



Development of Multiconfigurational Perturbation-adapted Perturbation Theory

TDK Thesis

Author: András Gombás¹

¹first year Chemistry MSc student

Advisors: Dr. Ágnes Szabados¹, Dr. Péter R. Surján²

¹ full professor, Department of Inorganic Chemistry

² full professor, Department of Physical Chemistry

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Summary

In 2022, a new approach in single-reference many body perturbation theory was introduced by Knowles [1]. The essence of the idea is to derive the zero-order Hamiltonian of a quantum many-body system from a model by requiring that the zero-order is as close to the exact Hamiltonian as possible. Knowles gave explicit formulae of the procedure that was termed *perturbation-adapted* perturbation theory. During the same year in the Laboratory of Theoretical Chemistry, Surján, Kóhalmi and Szabados [2] analyzed its connections to other previous partitioning techniques. After that, adaptation of the theory to the program system of the laboratory to obtain results for a variety of quantum chemical test systems followed by further analysis of the method’s applicability constituted my bachelor’s thesis work [3]. One of the conclusions of that thesis was that the efficiency of the single-reference method for weakly correlated systems encouraged towards the extension of the theory to the multi-reference approach in order to improve the description of strongly correlated systems by perturbation theory. During the summer and fall of 2023, I conducted further research, as a TDK project work the findings of which are presented in this thesis. The *single-but-multi* version of projected multi-configurational perturbation theory serves as the basis for the formulation of the multi-reference working formulae of perturbation-adapted perturbation theory as it requires the least alteration of the original approach introduced by Knowles. My work in the project entailed derivation, implementation and debugging of the extended approach. Currently we are at the stage of assessing the performance of the new method. Preliminary results indicate that the inherent pivot dependence of the projected multi-configurational method itself is enlarged by this formulation of the perturbation-adapted approach. Nonetheless, the multi-reference adaptation of the Knowles-partitioning is observed to be able to account for a significantly greater part of the correlation than that of the Møller-Plesset partitioning in strongly correlated cases. While it maintains the advance of the original single-reference version in the weakly correlated regime. Publication of the results are currently in progress. As for future research topics, it is found that the reformulation of the partitioning technique with reduced pivot-dependence might be a promising study.

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List of used acronyms

APSG: antisymmetrized product of strongly orthogonal geminals

CAS: complete active space

CAS(n_e, n_o): complete active space with n_e active electrons on n_o active orbitals

CAS-CI: complete active space configurational interaction

CASPT2: complete active space second-order perturbation theory

CBS: complete basis set

CCSD: coupled-cluster calculation with single and double excitations

CI: configurational interaction

CPU: central processing unit

FCI: full configurational interaction

GTO: Gaussian-type orbital

HF: Hartree-Fock

HOMO: highest occupied molecular orbital

LCAO-MO: molecular orbitals formed with linear combination of atomic orbitals

LUMO: lowest unoccupied molecular orbital

MBPT: Many Body Perturbation Theory

MCPT: multiconfigurational perturbation theory

MO: molecular orbital

MP: Møller-Plesset partitioning

MP n : order n in Møller-Plesset partitioning

MPI: message-passing interface

MR: multi-reference

MR-MP: multi-reference version of Møller-Plesset partitioning (formulated in the pMCPT framework if not declared otherwise)

MR-MP n : n th order energy calculated with the multi-reference version of Møller-Plesset partitioning (formulated in the pMCPT framework if not declared otherwise)

MR-PAPT: multi-reference version of Perturbation-adapted Perturbation Theory (formulated in the pMCPT framework if not declared otherwise)

MR-PAPT n : n th order energy calculated with the multi-reference version of Perturbation-adapted Perturbation Theory (formulated in the pMCPT framework if not declared otherwise)

NEVPT2: n -electron valence state perturbation theory

occ: occupied (orbital)

PAPT: Perturbation-adapted Perturbation Theory

pMCPT: projected version of multiconfigurational perturbation theory using Schmidt orthogonalization

PT: Perturbation Theory

RHF: restricted Hartree-Fock

RSPT: Rayleigh-Schrödinger Perturbation Theory

SCF: Self-consistent field

SR: single-reference

SR-MP: single-reference version of Møller-Plesset partitioning

SR-MP n : n th order energy calculated with the single-reference version of Møller-Plesset partitioning

SR-PAPT: single-reference version of Perturbation-adapted Perturbation Theory (as introduced by P. Knowles)

SR-PAPT n : n th order energy calculated with the single-reference version of Perturbation-adapted Perturbation Theory (as introduced by P. Knowles)

STO: Slater-type orbital

virt: virtual (orbital)

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1. Introduction

This Chapter presents the field of electronic structure theory in quantum chemistry from a bird’s eye view. The purpose of the following Sections is to give foundation to methods presented in detail in further Chapters. Additionally, the setting of the framework of the inevitable approximations of quantum chemistry is also aimed.

Understood is that if one wishes to achieve quantitative description of a challenging many-body system to a certain precision, some of the approximations and the limitations of simple methods presented in this Chapter are to be overcome by other, more complex methods that still exhibit feasible computational cost.

1.1 Approximations in Quantum Chemistry

Quantum chemistry is a scientific field that applies quantum mechanics to describe chemical systems in order to theoretically account for and predict physical and chemical properties, processes and other phenomena of interest to physical chemistry. A foremost task in quantum chemical modelling is the description of the electronic structure, that is achieved by a systematic application of approximations to make the solution of the time-independent Schrödinger equation

$$\hat{H} |\Psi\rangle = E |\Psi\rangle \tag{1.1}$$

feasible. Thus providing the Ψ wavefunctions as the eigenfunctions and E energy levels as the eigenvalues of the $\hat{H}$ Hamiltonian of a quantum many-body system in a stationary state from which the properties in question can be calculated e.g. as expectation values with the corresponding quantum mechanical operator of the physical quantity. Since Eq. (1.1) can only be solved analytically for simple, special cases e.g. two-body systems, in quantum chemistry one has no choice but to resort to some or all of the approximations discussed in this section.

Ab initio calculation methods for many-electron systems [4] aim to achieve the description to the desired accuracy without introducing empirical parameters, only using

fundamental physical principles. Besides *variational methods* that estimate the energy as an expectation value providing upper bound to the exact one, *many-body perturbation theory* is another ubiquitous tool to calculate approximate solutions to Eq. (1.1) [5].

1.1.1 The non-relativistic approximation

Let us consider the simplest, hydrogen-atom-like example of quantum physics, a stationary proton and an electron "orbiting" around at r distance. One would write the expression for the $\hat{H}$ Hamiltonian accounting for the $\hat{T}_e$ kinetic and $\hat{V}_{p-e}$ potential energy to calculate wavefunctions and energy levels in SI units as

$$\hat{H} = \hat{T}_e + \hat{V}_{p-e} = -\frac{\hbar^2}{2m_e}\Delta + \frac{e^2}{4\pi\epsilon_0 r} \quad (1.2)$$

with $\hbar$ being the reduced Planck-constant, m_e the mass of an electron, e the elementary charge and ϵ_0 the permittivity of vacuum. Transcribing this formula to atomic units, usual in *non-relativistic* quantum chemistry as

$$\hat{H} = \hat{T}_e + \hat{V}_{p-e} = -\frac{1}{2}\Delta + \frac{1}{r} \quad (1.3)$$

equivalently describes the system but hides some of the physical concepts behind it.

This formulation neglects the fact that instead of the classical m_e mass of the electron, the expression

$$m_{e,\text{rel}} = \frac{m_e}{\sqrt{1 - \left(\frac{v_e}{c}\right)^2}} \quad (1.4)$$

should be used, according to Einstein's theory of special relativity. Since the speed at which the electron would move in this model of the hydrogen atom at the Bohr radius, would be less than 1% of the speed of light[6], this effect can be considered negligible.

However, for atoms with heavier nucleus, the speed that can be associated with the electrons in this manner can be more comparable to the speed of light, possibly calling for the inclusion of relativistic corrections, e.g. through variational or perturbative procedures [7, 8].

The most frequently mentioned examples where relativistic effects are indeed non-negligible in the description of electronic structure to account for observations include the low melting point of mercury [6] and the colour of gold and caesium [9].

1.1.2 The Born-Oppenheimer approximation

The exact molecular many-electron $\hat{H}$ Hamiltonian

$$\hat{H}(\mathbf{r}, \mathbf{R}) = \underbrace{-\sum_i^n \frac{1}{2} \Delta_i}_{\hat{T}_e(\mathbf{r})} - \underbrace{\sum_I^N \frac{1}{2m_I} \Delta_I}_{\hat{T}_n(\mathbf{R})} + \underbrace{\sum_{i < j}^n \frac{1}{|r_i - r_j|}}_{\hat{V}_{e-e}(\mathbf{r})} + \underbrace{\sum_{I < J}^N \frac{Z_I Z_J}{|R_I - R_J|}}_{\hat{V}_{n-n}(\mathbf{R})} - \underbrace{\sum_i^n \sum_J^N \frac{Z_J}{|r_i - R_J|}}_{\hat{V}_{e-n}(\mathbf{r}, \mathbf{R})}, \quad (1.5)$$

where Z_I is the charge of nucleus I , depends inseparably on the $\mathbf{r}$ coordinates of the n electrons and $\mathbf{R}$, that of the N nuclei, because of the last $\hat{V}_{e-n}(\mathbf{r}, \mathbf{R})$ term accounting for the potential energy induced by the attraction between electrons and the nuclei. Similarly, the exact many-electron wavefunction of the molecule emerging as the solution of Eq. (1.1) also depends inseparably on these coordinates.

The *Born-Oppenheimer approximation* corresponding to chemical intuition separates the wavefunction of electrons and nuclei in a product form [10]

$$\Psi(\mathbf{r}, \mathbf{R}) = \Phi(\mathbf{r}; \mathbf{R}) \cdot \Xi(\mathbf{R}). \quad (1.6)$$

This *adiabatic approximation* assumes that electronic motion adjusts instantaneously to changes in nuclear positions. The reasoning behind this technique is justified by the difference of mass being of four orders of magnitude between the electrons and the nuclei. This allows for setting up a separate equation for $\Phi(\mathbf{r}; \mathbf{R})$ and $\Xi(\mathbf{R})$.

The $\Phi(\mathbf{r}; \mathbf{R})$ electronic wavefunction depends only parametrically on the position of the nuclei collected in $\mathbf{R}$. The $\hat{H}_e$ electronic Hamiltonian in the electronic Schrödinger equation

$$\hat{H}_e(\mathbf{r}; \mathbf{R}) |\Phi(\mathbf{r}; \mathbf{R})\rangle = E(\mathbf{R}) |\Phi(\mathbf{r}; \mathbf{R})\rangle, \quad (1.7)$$

is built of the terms indicated in Eq. (1.5), except for $\hat{T}_n$. Potential energy curves and surfaces can be constructed only in this approximation, by graphing the resulting solutions of Eq. (1.7), i.e. the $E(\mathbf{R})$ electronic energy at certain $\mathbf{R}$ parameters.

For the rest of this thesis, in which this approximation will be continuously upheld, let us refer to $\Phi(\mathbf{r}; \mathbf{R})$ simply by default as the Ψ exact many-electron wavefunction.

A well-known [10] limitation of the Born-Oppenheimer approximation is the description of molecular systems exhibiting *conical intersections* [11] or *avoided crossings* [12] in their potential energy surfaces.

1.1.3 The independent electron approximation

Whishful chemical thinking when describing the behavior of a single electron in a many-electron system would consider it independently and assume that it is only affected by the average field exerted by the nuclei and other electrons in the system. This could lead to a product of ϕ_i one-electron wavefunctions as an Ansatz for the Ψ many-electron molecular wavefunction (Hartree theory) [13]. However, the *Pauli principle* states that the multi-fermionic wavefunction should be antisymmetrical, i.e. change sign upon exchanging two of its fermions. Consequently, inherently antisymmetrical

$$\Psi(1, 2, \dots, n) = \frac{1}{\sqrt{n!}} \begin{vmatrix} \phi_1(1) & \phi_2(1) & \phi_3(1) & \cdots & \phi_n(1) \\ \phi_1(2) & \phi_2(2) & \phi_3(2) & \cdots & \phi_n(2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_1(n) & \phi_2(n) & \phi_3(n) & \cdots & \phi_n(n) \end{vmatrix} \quad (1.8)$$

Slater determinants [14] are used instead of the simple product in the *independent electron approximation*. These ϕ_i one-electron wavefunctions are called molecular *spin orbitals* as they depend not only on the spatial coordinates of that one electron but also on its spin coordinate. The many-electron wavefunction of an n -electron molecule written this way is merely an antisymmetrized product form of its n occupied spin orbitals.

1.1.4 The finite basis approximation

The ϕ_i molecular orbitals are constructed as

$$\phi_i = \sum_{\mu=1}^K c_{i\mu} \chi_{\mu} \quad (1.9)$$

linear combinations of a finite K number of χ_{μ} basis functions to make calculations feasible. This introduces the *finite basis approximation*. This restriction can be relieved by systematically expanding the basis set and extrapolating for the so-called complete basis set (CBS) limit.

