

Finite temperature dynamical correlation functions in the Ising Quantum Field Theory

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1. Introduction

The two-dimensional classical Ising model is one of the most influential theories in statistical physics [1]. As a canonical example of continuous phase transitions, it provides deep insights into the diverse world of critical phenomena. The model is integrable, allowing for analytical approaches that would be significantly more complicated or even impossible in most other theories. Its quantum mechanical implications also run deep [2], as the model can be mapped onto the one-dimensional quantum Ising spin chain, which, in line with the integrable property, can be transformed into a system of free fermions. However, this is a non-local transformation, which introduces severe difficulties in many calculations. As a result, determining all physically relevant quantities remains an actively researched area [3, 4].

In my paper, I examine the dynamical (two-time) correlation functions of the magnetization order parameter in the one-dimensional Ising quantum field theory at both zero and finite temperatures. This model can be considered as the scaling limit of the Ising spin chain that describes the vicinity of the quantum critical point. Unfortunately, the non-locality of the free fermions poses profound challenges in calculating these correlators. To circumvent this issue, I decided to use the Lehmann expansion since the form factors of the Ising quantum field theory are analytically known, and, unlike in most theories, this technique can be used even at finite temperatures [3, 5]. However, the numerical treatment of this procedure still presents significant challenges, as the expansion contains infinitely many integrals with ever-growing dimensions. Luckily, there are some exceptional techniques to handle these difficulties.

The relevant form factors of the theory can be expressed as specific matrix determinants [6], which yields a Fredholm determinant representation for the correlation functions [3, 5, 7]. Such representations are easy to handle numerically [8] and, as a result, lay the foundations of an efficient algorithm. For zero temperature, this algorithm works without any restrictions, but at finite temperatures, the underlying series only converges for space-like separations. Despite this limitation, the procedure allows for a comprehensive discussion and presents the opportunity to analyze these correlation functions to a depth still missing from the literature.

My paper is structured as follows. In Section 2, I review the fundamentals of the one-dimensional transverse Ising model and its scaling quantum field theory. I discuss the Lehmann expansion of the dynamical correlation functions and provide explicit results in both phases for zero and finite temperatures alike. In Section 3, I present the basics of the theory of Fredholm determinants and use it to derive such representations for the correlators. Section 4 contains my own results, in which I use these representations to create an efficient numerical algorithm. I discuss the zero and finite temperature cases separately and compare my simulations with several analytical asymptotic results. Furthermore, I derive a novel theoretical result for the asymptotic behaviors of the paramagnetic dynamical correlation functions, which aligns perfectly with my numerical method but exhibits a notable discrepancy with the recent low-temperature approximation presented in [4]. Finally, in Section 5, I summarize my results and give an outlook for my future work. When I talk about

my own work, I write in the first person singular, but I give the general discussion in the plural.

2. One-dimensional quantum Ising model

The one-dimensional transverse-field Ising model is a simplified but insightful quantum mechanical system used to study phase transitions and quantum magnetism. In this model, we consider a chain of interacting spins arranged in a one-dimensional lattice. Each spin interacts with its nearest neighbors and is subject to a transverse magnetic field. The Hamiltonian describing this system captures the interplay of these interactions and is given by the following expression:

$$H = -J \sum_i (g\sigma_i^x + \sigma_i^z \sigma_{i+1}^z). \quad (2.1)$$

In this equation, $J > 0$ is the energy scale of the problem, g is the relative strength of the external magnetic field compared to that of the nearest neighbor interaction, and $\sigma_i^{x,z}$ are the respective Pauli matrices implying that we are dealing with a spin-1/2 system. The index i runs over the lattice sites present in the one-dimensional chain. Note that the Hilbert space is the tensor product of the individual Hilbert spaces: $\mathcal{H} = \otimes_i \mathbb{C}^2$.

It is apparent that Eq. (2.1) describes a non-trivial quantum system. Since σ^x and σ^z do not commute, the eigenstates of either of the two interaction terms do not coincide with the eigenstates of H . However, with more involved considerations, utilizing both the Jordan–Wigner transformation and the Bogoliubov transformation, it is possible to give an exact solution for this many-body system [2]. For an infinite system, these manipulations yield the following diagonal Hamiltonian:

$$H = \int_{-\pi/a}^{\pi/a} \frac{dk}{2\pi} \epsilon(k) \gamma_k^\dagger \gamma_k \quad (2.2)$$

with the dispersion relation

$$\epsilon(k) = 2J\sqrt{1 - 2g \cos(ka) + g^2}. \quad (2.3)$$

Here, a represents the lattice spacing of the system, and the index k runs over the Brillouin zone, indicating that this result is understood in momentum space. The appearing γ_k operators are fermionic creation annihilation operators; they are related to the Pauli matrices of Eq. (2.1) in a complicated, non-local way. This pinpoints the difficulty in calculating physical correlation functions in this model.

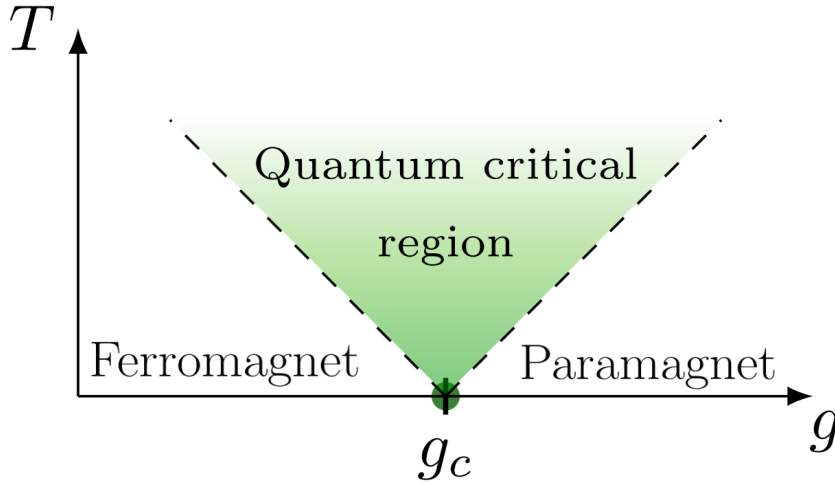


Figure 2.1: Phase diagram of the one-dimensional transverse-field Ising model. At zero temperatures, there is a second-order quantum phase transition between the ferromagnetic and paramagnetic phases separated by the $g_c = 1$ critical coupling. At finite temperatures, this phase transition disappears, but we still see its fingerprints near the quantum critical point. Based on a similar illustration in [9]

2.1. Scaling Quantum Field Theory

In the rest of this paper, I would like to analyze the scaling limit of the above-described one-dimensional lattice. To this end, let me review the phase diagram of the model using Figure 2.1 as an illustration.

It is well-established in the literature that the one-dimensional transverse-field Ising model features a zero temperature second-order quantum phase transition between the paramagnetic and ferromagnetic phases. The critical coupling separating these two regimes is $g_c = 1$. This is in harmony with Eq. 2.3 since it is at $g = 1$ when the gap of the system closes. In the vicinity of this critical point, the correlation length diverges and the system develops a universal behavior reminiscent of those seen in other second-order phase transitions. We can describe this regime with an appropriate scaling quantum field theory. However, even though the phase transition disappears at finite temperatures, we can extend this field theoretical description to encompass all scenarios where the correlation length greatly surpasses the lattice spacing. In this sense, we expect universal behavior to emerge even at finite temperatures given that we are close enough to the quantum critical point.

The scaling quantum field theory describing of the system can be represented by the Hamiltonian [2]

$$H = \int_{-\infty}^{\infty} dx \left[\frac{c}{2} \left(\Psi^\dagger(x) \frac{\partial \Psi^\dagger(x)}{\partial x} - \Psi(x) \frac{\partial \Psi(x)}{\partial x} \right) + \Delta \Psi^\dagger(x) \Psi(x) \right], \quad (2.4)$$

where Ψ is a fermionic quantum field, $\Delta = 2J(1 - g)$ is the (signed) gap of the model and $c = 2Ja$ is the sound velocity. Taking the scaling limit implies that we fix the parameters Δ and c and send the lattice spacing a to zero. In consequence, this field theoretical description is only valid at large scales. For example, when we consider spatial correlation functions for a separation x that naturally implies the relation $x \gg a$. This results in certain restrictions for the applicability range of the field theoretical approach that can be important if we want to match such calculations with actual measurement data. However, for a purely theoretical analysis, these limitations are not present.

Similarly to the chain, Eq. (2.4) can be transformed into a diagonal form:

$$H = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \epsilon(k) a_k^\dagger a_k, \quad (2.5)$$

where $a_k^\dagger$ and a_k are fermionic creation and annihilation operators and

$$\epsilon(k) = \sqrt{\Delta^2 + c^2 k^2}, \quad (2.6)$$

is the dispersion relation. It indicates that the theory is relativistically invariant, and it contains only one particle in the spectrum with a mass of

$$m = |\Delta|/c^2 = \frac{|1 - g|}{2Ja^2}. \quad (2.7)$$

In summary, we can describe the critical behavior of the one-dimensional transverse field Ising model with the scaling field theory given by Eq. (2.4). This description is only valid in a restricted phase space regime near the quantum critical point. There are three parameters in the resulting model: the mass m of the excitations, the sound (or light) velocity c , and the temperature T , which is unaffected by the scaling limit. Additionally, we also need to know whether the system is in the paramagnetic or the ferromagnetic phase since this information is not coded elsewhere. All further results of my paper are understood within this framework.

2.2. Form factor expansion of the correlation functions

The primary goal of this paper is to calculate the dynamical correlation functions of the magnetization in the Ising quantum field theory. The foundation of this calculation is the Lehmann expansion, also referred to as the form factor expansion, where we insert the identity between the operators under consideration using the completeness of the asymptotic multi-particle states. In this section, I review the necessary details of these considerations and give the results for both zero and finite temperatures.

In one-dimensional relativistically invariant systems, it is beneficial to introduce the rapidity parametrization. One of its advantages is that the conventional and convenient normalization of the one-particle states involves an additional energy contribution,

$$\langle p|q \rangle = 2\pi\epsilon(p)\delta(p - q), \quad (2.8)$$

which disappears if we use rapidities instead:

$$\langle \theta | \theta' \rangle = 2\pi \delta(\theta - \theta'). \quad (2.9)$$

The connection between the two is summarized by the relations

$$\begin{aligned} \epsilon(\theta) &= mc^2 \cosh(\theta), \\ k(\theta) &= mc \sinh(\theta), \end{aligned} \quad (2.10)$$

In the upcoming discussion, we set the speed of sound (light) to be unity. This is just a choice of units that can easily be restored in all further results using dimensional analysis.

2.2.1. Zero temperature

When we consider the zero temperature behavior of the scaling field theory, the system is always in the ground state. This means that the correlation functions of the magnetization can be written as

$$C(x, t) = \langle 0 | \sigma^z(x, t) \sigma^z(0, 0) | 0 \rangle, \quad (2.11)$$

where $|0\rangle$ is the ground state with no fermions and σ^z is a continuum version of the respective Pauli matrix defined on the chain. As implied previously, it is related to the more convenient Ψ or $a_k^\dagger$ operators in a complicated, non-local way. Therefore, calculating such correlators is highly non-trivial. Note that the space-time dependence of σ^z is understood in the Heisenberg picture.

