\Lambda - (E_i + E_j)/2}^\infty dE \frac{\mu_{nmk'}(E)}{E + (E_i + E_j)/2 - \mathcal{E}} , \quad (\text{A.16})$$

where

$$\mu_{nmk'}(E) = -(2 - \delta_{n,m}) k! \binom{n}{k} \binom{m}{k} \frac{\mu_k(E)}{E} \Big|_{k=(n+m-k')/2} . \quad (\text{A.17})$$

Neglecting the $E_i + E_j$ terms in comparison with the cut-off Λ , the result can be simplified further as

$$\Delta H = - \sum_k \kappa_k V_k, \quad (\text{A.18})$$

where

$$\kappa_k = \sum_{n \geq m} g_n g_m \int_{\Lambda}^{\infty} dE \frac{\mu_{nmk}(E)}{E - \mathcal{E}}. \quad (\text{A.19})$$

Note that this still contains the exact eigenvalue $\mathcal{E}$ of the level considered. Following [31], this can be replaced by a value which is chosen to lie in the expected range of the energy level(s), which are the objects of the numerical computation, henceforth referred to as *reference energy*.

The functions $\mu_{nmk}(E)$ relevant for $n, m = 2, 4, 6$ are given by

$$\begin{aligned} \mu_{220}(E) &= \frac{1}{\pi E^2}, & \mu_{422}(E) &= \frac{12}{\pi E^2}, & \mu_{440}(E) &= \frac{3}{\pi^3 E^2} \left[6(\log E/m)^2 - \frac{\pi^2}{2} \right], \\ \mu_{442}(E) &= \frac{72 \log E/m}{\pi^2 E^2}, & \mu_{444}(E) &= \frac{36}{\pi E^2}, \\ \mu_{624}(E) &= \frac{30}{\pi E^2}, & \mu_{642}(E) &= \frac{90}{\pi^3 E^2} \left[6(\log E/m)^2 - \frac{\pi^2}{2} \right], \\ \mu_{644}(E) &= \frac{720 \log E/m}{\pi^2 E^2}, & \mu_{646}(E) &= \frac{180}{\pi E^2}, \\ \mu_{660}(E) &= \frac{675}{2\pi^5 E^2} \left[(\log E/m)^4 - \frac{\pi^2}{2} (\log E/m)^2 + 2\zeta(3) \log E/m + \frac{\pi^4}{80} \right], \\ \mu_{662}(E) &= \frac{1350}{\pi^4 E^2} \left[2(\log E/m)^3 - \frac{\pi^2}{2} \log E/m + \zeta(3) \right], \\ \mu_{664}(E) &= \frac{675}{\pi^3 E^2} \left[6(\log E/m)^2 - \frac{\pi^2}{2} \right], \\ \mu_{666}(E) &= \frac{1800 \log E/m}{\pi^2 E^2}, & \mu_{668}(E) &= \frac{225}{\pi E^2}. \end{aligned} \quad (\text{A.20})$$

The terms involving $n, m = 2, 4$ agree with those of Ref. [31], while the rest are my own results. We can verify the latter by computing the vacuum energy and the mass gap in ϕ^6 theory at different cutoff values, with the results shown in Fig. A.1. Comparing the numerical results obtained from the naive truncated (bare) Hamiltonian H_l and those obtained from the renormalised Hamiltonian $H_l + \Delta H$, we find that the counter terms significantly suppress the dependence on Λ , making THA converge much faster as the cutoff is increased.

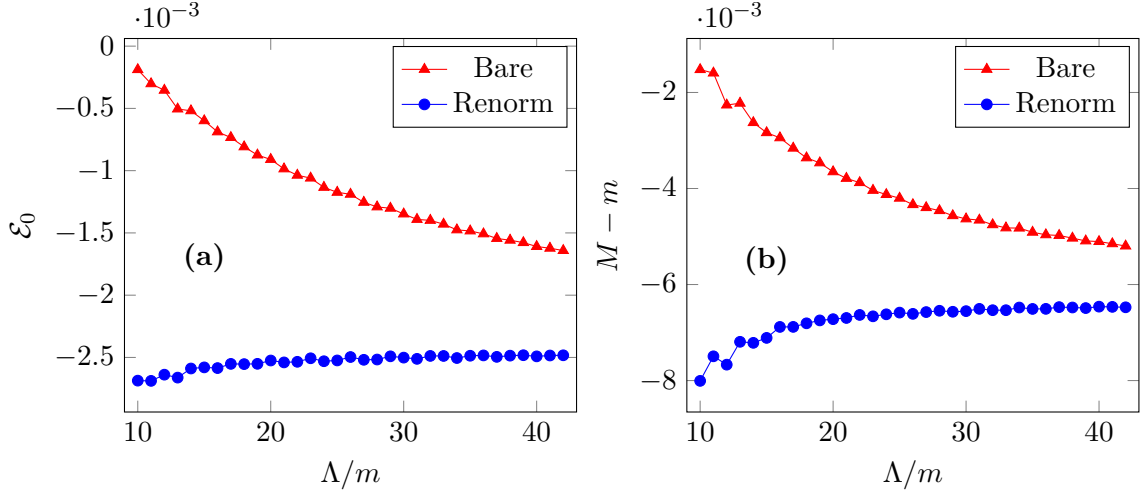


Figure A.1: **(a)** The vacuum energy and **(b)** the mass gap ($M = \mathcal{E}_1 - \mathcal{E}_0$) as a function of the cutoff energy in the ϕ^6 theory using the naive (*bare*) and the RG-improved (*renorm*) Hamiltonians. Parameters: $mL = 4$, $\tilde{g}_6 = 0.05$, the reference energy $\mathcal{E}$ was set to 0 for the vacuum energy and m for the mass gap calculation.

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