It is customary in quantum chemistry to use atomic orbitals centered at each nucleus within the molecule as basis functions. From a theoretical point of view, Slater-type orbitals (STOs) would be advantageous [15] as they both exhibit exponential decay at large distances from the nucleus and fulfill Kato's cusp conditions near the centre [16]. However, integral calculations with STOs are computationally not cost effective. Consequently, upon Boys' suggestion [17], most often Gaussian expansions of Slater-type atomic orbitals are used. Pople [18] gave closed form for the integrals by expressing products of

Gaussian-type orbitals (GTOs) as linear combination of GTOs thereby significantly reducing computational costs.

Minimal basis sets [4, 19] consider only one basis function for fully filled shells and the entire valence shell. That involves 1 function for s , 3 functions for p shells etc. Each atomic orbital in itself is constructed as a linear combination of primitive Gaussians in most cases. *Split valence basis sets* add flexibility that enables the orbitals to adjust to the molecular environment by representing each valence shell by more than one basis functions. Depending on the number of basis functions per shell, one might use so-called double-, triple-, quadruple-zeta etc. basis sets. That means 2 functions for s , 6 functions for p shells, etc. for double-zeta basis sets. The naming originates from the exponent of STOs, that are usually denoted with ζ . *Polarization functions* alter the spatial distribution of orbitals allowing for the description of chemical bonds that are often indeed polarized. This is achieved by adding a set of l -functions, where l is the smallest azimuthal quantum number not occurring in the valence shell of the atom. Gaussian basis functions with small exponents used as *diffuse functions* to represent the "tail" part of the orbitals can be important to describe anions or dipole moments.

1.2 Hartree-Fock Theory

The simplest, most elemental method that follows from the approximations introduced so far, the *Hartree-Fock Theory* [20, 21] forms the basis of the description of molecular electronic structure theory. Here we consider the *restricted* Hartree-Fock (RHF) method that is formulated for closed shell systems. This restriction allows for the introduction of $n/2$ φ_i molecular *spatial orbitals* that can accomodate two electrons with opposite spin, so the molecular many-electron wavefunction can be simplified to

$$\Psi(1, 2, \dots, n) = \hat{\mathcal{A}} \left[\underbrace{\varphi_1(1)\alpha(1)}_{\phi_1(1)} \underbrace{\varphi_1(2)\beta(2)}_{\phi_2(2)} \dots \underbrace{\varphi_{n/2}(n-1)\alpha(n-1)}_{\phi_{n-1}(n-1)} \underbrace{\varphi_{n/2}(n)\beta(n)}_{\phi_n(n)} \right], \quad (1.10)$$

where $\hat{\mathcal{A}}$ is the operator ensuring the antisymmetry of the wavefunction (cf. Eq. (1.8)). From this formulation it follows that the RHF determinant always corresponds to a singlet state.

Hartree-Fock theory also decouples the electronic Schrödinger equation of an n -electron system (Eq. (1.7)) to a

$$\hat{F}(i) |\varphi_i(i)\rangle = \varepsilon_i |\varphi_i(i)\rangle \quad \text{for } i \in 1, 2, \dots, n/2 \quad (1.11)$$

system of $n/2$ equations in the restricted, i.e. closed shell case.

The $\hat{F}(i)$ Fockian is an effective one-electron Hamiltonian operator that only contains the average effective field that the electrons are acted upon by, i.e.

$$\hat{F}(i) = \hat{h}(i) + \sum_{j=1}^{n/2} (2\hat{J}_j(i) - \hat{K}_j(i)) \quad (1.12)$$

where $\hat{h}$ contains the one-electron part of the total Hamiltonian, i.e. the kinetic energy of electron i (cf. Eq. (1.5)), $\hat{J}_j$ is the one-electron Coulomb operator accounting for the repulsion of electron i and the charge density corresponding to an electron on orbital j and $\hat{K}$ is the so-called exchange operator stemming from the antisymmetric nature of the determinant. Matrix elements of the Fock operator on the orbitals used as basis, denoted with Greek letters, in accordance with Eq. (1.12) are

$$F_{\mu\nu} = h_{\mu\nu} + \sum_{\lambda\sigma} P_{\lambda\sigma} \left([\mu\lambda|\nu\sigma] - \frac{1}{2}[\mu\lambda|\sigma\nu] \right), \quad (1.13)$$

where $P_{\lambda\sigma}$ is the

$$P_{\lambda\sigma} = 2 \sum_i^{\text{occ}} c_{i\lambda} c_{i\sigma}^* \quad (1.14)$$

density matrix (cf. Eq. (1.9)) and occ refers to the $n/2$ occupied molecular orbitals. In this formulation, one-electron integrals are represented as $h_{\mu\nu}$, whereas for the two-electron integrals, the [12|12] Dirac convention is used.

The HF equations arise by applying the variational principle for the expectation value of $\hat{H}$ taken with the determinant of Eq. (1.10). The variations of the wavefunction are obtained by varying expansion parameters $c_{i\mu}$ corresponding to the μ th atomic orbital and the i th spatial molecular orbital. In the RHF framework, the number of variational parameters is half of that introduced in Eq. (1.9) that was given for spin orbitals. The solution of Eq. (1.11) can be achieved iteratively. First, a trial wavefunction is constructed by creating a set of φ_i molecular orbitals with some initial coefficients applied in Eq. (1.9). These coefficients can be used to construct the density matrix of Eq. (1.14). Then the matrix representation of the Fockian can be constructed according to Eq. (1.13) using the one- and two-electron integrals. Consequently, the solution of the eigenvalueproblem of the Fockian yields a new set of coefficients to construct the next trial wavefunction and the procedure is repeated until self-consistency is reached.

Orbitals φ_i that diagonalize the Fockian are called canonical molecular orbitals. Orbital energies ε_i appear e.g. in the Koopmans theorem as approximate ionization potentials and electron affinities, and also in the denominators of many-body perturbative

formulae. The total electronic energy is not the sum of these orbital energies but can be calculated as an expectation value of the $\hat{H}$ Hamiltonian with the Ψ_{HF} one-determinant molecular wavefunction instead

$$E_{\text{HF}} = \langle \Psi_{\text{HF}} | \hat{H} | \Psi_{\text{HF}} \rangle = \sum_i^{\text{occ}} (h_{ii} + F_{ii}). \quad (1.15)$$

given in spatial orbitals, too.

Brillouin's theorem [22] states that matrix elements are

$$\langle \Psi_0^{\text{HF}} | \hat{H} | K_a^p \rangle = 0, \quad (1.16)$$

where Ψ_0^{HF} is the ground state SCF optimized HF wavefunction and K_a^p is any single excitation from it. Simple algebraic proof for that using the Slater-Condon rules yields that such matrix element corresponds to an offdiagonal matrix element of the Fockian that was already diagonalized when the wavefunctions were obtained.

1.3 Post-Hartree-Fock methods

The HF mean-field approach has a foundational role in electronic structure theory, neglecting the correlated behaviour of electrons. *Post-Hartree-Fock* methods, on the other hand, aim to account for the so-called *correlation energy*, the difference of the exact and the Hartree-Fock energy.

The Ψ_{HF} Hartree-Fock determinant classifies the molecular spin orbitals [4] as either occupied or unoccupied, i.e. virtual. Excited determinants, Ψ_K can thusly be formed by placing one or more electrons from occupied orbitals to virtual ones. The exact molecular wavefunction can be expanded on the basis of these n_{CI} determinants

$$|\Phi_{\text{CI}}\rangle = c_{\text{HF}}|\Psi_{\text{HF}}\rangle + \sum_{K=1}^{n_{\text{CI}}} c_K |\Psi_K\rangle, \quad (1.17)$$

the configurational interactions' (CI) space. Note that the variational obtaining of the c coefficient for all possible determinants of the CI-space, the so-called *Full-CI* (FCI) solution indeed yields the exact expansion of the wavefunction in the given basis. Therefore, from a practical point of view, correlation energy can be considered the difference of E_{HF} (see Eq. (1.15)) and E_{FCI} , i.e.

$$E_{\text{FCI}} = \langle \Phi_{\text{FCI}} | \hat{H} | \Phi_{\text{FCI}} \rangle. \quad (1.18)$$

We speak of *weakly correlated* cases where in this expansion, the c_K weight of non-Hartree-Fock determinants is significantly smaller than c_{HF} , that of the HF determinant.

In such cases the so-called *dynamic correlation* can be calculated e.g. by single-reference perturbative methods (see Ch. 2) using the Hartree-Fock determinant as reference.

However, when there are more than one dominant determinants in the expansion of Eq. (1.17), then merely to achieve qualitative description, the so-called *static correlation* should also be accounted for. This necessitates the inclusion of all determinants of large and comparable weight in the reference function and leads to the multi-reference approach (see Ch. 3). We note here that this should be considered only as a *medium level* of correlation as Scuseria emphasizes [23] that *strong correlation*, a term frequently used in quantum chemical jargon, should refer to those special cases where most or all determinants in the expansion have comparable weight. In this thesis, the nomenclature of Scuseria is not completely adapted. For simplicity, the scenario of multiple dominant determinants in the CI expansion is referred to as strong correlation, i.e. distinguishing cases that call for the consideration of static correlation via application of multi-reference methods for sufficient accuracy when describing electronic structure from cases where single-reference methods are adequate.

Suitable reference functions in the presence of static correlation may be unrestricted Hartree-Fock, as well as multi-reference, geminal-based and complete active space functions. Systems for which the one-determinant RHF solution is insufficient encompass ionized, excited, near-degenerate and far from equilibrium geometry cases. Correspondingly, to describe many of the systems and processes interesting to chemistry, the method used is required to treat both dynamic and static correlation in balance.

2. Single-Reference Approach

As mentioned in Sec. 1.3, for weakly correlated systems, the post-HF consideration of dynamic correlation can be sufficient to achieve the desired quantitative precision. In this Chapter, single-reference perturbative methods are discussed.

The original formulation of the Knowles-partitioning termed Perturbation-adapted Perturbation Theory refers to this framework. Additionally, those features of PAPT are also stressed here that can play an important role when constructing the working formulae of the multi-reference extension.

2.1 Rayleigh-Schrödinger Perturbation Theory

Perturbation theory is a tool for generating approximate solution to differential equations [24] by expanding the solution in terms of power series. In this section, non-degenerate time-independent *Rayleigh-Schrödinger perturbation theory* is discussed. The efficiency arises from a well-chosen partitioning of the total Hamiltonian $\hat{H}$ of the system by identifying a zero-order Hamiltonian of most frequently a model problem for which the eigenvalueproblem is feasible and thus, its complete spectrum can be calculated. All other effects are obtained by introducing a scaling parameter λ in front of the remainder, called perturbation $\hat{W}$, i.e.

$$\hat{H} = \hat{H}^{(0)} + \lambda \hat{W}. \quad (2.1)$$

The emerging Schrödinger equation

$$[\hat{H}^{(0)} + \lambda \hat{W}]|\Psi\rangle = E |\Psi\rangle, \quad (2.2)$$

then can be solved by assuming a series expansion for E and Ψ in increasing powers of λ . The exact k th wavefunction is written as

$$|\Psi_k\rangle = |\Psi_k^{(0)}\rangle + \sum_{i=1}^{\infty} \lambda^i |\Psi_k^{(i)}\rangle. \quad (2.3)$$

The i th order correction, $\Psi_k^{(i)}$ is conveniently expanded in terms of the zero-order solution $\Psi_j^{(0)}$, leading to

$$|\Psi_k\rangle = |\Psi_k^{(0)}\rangle + \sum_{i=1}^{\infty} \sum_{j \neq k} \lambda^i c_{kj}^{(i)} |\Psi_j^{(0)}\rangle. \quad (2.4)$$

The order by order expansion of Eq. (2.1) yields recursive formulae for the $c_{kj}^{(i)}$ coefficient accounting for the contribution of the j th zero-order wavefunction to the k th i -order wavefunction. The exact energy of such a Ψ_k wavefunction can be evaluated as

$$E_k = E_k^{(0)} + \sum_{i=1} \lambda^i E_k^{(i)} = E_k^{(0)} + \sum_{i=1} \lambda^i \langle \Psi_k^{(0)} | \hat{W} | \Psi_k^{(i-1)} \rangle \quad (2.5)$$

Let us now comment on calculating Ψ_k and E_k according to Eqs. (2.4) and (2.5). In practice, the approximation is constituted by stopping at a finite order. When the expressions are generated to all orders, it is not necessary, that the series indeed provides the exact quantity. A convergence study is necessary to decide on the latter question.

Overall, PT methods are useful, when sufficiently accurate results are obtained by low-order truncation of perturbation series.

2.2 Møller-Plesset Perturbation Theory

When applying RSPT for the many-electron problem, *Møller-Plesset partitioning* [25] is used most frequently. It entails choosing the Fockian as zero-order

$$\hat{H}^{(0)} = \hat{F}, \quad (2.6)$$

and treating the rest, i.e. electron correlation as perturbation.