One way to make progress in this calculation is to insert the asymptotic multi-particle resolution of the identity operator between the two σ^z operators. This resolution is given by

$$\mathbb{I} = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{-\infty}^{\infty} \frac{d\theta_1}{2\pi} \cdots \int_{-\infty}^{\infty} \frac{d\theta_k}{2\pi} |\theta_1, \dots, \theta_k\rangle \langle \theta_1, \dots, \theta_k|. \quad (2.12)$$

If we combine this with the Heisenberg space-time evolution,

$$\sigma^z(x, t) = e^{iHt} e^{-iPx} \sigma^z(0, 0) e^{iPx} e^{-iHt}, \quad (2.13)$$

and the action of the energy and momentum operators on the Fock states,

$$\begin{aligned} H |\theta_1, \dots, \theta_k\rangle &= \left(\sum_{i=1}^k m \cosh(\theta_i) \right) |\theta_1, \dots, \theta_k\rangle, \\ P |\theta_1, \dots, \theta_k\rangle &= \left(\sum_{i=1}^k m \sinh(\theta_i) \right) |\theta_1, \dots, \theta_k\rangle, \end{aligned} \quad (2.14)$$

we arrive at the form factor representation of Eq. (2.11):

$$C(x, t) = \sum_{k=0}^{\infty} \frac{1}{k!} \prod_{i=1}^k \left(\int_{-\infty}^{\infty} \frac{d\theta_i}{2\pi} e^{imx \sinh \theta_i - imt \cosh \theta_i} \right) |\langle \theta_1, \dots, \theta_k | \sigma^z(0, 0) | 0 \rangle|^2. \quad (2.15)$$

Here, the term

$$\langle \theta_1, \dots, \theta_k | \sigma^z(0, 0) | 0 \rangle \quad (2.16)$$

is called a form factor, hence the name form factor expansion. We see that this is the only model-dependent quantity that we need to know in order to get a useful expression for the correlation functions.

Here, however, a little complication arises. So far, we did not differentiate between the ferromagnetic and paramagnetic phases. In fact, Eq. (2.5) is invariant under the $\Delta \leftrightarrow -\Delta$ transformation, so it is unclear how to get two separate theories. The hidden difference comes from the Neveu–Schwarz and Ramond sectors inherited from the chain, but the exact details are beyond the scope of this discussion. It is sufficient for our purposes to study the ferromagnetic phase, calculate the correlation functions here, and then use the Kramers–Wannier duality to translate these results into the paramagnetic regime. Then, the order operator σ gives the form factors for the ferromagnetic case, while the disorder operator μ supplies the paramagnetic results. Luckily, these are known analytically and have the form [6]

$$\begin{aligned} \langle \theta_1, \dots, \theta_k | \sigma | 0 \rangle &= m^{1/8} \prod_{1 \leq i < j \leq k} \tanh \left(\frac{\theta_i - \theta_j}{2} \right) \times \begin{cases} H_k, & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is odd;} \end{cases} \\ \langle \theta_1, \dots, \theta_k | \mu | 0 \rangle &= m^{1/8} \prod_{1 \leq i < j \leq k} \tanh \left(\frac{\theta_i - \theta_j}{2} \right) \times \begin{cases} 0, & \text{if } k \text{ is even,} \\ \tilde{H}_k, & \text{if } k \text{ is odd.} \end{cases} \end{aligned} \quad (2.17)$$

The appearing H_k and $\tilde{H}_k$ are dimensionless constants that depend on the normalization of the operators. In our convention, $\langle \sigma \rangle = m^{1/4}$ in the ferromagnetic ground state, so they are just complex phases and drop out from the calculations. Crucially, σ only has even, while μ only has odd matrix elements.

In conclusion, if we combine Eqs. (2.15) and (2.17) with the above discussion, we see that the correlation functions of the magnetization can be expressed as

$$C_f(x, t) = m^{1/4} \sum_{\substack{k=0 \\ k \text{ even}}}^{\infty} \frac{1}{k!} \prod_{i=1}^k \left(\int_{-\infty}^{\infty} \frac{d\theta_i}{2\pi} e^{imx \sinh \theta_i - imt \cosh \theta_i} \right) \prod_{1 \leq i < j \leq k} \tanh^2 \left(\frac{\theta_i - \theta_j}{2} \right) \quad (2.18)$$

in the ferromagnetic phase and as

$$C_p(x, t) = m^{1/4} \sum_{\substack{k=0 \\ k \text{ odd}}}^{\infty} \frac{1}{k!} \prod_{i=1}^k \left(\int_{-\infty}^{\infty} \frac{d\theta_i}{2\pi} e^{imx \sinh \theta_i - imt \cosh \theta_i} \right) \prod_{1 \leq i < j \leq k} \tanh^2 \left(\frac{\theta_i - \theta_j}{2} \right) \quad (2.19)$$

in the paramagnetic phase.

2.2.2. Finite temperature

The finite temperature generalization of the form factor expansion is a notoriously difficult task [10]. In general, there are no explicit results or techniques regarding such calculations. Indeed, at finite temperature, the system is in a thermal state, so the correlation functions are defined as

$$C(x, t; \beta) = \frac{\text{Tr} (e^{-\beta H} \sigma^z(x, t) \sigma^z(0, 0))}{\text{Tr} (e^{-\beta H})}, \quad (2.20)$$

where $\beta = 1/T$ is the inverse temperature of the system. Then, due to the trace, if we insert Eq. (2.12) between the operators, the expansion contains multi-particle matrix elements, like

$$\langle \theta_1, \dots, \theta_k | \sigma^z | \theta'_1, \dots, \theta'_{k'} \rangle. \quad (2.21)$$

However, these form factors have singularities, so-called kinematic poles, when two rapidities in the bra and the ket states are very close in value. This implies that the naive finite temperature form factor expansion becomes divergent, and a more careful analysis is needed to give meaningful results.

Since our quantum field theory has an underlying spin chain, we can try to regularize this form factor expansion by looking at the finite temperature correlation functions in the chain and taking the scaling limit of these results. This is a well-defined procedure, but it turns out that as one considers higher-order terms in the form factor expansion, such calculations become more and more involved. In any other model, no efficient technology is known to give the complete finite temperature form factor expansion in the continuum theory. However, for the Ising model, these issues can be solved. Indeed, there are analytical results in the literature for the finite temperature dynamical correlation functions of the magnetization operator that surpass all the above-described problems.

Following [3], the finite temperature two-point functions in the ferromagnetic and paramagnetic phases can be calculated as

$$C_f(x, t; \beta) = 2\pi C(\beta)^2 e^{\Delta \mathcal{E} x} m^{1/4} \sum_{\substack{k=0 \\ k \text{ even}}}^{\infty} \sum_{\epsilon_1, \dots, \epsilon_k = \pm} \int_{-\infty + i\epsilon_j \delta}^{\infty + i\epsilon_j \delta} \frac{d\theta_1 \dots d\theta_k}{k!} \times \quad (2.22)$$

$$\times \prod_{j=1}^k \left(\frac{e^{im\epsilon_j(x \sinh \theta_j - t \cosh \theta_j)}}{1 - e^{-\epsilon_j m \beta \cosh \theta_j}} i \left(\frac{\omega F^{\mu+}(\theta_j)}{\sqrt{2\pi} C(\beta)} \right)^2 \right) \prod_{1 \leq i < j \leq k} \tanh \left(\frac{\theta_j - \theta_i}{2} \right)^{2\epsilon_i \epsilon_j}$$

and

$$C_p(x, t; \beta) = -2\pi C(\beta)^2 e^{\Delta \mathcal{E} x} m^{1/4} \sum_{\substack{k=0 \\ k \text{ odd}}}^{\infty} \sum_{\epsilon_1, \dots, \epsilon_k = \pm 1} \int_{-\infty + i\epsilon_j \delta}^{\infty + i\epsilon_j \delta} \frac{d\theta_1 \dots d\theta_k}{k!} \times \\ \times \prod_{j=1}^k \left(\frac{e^{im\epsilon_j(x \sinh \theta_j - t \cosh \theta_j)}}{1 - e^{-\epsilon_j m \beta \cosh \theta_j}} i \left(\frac{\omega F_{\epsilon_j}^{\mu+}(\theta_j)}{\sqrt{2\pi} C(\beta)} \right)^2 \right) \prod_{1 \leq i < j \leq k} \tanh \left(\frac{\theta_j - \theta_i}{2} \right)^{2\epsilon_i \epsilon_j} \quad (2.23)$$

respectively. The appearing extra factors are given by the integrals

$$C(\beta) = \frac{1}{\sqrt{2\pi}} \exp \left[\frac{(m\beta)^2}{2} \int_{-\infty}^{\infty} \frac{d\theta_1 d\theta_2}{(2\pi)^2} \frac{\sinh \theta_1 \sinh \theta_2}{\sinh(m\beta \cosh \theta_1) \sinh(m\beta \cosh \theta_2)} \times \right. \\ \left. \times \log \left| \coth \left(\frac{\theta_1 - \theta_2}{2} \right) \right| \right] \quad (2.24)$$

and

$$\Delta \mathcal{E} = - \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} m \cosh \theta \log \left[\frac{1 + e^{-m\beta \cosh \theta}}{1 - e^{-m\beta \cosh \theta}} \right]. \quad (2.25)$$

Furthermore, the F terms satisfy

$$i \left(\frac{\omega F_{\epsilon}^{\mu+}(\theta)}{\sqrt{2\pi} C(\beta)} \right)^2 = -\frac{\epsilon}{2\pi} \exp(\epsilon \eta(\theta)), \quad (2.26)$$

where

$$\eta(\theta) = \int_{-\infty}^{\infty} \frac{d\theta'}{\pi i} \frac{1}{\sinh(\theta - \theta')} \log \left[\frac{1 + e^{-m\beta \cosh \theta'}}{1 - e^{-m\beta \cosh \theta'}} \right]. \quad (2.27)$$

Finally, $\delta > 0$ is a small positive constant governing the contour prescription.

The finite temperature results are much more complicated than their zero temperature counterparts given in Eqs. (2.18) and (2.19). The most crucial difference between the two is the appearance of the ϵ signs: if $\epsilon_i = +1(-1)$, then the i th rapidity is in the bra (ket) state in the form factors analogous to Eq. (2.21). This can also be interpreted as the i th rapidity corresponding to a particle (antiparticle). Note that naive singularities, the kinematic poles, occur due to the negative power of the tanh form factor terms when a particle and an antiparticle have the same rapidity. This is circumvented in Eqs. (2.22) and (2.23) by the ϵ -dependent complex shifts in the rapidities. This ensures that no divergences occur in the above expansions, and they give well-defined values for the correlators.

The other differences from the zero temperature case can be thought of as the elaborate consequences of taking the scaling limit of the chain results. This is essentially a complete

resummation of the naive form factor expansion. Most of these differences, like the $C(\beta)$ and ${}^\omega F_\epsilon^{\mu+}(\theta)$ terms are just technical details, but the $e^{\Delta\mathcal{E}x}$ term gives a qualitatively important feature: it encodes an exponential decay in the x coordinate having a purely thermal origin. This will be very important when considering the asymptotic behaviors. (For example, in the $t = 0$ static case, this term governs the large-distance behavior and connects to the Toeplitz determinant results of Sachdev [11, 12].) This is a consequence of a more general feature of the above results: for space-like separations, they are large-distance and large-time expansions, so if we are only interested in the asymptotics, it is enough to consider only one- or two-particle contributions.

Equations (2.18) and (2.19) have one vital restriction: they can only be used outside the light cone, i.e. in the $|x| > |t|$ regime. It turns out that inside the light cone, the above series are divergent, so in my work, I was restricted to the space-like separated regime.

3. Theory of Fredholm determinants

In this chapter, we show that the form factor expansions given in Eqs. (2.18), (2.19), (2.22) and (2.23) can be represented as so-called Fredholm determinants. First, we review the theory of these determinants and then show that their formalism matches that of the form factor expansions perfectly. It leads to new expressions for the correlation functions that will give the foundation for a highly efficient numerical algorithm in Section 4.

To develop the theory of Fredholm determinants, let us first consider the following integral equation:

$$u(\theta) + \int_a^b d\theta' K(\theta, \theta') u(\theta') = f(\theta), \quad (3.1)$$

where $u(\theta)$ and $f(\theta)$ are complex functions and $K(\theta, \theta')$ is a complex kernel. This is the continuum version of a matrix equation for u , a viewpoint that soon becomes very useful for our purposes. In an abstract notation, we can write this as

$$(\mathbf{1} + \mathbf{K})u = f, \quad (3.2)$$

where $\mathbf{1}$ is the identity and $\mathbf{K}$ is an appropriate integral operator. Now, we can ask whether such an equation has a solution, i.e., what is the determinant of the appearing $\mathbf{1} + \mathbf{K}$ transformation. However, since it is not a finite-dimensional object, this calculation raises a few obstacles. To surpass these, the strategy is the following: we discretize Eq. (3.1) by splitting the $[a, b]$ interval into N equal pieces, calculate the determinant in this finite space and then take the $N \rightarrow \infty$ limit. After the discretization, we get the following matrix equation:

$$\left(\sum_{j=1}^N \delta_{ij} + h \sum_{j=1}^N K(\theta_i, \theta_j) \right) u(\theta_j) = f(\theta_i), \quad (3.3)$$

where $h = (b - a)/N$. In this case, we can approximate the desired determinant with

$$D(h) = \det(\delta_{ij} + hK(\theta_i, \theta_j)). \quad (3.4)$$

As described above, we expect that $D(h) \rightarrow \det(\mathbf{1} + \mathbf{K})$ as $h \rightarrow 0$.

Next, we can proceed by expressing $D(h)$ as a polynomial in h :

$$D(h) = \sum_{m=0}^N a_m h^m. \quad (3.5)$$

It turns out that the a_m coefficients can be expressed as subdeterminants of the matrix $K_{ij} \equiv K(\theta_i, \theta_j)$ with the following identity:

$$D(h) = 1 + h \sum_{i=1}^N |K_{ii}| + \frac{h^2}{2} \sum_{i,j=1}^N \begin{vmatrix} K_{ii} & K_{ij} \\ K_{ji} & K_{jj} \end{vmatrix} + \frac{h^3}{6} \sum_{i,j,k=1}^N \begin{vmatrix} K_{ii} & K_{ij} & K_{ik} \\ K_{ji} & K_{jj} & K_{jk} \\ K_{ki} & K_{kj} & K_{kk} \end{vmatrix} + \dots \quad (3.6)$$

Now, we can easily take the $h \rightarrow 0$ limit of this expansion to arrive at

$$\det(\mathbf{1} + \mathbf{K}) = \sum_{k=0}^{\infty} \int_a^b \frac{d\theta_1 \dots d\theta_k}{k!} \det(K(\theta_i, \theta_j))_{i,j=1}^k. \quad (3.7)$$

This can be thought of as the definition of a Fredholm determinant. Notice the similarity between this result and the zero temperature form factor expressions of Eqs. (2.18) and (2.19). We could arrive at a Fredholm determinant representation of the correlators if we could find such kernel functions that generate the form factors as their subdeterminants. In the next section, we will do precisely that, but first, we have to generalize the theory a bit further until it becomes applicable to finite temperatures.

The Fredholm determinant representations differ between zero and finite temperatures due to an extra summation for the ϵ signs in Eqs. (2.22) and (2.23). However, we can adapt our previous formalism to this case too, if we slightly modify the initial integral equation:

$$u_{\epsilon}(\theta) + \sum_{\epsilon'=\pm} \int_a^b d\theta' K_{\epsilon,\epsilon'}(\theta, \theta') u_{\epsilon'}(\theta') = f_{\epsilon}(\theta). \quad (3.8)$$

It is reminiscent of a matrix equation of size $2N$, so it is natural to introduce the notations $\tilde{u} = (\{u_+(\theta)\}, \{u_-(\theta)\})$, $\tilde{f} = (\{f_+(\theta)\}, \{f_-(\theta)\})$, $\xi = (\{\theta_i\}, \{\theta_i\})$ and

$$\tilde{K} = \begin{pmatrix} \{K_{++}(\theta_i, \theta_j)\} & \{K_{+-}(\theta_i, \theta_j)\} \\ \{K_{-+}(\theta_i, \theta_j)\} & \{K_{--}(\theta_i, \theta_j)\} \end{pmatrix}. \quad (3.9)$$

In this language, the discretized equation becomes

$$\tilde{u}(\xi_i) + h \sum_{j=1}^{2N} \tilde{K}(\xi_i, \xi_j) \tilde{u}(\xi_j) = \tilde{f}(\xi_i), \quad (3.10)$$

which means that we can apply Eq. (3.6) to get the following approximation for the Fredholm determinant:

$$D(h) = 1 + h \sum_{i=1}^{2N} |\tilde{K}_{ii}| + \frac{h^2}{2} \sum_{i,j=1}^{2N} \begin{vmatrix} \tilde{K}_{ii} & \tilde{K}_{ij} \\ \tilde{K}_{ji} & \tilde{K}_{jj} \end{vmatrix} + \dots \quad (3.11)$$

Now, using the fact that for a general g function

$$\sum_{i=1}^{2N} \tilde{g}(\xi_i) = \sum_{\epsilon=\pm} \sum_{i=1}^N g_{\epsilon}(\theta_i), \quad (3.12)$$

we get the following final result:

$$\det(\mathbf{1} + \tilde{\mathbf{K}}) = \sum_{k=0}^{\infty} \sum_{\epsilon_1, \dots, \epsilon_k = \pm} \int_a^b \frac{d\theta_1 \dots d\theta_k}{k!} \det(K_{\epsilon_i, \epsilon_j}(\theta_i, \theta_j))_{i,j=1}^k, \quad (3.13)$$

which has exactly the desired structure. As we will see in the next section, this result gives the foundations for the Fredholm determinant representation of the finite temperature correlation functions.

3.1. Representing the correlation functions

As previously noted, there is a striking similarity between the formalism of Fredholm determinants and form factor expansions. When we align the kernel functions with the form factors, we can discover Fredholm determinant representations for the correlation functions, which prove highly beneficial for numerical analysis. In this section, we carry out the computations necessary for this purpose.

First, let us consider the zero temperature case, as it has a much simpler formalism. We would like to give Fredholm determinant representations for Eqs. (2.18) and (2.19) using Eq. (3.7). Following [7], it is useful to introduce the

$$C_{\pm}(x, t) = C_f(x, t) \pm C_p(x, t) = m^{1/4} \sum_{k=0}^{\infty} \frac{(\pm 1)^k}{k!} \prod_{i=1}^k \left(\int_{-\infty}^{\infty} \frac{d\theta_i}{2\pi} e^{imx \sinh \theta_i - imt \cosh \theta_i} \right) \times \prod_{1 \leq i < j \leq k} \tanh^2 \left(\frac{\theta_i - \theta_j}{2} \right) \quad (3.14)$$

combinations, since it removes any restrictions from the domain of k . Here, however, we have to deal with a little complication. While this formula is perfectly well-defined, during a numerical analysis, it fails to converge since the exponential terms oscillate very fast for large rapidities. However, by extending the rapidities to the complex plane in a clever way, we can circumvent

these issues, although we have to use different contours for space-like and time-like separated (x, t) points. For space-like separations, a simple contour shift will suffice:

$$\Gamma : \mathbb{R} \rightarrow \mathbb{C} \quad \theta \rightarrow \theta + i\delta, \quad (3.15)$$

while for time-like separations we need to use an appropriate rapidity-dependent shift, e.g.

$$\Gamma : \mathbb{R} \rightarrow \mathbb{C} \quad \theta \rightarrow \theta - i\delta \tanh(\theta). \quad (3.16)$$

In both cases, δ is a sufficiently small positive constant. Implementing these contour shifts, we get the following result:

$$C_{\pm}(x, t) = m^{1/4} \sum_{k=0}^{\infty} \frac{(\pm 1)^k}{k!} \prod_{i=1}^k \left(\int_{-\infty}^{\infty} \frac{d\theta_i}{2\pi} e^{imx \sinh \Gamma(\theta_i) - imt \cosh \Gamma(\theta_i)} \Gamma'(\theta_i) \right) \times \prod_{1 \leq i < j \leq k} \tanh^2 \left(\frac{\Gamma(\theta_i) - \Gamma(\theta_j)}{2} \right), \quad (3.17)$$

where the Γ function is given by Eq. (3.15) outside, and by Eq. (3.16) inside the light cone. Note that in the first case, the derivative term simply gives unity, since the contour is just a constant shift.

Now, we are ready to tackle the task of giving a Fredholm determinant representation for Eq. (3.17). The heart of this calculation is the determinant identity

$$\prod_{1 \leq i < j \leq k} \tanh^2 \left(\frac{\Gamma(\theta_i) - \Gamma(\theta_j)}{2} \right) = \det \left(\frac{2e^{\Gamma(\theta_i)}}{e^{\Gamma(\theta_i)} + e^{\Gamma(\theta_j)}} \right)_{i,j=1}^k. \quad (3.18)$$

To prove this, we just need to introduce the $u_i = e^{\Gamma(\theta_i)}$ variables and notice that the resulting

$$\prod_{1 \leq i < j \leq k} \left(\frac{u_i - u_j}{u_i + u_j} \right)^2 = \det \left(\frac{2u_i}{u_i + u_j} \right)_{i,j=1}^k \quad (3.19)$$

expression is trivially true by the famous Cauchy determinant identity

$$\det \left(\frac{1}{x_i + y_j} \right)_{i,j=1}^k = \frac{\prod_{1 \leq i < j \leq k} (x_j - x_i)(y_j - y_i)}{\prod_{1 \leq i, j \leq k} (x_i + y_j)} \quad (3.20)$$

applied for the special case $x_i = y_i = u_i$. If we combine Eqs. (3.18), (3.17) and (3.7), we arrive at the Fredholm determinant representations

$$C_{\pm}(x, t) = m^{1/4} \det \left(\mathbf{1} + \mathbf{K}_{x,t}^{\pm} \right), \quad (3.21)$$

where the kernels are given by

$$K_{x,t}^{\pm}(\theta, \theta') = \frac{\pm 1}{2\pi} e^{imx \sinh \Gamma(\theta) - imt \cosh \Gamma(\theta)} \Gamma'(\theta) \frac{2e^{\Gamma(\theta)}}{e^{\Gamma(\theta)} + e^{\Gamma(\theta')}}. \quad (3.22)$$

Finally, using these results, we can express the ferromagnetic and paramagnetic correlation functions in the following way:

$$\begin{aligned} C_f(x, t) &= \frac{1}{2} m^{1/4} \left(\det(\mathbf{1} + \mathbf{K}_{x,t}^+) + \det(\mathbf{1} + \mathbf{K}_{x,t}^-) \right), \\ C_p(x, t) &= \frac{1}{2} m^{1/4} \left(\det(\mathbf{1} + \mathbf{K}_{x,t}^+) - \det(\mathbf{1} + \mathbf{K}_{x,t}^-) \right). \end{aligned} \quad (3.23)$$

Eqs. (3.23) and (3.22), complemented by (3.15) and (3.16), are the most important results of this chapter regarding the zero temperature case.