The sum of occupied orbital energies constitutes the zero-order ground state energy value

$$E_0^{(0)} = \sum_i^{\text{occ}} \varepsilon_i \quad (2.7)$$

associated with the $\Psi_0^{(0)}$ eigenfunction, which is the HF determinant. The Hartree-Fock energy is, therefore, obtained as the total energy calculated to the first order, i.e.

$$\begin{aligned} E_0^{(0)} + E_0^{(1)} &= \langle \Psi_0^{(0)} | \hat{H}^{(0)} | \Psi_0^{(0)} \rangle + \langle \Psi_0^{(0)} | \hat{W} | \Psi_0^{(0)} \rangle \\ &= \langle \Psi_0^{(0)} | \hat{H} | \Psi_0^{(0)} \rangle = \langle \hat{H} \rangle = E_{\text{HF}}. \end{aligned} \quad (2.8)$$

Assuming canonical molecular orbitals, the formula of the ground state first-order correction wavefunction can be expressed with all possible double excitations from the Hartree-Fock determinant

$$\begin{aligned} |\Psi_0^{(1)}\rangle &= -\frac{1}{2} \sum_{a,b}^{\text{occ}} \sum_{p,q}^{\text{virt}} \sum_{\sigma,\sigma'} \frac{[ab|pq]}{\varepsilon_p + \varepsilon_q - \varepsilon_a - \varepsilon_b} p_{\sigma}^{+} q_{\sigma'}^{+} b_{\sigma'}^{-} a_{\sigma}^{-} |\Psi_0^{(0)}\rangle \\ &= \sum_{a,b}^{\text{occ}} \sum_{p,q}^{\text{virt}} \sum_{\sigma,\sigma'} c_{pq}^{ab} p_{\sigma}^{+} q_{\sigma'}^{+} b_{\sigma'}^{-} a_{\sigma}^{-} |\Psi_0^{(0)}\rangle, \end{aligned} \quad (2.9)$$

where the c_{pq}^{ab} coefficient is introduced as

$$c_{pq}^{ab} = -\frac{1}{2} \frac{[ab|pq]}{\varepsilon_p + \varepsilon_q - \varepsilon_a - \varepsilon_b}, \quad (2.10)$$

and the ε_i canonical orbital energies appear in the denominator. The equations are written using second quantized formalism, with latin letters denoting the spatial molecular orbitals.

The second-order energy correction in this partitioning also resorts to double excitations reading

$$E_0^{(2)} = - \sum_{a,b}^{\text{occ}} \sum_{p,q}^{\text{virt}} \frac{[ab|pq] (2[ab|pq] - [ab|qp])}{\varepsilon_p + \varepsilon_q - \varepsilon_a - \varepsilon_b}. \quad (2.11)$$

This partitioning, although being robust, underestimates the correlation energy, it is prone to slow convergence or even divergence [26] and fails with excitations involving near-degenerate (eg. HOMO-LUMO) orbitals (cf. orbital energies in the denominator of Eqs. (2.9) and (2.11)), that are well-known phenomena.

2.3 Repartitioning

To enhance the efficiency of perturbative procedures, one can reconsider the partitioning. Let us first investigate this question out of the scope of perturbation theory, and consider simply the partitioning of the electronic Hamiltonian to one-electron part and a two-electron part as

$$\hat{H} = \sum_i^n \hat{h}(i) + \sum_{i<j}^n \hat{V}(i,j). \quad (2.12)$$

In second-quantized formalism, Eq. (2.12) corresponds to

$$\hat{H} = \sum_{i,j} \sum_{\sigma} h_{ij} i_{\sigma}^{+} j_{\sigma}^{-} + \frac{1}{2} \sum_{i,j,k,l} \sum_{\sigma,\sigma'} [ik|jl] i_{\sigma}^{+} k_{\sigma'}^{+} l_{\sigma'}^{-} j_{\sigma}^{-} \quad (2.13)$$

with one- and two-electron integrals depending on the basis.

Davidson pointed out [27] that the canonical molecular orbitals, which diagonalize the matrix representation of the Fockian (see. Eq. (1.13)), “are still arbitrary”. Since to define a transformed Fockian, one may change the partitioning of Eq. (2.12) by adding an arbitrary $\hat{\alpha}$ one-electron, preferably hermitian operator to $\hat{h}$ and correct for this in the original two-electron term according to

$$\hat{H} = \sum_i^n \underbrace{[\hat{h}(i) + (n-1) \cdot \hat{\alpha}(i)]}_{\hat{\bar{h}}(i)} + \sum_{i<j}^n \underbrace{[\hat{V}(i,j) - \hat{\alpha}(i) - \hat{\alpha}(j)]}_{\hat{\bar{V}}(i,j)}. \quad (2.14)$$

The transformed Fockian stemming from Eq. (2.14) is constructed with dressed integrals

$$\bar{F}_{ij} = \underbrace{h_{ij} + A_{ij}}_{\bar{h}_{ij}} + \sum_{k,l} P_{kl} \left[\underbrace{\left([ik|jl] - \frac{A_{ij}\delta_{kl} + A_{kl}\delta_{ij}}{n-1} \right)}_{\bar{[ik|jl]}} - \frac{1}{2} \underbrace{\left([ij|lk] - \frac{A_{il}\delta_{jk} + A_{jk}\delta_{il}}{n-1} \right)}_{\bar{[ij|lk]}} \right], \quad (2.15)$$

where A_{ij} is the matrix representation of

$$\hat{A} = (n-1) \cdot \hat{\alpha}. \quad (2.16)$$

This transformation leaves the density matrix, the Hartree-Fock energy and wavefunction unchanged as it generates non-zero elements in the occ-occ and virt-virt block of $\bar{F}$. Consequently, $\hat{A}$ defines a new set of orbitals that can be constructed by rotating the original canonical orbitals to obtain $\bar{\varphi}_i$ *proper canonicals*, on the basis of which $\bar{F}$ becomes diagonal.

When applying only the diagonal elements of A , the change affects only the ε_i orbital energies associated with F . Diagonals of A generate *level shifts* according to

$$\bar{\varepsilon}_a = \varepsilon_a + \frac{A_{aa} - 2 \cdot \text{Tr}_{\text{occ}}(A)}{n-1} \quad (2.17)$$

for occupied and

$$\bar{\varepsilon}_p = \varepsilon_p - \frac{A_{pp} - 2 \cdot \text{Tr}_{\text{occ}}(A)}{n-1} \quad (2.18)$$

for virtual levels.

The choice of $\hat{\alpha}$ and the underlying orbital rotations and level shifts may be guided by a specific interest. One may use it to get localized orbitals, or orbitals optimized for calculating ionization potentials or excitation energies. It is also possible to harness the flexibility provided by $\hat{\alpha}$ to enhance efficiency of many-body perturbation theory as it is shown in the upcoming Section.

2.4 Perturbation-adapted Perturbation Theory

One might argue that in many-body perturbation theory, the model problem is not necessarily introduced to aid physical interpretation. Following this reasoning, the only purpose of the model used is to have a one-body zero-order operator with the ground state zero-order wavefunction being a determinant as close to the target problem as possible and the excited eigenstates being simple determinants, too.

Introducing more parameters at zero-order by using a $\hat{\bar{F}}$ one-electron operator, whose representation is not diagonal but block-diagonal on the canonical basis could possibly drive the zero-order problem closer to the total problem in question. What is of greater difficulty, is to give a general, straightforwardly applicable formula to obtain these parameters.

This is what Knowles has suggested in 2022 [1], by giving projection functions as

$$|\Theta_{ab}\rangle = \sum_c^{\text{occ}} \sum_{p,q}^{\text{virt}} c_{ac}^{pq} p^+ q^+ c^- b^- |0\rangle \quad (2.19)$$

$$|\Theta_{pq}\rangle = \sum_{a,b}^{\text{occ}} \sum_r^{\text{virt}} c_{ab}^{pr} q^+ r^+ b^- a^- |0\rangle \quad (2.20)$$

where $|0\rangle$ denotes the Hartree-Fock determinant. In the above and for the rest of the thesis orbital indices refer to spin orbitals when not declared otherwise. We also introduce an index convention:

$$a, b, c, \dots, h \in \{\text{occupied}\}$$

$$i, j, k, \dots, o \in \{\text{generic}\}$$

$$p, q, r, \dots, z \in \{\text{virtual}\}.$$

Functions Θ_{aa} and Θ_{pp} emerge as partial sums of the terms of the Møller-Plesset first-order correction wavefunction (cf. Eq. (2.9)). The number of Θ_{ab} and Θ_{pq} functions altogether is the same as the number of parameters Λ_{ab} and Λ_{pq} in

$$\hat{\Lambda} = \sum_{a,b}^{\text{occ}} \Lambda_{ab} b^+ a^- + \sum_{p,q}^{\text{virt}} \Lambda_{pq} q^+ p^-. \quad (2.21)$$

Knowles introduced operator $\hat{\Lambda}$ in order to serve as zero-order. Note that the matrix representation of $\hat{\Lambda}$ is blockdiagonal in the occ-occ and virt-virt sense. According to Sec. 2.3, when obtained, Λ_{ab} and Λ_{pq} parameters correspond to level shifts and orbital rotations, i.e. they can be interpreted as the matrix elements of the $\hat{\bar{F}}$ transformed Fockian using the Davidson technique [2].

Using the projection functions of Eqs. (2.19) and (2.20), the

$$\hat{\Lambda} |\Psi^{(1)}\rangle \approx \hat{H} |\Psi^{(1)}\rangle, \quad (2.22)$$

operation equation, where $|\Psi^{(1)}\rangle$ is the MP1 wavefunction, can be projected to the first-order interacting space of double excitations to obtain the equation

$$\langle \Theta_{kl}^+ | \hat{\Lambda} - \hat{H} | \Psi^{(1)} \rangle = 0, \quad (2.23)$$

that defines $\hat{\Lambda}$ so that it can be used as a zero-order Hamiltonian in RSPT. To preserve hermiticity of the resulting operator, the

$$|\Theta_{kl}^+\rangle = |\Theta_{kl}\rangle + |\Theta_{lk}\rangle, \quad \text{for } k \geq l \quad (2.24)$$

symmetrization is introduced.

Knowles named this method “perturbation-adapted” perturbation theory, because in a sense the zero-order Hamiltonian is not directly taken from a model but constructed with the help of the perturbed function to be as close to the total Hamiltonian as possible within the first-order interaction space.

It is important to note here that since partial sums of the MP1 wavefunction can be found among the projection functions, they are linearly dependent. This results in the $\mathbf{T}^\Lambda$ coefficient matrix of the

$$\sum_{i \geq j} \underbrace{\langle \Theta_{kl}^+ | [j^+ i^- + i^+ j^-] | \Psi^{(1)} \rangle}_{\mathbf{T}_{kl,ij}^\Lambda} \Lambda_{ij} = \underbrace{\langle \Theta_{kl}^+ | \hat{H} | \Psi^{(1)} \rangle}_{\mathbf{Y}_{kl}^\Lambda} \quad (2.25)$$

PAPT equations being singular.

It was shown [3] that using normal ordered operators, thus constructing

$$\mathbf{T}_{kl,ij}^{\Lambda_N} = \langle \Theta_{kl}^+ | \left[[j^+ i^- + i^+ j^-] - \langle 0 | [j^+ i^- + i^+ j^-] | 0 \rangle \right] | \Psi^{(1)} \rangle, \quad (2.26)$$

and

$$\mathbf{Y}_{kl}^{\Lambda_N} = \langle \Theta_{kl}^+ | [\hat{H} - \langle 0 | \hat{H} | 0 \rangle] | \Psi^{(1)} \rangle, \quad (2.27)$$

there is only one zero singular value of $\mathbf{T}^{\Lambda_N}$ which always corresponds to that linear combination of the Λ_{ij} parameters being undefined that is anyway immaterial from the point of view of perturbation.