The remaining step is to develop analogous results for finite temperatures. It is a similar but more tedious calculation, therefore I give the details in Appendix A and here I only summarize the main results. In conclusion, the finite temperature correlation functions can be expressed as

$$\begin{aligned} C_f(x, t) &= \pi C(\beta)^2 e^{\Delta \mathcal{E} x} m^{1/4} \left(\det(\mathbf{1} + \tilde{\mathbf{K}}_{x,t;\beta}^+) + \det(\mathbf{1} + \tilde{\mathbf{K}}_{x,t;\beta}^-) \right), \\ C_p(x, t) &= \pi C(\beta)^2 e^{\Delta \mathcal{E} x} m^{1/4} \left(\det(\mathbf{1} + \tilde{\mathbf{K}}_{x,t;\beta}^+) - \det(\mathbf{1} + \tilde{\mathbf{K}}_{x,t;\beta}^-) \right), \end{aligned} \quad (3.24)$$

where the operators $\tilde{\mathbf{K}}_{x,t;\beta}^{\pm}$ are given by the kernel

$$K_{\epsilon, \epsilon' | x, t; \beta}^{\pm}(\theta, \theta') = \frac{\pm e^{im\epsilon(x \sinh(\theta + i\tilde{\epsilon}) - t \cosh(\theta + i\tilde{\epsilon})) + \epsilon\eta(\theta + i\tilde{\epsilon})}}{2\pi(1 - e^{-\epsilon m \beta \cosh(\theta + i\tilde{\epsilon})})} \left(\frac{2e^{\theta + i\tilde{\epsilon}}}{\epsilon e^{\theta + i\tilde{\epsilon}} + \epsilon' e^{\theta' + i\tilde{\epsilon}'}} \right) \quad (3.25)$$

with the shorthand notation $\tilde{\epsilon} = \epsilon\delta$. These results are equivalent to that presented in [3]. Furthermore, as mentioned earlier, they are only applicable in the space-like separated regime. The main challenge in generalizing to the time-like domain is that, unlike the zero-temperature scenario, the contour prescription provided in Eq. (3.16) does not work due to the presence of kinematic poles in the form factors.

4. Numerical Simulations

In this section, I present my numerical findings on the correlation functions. First, I explain my algorithm for calculating Fredholm determinants, which is applicable for both zero and finite temperatures. Then, I compare my results with known analytical predictions. For simplicity, I use the $m = 1$ convention in my figures but leave it in the equations for theoretical completeness.

4.1. Evaluating Fredholm determinants

Following [8], Eq. (3.4) lays the foundations of my numerical algorithm. It suggests that as the discretization of the integral operators gets finer and finer, $h \rightarrow 0$, the discretized determinant gets closer and closer to the Fredholm determinant, $D(h) \rightarrow \det(\mathbf{1} + \mathbf{K})$. Therefore, we can evaluate Fredholm determinants by calculating $D(h)$ for a sufficiently small h .

The kernels in Eqs. (3.22) and (3.25) contain small, perhaps infinitesimal complex shifts, which could lead to numerical instabilities. However, it is possible to show that we can choose these parameters to be finite as long we remain in specific domains. For zero temperature, a simple asymptotic analysis yields that the condition for convergence is $\delta \in (0, \pi)$ for both space-like and time-like separations. For finite temperatures, the argument is more involved and mostly based on the techniques presented in [5]; it turns out that here, the condition is $\delta \in (0, \pi/2)$. These results impose additional freedoms for the choice of kernels. In practice, however, there is no substantial difference between the efficiency of these methods as long as δ is not too close to the boundary of the respective domain. Therefore, in all cases, I made the $\delta = \pi/4$, as it proved to be sufficient for all practical purposes.

It is also important that the integrals in the form factor representations run over the $\theta \in (-\infty, \infty)$ values, so, in principle, I have to discretize an infinitely large interval. However, due to the above-mentioned complex shifts, the kernels vanish exponentially in one of their arguments, which implies that I can reduce the integration domain to a sufficiently large finite symmetric interval $[-L, L]$ without causing any significant errors. Then, I can perform the discretization by separating this into M equal pieces, i.e. by using a discretization parameter $h = 2L/M$. Typical values of these parameters are $L = 20$ and $M = 1000$, but for specific choices of x, t and β , additional adjustments are necessary for sufficient convergence. In all cases, however, calculating the correlation functions at a specific point takes less than a minute.

4.2. Zero temperature results

First, let me analyze the zero temperature case based on Eqs. (3.23) and (3.22). It is important to realize that here, the correlation functions are Lorentz invariant, i.e., they only depend on $x^2 - t^2$. We can prove this using Eqs. (2.18) and (2.19) by shifting the θ_i integrals with a constant θ_0 that satisfies

$$\tanh \theta_0 = \min\{|x/t|, |t/x|\}. \quad (4.1)$$

Therefore, it is enough to consider the purely space-like and time-like correlators $C(x, 0)$ and $C(0, t)$, since then we can obtain them at any other space-time point by Lorentz invariance. In the following, I investigate these functions in both phases.

It is also important to note that in the space-like regime, the correlators are real functions. It is simply a consequence of Lorentz invariance and the $C(x, t) = C(x, -t)^*$ symmetry applied for $t = 0$. Therefore, the imaginary parts of these functions are only non-trivial for time-like separations.

4.2.1. Large-separation asymptotics

First, let me discuss the asymptotic behaviors of the correlators. As mentioned earlier, for space-like separations, the form factor expansions are large-distance and large-time expansions, so we can extract these asymptotics by calculating the first non-trivial terms in Eqs. (2.18) and (2.19). (Although such a general statement is not true for time-like separations, I conjectured that in the Ising model, it indeed holds.) In the paramagnetic phase, it is a one-dimensional integral that can be evaluated analytically using the identity

$$\int_{-\infty}^{\infty} \frac{d\theta}{2\pi} e^{imx \sinh \theta - imt \cosh \theta} = \frac{1}{\pi} K_0 \left(m \sqrt{x^2 - t^2} \right), \quad (4.2)$$

where K_0 is the zeroth modified Bessel function of the second kind. Therefore, the large-separation behavior of the paramagnetic correlators is given by

$$\begin{aligned} C_p(x, 0) &\xrightarrow{x \rightarrow \infty} \frac{m^{1/4}}{\pi} K_0(mx) \xrightarrow{x \rightarrow \infty} \frac{m^{1/4}}{\sqrt{2\pi}} \frac{e^{-mx}}{\sqrt{mx}}, \\ C_p(0, t) &\xrightarrow{t \rightarrow \infty} \frac{m^{1/4}}{\pi} K_0(imt) \xrightarrow{t \rightarrow \infty} \frac{m^{1/4}}{\sqrt{2\pi}} \frac{e^{-imt}}{\sqrt{imt}}. \end{aligned} \quad (4.3)$$

Calculating the ferromagnetic asymptotics is similar; they are given by the two-dimensional integral

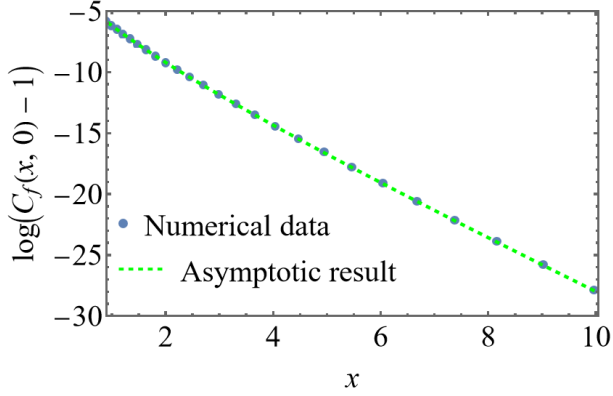
$$C_f(x, t) \rightarrow m^{1/4} \left[1 + \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\theta_1 d\theta_2}{(2\pi)^2} e^{imx(\sinh \theta_1 + \sinh \theta_2) - imt(\cosh \theta_1 + \cosh \theta_2)} \tanh^2 \left(\frac{\theta_1 - \theta_2}{2} \right) \right]. \quad (4.4)$$

Upon introducing the $u = \theta_1 + \theta_2$ and $v = \frac{1}{2}(\theta_1 - \theta_2)$ variables, we can carry out the u integral using Eq. (4.2). This calculation yields the following asymptotics:

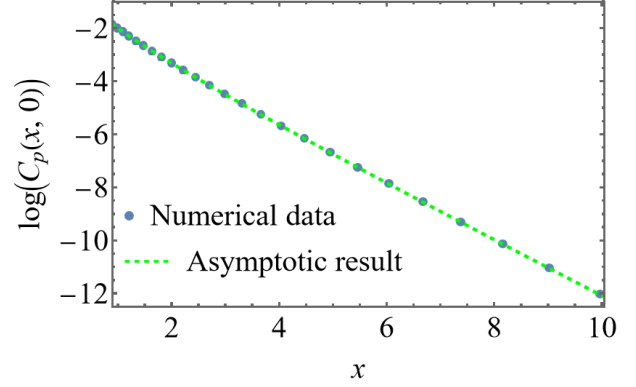
$$\begin{aligned} C_f(x, 0) &\xrightarrow{x \rightarrow \infty} m^{1/4} \left[1 + \frac{1}{2\pi^2} \int_{-\infty}^{\infty} dv K_0(2mx \cosh v) \tanh^2(v) \right], \\ C_f(0, t) &\xrightarrow{t \rightarrow \infty} m^{1/4} \left[1 + \frac{1}{2\pi^2} \int_{-\infty}^{\infty} dv K_0(2imt \cosh v) \tanh^2(v) \right]. \end{aligned} \quad (4.5)$$

These integrals are easy to handle numerically.

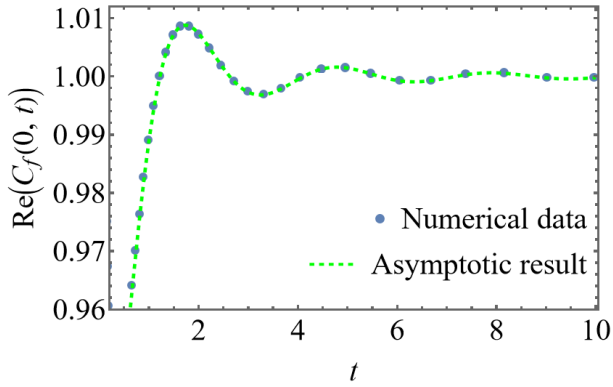
I compared the predictions of Eqs. (4.3) and (4.5) with my numerical simulations, the results are depicted in Figure 4.1. For space-like separations, the correlation functions are real, but for the time-like results, I had to consider the real and imaginary parts separately. As a result, Figure 4.1 contains six plots. In all cases, the results of the simulations follow the theory remarkably well. (This shows that it was reasonable to assume that the first terms in the form factors expansions give the time-like asymptotics.) Note that the real part of the ferromagnetic correlators approach one as a result of their vacuum expectation value (and the $m = 1$ convention).



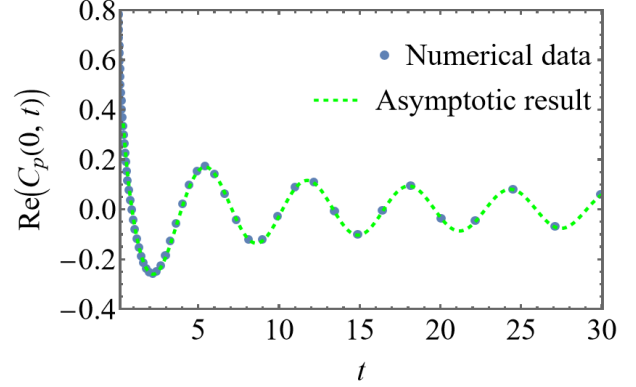
(a) Ferromagnetic phase, space-like separation.