Fig. 2.1 shows an overview of the workflow of the PAPT procedure that has been implemented to our program-system. It starts with solving the Hartree-Fock equations, that way one obtains the Fockian and the φ_i canonical orbitals with ε_i canonical orbital energies. From the Fockian the first-order Møller-Plesset wavefunction can be constructed and then the PAPT equations can be solved yielding the Λ_{ij} parameters. The new set of the $\bar{\varphi}_i$ proper canonical orbitals then emerge as its eigenbasis with the level-shifted $\bar{\varepsilon}_i$ orbital energies as the corresponding eigenvalues. Using $\hat{\Lambda}$, a new first-order correction

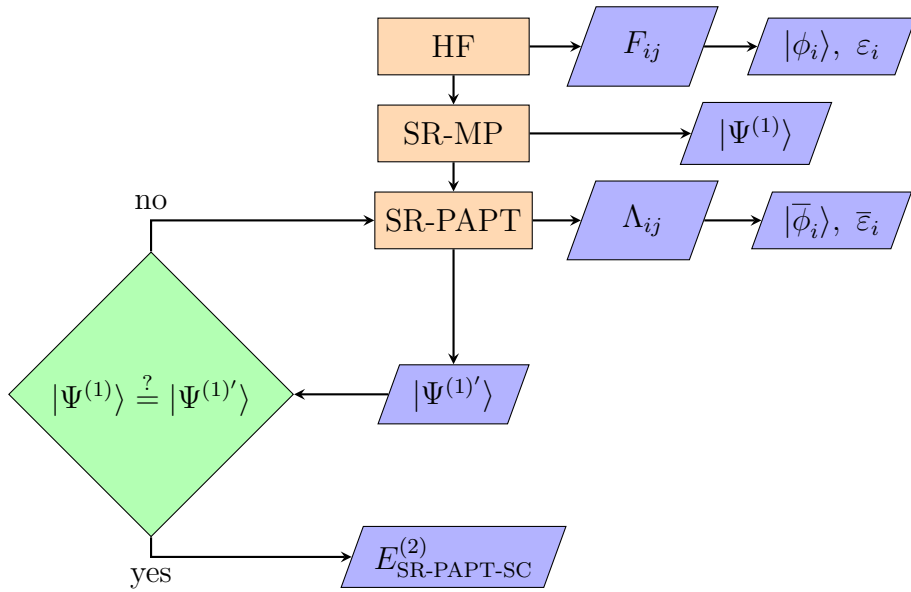


Figure 2.1: The workflow of the SR-PAPT procedure

wavefunction can be constructed in the usual manner and the regular many-body formulae can be used to obtain the second- and third-order energies etc. Our code was written in a way that also allows one to separate the two components of the repartitioning, the level-shifts and the orbital rotations and it was concluded that their effect is synergic [3].

Knowles also suggested that the PAPT procedure might be used iteratively. As Fig. 2.1 indicates, not only the original MP1, but also the newly obtained correction wavefunction can be used to construct new parameters and thus a new zero-order until self-consistency is achieved. We also showed that when doing so with normal ordered operators, the third order energy correction vanishes and the first non-zero correction to the second order energy appears at the fourth order [3]. This feature has a resemblance to Feenberg scaling [28], with the difference that self-consistent PAPT in the single-reference approach is size-consistent whereas the former is not.

Performance assessment of PAPT in the single-reference approach, revealed that the method is able to provide results for total energies at third order that are an order of magnitude closer to the Full-CI benchmark than traditional MP [3]. In fact, PAPT2 is comparable to CCSD in most cases. With appropriate computer implementation (see Sec. B in Appendix), computational cost of PAPT scales with the sixth power of the number of basis functions used. The cost and accuracy enables PAPT to stand out from the family of optimized partitionings that have previously been investigated [2].

3. Multi-Reference Approach

For systems entailing medium-level or strong correlation, single-reference methods are insufficient as they cannot account for the static correlation as discussed in Sec. 1.3. In this Chapter, some examples of variational as well as perturbative multi-reference methods are discussed before a detailed presentation of Multiconfigurational Perturbation Theory that is used in Ch. 4 to construct the working formulae of Multi-reference Perturbation-adapted Perturbation Theory.

3.1 Static correlation

3.1.1 Geminal Theory

Two-electron functions, i.e. so-called *geminals* have been introduced in quantum chemistry as early as the 1950's [29, 30] to tackle intrapair electron correlation. Starting from the determinant of Eq. (1.10), it is possible to deduce $n/2$ Φ_i two-electron functions that are antisymmetric. They can be considered as *separated pairs* [31] constituting the n -electron system function [32, 33]

$$\Psi(1, 2, \dots, n) = \hat{\mathcal{A}}[\Phi_1(1, 2) \Phi_2(3, 4) \dots \Phi_{n/2}(n-1, n)]. \quad (3.1)$$

The expansion of geminals in a one-electron basis can be constructed as

$$|\Phi_i(1, 2)\rangle = \sum_{m,n} c_{mn}^i \hat{\mathcal{A}}[\chi_m(1)\chi_n(2)]. \quad (3.2)$$

In the RHF case, the extension of Eq. (3.2) extends over $\varphi_{i\alpha}$ and $\varphi_{i\beta}$. Intrapair correlation can be described by including more orbitals in the expansion. The geminal of Eq. (3.2) can be represented in second-quantized formalism in a concise manner as

$$|\Phi_i\rangle = \sum_{m,n} c_{mn}^i m^+ n^+ |\text{vac}\rangle = \Phi_i^+ |\text{vac}\rangle. \quad (3.3)$$

It is important to point out here that this implies that the coefficient c_{mn}^i describing the contribution of the m th and n th basis functions to the i th geminal is given in a matrix form.

The geminal-based many-electron wavefunction with the above introduced Φ_i^+ composite creation operators, can be simply given as

$$|\Psi\rangle = \Phi_{n/2}^+ \dots \Phi_2^+ \Phi_1^+ |\text{vac}\rangle \quad (3.4)$$

due to the inherent antisymmetry of the second quantized formalism.

Optimization of the Φ_i two-electron functions with respect to the energy, i.e. obtaining the c_{mn}^i coefficients variationally is a much more elaborate task than optimizing one-electron orbitals in the HF method. At the same time, one has the benefit that geminal-based calculations are size-consistent, provide an upper bound to the energy of the many-electron system and even provide the exact description of two-electron systems.

Orthogonality of geminals reads

$$\langle \Phi_i | \Phi_j \rangle = \int dx_1 \int dx_2 \Phi_i(x_1, x_2) \Phi_j(x_1, x_2) = 0. \quad (3.5)$$

Geminals can be orthogonal not only if their spatial parts do not interfere, but also when they have non-zero contributions at the same region of the space, their overlap integral can still be zero due to the nodal structure.

The so-called *strong orthogonality* occurs when

$$\int dx_1 \Phi_i(x_1, x_2) \Phi_j(x_1, x_2) = 0 \quad (3.6)$$

is fulfilled.

Arai showed [34] that the set of geminals that obey strong orthogonality can be expanded in disjunct subspaces of one-electron functions, modifying the expression in Eq. (3.3) to

$$\Phi_i^+ = \sum_{m,n \in s_i} c_{mn}^i m^+ n^+ \quad (3.7)$$

where s_i is the subspace of the one-electron basis functions assigned to the i th geminal.

To prove Arai's theorem, one needs to show implications in two directions. One is rather involved but the other is relatively easy. When geminal coefficient matrices correspond to disjunct orbital sets, as in Eq. (3.7), the relation for $i \neq j$

$$\sum_n c_{mn}^i c_{ln}^j = 0 \quad (3.8)$$

holds for all m th and l th basis functions. This ensures Eq. (3.6).

Using antisymmetrized product of strongly orthogonal geminals (APSG) in Eq. (3.4) simplifies calculations as matrix elements between APSG-type wavefunctions can be evaluated in a similar manner to those between single determinants.

The APSG calculation can be initiated after choosing the necessary s_i subspaces of one-electron orbitals. This step is usually performed relying on localization procedures and analyzing the spatial distribution of localized orbitals. The optimal orbitals and c_{mn}^i coefficients can be computed e.g. by solving local two-electron effective Schrödinger equations.

All in all, APSG calculations can account for static correlation while several important correlation-based phenomena cannot be described by this theory as it lacks intergeminal dynamic correlation. Yet, computational economy of the APSG wavefunction makes it a viable candidate to serve as the underlying reference function in a multiconfigurational perturbational treatment e.g. via the single-but-multi method discussed in Sec. 3.2.2.

3.1.2 Complete Active Space

To construct a multiconfigurational reference wavefunction, it is customary [4, 35] to separate the space of molecular spatial orbitals into 3 subsets in the following manner: (i) inactive (occupied) orbitals that are occupied by 2 electrons of opposite spin in all configurations, (ii) virtual orbitals that are unoccupied in all configurations, and (iii) active (variably occupied) orbitals that have different occupancy in different configurations. The orbital energies associated with these orbitals follow the order

$$\varepsilon_i^{\text{inactive}} < \varepsilon_j^{\text{active}} < \varepsilon_k^{\text{virtual}},$$

respectively.

This separation is in accordance with chemical intuition suggesting that occupied HF orbitals that are lower in energy remain occupied during chemical processes, while those closer to the Fermi-level may vary in occupancy as reactions take place such as breaking or forming of chemical bonds etc.

The CI expansion

$$|\Phi_{\text{CAS}}\rangle = \sum_K c_K |K\rangle \quad \forall K \in \{\text{CAS-CI}\} \quad (3.9)$$

of the reference function considers all possible K determinants with n_e^{active} active electrons on the n_o^{active} active orbitals, preserving the total spin quantum number of the system distributed. These determinants span the CAS-CI space. The name Complete Active Space

(CAS) refers to including all determinants of the CAS-CI space in Eq. (3.9). Number of active electrons and orbitals are options of the method usually given in a short-hand form as $\text{CAS}(n_e^{\text{active}}, n_o^{\text{active}})$. Correspondingly, $\text{CAS}(2,2)$ has two active electrons on two active orbitals, which makes it identical to the previously discussed APSG method for the case of a single correlated geminal.

Since the number of the determinants in CAS-CI increases exponentially with n_e^{active} and n_o^{active} , computation time limits allow for calculations with a couple of tens of active orbitals and electrons. The necessity of CAS calculations with relatively large active subsets can be justified by the fact that for many strongly correlated systems, the importance of each determinant can significantly vary across the potential surface, while at the same time predicting which determinants will be of greater weight at a given point in the expansion can be difficult.

3.2 Dynamic correlation

Augmenting the CAS-CI space to include all determinants – which may become necessary to describe a chemical process – is not practical due to the prohibitive computational cost. As an alternative, it has been suggested [36] to resort to a rather minimal CAS expansion and apply a more cost-economical scheme to include the individually small effect of the many determinants left out from CAS-CI.

Dynamic correlation techniques developed for the HF reference have been generalized [37] with this purpose for the multi-reference situation. Among these, let us focus on a few perturbative methods.

3.2.1 CAS-based Perturbation Theory

3.2.1.1 CASPT2

Based on the philosophy mentioned above, Roos et al. devised [36] the Complete Active Space Second Order Perturbation Theory (CASPT2) method in 1992. It became widely used and got implemented into the MOLCAS and MOLPRO quantum chemical program systems. The CASPT2 approach assumes a CAS wavefunction as zero-order function $\Psi^{(0)}$.

The first-order correction wavefunction spans the first-order interacting space of

double excitations similarly to Eq. (2.9) as

$$|\Psi^{(1)}\rangle = \sum_{a,b}' \sum_{p,q}' \sum_{\sigma,\sigma'} c_{pq}^{ab} p_{\sigma}^{+} q_{\sigma'}^{+} b_{\sigma}^{-} a_{\sigma'}^{-} |\Psi^{(0)}\rangle, \quad (3.10)$$

but the restrictions of the summations (denoted by $'$) are different in this case, i.e.

$$a, b \in \{\text{inactive}, \text{active}\}$$

$$p, q \in \{\text{active}, \text{virtual}\}.$$

The case where all orbitals would be active is excluded from the expansion of Eq. (3.10). The coefficients in Eq. (3.10) can be determined by solving the

$$[\hat{H}^{(0)} - E^{(0)}]|\Psi^{(1)}\rangle = [\hat{H}^{(0)} - \hat{H}]|\Psi^{(0)}\rangle \quad (3.11)$$

system of linear equations e.g. iteratively. The energy at second-order can be obtained by the standard RSPT formula

$$E^{(2)} = \langle \Psi^{(0)} | \hat{H} | \Psi^{(1)} \rangle, \quad (3.12)$$

in accordance with Eq. (2.5).

The partitioning underlying CASPT2 is formulated with the Fockian as

$$\hat{H}^{(0)} = \hat{O}\hat{F}\hat{O} + \hat{P}\hat{F}\hat{P}. \quad (3.13)$$

Writing two separate terms in Eq. (3.13) is necessary since the reference function is usually not an eigenfunction of the Fockian. In the above $\hat{O}$ is the projector to the model space formed by the CAS reference function, and $\hat{P}$ projects to the complementary space, orthogonal to the model space. For the second-order energy, the inclusion of the single and double excitations of $\Psi^{(0)}$ are sufficient. Note, that excited functions obtained by replacement operators acting on $\Psi^{(0)}$ are not orthogonal. This must be taken care of when constructing a matrix representation of Eq. (3.13). The Fockian, constructed from the spin-averaged first-order density matrix and usually modified with small occupation-dependent level-shifts [38], has non-diagonal elements that were not included in $\hat{H}^{(0)}$ in the first version of CASPT2 [39]. It was advantageous regarding computational costs, but caused the energies to change discontinuously for rearranging orbital subspaces even if the occupancies remained unchanged. Later implementation of the CASPT2 method [36] that became e.g. part of the MOLPRO program system, therefore included off-diagonals of the Fockian in the zero-order as well.

In 2011, Pulay concluded [40] that the greatest drawback of CASPT2 is its performance being highly dependent on the declaration of orbital subsets. A minimal choice

can be easily identified but unfortunately, it also implies insufficient description of the dynamic correlation. The optimal choice for the system in question, on the other hand, requires great insight to the method itself, i.e. overall, CASPT2 is not fail safe to use as a black-box.

3.2.1.2 NEVPT2

In 2001, Malrieu et al. [41] introduced the n -electron valence state perturbation theory (NEVPT2) that starts from the CAS wavefunction written as the antisymmetrized product form Ansatz

$$\Phi_{\text{CAS}} = \hat{\mathcal{A}} \left[\Phi_{\text{core}}^{(c)} \Phi_{\text{valence}}^{(v)} \right] \quad (3.14)$$

indicating separately the c core (inactive) and v valence (active) electrons. The CAS function is obtained by

$$\hat{P}_{\text{CAS}} \hat{H} \hat{P}_{\text{CAS}} |\Phi_{\text{CAS}}\rangle = E^{(0)} |\Phi_{\text{CAS}}\rangle, \quad (3.15)$$

i.e. diagonalizing the Hamiltonian in the CAS-CI space.

Theoretical development proceeds by formulating subspaces of determinants in the CI space. In particular, determinants

$$\hat{\mathcal{A}} \left[\Phi_{\text{core \& inactive}, l}^{-k} \Phi_{\text{valence}}^{+k} \right] \quad (3.16)$$

constitute subspace s_{kl} . Determinants in Eq. (3.16) feature a common occupancy of $c - k$ electrons distributed over the core and inactive orbitals and vary in the distribution of $v + k$ valence electrons over the active orbitals.

In the so-called totally uncontracted version of the method, the zero-order Hamiltonian is composed by considering the full Hamiltonian in subspaces s_{kl} and neglecting any interaction element between functions of different subspaces. This corresponds to

$$\hat{H}^{(0)} = \sum_{k,l} \hat{P}_{s_{kl}} \hat{H} \hat{P}_{s_{kl}} \quad (3.17)$$

where $\hat{P}_{s_{kl}}$ projects onto subspace s_{kl} .

Due to the possibly large number of the valence part of the determinants, the dimension of subsets s_{kl} can be computationally prohibitive in spite of the fact that $-2 \leq k \leq 2$ is sufficient e.g. for the second-order energy. This initiated the development of various contracted versions of the theory, that are not to be detailed here further.

The NEVPT2 method is available in the ORCA [42] and MOLPRO [43] program systems. Its advantageous features include size-consistency, invariance to rotations among active space orbitals, spin-conservation, and insensitivity to intruder states.

3.2.2 Multiconfigurational Perturbation Theory

The perturbative frameworks discussed above assume a CAS reference. This Section presents a method that is applicable to a reference function of arbitrary structure. The approach termed *multiconfigurational perturbation theory* (MCPT) was devised in the Laboratory of Theoretical Chemistry as a black box tool to obtain perturbative corrections to any reference function that is provided as an expansion over determinants of the CI space. Here, the general working formulae of MCPT are discussed as well as the Møller-Plesset partitioning within MCPT (see Sec. 2.2). The MCPT framework introduced in this Section will be extensively used later in Ch. 4 to formulate the multi-reference version of PAPT (see Sec. 2.4).

3.2.2.1 The reference function

A common feature of MCPT and the previously discussed methods is their *single-but-multi* nature. This means that a one-dimensional model space is associated with the projector

$$\hat{O} = |\Phi\rangle\langle\Phi|. \quad (3.18)$$

The projector of the complementary space becomes

$$\hat{P} = 1 - |\Phi\rangle\langle\Phi|. \quad (3.19)$$

The Φ reference model function is taken as not just one but a linear combination of multiple determinants

$$|\Phi\rangle = \sum_K c_K |K\rangle = c_0 |0\rangle + \sum_{K \neq 0} c_K |K\rangle, \quad (3.20)$$

with c_0 being the coefficient of the $|0\rangle$ *pivot determinant*, usually but not necessarily corresponding to the determinant of largest weight (i.e. c_K^2) in Φ . With this separation made, the remaining $K \neq 0$ terms are considered *excited determinants* with respect to the pivot. The reference is normalized as

$$\langle\Phi|\Phi\rangle = \sum_{K=0} c_K^2 = 1. \quad (3.21)$$

Firstly, it is supposed that the

$$\langle\Psi|\Phi\rangle \neq 0 \quad (3.22)$$

overlap of the Φ reference and the Ψ exact wavefunction is non-zero, i.e. Φ is not a singularly bad approximation to Ψ .

Secondly, it is also supposed that the expansion of the Φ reference function upon the K determinants is known, i.e. the values of c_K are obtained beforehand e.g. from generalized valence bond, geminal-based or complete active space calculation.

3.2.2.2 The partitioning

The zero order energy associated with Φ is fixed to be the eigenvalue of the total Hamiltonian taken with the multi-reference function. The zero-order equation hence reads

$$\hat{H}^{(0)}|\Phi\rangle = E^{(0)}|\Phi\rangle, \quad (3.23)$$

where

$$E^{(0)} = \langle\Phi|\hat{H}|\Phi\rangle. \quad (3.24)$$

Since to fulfill Eq. (3.23), one element of the basis in the CI space basis must be Φ , the appropriate choice for the set of many-electron basis functions is

$$\{|\Phi\rangle\} \cup \{|K\rangle \mid K \neq 0\}. \quad (3.25)$$

Vectors of the space given in Eq. (3.25) are used to to construct the spectrum of $\hat{H}^{(0)}$. These vectors are neither redundant nor orthogonal since

$$\langle K|\Phi\rangle = \sum_L c_L \langle K|L\rangle = c_K. \quad (3.26)$$

For this reason it is practical to perform Schmidt orthogonalization in order to obtain vectors

$$|K'\rangle = \hat{P}|K\rangle = |K\rangle - |\Phi\rangle \underbrace{\langle\Phi|K\rangle}_{c_K} = |K\rangle - c_K|\Phi\rangle \quad (3.27)$$

that are orthogonal to Φ . This version of the theory is termed *projected* MCPT [44] as opposed to *unprojected* MCPT where the Schmidt orthogonalization step is skipped.

The $|K'\rangle$ projected determinants still overlap with each other. This is treated by introducing $\langle\widetilde{K}'|$ reciprocal vectors fulfilling

$$\langle\widetilde{K}'|L'\rangle = \delta_{KL}. \quad (3.28)$$

It can be verified by substitution in Eq. (3.28) and using Eq. (3.27) that the reciprocal vectors read

$$\langle\widetilde{K}'| = \langle K| - \frac{c_K}{c_0}\langle 0|. \quad (3.29)$$

(See Sec. A in Appendix for confirmation.)

As for the ground state energy up to the first-order is simply

$$E_0^{(0)} + E_0^{(1)} = \langle\Phi|\hat{H}|\Phi\rangle, \quad (3.30)$$

and the spectral resolution of the zero-order Hamiltonian in the biorthogonal formalism can be expressed as

$$\hat{H}^{(0)} = E_0^{(0)} |\Phi\rangle\langle\Phi| + \sum_{K \neq 0} E_K^{(0)} |K'\rangle\langle\widetilde{K}'|. \quad (3.31)$$

The partitioning in MCPT is fixed by choosing a set of values for $E_K^{(0)}$.

As mentioned before, $E_0^{(0)}$ is usually fixed as the reference energy

$$E_{\text{ref}} = E_0^{(0)} = \langle\Phi|\hat{H}|\Phi\rangle \quad (3.32)$$

leaving

$$E_0^{(1)} = 0 \quad (3.33)$$

in accordance with Eq. (3.30).

The $E_K^{(0)}$ energies associated with excited determinants may be obtained e.g. by generalizing the well-known Møller-Plesset or Epstein-Nesbet partitionings of the single reference framework.

For example, it may appear plausible to generalize the MP partitioning as

$$E_K^{(0)} = \langle K|\hat{F}|K\rangle = \sum_i^{\text{occ in } K} F_{ii}, \quad (3.34)$$

where $\hat{F}$ may be a generalized Fockian, reflecting the multi-reference nature of Φ [37].

While the expression in Eq. (3.34) falls back to the sum of orbital energies in the single-reference case, in multi-reference scenario, $\hat{F}$ usually exhibits off-diagonal elements, too.

For this reason, the partitioning based on Eq. (3.34) is named after Davidson and Kapuy, who advocated the use of the diagonal elements of F at zero-order [45].

In the next section, a generalization of the MP partitioning in the MCPT framework is considered, that uses the full matrix representation of $\hat{F}$, in line with CASPT2.

Once the partitioning is fixed, PT corrections are generated by the case of biorthogonal PT formulae.

The first-order wavefunction, orthogonal to the reference function reads

$$|\Psi^{(1)}\rangle = \sum_{K \neq 0} c_K^{(1)} |K'\rangle, \quad (3.35)$$

where the $c_K^{(1)}$ coefficients are expressed as

$$c_K^{(1)} = -\frac{\langle\widetilde{K}'|\hat{H}|\Phi\rangle}{E_K^{(0)} - E_{\text{ref}}} \quad (3.36)$$

in case of the diagonal zero-order Hamiltonian of Eq. (3.31).

The second-order energy formula is obtained as

$$E_{\text{pMCPT}}^{(2)} = -\langle \Phi | \hat{H} \hat{Q} \hat{H} | \Phi \rangle, \quad (3.37)$$

where the $\hat{Q}$ reduced resolvent was introduced as

$$\hat{Q} = \sum_{K \neq 0} \frac{|K'\rangle \langle \widetilde{K}'|}{E_K^{(0)} - E_{\text{ref}}}. \quad (3.38)$$

3.2.2.3 Møller-Plesset partitioning in pMCPT

In analogy with Eq. (3.13) the zero-order Hamiltonian in the MP partitioning [46] is

$$\hat{H}^{(0)} = E^{(0)} \hat{O} + \hat{P} \hat{F} \hat{P}. \quad (3.39)$$

Similarly to Eq. (3.31), projection to the

$$\hat{P} = \sum_{K \neq 0} |K'\rangle \langle \widetilde{K}'|, \quad (3.40)$$

complementary space is necessary in order to ensure that the zero-order equation is fulfilled, cf. Eq. (3.23).

The *generalized Fockian* appearing in Eq. (3.39) reads as

$$F_{pq} = h_{pq} + \sum_{rs} P_{sr} [pr || qs], \quad (3.41)$$

where

$$P_{sr} = \langle 0 | r^+ s^- | 0 \rangle \quad (3.42)$$

is the *first-order density matrix*.

Again in line with CASPT2, the first-order wavefunction can be obtained in analogy to Eq. (3.11) as

$$[\hat{H}^{(0)} - E^{(0)}] |\Psi^{(1)}\rangle = [E^{(1)} - \hat{W}] |\Phi\rangle. \quad (3.43)$$

At difference with the previous section, the $c_K^{(1)}$ coefficients cannot, in this case, simply be calculated by the formula in Eq. (3.36) as the Fockian is not diagonal in the pMCPT basis. One therefore has to project Eq. (3.43) with reciprocal vectors $\langle \widetilde{L}' |$ and solve the resulting

$$\sum_{K \neq 0} \langle \widetilde{L}' | \hat{F} - E^{(0)} | K' \rangle c_K^{(1)} = -\langle \widetilde{L}' | \hat{H} | \Phi \rangle \quad (3.44)$$

system of linear equations. Let us introduce in the above elements of the $\mathbf{T}^{\Psi^{(1)}}$ coefficient matrix as

$$\begin{aligned} T_{LK}^{\Psi^{(1)}} &= \langle \widetilde{L}' | \hat{F} - E^{(0)} | K' \rangle \\ &= \langle L | \hat{F} - E^{(0)} | K \rangle - \frac{c_L}{c_0} \langle 0 | \hat{F} - E^{(0)} | K \rangle \\ &\quad - c_K \langle L | \hat{F} - E^{(0)} | \Phi \rangle + \frac{c_K c_L}{c_0} \langle 0 | \hat{F} - E^{(0)} | \Phi \rangle \end{aligned} \quad (3.45)$$

and components of the $Y^{\Psi^{(1)}}$ inhomogenous vector as

$$\begin{aligned} Y_L^{\Psi^{(1)}} &= -\langle \widetilde{L}' | \hat{H} | \Phi \rangle \\ &= -\langle L | \hat{H} | \Phi \rangle + \frac{c_L}{c_0} \langle 0 | \hat{H} | \Phi \rangle. \end{aligned} \quad (3.46)$$

Using the above, Eq. (3.44) can be concisely written as the matrix equation

$$\sum_{K \neq 0} T_{LK}^{\Psi^{(1)}} c_K^{(1)} = Y_L^{\Psi^{(1)}}, \quad (3.47)$$

from which the $c_K^{(1)}$ coefficients can be obtained by inverting $\mathbf{T}^{\Psi^{(1)}}$.