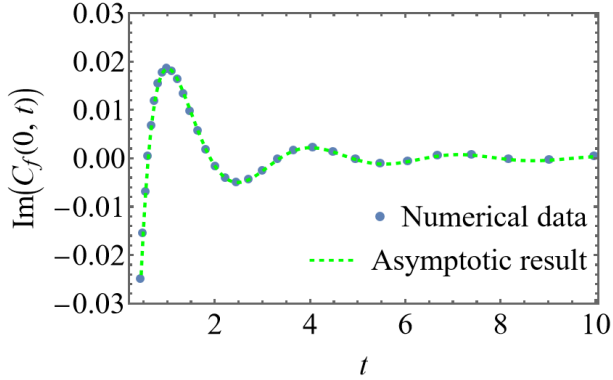
(b) Paramagnetic phase, space-like separation.



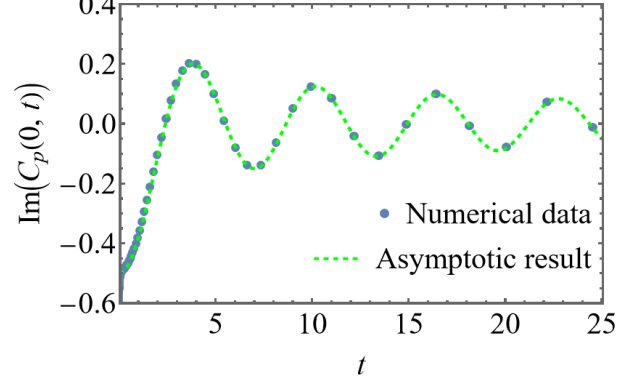
(c) Ferromagnetic phase, time-like separation, real part.



(d) Paramagnetic phase, time-like separation, real part.



(e) Ferromagnetic phase, time-like separation, imaginary part.



(f) Paramagnetic phase, time-like separation, imaginary part.

Figure 4.1: Asymptotic behaviors of the zero temperature correlation functions. The left (right) column corresponds to the ferromagnetic (paramagnetic) phases. For space-like separations, the correlators are real, but in the time-like regime, they have a non-trivial imaginary part. All six cases follow the theoretical asymptotics given by Eqs. (4.3) and (4.5) extremely well, even for relatively small separations.

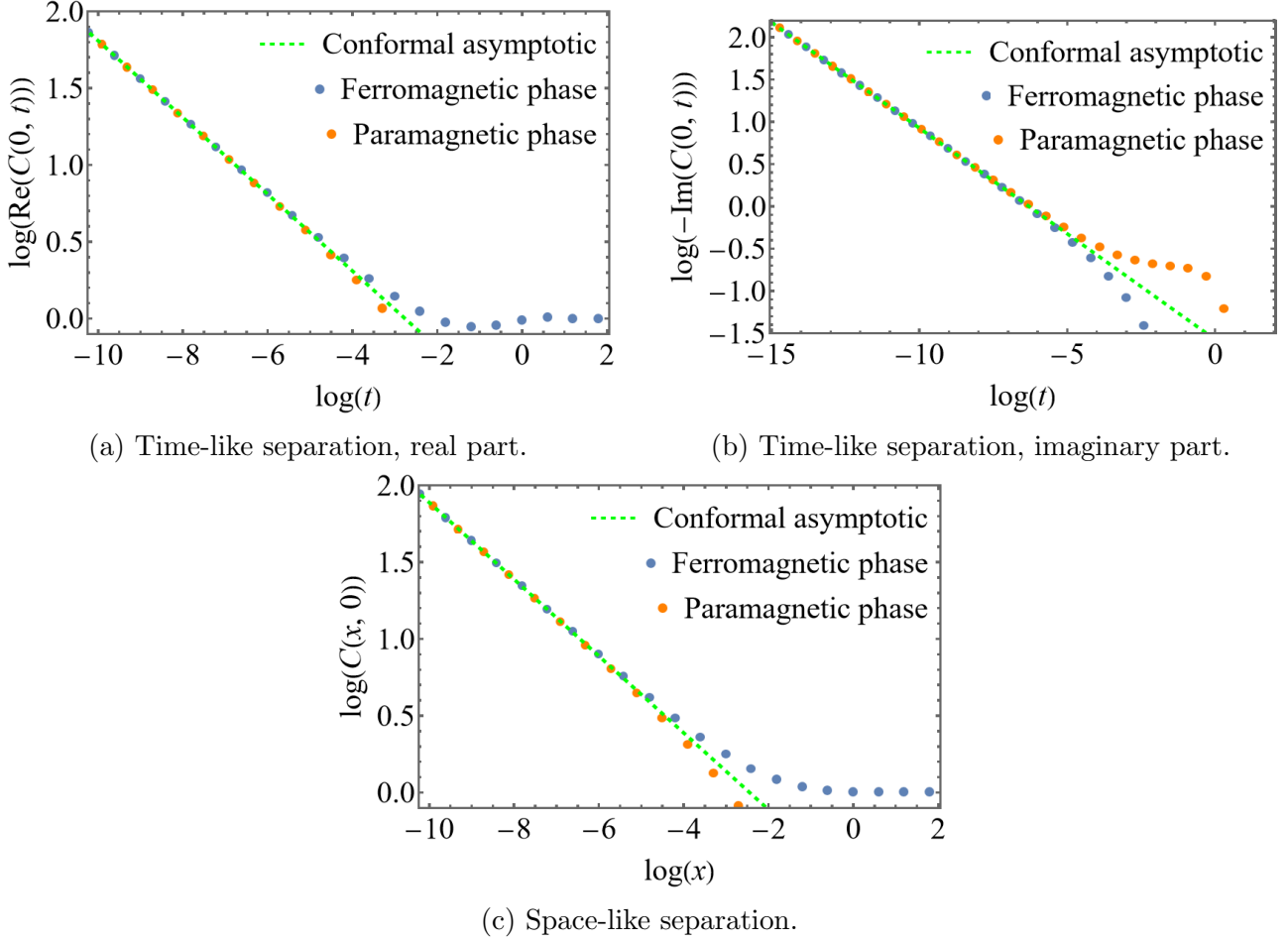


Figure 4.2: Zero temperature correlation functions for small separations. In the time-like domain, both the real and imaginary parts are important, while in the space-like regime, only the real part is non-vanishing. In all three cases, the paramagnetic and ferromagnetic curves collapse into the same function in agreement with Eq. (4.6).

4.2.2. Small separations

Next, let me discuss what happens in the $x \rightarrow 0$ and $t \rightarrow 0$ limits. Since small separations correspond to large energies, these correlators are basically unaffected by the mass gap of the system. This means that both the paramagnetic and ferromagnetic correlation functions tend to the conformal results [1]:

$$\begin{aligned} C_{p,f}(x, 0) &\xrightarrow{x \rightarrow 0} 2^{-1/6} e^{1/4} A^{-3} x^{-1/4}, \\ C_{p,f}(0, t) &\xrightarrow{t \rightarrow 0} 2^{-1/6} e^{1/4} A^{-3} (it)^{-1/4}, \end{aligned} \quad (4.6)$$

where $A \approx 1.282$ is the Glaisher constant. Note that these results imply a power law divergence near the origin and, as a result of Lorentz invariance, the light cone.

In Figure 4.2, I compared my numerical findings with the prediction of Eq. (4.6). I found that for very small separations, both the ferromagnetic and paramagnetic correlators followed the same linear function on the log-log plot, indicating that the effects of the mass gap are indeed negligible in this regime. These results coincided with the theoretical predictions as well.

4.2.3. Dynamical structure factors

To conclude the zero temperature discussion, I calculate the dynamical structure factors of the magnetization operator. These are defined as the Fourier transforms of the connected correlation functions:

$$S_{p,f}(\omega, q) = \int_{-\infty}^{\infty} dx dt e^{i\omega t - iqx} C_{p,f}^{(c)}(x, t), \quad (4.7)$$

where $C_p^{(c)}(x, t) = C_p(x, t)$ and $C_f(x, t) = C_f^{(c)}(x, t) - m^{1/4}$. As a result of Lorentz invariance, it is sufficient to consider the $q = 0$ case,

$$S_{p,f}(\omega, 0) = \int_{-\infty}^{\infty} dx dt e^{i\omega t} C_{p,f}^{(c)}(x, t), \quad (4.8)$$

since it contains all the information about the structure factors. Interestingly, this quantity can be calculated exactly for small energies [6]. Indeed, in the ferromagnetic phase

$$S_f(\omega, 0) = \frac{2m^{1/4}}{\omega^3} \sqrt{\omega^2 - 4m^2} \Theta(\omega - 2m) \quad (4.9)$$

for $\omega \leq 4m$, while in the paramagnetic phase

$$S_p(\omega, 0) = 2\pi m^{-3/4} \delta(\omega - m) \quad (4.10)$$

for $\omega \leq 3m$. Note that there is a factor of 2 discrepancy between Eq. (4.9) and Ref. [6], but my calculations in Appendix B suggest that this modification is necessary.

I wanted to check the validity of Eqs. (4.9) and (4.10), but in the paramagnetic case I could not obtain conclusive results, due to the Dirac delta contribution. However, in the ferromagnetic case, I was able to carry out the necessary numerical integrals. Figure 4.3 depicts my results near the $\omega = 2m$ threshold alongside the exact result coming from Eq. (4.9). The two curves match remarkably well, which confirms the necessity of the extra factor of 2 in the exact result.

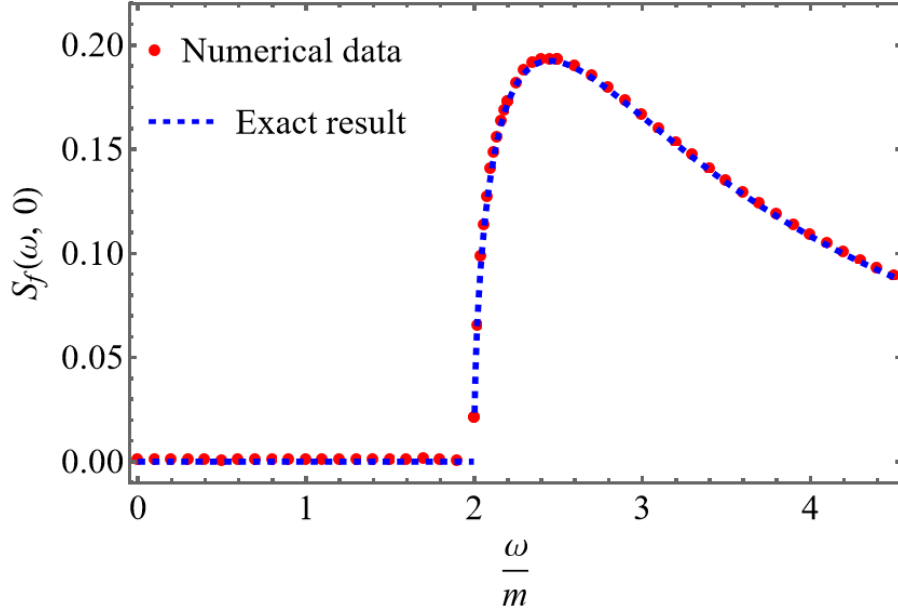


Figure 4.3: Zero temperature dynamical structure factors of the magnetization operator in the ferromagnetic case. To obtain the numerical data, Eq. (4.8) was evaluated in the $(x, t) \in [-100, 100]^2$ domain. The dashed line corresponds to the exact result given in Eq. (4.9).

4.3. Finite temperature results

After discussing the zero temperature behavior of the correlation functions, let me investigate the finite temperature case. Here, the correlators have an additional temperature dependence, and they are not Lorentz invariant anymore, which means that the results exhibit a much richer set of characteristics. In the following subsections, I examine these features. It is important to note that this discussion is limited to the space-like regime, and like the zero temperature case, these correlators are real functions.

4.3.1. Small separations

Similarly to the zero temperature case, small separations correspond to high energies, so the effects of both the mass gap and the temperature are negligible in this regime. This is expected to hold even if we consider the Lorentz invariant distance, i.e., the correlators are governed by the zero temperature conformal result [11]:

$$C_{p,f}(x, t; \beta) \xrightarrow{x \rightarrow t} 2^{-1/6} e^{1/4} A^{-3} \sqrt{x^2 - t^2}^{-1/4}. \quad (4.11)$$

First, I checked the validity of Eq. (4.11) for the static ($t = 0$) case in Figure 4.4. I plotted the correlators at $m\beta = 1$ inverse temperature in both phases alongside the conformal prediction.

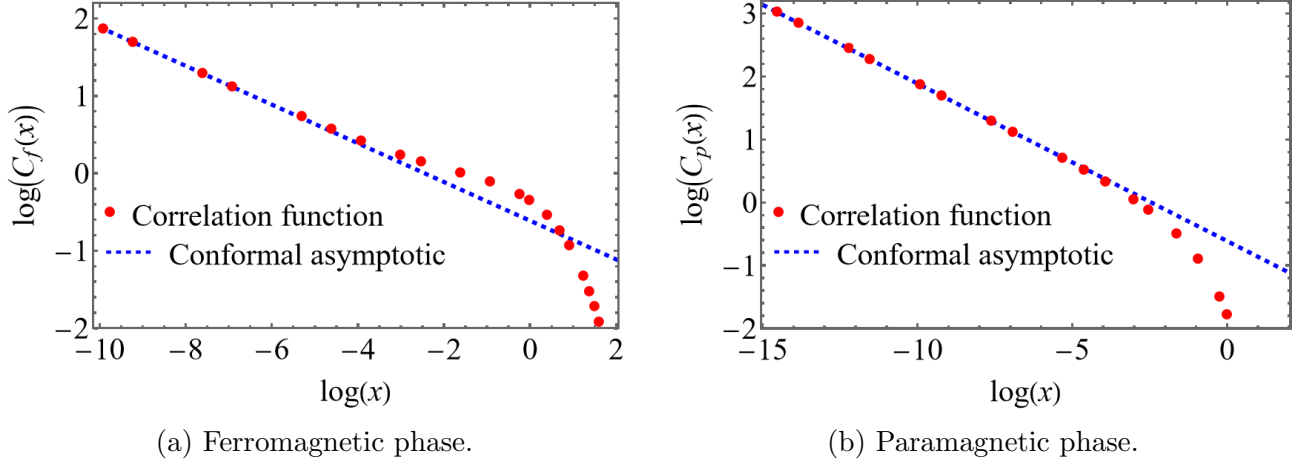


Figure 4.4: Small-distance behavior of the static correlation functions at $m\beta = 1$. In both phases, the results follow the conformal asymptotics perfectly.

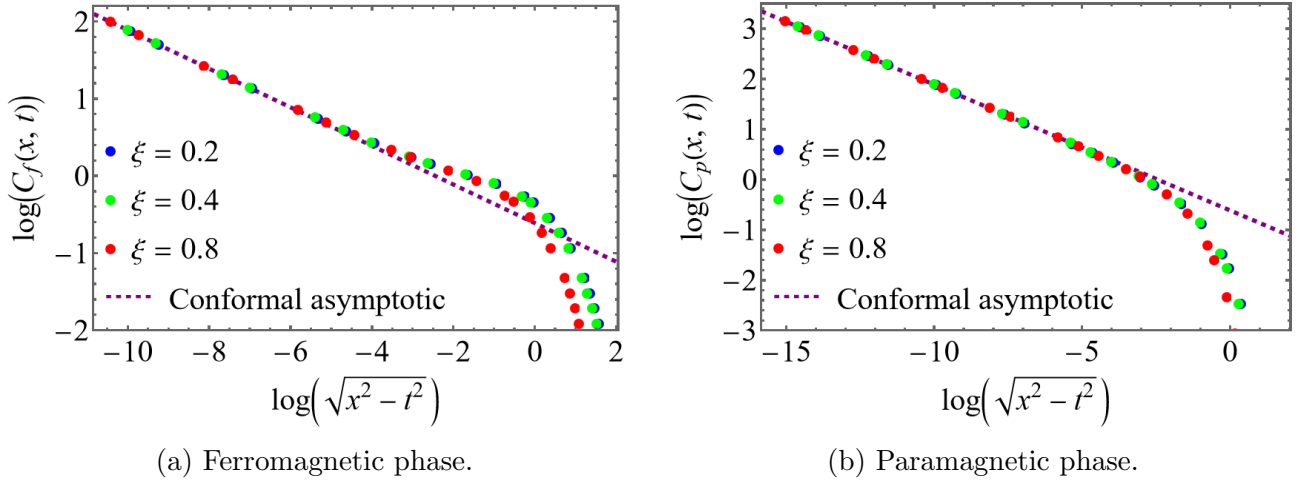


Figure 4.5: Dynamical correlation functions for different $\xi = t/x$ rays plotted against the Lorentz invariant separation from the origin. For small distances, the correlators are Lorentz invariant as predicted by the underlying conformal field theory, while for larger separations, this Lorentz invariance breaks. These simulations correspond to $m\beta = 1$.

The numerical data matched the theory perfectly, which was an important crosscheck for the validity of my finite temperature algorithm.

The conformal result also predicts that the correlators become Lorentz invariant for small separations. In Figure 4.5, I checked this by plotting multiple dynamical correlation functions corresponding to different

$$\xi = t/x \quad (4.12)$$

space-time rays against the Lorentz invariant distance from the origin. All curves confirm Eq. (4.11): they follow the same asymptotics for small separations. It is also noteworthy that for large distances, this Lorentz invariance breaks, and the space-time dependence of the correlators becomes more complicated.

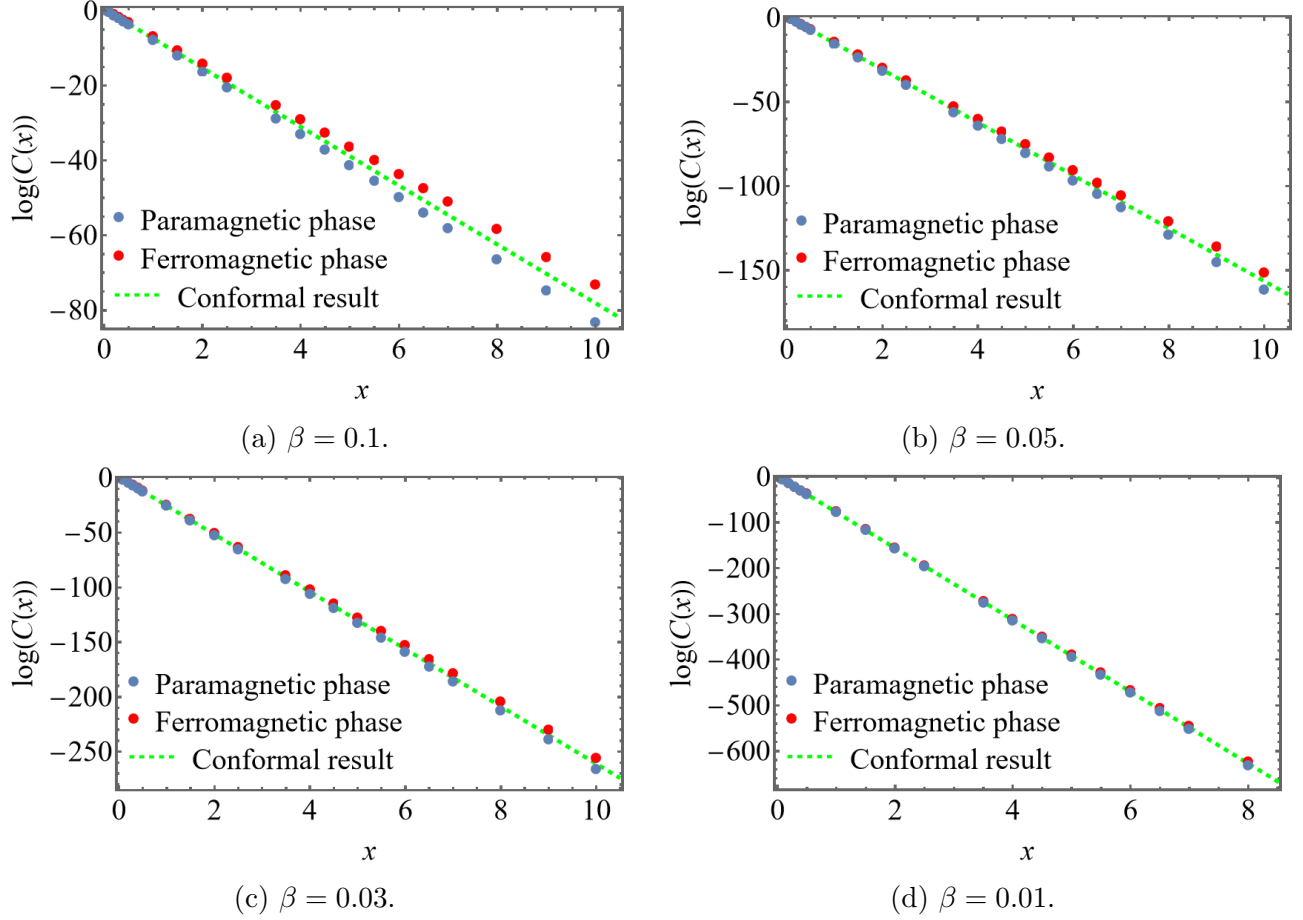


Figure 4.6: Static correlation functions at high temperatures. The dashed lines represent the conformal asymptotics. In both phases, as the temperature increases, the correlators tend toward this curve.

4.3.2. High temperatures

Next, I investigate what happens to the correlators at high temperatures. I restrict my discussion to the static case because the effects of additional temporal separations are completely negligible in this case (c.f. Section 4.3.4). Once again, when the temperature is high, the effects of the mass gap turn out to be negligible, and hence, the correlators follow

the finite temperature conformal result [2, 13]:

$$C_{p,f}(x, 0; \beta) \xrightarrow{m\beta \rightarrow 0} 2^{-1/6} e^{1/4} A^{-3} \left[\frac{\beta}{\pi} \sinh \left(\frac{\pi x}{\beta} \right) \right]^{-1/4}. \quad (4.13)$$

In Figure 4.6, I depicted the correlation functions at $m\beta = 0.1, 0.05, 0.03$ and 0.01 temperatures alongside the prediction of Eq. (4.13). As we can see, even at $m\beta = 0.1$, the correlators are very close to the conformal result, and as the temperature increases, this approximation gets more and more accurate.