In the study where MP partitioning was developed in MCPT [37], excited determinants contributing to $\Psi^{(1)}$ were restricted for computational economy as doubly excited with respect to the pivot determinant. In this work, double and single excitations are also allowed with respect to the pivot, leading to

$$|\Psi^{(1)}\rangle = \sum_K^{2x \text{ exc}} c_K^{(1)} |K'\rangle = \frac{1}{4} \sum_{ab}^{\text{occ}} \sum_{pq}^{\text{virt}} c_{pq}^{ab(1)} |K_{ab}^{pq'}\rangle + \sum_a^{\text{occ}} \sum_p^{\text{virt}} c_p^{a(1)} |K_a^{p'}\rangle. \quad (3.48)$$

The notation introduced in the above is used extensively in this thesis, where K_a^p stands for the determinant in which the electron from the a occupied spin orbital of the pivot determinant is excited to the p virtual one. In an analogous manner, K_{ab}^{pq} is the determinant in which, besides the aforementioned a to p replacement, the electron from the b occupied orbital was additionally moved to the q virtual one. The excitations obtained this way preserve the net spin z -component of the pivot determinant, when spin indices of spin orbitals a and b agree with those of p and q .

The inclusion of single excitations with respect to the pivot determinant can be justified by the fact that as one moves away from the single-reference Hartree-Fock SCF solution, *Brillouin's theorem* no longer holds.

4. Development of MR-PAPT

4.1 Constructing the Working Formulae

An important feature of the partitioning introduced by Knowles in the single-reference approach (see Sec. 2.4) is that the construction of the zero-order Hamiltonian is based upon the projection of an operator equation to the first-order interacting space of double excitations with respect to the Hartree-Fock determinant. When extending this philosophy to the multi-reference case, single-but-multi projected multiconfigurational perturbation theory (see Sec. 3.2.2) is a suitable choice as it implies the concept of a pivot determinant offering a plausible way to construct singly, doubly etc. excited determinants. Methods presented in Sec. 3.2.1 treat all determinants of the reference function on an equal footing, that would call for more elaborate considerations on the construction and treatment of single, double etc. excitations.

The MCPT method assumes no special structure of the Φ multideterminantal reference wavefunction (see Eq. (3.20)) and this feature is conserved for the MR-PAPT method based upon it. In accordance with Eq. (3.32), the $E_{\text{ref},\Phi}$ reference energy is calculated as

$$E_{\text{ref},\Phi} = \langle \Phi | \hat{H} | \Phi \rangle, \quad (4.1)$$

which – as usual in MCPT – is fixed as the zero-order ground state energy

$$E_0^{(0)} = E_{\text{ref},\Phi}, \quad (4.2)$$

consequently the first-order correction to the ground state energy vanishes

$$E_0^{(1)} = 0. \quad (4.3)$$

The zero-order Hamiltonian is formulated in accordance with the pMCPT version of the Møller-Plesset partitioning (see Eq. (3.39)). In the spirit of PAPT, matrix elements of the one-body operator are considered parameters to be optimized. For this reason the $\hat{F}$ generalized Fockian is replaced with the parameterized, perturbation-adapted $\hat{\Lambda}$ operator

in Eq. (3.39) to obtain

$$\hat{H}^{(0)} = E_0^{(0)} \hat{O} + \hat{P} \hat{\Lambda} \hat{P}, \quad (4.4)$$

where the projection operators of the model and complementary space are given in Eqs. (3.18) and (3.40), correspondingly. In terms of projected excited determinants and their reciprocal, the zero-order Hamiltonian reads

$$\hat{H}^{(0)} = E_0^{(0)} |\Phi\rangle\langle\Phi| + \sum_{\substack{K,L \\ \neq 0}} \sum_{i,j} \Lambda_{ij} |K'\rangle\langle\widetilde{K}'| j^+ i^- |L'\rangle\langle\widetilde{L}'|. \quad (4.5)$$

The degree of freedom when it comes to the partitioning is associated with Λ_{ij} in Eq. (4.5) whereas in Eq. (3.31) these are the $E_K^{(0)}$ values that are arbitrary. The role of PAPT in pMCPT is to fix the parameters of $\hat{\Lambda}$ to values that possibly differ from the elements of the $\hat{F}$ generalized Fockian utilized in the MP partitioning of pMCPT (see Eq. (3.39)).

Formulation of the MR extended version of the PAPT equations (see Eq. (2.25) for the single-reference version) that eventually define the set of $\{\Lambda_{ij}\}$ parameters necessitates functions to project Eq. (2.22), that represents the essence of the PAPT idea. For this we aim to formulate the equations in a way that the most prominent feature of SR-PAPT, i.e. vanishing of the third-order energy correction in the self-consistent case is maintained.

Though a generalized Brillouin theorem can be formulated for the MR case [37], it does not set the occ-virt block of the generalized Fockian zero. It therefore generates no ground to confine the matrix representation of $\hat{\Lambda}$ blockdiagonal in the occ-occ and virt-virt sense. The second quantized expression of $\hat{\Lambda}$ is given simply as

$$\hat{\Lambda} = \sum_{i,j} \Lambda_{ij} j^+ i^-. \quad (4.6)$$

Projection functions of SR-PAPT can be generalized in a straightforward manner as

$$\langle\widetilde{\Theta}_{ab}| = \sum_c^{\text{occ}} \sum_{p,q}^{\text{virt}} \widetilde{c}_{pq}^{ac(1)} \langle\widetilde{K}_{bc}^{pq'}| \quad (4.7)$$

$$\langle\widetilde{\Theta}_{pq}| = \sum_{a,b}^{\text{occ}} \sum_r^{\text{virt}} \widetilde{c}_{pr}^{ab(1)} \langle\widetilde{K}_{ab}^{qr'}| \quad (4.8)$$

they are however not sufficient in number since $\{\Lambda_{ij}\}$ parameters need to be defined in the occ-virt block, too. For this end, singly excited functions

$$\langle\widetilde{\Theta}_{ap}| = \langle\widetilde{K}_a^{p'}| \quad (4.9)$$

with occ-virt indices a and p are also considered. Since the original SR-PAPT equations (see Eq. (2.25)) contain the projection functions in bra vectors, the $\langle\widetilde{K}'|$ reciprocal vectors,

introduced in Eq. (3.29) figure in the projection functions. Additionally, in Eqs. (4.7) and (4.8), the $\tilde{c}_K^{(1)}$ coefficients are introduced as

$$\tilde{c}_K^{(1)} = - \sum_{L \neq 0} \langle \Phi | \hat{H} | L' \rangle G_{LK}, \quad (4.10)$$

where

$$\mathbf{G} = \left(\mathbf{T}^{\Psi^{(1)}} \right)^{-1} \quad (4.11)$$

is the reduced resolvent and the elements of the $\mathbf{T}^{\Psi^{(1)}}$ coefficient matrix are

$$T_{KL}^{\Psi^{(1)}} = \langle \tilde{K}' | \hat{H}^{(0)} - E^{(0)} | L' \rangle, \quad (4.12)$$

in accordance with Eq. (3.45). Specification of projection functions according to Eqs. (4.7), (4.8) and (4.9) ensures the vanishing of the third-order energy correction as shown in Sec. 4.1.1.

To preserve hermiticity, the introduction of

$$\langle \tilde{\Theta}_{ij}^+ | = \langle \tilde{\Theta}_{ij} | + \langle \tilde{\Theta}_{ji} | \quad (4.13)$$

for $i \geq j$ is still necessary.

Similarly to the single-reference version of PAPT, the set of $c_K^{(1)}$ coefficients, that are obtained from a preceding PT correction step in Møller-Plesset partitioning – in this MR case with the same pMCPT technique (see Sec. 3.2.2.3) –, define the $\Psi^{(1)}$ wavefunction and the set of $\{\Theta_{kl}^+\}$ projection functions. With these quantities, the MR-PAPT equations are set up as

$$\sum_{i \geq j} \underbrace{\langle \tilde{\Theta}_{kl}^+ | [j^+ i^- + i^+ j^-] | \Psi^{(1)} \rangle}_{T_{ij,kl}^\Lambda} \Lambda_{kl} = \underbrace{\langle \tilde{\Theta}_{kl}^+ | \hat{H} | \Psi^{(1)} \rangle}_{Y_{kl}^\Lambda} \quad (4.14)$$

in analogy with the single-reference version (cf. Eq. (2.23)). The MR-PAPT equations of Eq. (4.14) that can be solved to obtain the parameters of $\hat{\Lambda}$ (see Sec. 4.1.2 for elaboration on the singularity of the coefficient matrix).

A working code was constructed in FORTRAN90 language, the workflow of which is presented on Fig. 4.1. The input of the code is the c_K coefficients and the specification of the associated K determinants as well as the one- and two-electron integrals on the MO basis. The code applies the pMCPT multi-reference adaptation of PAPT and solves the emerging equations iteratively until self-consistency is reached. The ground state energy is calculated in each iteration step up to second-order.

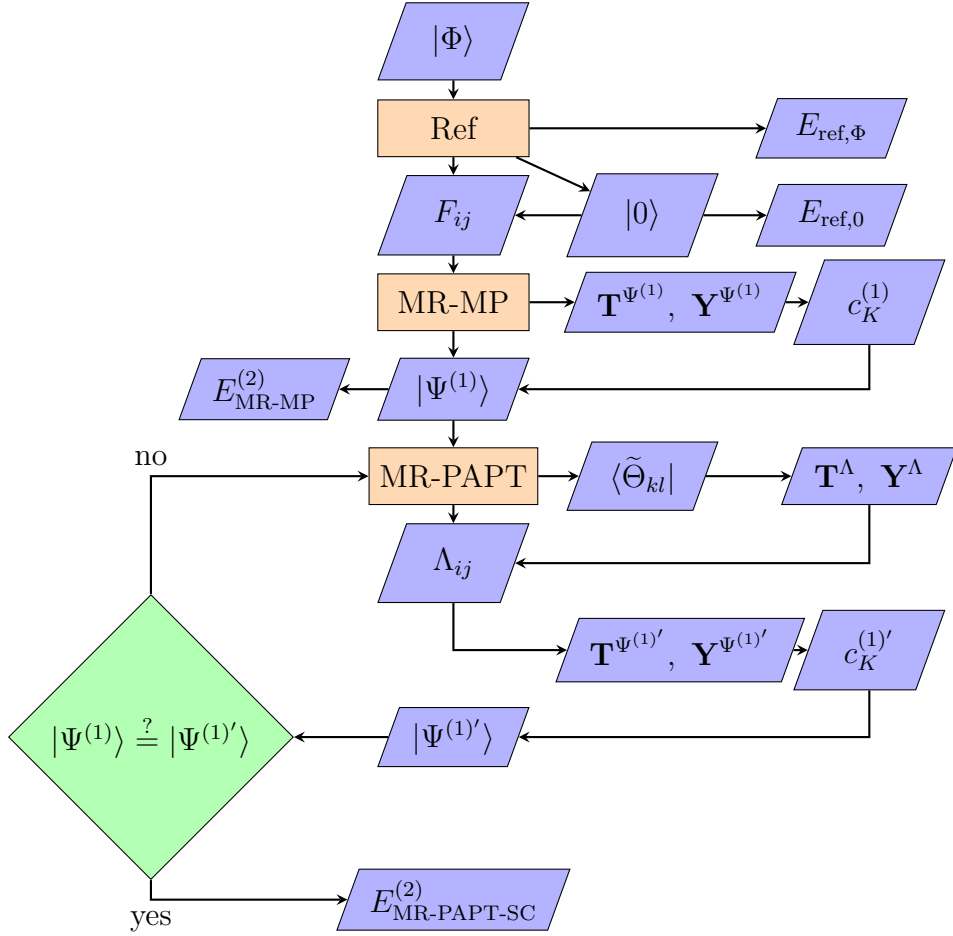


Figure 4.1: The workflow of the MR-PAPT working code

As it is illustrated by Fig. 4.1, upon the selection of the pivot determinant within the Φ reference function, reference energies are calculated and the Fockian is constructed using Eqs. (3.41) and (3.42). Then the MR-MP procedure is initialized using the Fockian, which is constructed using the pMCPT method described in Sec. 3.2.2.3. The MR-MP procedure is used to create the $\Psi^{(1)}$ first-order correction wavefunction and obtain the underlying $c_K^{(1)}$ coefficients by solving the matrix equation involving $\mathbf{T}^{\Psi^{(1)}}$ and $\mathbf{Y}^{\Psi^{(1)}}$ as shown in Eq. (3.47). Then $\Psi^{(1)}$ is used to initiate the MR-PAPT procedure. It entails construction of the $\langle\tilde{\Theta}_{kl}|$ projection functions followed by the solution of the matrix equation involving $\mathbf{T}^\Lambda$ and $\mathbf{Y}^\Lambda$ – e.g. via the singular-value decomposition of the former – to obtain the set of Λ_{ij} parameters. This defines the PAPT partitioning and allows for the calculation of the $c_K^{(1)'}$ coefficients yielding the $\Psi^{(1)'}$ first-order PAPT correction wavefunction, again

through the solution of the corresponding matrix equation

$$\sum_{K \neq 0} \underbrace{\langle \widetilde{L}' | \hat{\Lambda} - E^{(0)} | K' \rangle}_{\mathbf{T}_{KL}^{\Psi^{(1)'}}} c_K^{(1)'} = - \underbrace{\langle \widetilde{L}' | \hat{H} | \Phi \rangle}_{\mathbf{Y}_L^{\Psi^{(1)'}}} \quad (4.15)$$

involving $\mathbf{T}^{\Psi^{(1)'}}$ and $\mathbf{Y}^{\Psi^{(1)'}}$. The solution of the PAPT equations is set to be self-consistent iteratively by constructing the projection functions from $\Psi^{(1)'}$ in a consecutive PAPT procedure until the first-order correction wavefunction and correspondingly, the second-order energy converges.