4.3.3. Large separations in the static case

Now, I discuss the large-separation behavior of the correlation functions. In this subsection, I focus on the static case, where there are precise theoretical results for these asymptotics [11] that turn out to be purely exponential decays:

$$C(x, 0; \beta) \xrightarrow{x \rightarrow \infty} m^{1/4} \Omega(m\beta) \exp(-mx/\Gamma(m\beta)), \quad (4.14)$$

but the correlation length Γ and the constant prefactor Ω are different in the ferromagnetic and paramagnetic phases. Upon introducing the

$$f(s) = \int_0^\infty \frac{dy}{\pi} \log \coth \left(\frac{1}{2} \sqrt{s^2 + y^2} \right) + |s| \theta(-s) \quad (4.15)$$

and

$$g(s) = \exp \left[\int_s^1 \frac{dy}{y} \left(f'(y)^2 - \frac{1}{4} \right) + \int_1^\infty \frac{dy}{y} f'(y)^2 \right] \quad (4.16)$$

functions, $\Gamma(m\beta)$ and $\Omega(m\beta)$ can be calculated as

$$\Gamma(m\beta)^{-1} = \frac{1}{m\beta} f(m\beta), \quad \Omega(m\beta) = (m\beta)^{-1/4} g(m\beta) \quad (4.17)$$

and

$$\Gamma(m\beta)^{-1} = \frac{1}{m\beta} f(-m\beta), \quad \Omega(m\beta) = (m\beta)^{-1/4} g(-m\beta) \quad (4.18)$$

in the ferromagnetic and paramagnetic phases respectively. Note that Eq. (4.14) seemingly contradict Eqs. (4.3) and (4.5), but it is not the case since the $m\beta \rightarrow \infty$ and $x \rightarrow \infty$ limits do not commute.

In Figure 4.7, I depicted the asymptotic behaviors of the static correlators for three different temperatures with the predictions of Eq. (4.14). In all cases, my results matched the theoretical expectations perfectly.

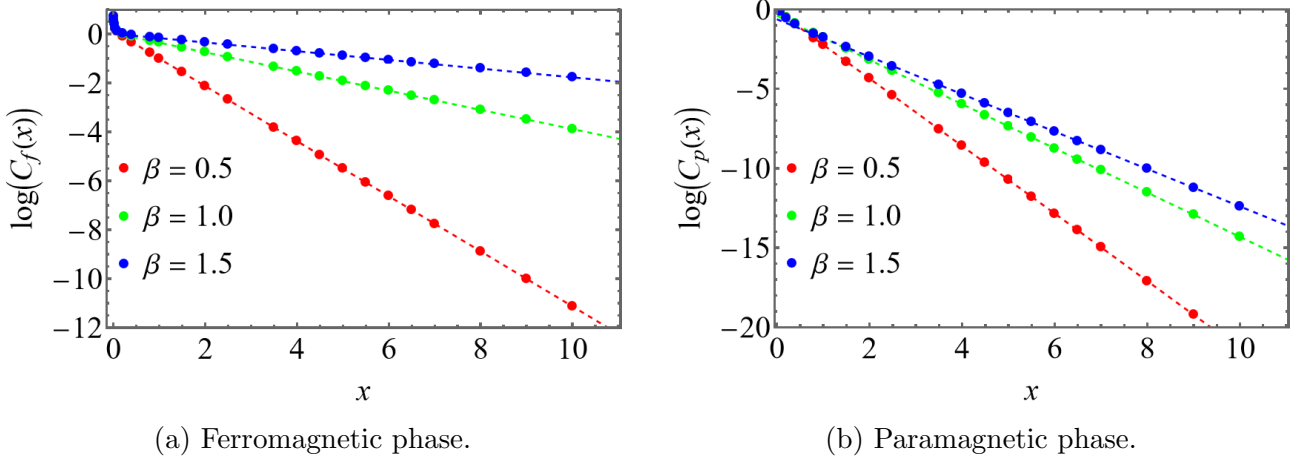


Figure 4.7: Asymptotic behavior of static correlation functions at three different temperatures. The dashed lines represent the analytical results. We see that our results are in perfect agreement with the theoretical expectations.

4.3.4. Large separations in the dynamical case

Finally, I consider the large-separation behavior of the dynamical correlation functions. I focus on the $x \rightarrow \infty$ limit for a fixed $\xi = t/x$ ray. This problem is only interesting in the paramagnetic phase since, in the ferromagnetic regime, the asymptotic behavior is independent of ξ [4, 14]. My simulations confirmed this result: for any reasonable choices of β , changing the value of ξ did not change the numerical value of the correlators at large distances. Therefore, I only consider the paramagnetic case. Here, however, the asymptotic behaviors have a non-trivial ξ -dependence. In fact, to my knowledge, there are no complete theoretical results for this dependence. Furthermore, previous works [4, 14] that performed low-temperature calculations provide results that contradict my numerical findings. To settle this discrepancy, I conduct a novel asymptotic analysis and prove that my simulations are consistent with this calculation.

Since Eq. (2.23) is a large-distance and large-time expansion, the asymptotic behavior can be obtained by its first term. This implies that for large separations, the paramagnetic correlation functions can be calculated as

$$C_p(x, t; \beta) \xrightarrow{x, t \rightarrow \infty} -2\pi C(\beta)^2 e^{\Delta \mathcal{E} x} m^{1/4} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \left(\frac{e^{im(x \sinh(\theta+i\delta) - t \cosh(\theta+i\delta))}}{e^{-m\beta \cosh(\theta+i\delta)} - 1} e^{\eta(\theta+i\delta)} + \frac{e^{-im(x \sinh(\theta-i\delta) - t \cosh(\theta-i\delta))}}{1 - e^{m\beta \cosh(\theta-i\delta)}} e^{-\eta(\theta-i\delta)} \right). \quad (4.19)$$

Next, I perform the asymptotic analysis of Eq. (4.19) using the techniques discussed in [5].

First, I write the two separate integrals as a single complex contour integral:

$$C_p(x, t; \beta) \xrightarrow{x, t \rightarrow \infty} -2\pi C(\beta)^2 e^{\Delta \mathcal{E} x} m^{1/4} \int_C \frac{d\theta}{2\pi} \frac{e^{im(x \sinh(\theta) - t \cosh(\theta)) + \eta(\theta)}}{e^{-m\beta \cosh(\theta)} - 1}, \quad (4.20)$$

where the contour C is a counterclockwise curve consisting of the $\text{Im}(\theta) = \delta$ and $\text{Im}(\theta) = \pi - \delta$ lines. The poles of the integrand are given by

$$\theta_k = \sinh^{-1} \left(\frac{2\pi}{m\beta} k \right) + i\pi/2, \quad (4.21)$$

$k \in \mathbb{Z}$, i.e., they all have a $\pi/2$ imaginary part. Therefore I use the residue theorem,

$$\int_C \frac{d\theta}{2\pi} \frac{e^{im(x \sinh(\theta) - t \cosh(\theta)) + \eta(\theta)}}{e^{-m\beta \cosh(\theta)} - 1} = 2\pi i \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} \frac{e^{im(x \sinh(\theta_k) - t \cosh(\theta_k)) + \eta(\theta_k)}}{-m\beta \sinh(\theta_k) e^{-m\beta \cosh \theta_k}}, \quad (4.22)$$

to obtain a more convenient result for the asymptotic behavior of the paramagnetic correlation functions:

$$C_p(x, t; \beta) \xrightarrow{x, t \rightarrow \infty} 2\pi C(\beta)^2 e^{\Delta \mathcal{E} x} m^{1/4} \sum_{k=-\infty}^{\infty} \frac{\exp \left[-mx \sqrt{1 + \frac{4\pi^2}{m^2 \beta^2} k^2} + t \frac{2\pi}{\beta} k \right] e^{\eta(\sinh^{-1}(\frac{2\pi}{m\beta} k) + i\pi/2)}}{m\beta \sqrt{1 + \frac{4\pi^2}{m^2 \beta^2} k^2}}. \quad (4.23)$$

Now it is elementary to see that the asymptotic behavior for a given ξ and β is

$$C_p(x, t; \beta) \xrightarrow{x, t \rightarrow \infty} 2\pi C(\beta)^2 e^{\Delta \mathcal{E} x} m^{1/4} \frac{\exp \left[-mx \sqrt{1 + \frac{4\pi^2}{m^2 \beta^2} k_0^2} + t \frac{2\pi}{\beta} k_0 \right] e^{\eta(\sinh^{-1}(\frac{2\pi}{m\beta} k_0) + i\pi/2)}}{m\beta \sqrt{1 + \frac{4\pi^2}{m^2 \beta^2} k_0^2}}, \quad (4.24)$$

where

$$k_0 = \underset{k \in \mathbb{Z}}{\text{argmin}} \left[\sqrt{1 + \frac{4\pi^2}{m^2 \beta^2} k^2} - \xi \frac{2\pi}{m\beta} k \right]. \quad (4.25)$$

This means that the correlation length given by

$$\zeta(\xi; \beta) = \left[\sqrt{m^2 + \frac{4\pi^2}{\beta^2} k_0^2} - \xi \frac{2\pi}{\beta} k_0 - \Delta \mathcal{E}(\beta) \right]^{-1} \quad (4.26)$$

is not an analytical function of ξ and β : it is continuous, but at specific parameters it has cusps. To demonstrate this, I plotted ζ as a function of ξ for $\beta = 5$ and as a function of β for $\xi = 0.5$. These are depicted in Figure 4.8.

Let me now check the validity of Eq. (4.26) using my numerical algorithm. I calculated the paramagnetic correlation functions at temperatures $\beta = 5$ and $\beta = 1$ for four different

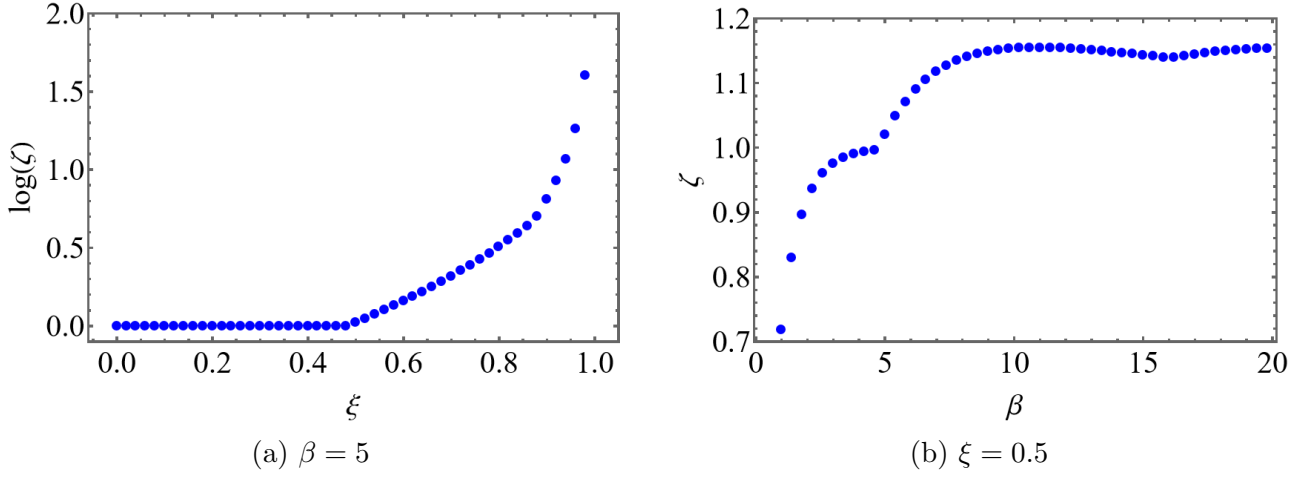


Figure 4.8: Theoretical result for the correlation length of the paramagnetic correlation functions as a function of ξ and β . The left figure corresponds to a constant inverse temperature, $\beta = 5$, while the right one for a specific space-time, $\xi = 0.5$. We see that the functions have cusps corresponding to a change in the k_0 value of Eq. (4.25).

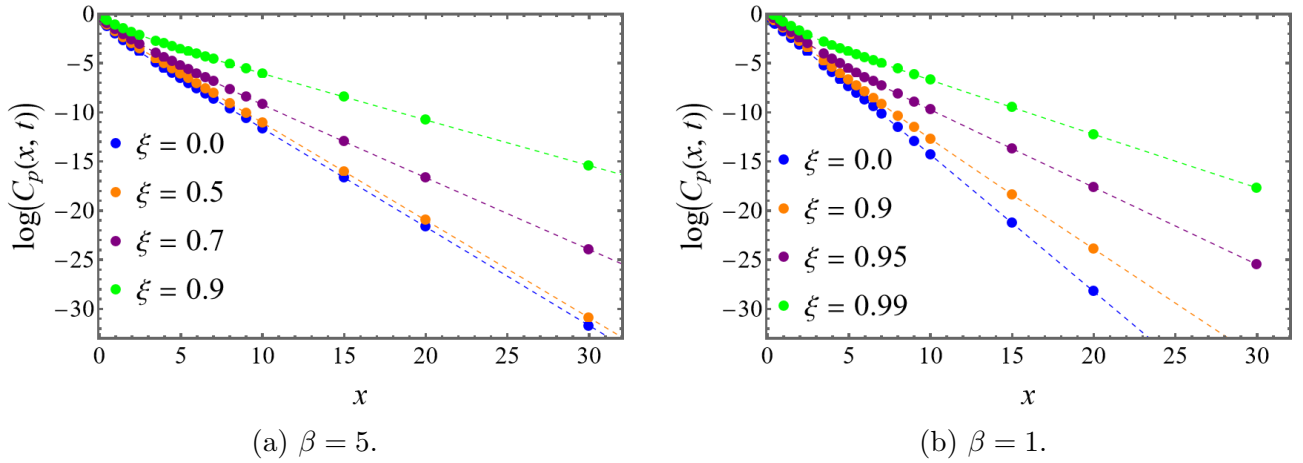


Figure 4.9: Numerical results for the dynamical correlation functions in the paramagnetic phase at temperatures $\beta = 5$ and $\beta = 1$. Different colors indicate different $\xi = t/x$ rays. All of the correlation functions follow the dashed asymptotics coming from Eq. (4.26). Note that as the temperature increases, the behavior of the correlators is governed more and more by the static case.

rays; these results are depicted in Figure 4.9. We can see that the simulated data matches the theoretical results perfectly. Note that at $\beta = 5$, the asymptotic correlators differ from the static ($\xi = 0$) curve only above $\xi \approx 0.5$, in harmony with the first cusp in Figure 4.8a, while for $\beta = 1$, this threshold increases to $\xi \approx 0.85$. This is a general feature of Eq. (4.26): the effects

of a temporal separation are only significant at low temperatures; at high temperatures, only those regions are affected that are close to the light cone.

As a final note, Eq. (4.26) highlights the discrepancy between my results and that obtained in [14] or in the recent work [4]. Their formulas always contain a square root in the denominator, reminiscent of that in Eq. (4.3), which is absent in my case. The issue probably comes from the non-commutativity of the $m\beta \rightarrow \infty$ and $x \rightarrow \infty$ limits, although the exact details are not yet understood.

5. Summary and outlook

In my work, I studied the zero and finite temperature dynamical correlation functions of the magnetization operator in the one-dimensional Ising quantum field theory. My approach was based on the Lehmann expansion and the simple structure of the relevant form factors. These allowed for a Fredholm determinant representation of the correlators that I could evaluate numerically. This technique worked exceptionally well for a wide range of parameters; the only exception was the time-like regime in the finite temperature case.

With my numerical algorithm, I explored various intriguing characteristics of the correlation functions. I placed particular emphasis on small and large space-time separations since, in these cases, several analytical predictions were available. In addition, I derived a novel asymptotic result for the large-space and large-time behavior in the paramagnetic phase. I was also able to calculate the zero temperature dynamical structure factors in the ferromagnetic regime. In the future, it would be interesting to extend the finite temperature approach to the time-like domain as well to obtain a complete description of these correlators.

A Finite temperature kernel functions

In this appendix, I derive Eqs. (3.24) and (3.25) following [3] and [7]. First, I introduce the finite temperature analogy of Eq. (3.14) using Eqs. (2.22), (2.23) and (2.26):

$$C_{\pm}(x, t; \beta) = 2\pi C(\beta)^2 e^{\Delta \mathcal{E} x} m^{1/4} \sum_{k=0}^{\infty} \frac{(\pm 1)^k}{k!} \sum_{\epsilon_1, \dots, \epsilon_k = \pm} \int_{-\infty + i\epsilon_j \delta}^{\infty + i\epsilon_j \delta} \frac{d\theta_1 \dots d\theta_k}{(2\pi)^k} \times \prod_{i=1}^k \left(\frac{e^{im\epsilon_i(x \sinh \theta_i - t \cosh \theta_i)}}{1 - e^{-\epsilon_i m \beta \cosh \theta_i}} \epsilon_i e^{\epsilon_i \eta(\theta_i)} \right) \prod_{1 \leq i < j \leq k} \tanh \left(\frac{\theta_j - \theta_i}{2} \right)^{2\epsilon_i \epsilon_j}. \quad (\text{A.1})$$

Introducing the $u_i = e^{\theta_i}$ variables, the form factor term can be expressed as a determinant:

$$\prod_{1 \leq i < j \leq k} \tanh \left(\frac{\theta_j - \theta_i}{2} \right)^{2\epsilon_i \epsilon_j} = \prod_{1 \leq i < j \leq k} \left(\frac{u_i - u_j}{u_i + u_j} \right)^{2\epsilon_i \epsilon_j} = \prod_{1 \leq i < j \leq k} \left(\frac{\epsilon_i u_i - \epsilon_j u_j}{\epsilon_i u_i + \epsilon_j u_j} \right)^2 = \det \left(\frac{2\epsilon_i u_i}{\epsilon_i u_i + \epsilon_j u_j} \right)_{i,j=1}^k = \det \left(\frac{2\epsilon_i e^{\theta_i}}{\epsilon_i e^{\theta_i} + \epsilon_j e^{\theta_j}} \right)_{i,j=1}^k, \quad (\text{A.2})$$

where the third equality holds because it holds for all four $\epsilon_{i,j} = \pm 1$ combinations, and the fourth one is the result of the Cauchy identity given in Eq. (3.20).

Upon substituting Eq. (A.2) into Eq. (A.1) and introducing the new variables $\tilde{\theta}_i = \theta_i - i\epsilon_i \delta$, the correlators become

$$C_{\pm}(x, t; \beta) = 2\pi C(\beta)^2 e^{\Delta \mathcal{E} x} m^{1/4} \sum_{k=0}^{\infty} \frac{(\pm 1)^k}{k!} \sum_{\epsilon_1, \dots, \epsilon_k = \pm} \int_{-\infty}^{\infty} \frac{d\tilde{\theta}_1 \dots d\tilde{\theta}_k}{(2\pi)^k} \times \prod_{i=1}^k \left(\frac{e^{im\epsilon_i(x \sinh(\tilde{\theta}_i + i\epsilon_i \delta) - t \cosh(\tilde{\theta}_i + i\epsilon_i \delta))}}{1 - e^{-\epsilon_i m \beta \cosh(\tilde{\theta}_i + i\epsilon_i \delta)}} \epsilon_i e^{\epsilon_i \eta(\tilde{\theta}_i + i\epsilon_i \delta)} \right) \det \left(\frac{2\epsilon_i e^{\tilde{\theta}_i + i\epsilon_i \delta}}{\epsilon_i e^{\tilde{\theta}_i + i\epsilon_i \delta} + \epsilon_j e^{\tilde{\theta}_j + i\epsilon_j \delta}} \right)_{i,j=1}^k. \quad (\text{A.3})$$

Then, using Eq. (3.13), we see that

$$C_{\pm}(x, t; \beta) = 2\pi C(\beta)^2 e^{\Delta \mathcal{E} x} m^{1/4} \det \left(\mathbf{1} + \tilde{\mathbf{K}}_{x,t;\beta}^{\pm} \right), \quad (\text{A.4})$$

where the kernels are given by

$$K_{\epsilon, \epsilon' | x, t; \beta}^{\pm}(\theta, \theta') = \frac{\pm \epsilon e^{im\epsilon(x \sinh(\theta + i\epsilon \delta) - t \cosh(\theta + i\epsilon \delta)) + \epsilon \eta(\theta + i\epsilon \delta)}}{2\pi(1 - e^{-\epsilon m \beta \cosh(\theta + i\epsilon \delta)})} \left(\frac{2\epsilon e^{\theta + i\epsilon \delta}}{\epsilon e^{\theta + i\epsilon \delta} + \epsilon' e^{\theta' + i\epsilon' \delta}} \right). \quad (\text{A.5})$$

This is equivalent to Eq. (3.25). Furthermore, Eq. (3.24) now follows trivially if we express $C_p(x, t; \beta)$ and $C_f(x, t; \beta)$ from $C_{\pm}(x, t; \beta)$.

B Ferromagnetic structure factors

In this appendix, I compute the dynamical structure factors $S_f(\omega, 0)$ in the ferromagnetic case near the two-particle threshold, $\omega = 2m$. Using Eqs (2.18) and (4.7), this can be expressed as

$$S_f(\omega, 0) = m^{1/4} \int_{-\infty}^{\infty} dx dt e^{i\omega t} \frac{1}{2} \left(\int_{-\infty}^{\infty} \frac{d\theta_1 d\theta_2}{(2\pi)^2} e^{imx(\sinh \theta_1 + \sinh \theta_2) - imt(\cosh \theta_1 + \cosh \theta_2)} \right) \times \quad (\text{B.1})$$

$$\times \tanh^2 \left(\frac{\theta_1 - \theta_2}{2} \right).$$

We can evaluate these integrals since only the exponential terms depend on space-time:

$$S_f(\omega, 0) = \frac{1}{2} m^{1/4} \left(\int_{-\infty}^{\infty} \frac{d\theta_1 d\theta_2}{(2\pi)^2} (2\pi)^2 \delta(m(\sinh \theta_1 + \sinh \theta_2)) \delta(\omega - m(\cosh \theta_1 + \cosh \theta_2)) \right) \times \quad (\text{B.2})$$

$$\times \tanh^2 \left(\frac{\theta_1 - \theta_2}{2} \right).$$

The momentum Dirac delta can be simplified to

$$\delta(m(\sinh \theta_1 + \sinh \theta_2)) = \frac{\delta(\theta_1 + \theta_2)}{m \cosh(\theta_1)}. \quad (\text{B.3})$$

This means that the θ_2 integral can be performed trivially:

$$S_f(\omega, 0) = \frac{1}{2} m^{1/4} \int_{-\infty}^{\infty} d\theta_1 \frac{\delta(\omega - 2m \cosh \theta_1)}{m \cosh \theta_1} \tanh^2(\theta_1). \quad (\text{B.4})$$

In this form, the energy Dirac delta can be expressed as

$$\delta(\omega - 2m \cosh \theta_1) = \left(\frac{\delta(\theta_1 - \varphi)}{2m |\sinh \varphi|} + \frac{\delta(\theta_1 + \varphi)}{2m |\sinh(-\varphi)|} \right) \Theta(\omega/m - 2), \quad (\text{B.5})$$

where φ satisfies $\cosh \varphi = \omega/2m$. The Heaviside step function indicates that this equation has no real solutions for $\omega \leq 2m$. Inserting this form back to Eq. (B.4), the structure factors become

$$S_f(\omega, 0) = \frac{1}{2} m^{1/4} \frac{2}{2m^2 |\sinh \varphi| \cosh \varphi} \tanh^2(\varphi) \Theta(\omega/m - 2), \quad (\text{B.6})$$

which, after simplification, yields

$$S_f(\omega, 0) = \frac{2m^{1/4}}{\omega^3} \sqrt{\omega^2 - 4m^2} \Theta(\omega/m - 2). \quad (\text{B.7})$$

This agrees with Eq. (4.9) but differs from the result in [6] by a factor of 2.

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