The MR-PAPT working code was tested against the SR-PAPT working code that was in return tested against the implementation of P. Knowles. The bug-free character of the MR extension was further checked by inspecting and cross-checking the intermediate quantities generated during the MR-PAPT procedure for simple, 2- and 4-electron systems in minimal basis set. Since the routines underlying the MR-PAPT implementations evaluate matrix elements taken with determinants of any number of electron, it can be positively expected that correcting bugs for few-electron systems ensures bug-free nature for system with more electrons and/or orbitals, too.

4.1.1 On the vanishing third-order energy correction

Let us inspect the third-order energy correction since the guideline of extending the PAPT procedure to the MR case is to set it zero upon self-consistency analogously to SR-PAPT. Starting with the traditional RSPT formula

$$E^{(3)} = \langle \Psi^{(0)} | \hat{H} | \Psi^{(2)} \rangle, \quad (4.16)$$

the reference function Φ serves as the zero-order wavefunction and the second-order correction vector, orthogonal to Φ , expanded on the basis of K' projected excited determinants reads

$$|\Psi^{(2)}\rangle = \sum_{K \neq 0} c_K^{(2)} |K'\rangle. \quad (4.17)$$

The $c_K^{(2)}$ second-order coefficients are expressed as

$$c_K^{(2)} = - \sum_{L \neq 0} G_{KL} \langle \widetilde{L}' | \hat{W} - E^{(1)} | \Psi^{(1)} \rangle, \quad (4.18)$$

with matrix $\mathbf{G}$ introduced in Eq. (4.11). Substituting Eqs. (4.17) and (4.18) into Eq. (4.16) yields

$$E^{(3)} = \sum_{L \neq 0} \left[- \underbrace{\sum_{K \neq 0} \langle \Phi | \hat{H} | K' \rangle G_{KL}}_{\widetilde{c}_L^{(1)}} \right] \langle \widetilde{L}' | \hat{W} - E^{(1)} | \Psi^{(1)} \rangle, \quad (4.19)$$

where the $\tilde{c}_L^{(1)}$ coefficient introduced in Eq. (4.10) is recognized. Consequently, Eq. (4.20) simply reduces to

$$E^{(3)} = \underbrace{\sum_{L \neq 0} \tilde{c}_L^{(1)} \langle \tilde{L}' |}_{\langle \tilde{\Psi}^{(1)} |} \hat{W} - E^{(1)} | \Psi^{(1)} \rangle, \quad (4.20)$$

where the bra counterpart of $\Psi^{(1)}$ is introduced as

$$\langle \tilde{\Psi}^{(1)} | = \sum_{L \neq 0} \tilde{c}_L^{(1)} \langle \tilde{L}' |. \quad (4.21)$$

This allows to simplify Eq. (4.20) to

$$E^{(3)} = \langle \tilde{\Psi}^{(1)} | \hat{W} - E^{(1)} | \Psi^{(1)} \rangle. \quad (4.22)$$

Since in Eq. (4.22), $\hat{W}$ can be replaced with $\hat{H} - \hat{H}^{(0)}$, the third-order energy correction vanishes if

$$\langle \tilde{\Psi}^{(1)} | \hat{H}^{(0)} | \Psi^{(1)} \rangle = \langle \tilde{\Psi}^{(1)} | \hat{H} - E^{(1)} | \Psi^{(1)} \rangle \quad (4.23)$$

holds. However, such a zero-order Hamiltonian cannot be given in a straightforward manner as there is only one equation, i.e. the problem is underdefined for the number of n_{basis}^2 parameters in the zero-order. One could separate each term of the summation for the singly and doubly excited determinants and set up an equation for each determinant, but then the number of equations would not match the number of parameters in general. In typical applications, the former would be larger than the latter, i.e. the problem would be overdefined.

This logical barrier is elevated by following the philosophy introduced by Knowles in the single-reference version. In the same manner, projection functions

$$\langle \tilde{\Theta}_K | = \sum_{L \neq 0} \tilde{c}_L^{(1)} \langle \tilde{L}' | \quad (4.24)$$

with restricted summations are introduced. These correspond to $\langle \tilde{\Theta}_{aa} |$ and $\langle \tilde{\Theta}_{pp} |$ in Eqs. (4.7) and (4.8), respectively.

Once projection functions are expressed with MO indices for the $a = b$ and $p = q$ case, it is straightforward to obtain the expression for $a \neq b$ and $p \neq q$. In analogy with SR-PAPT, the outer MO indices associated with $\tilde{c}_L^{(1)}$ and $\langle \tilde{L}' |$ are allowed to differ. Note, that projection functions with $a \neq b$ and $p \neq q$ have no contribution to $E^{(3)}$. Extending the set of projection functions with singly excited determinants, cf. Eq. (4.9), equations are introduced in the same number as the Λ_{ij} parameters to construct the zero-order Hamiltonian with. Replacing $\langle \tilde{\Psi}^{(1)} |$ in Eq. (4.23) with the $\langle \tilde{\Theta}_K |$ projection functions of Eqs. (4.7), (4.8) and (4.9) yields the MR version of the PAPT equations (see Eq. (4.14)).

4.1.2 On the redundancy of projection functions

Similarly to the single-reference case, the set of projection functions is affected by redundancy since

$$\sum_a^{\text{occ}} \langle \tilde{\Theta}_{aa}^+ | = \sum_p^{\text{virt}} \langle \tilde{\Theta}_{pp}^+ |. \quad (4.25)$$

This sets the coefficient matrix of Eq. (4.14) singular. In particular, one singular value of matrix $\mathbf{T}^\Lambda$ is zero as a consequence.

This is concluded to have no effect on the PT corrections obtained with the Λ_{ij} parameters set by PAPT [3], if the P -space contribution of the zero-order Hamiltonian is shifted by its expectation value taken with the pivot determinant. Accordingly, Eq. (4.4) is replaced by a shifted zero-order Hamiltonian reading as

$$\hat{H}_s^{(0)} = E_{0,s}^{(0)} \hat{O} - \hat{P} \left[\hat{\bar{F}} - \langle 0 | \hat{\bar{F}} | 0 \rangle \right] \hat{P}. \quad (4.26)$$

In the above, $\hat{\bar{F}}$ represents both the $\hat{F}$ generalized Fockian applied in MR-MP and the $\hat{\Lambda}$ operator appearing in the MR-PAPT working formulae.

Since the P -space component of $\hat{H}_s^{(0)}$ involves an excitation energy at the one-electron level, it is plausible to set

$$E_{0,s}^{(0)} = 0. \quad (4.27)$$

This ensures that

$$\hat{\bar{F}} - \langle 0 | \hat{\bar{F}} | 0 \rangle \quad (4.28)$$

enters without modification in the zero-order reduced resolvent. A non-zero $E_0^{(0)}$ would introduce a further shift, c.f. the LHS of Eq. (3.43).

It is noted here, that one may envisage subtracting a quantity different from $\langle 0 | \hat{\bar{F}} | 0 \rangle$ from $\hat{H}^{(0)}$. The version according to Eq. (4.26) was implemented in order to develop the working formulae of MR-PAPT with the least modification of the single-reference variant of PAPT, that is constructed with normal-ordered operators. In the single-reference case, as presented in Eqs. (2.26) and (2.27), the expectation value that was subtracted from the operators was taken with the HF determinant. Eq. (4.26) is analogously constructed with the subtraction of the expectation value taken with the pivot determinant of the pMCPT method.

Up to this point the freedom in specifying $\hat{H}^{(0)}$ was considered. Another scalar of energy dimension is up to choice, that is the zero-point of the energy scale. This quantity

is set to ensure

$$E^{(1)} = 0, \quad (4.29)$$

in parallel with the MCPT formulae presented in Sec. 3.2.2.

A constant shift in the total Hamiltonian is accordingly introduced as

$$\hat{H}_s = \hat{H} - \langle \Phi | \hat{H} | \Phi \rangle. \quad (4.30)$$

Measuring the energy from $\langle \Phi | \hat{H} | \Phi \rangle$, the shifted zero-order ground state energy brings no contribution in agreement with Eq. (4.26).

Obviously,

$$E^{(1)} = \langle \Phi | \hat{H}_s - \hat{H}_s^{(0)} | \Phi \rangle = 0 \quad (4.31)$$

since both operators have vanishing expectation value with Φ .

For the shifted version of Eqs. (3.44) and (4.15), shifted Fockian and $\hat{\Lambda}$ operators need to be expressed as

$$\begin{aligned} \hat{\bar{F}}_s &= \hat{\bar{F}} - \langle 0 | \hat{\bar{F}} | 0 \rangle \\ &= \sum_{i,j} [j^+ i^- - \langle 0 | j^+ i^- | 0 \rangle] \bar{F}_{ji}. \end{aligned} \quad (4.32)$$

The shifted $c_{K,s}^{(1)}$ coefficients of the first-order wavefunction correction of MR-MP and PAPT arise as

$$\sum_{K \neq 0} \underbrace{\langle \tilde{L} | \hat{\bar{F}}_s | K' \rangle}_{\mathbf{T}_{KL}^{\Psi(1),s}} c_{K,s}^{(1)} = - \underbrace{\langle \tilde{L} | \hat{H} | \Phi \rangle}_{\mathbf{Y}_L^{\Psi(1),s}}, \quad (4.33)$$

where the shifted $\mathbf{T}^{\Psi(1),s}$ coefficient matrix and $\mathbf{Y}^{\Psi(1),s}$ inhomogeneous vector can be introduced. Note that s is omitted from the subscript of $\hat{H}$ on the RHS of Eq. (4.33) on purpose, since a constant has no effect in between orthogonal bra and ket vectors.

As for the MR-PAPT equations that define the PAPT parameters, the shifted version of Eq. (4.14) is introduced as

$$\sum_{i \geq j} \mathbf{T}_{ij,kl}^{\Lambda_s} \Lambda_{kl} = \mathbf{Y}_{kl}^{\Lambda_s}. \quad (4.34)$$

The shifted coefficient matrix is formulated as

$$\mathbf{T}_{ij,kl}^{\Lambda_s} = \sum_{i \geq j} \langle \tilde{\Theta}_{kl}^+ | [j^+ i^- - \langle 0 | j^+ i^- | 0 \rangle + i^+ j^- - \langle 0 | i^+ j^- | 0 \rangle] | \Psi^{(1)} \rangle. \quad (4.35)$$

At difference with the $\mathbf{Y}^{\Psi^{(1)}}$ inhomogeneous vector of the equations that define $\Psi^{(1)}$, the shift of $\hat{H}$ does appear in the inhomogeneous vector of the PAPT-equations reading as

$$Y_{kl}^{\Lambda_s} = \langle \tilde{\Theta}_{kl}^+ | \hat{H}_s | \Psi^{(1)} \rangle. \quad (4.36)$$

Note however, that $E^{(1)}$ can be omitted in Eq. (4.36), due to Eq. (4.31). This leads to

$$Y_{kl}^{\Lambda_s} = \langle \tilde{\Theta}_{kl}^+ | \left[\hat{H} - \langle \Phi | \hat{H} | \Phi \rangle \right] | \Psi^{(1)} \rangle. \quad (4.37)$$

The constant shifts discussed in this Section are intentionally omitted from the formulae introduced at the beginning of Sec. 4.1 in order to keep the presentation transparent.

4.2 Parallelizing the Working Code

The working code that was written in FORTRAN90 language is a modular program system that calculates matrix elements of operators with determinants on the fly. On the one hand, this allows for the straightforward transcription of the developed working formulae into a demo code. This is advantageous as one does not have to deal with rather tedious derivations of matrix elements based on Wick's theorem. On the other hand, such a programming technique results in high computational cost, since instead of direct formulae expressed with quantities indexed by MOs, one has expressions in terms of determinants of the Hilbert space. The latter usually involve extra do loop when compared with the former. The ideal scaling of computation time with system size is achievable only with the former approach.

A standard FORTRAN90 code compiled with e.g. GNUFORTRAN can be considered serial as it runs on only one thread of the CPU and executes commands one by one, consecutively. A simple time-saving approach e.g. when one conducts a numerical test involving a few sterical configurations of a certain many-body test system, is to launch multiple serial calculations at once to utilize not just one but more threads of the CPU.

Another approach to achieve reduced computational time by parallelization is to use message-passing interface (MPI). This is helpful when a section of the serial code can be identified that does not require consecutive serial evaluation but instead can be separated into parts allowing evaluation on multiple threads simultaneously. The message-passing necessary in this case entails communication between the multiple threads of the CPU. Following this latter approach, the MR-PAPT working code was parallelized using the MPICH [47] implementation of MPI that is to be detailed in this Section.

After conducting a few preliminary tests of the serial working code, it was concluded that the construction of the $\mathbf{T}^\Lambda$ coefficient matrix for the MR-PAPT equations (see Eq. (4.14)) is the slowest. It is one of the arrays used in the MR-PAPT procedure that would exhibit computational time scaling with the sixth power of the number of basis functions in the ideal case. Therefore, in order to be able to calculate results for simple test systems with chemically reasonable basis sets in feasible time, the parallelization of the routine that constructs $\mathbf{T}^\Lambda$ was accomplished by MPICH. Each row of the $\mathbf{T}^\Lambda$ coefficient matrix corresponds to the LHS of an equation with one particular $\langle \tilde{\Theta}_{ij}^+ |$ projection function, so distributing the calculation by rows is an adequate way of parallelization.

Fig. 4.2 illustrates the parallelized part of the MR-PAPT working code. One among the threads is usually distinguished, having the role of the "master". The master thread coordinates the others, classified as "slaves". Serial parts of the code preceding and following the parallelized part presented on Fig. 4.2, are evaluated only by the master thread. Slave processes are sent to the beginning of the MR-PAPT routine at the start of the program and are commanded to wait until the master, having calculated all the preceding parts, reaches the parallelized part as well.

Construction of the projection functions and the inhomogeneous vector of the PAPT-equations is also done only by the master thread. Following this, a communication between the threads, coordinated by the master thread takes place, with the aim of passing the necessary variables to the slave threads.

Since slave threads calculate all elements of the $\mathbf{T}^\Lambda$ coefficient matrix in a row, all the determinants within the first-order correction vector need to be available for all slave threads as well as the master thread. Therefore, at the beginning of the parallel execution part, the first-order correction wavefunction that serves as an input to the $\mathbf{T}^\Lambda$ construction routine has to be *broadcasted*, i.e. all slave threads need to make a copy of the arrays and variables obtained from the master thread to their own memory area for their own use.

As for the set of projection functions, each thread requires only that subset of the functions that correspond to the set of rows the given thread is commanded to calculate. Knowing the number of available threads and the number of rows in the coefficient matrix, the master thread *scatters* the total set of projection functions into subsets accordingly. Consequently, scatter-type inter-thread communication entails the distribution of the arrays and variables constituting the subsets of projection functions among all available threads.

When all threads have finished calculation of their batch of the rows of the coefficient matrix, the concluding message-passing task is to *gather* all rows to the memory area of the master thread. Computation then proceeds by calling the DGELSD LAPACK FORTRAN

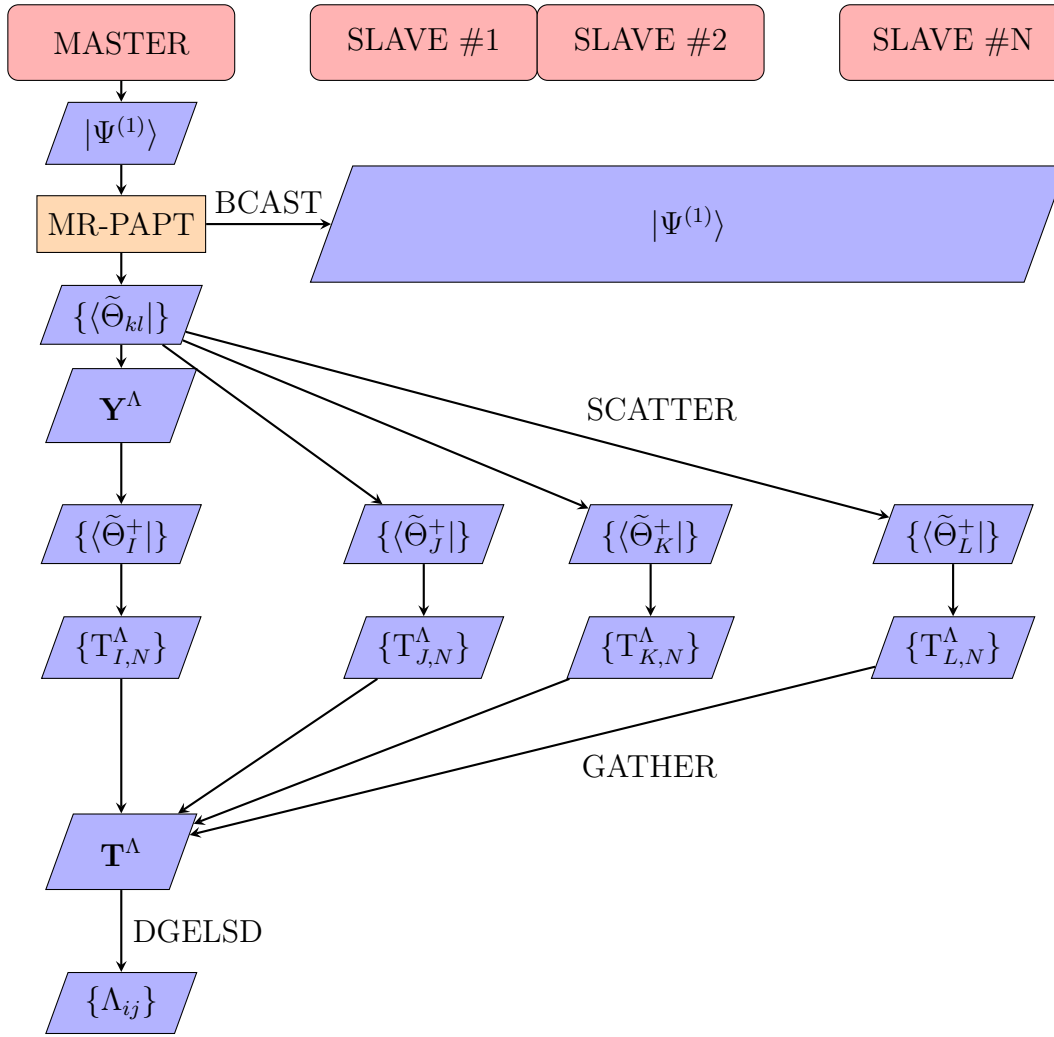


Figure 4.2: The workflow of the parallelized part of the MR-PAPT working code

routine to solve the singular matrix equation and obtain the parameters of MR-PAPT.

To illustrate computational time savings accomplished by the parallelization of one particular section of the working code, Fig. 4.3 presents the decrease of computational time on a logarithmic scale by increasing the number of threads used for the calculations. The test presented was conducted on the ATLAS high-performance computer of ELTE that allows for a maximum number of 128 threads for calculating rows of the $\mathbf{T}^\Lambda$ matrix simultaneously.

In an ideal case, where the time spent on message-passing between the threads is negligible with respect to the total computational time, one expects a reciprocal depen-

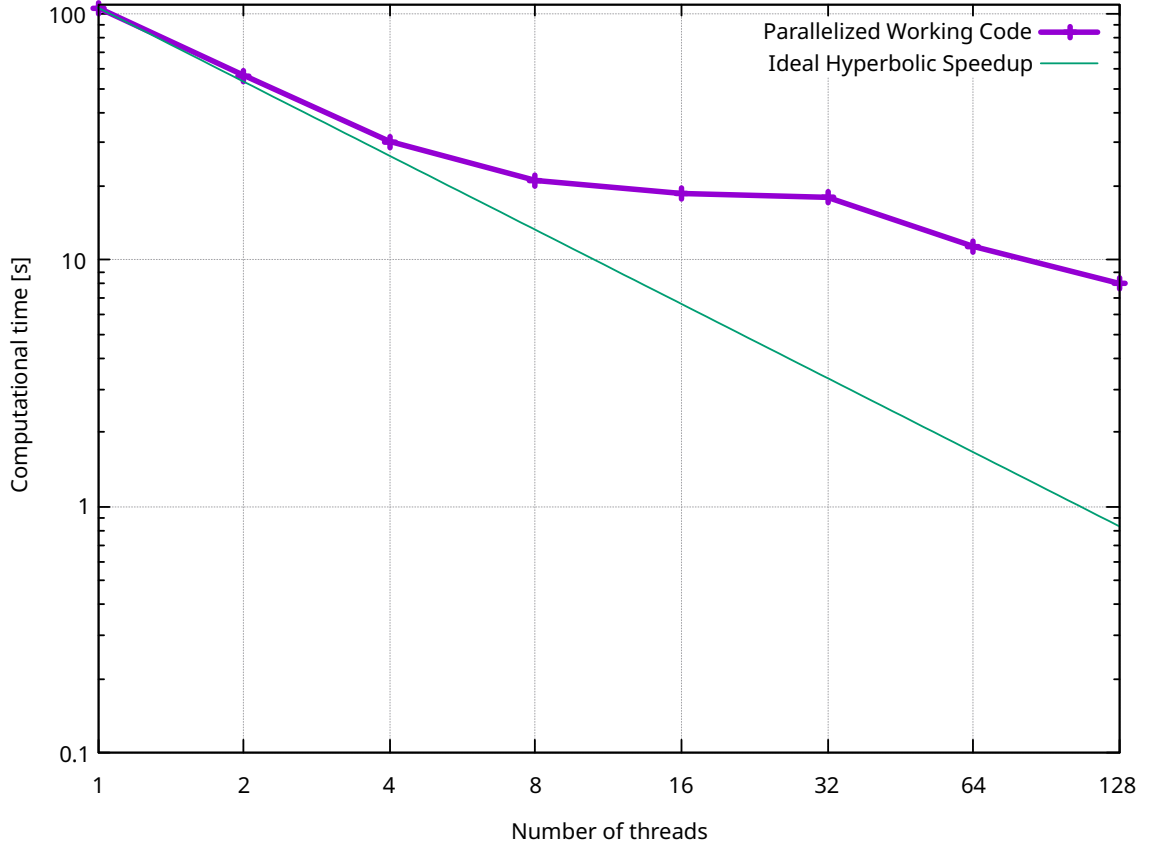


Figure 4.3: Comparison of ideal and actual computational times of the parallelized working code for a BeH_2 calculation in 6-311G basis using various number of threads.

dence of the t computational time on the n_t number of threads as

$$t(n_t) = \frac{t(1)}{n_t}. \quad (4.38)$$

It is correspondingly observed on Fig. 4.3 that computational time reduces quite significantly and in an ideal-like manner by increasing the number of threads from 1 to cca. 8 reassuring that parallelization is executed at an adequate section of the working code.

5. Results and Discussion

Construction and parallelization of the working code was conducted simultaneously with development of the working formulae in order to allow for preliminary test results. In this Chapter, three different kinds of tests of the pMCPT adaptation of PAPT are discussed entailing bond dissociation potential curves, size-consistency as well as transition from a three-atom molecule configuration to a two-atom molecule and a distant single-atom configuration.

All the tests presented here had been carried out with a version of the parallelized working code that constructs projection functions of Eqs. (4.7) and (4.8) with the regular $c_K^{(1)}$ coefficients replacing $\tilde{c}_K^{(1)}$ coefficients, i.e.

$$\langle \tilde{\Theta}_{ab} | = \sum_c^{\text{occ}} \sum_{p,q}^{\text{virt}} c_{pq}^{ac(1)} \langle \tilde{K}_{bc}^{pq'} |, \quad (5.1)$$

$$\langle \tilde{\Theta}_{pq} | = \sum_{a,b}^{\text{occ}} \sum_r^{\text{virt}} c_{pr}^{ab(1)} \langle \tilde{K}_{ab}^{qr'} |. \quad (5.2)$$

Projection functions with singly excited determinants, in accordance with Eq. (4.9) are unaffected by this.

Implementation of the $\tilde{c}_K^{(1)}$ coefficients to the working code is yet to be completed at the time of writing this thesis. In numerical terms, the effect of tilde on $c_K^{(1)}$ is not anticipated to be substantial on the non-iterated MR-PAPT results, while it is inevitable for vanishing of the third-order energy upon iteration.

5.1 Potential energy curve of lithium-hydride

First, let us consider the potential curve for the dissociation of the heteronuclear diatomic lithium-hydride molecule. In the simple LCAO-MO picture, the 4-electron system in equilibrium geometry can be described by a doubly occupied core MO and a doubly occupied bonding MO that is polarized towards the lithium atom. The molecule, being stable but highly reactive, can be associated with a bond order of 1 and a bonding energy

of 243 kJ/mol, correspondingly.

The system is weakly correlated at interatomic distances close to the equilibrium, where the HF determinant is the only dominant term of the CI expansion. At larger distances, the system starts to exhibit some strongly correlated characteristics as the weight of the first (lowest in energy) doubly excited determinant becomes comparable to the weight of the HF determinant.

Fig. 5.1 shows potential energy curves of LiH calculated in correlation-consistent triple-zeta (cc-pVTZ) basis set. It can be observed that the equilibrium bond length is around 1.6 Å. Medium level of correlation is exhibited above cca. 2.5 Å, indicated by the increasing distance between the HF and the FCI curves. Single-reference second-order PT calculations (SR-MP2 and SR-PAPT2) can account for dynamic correlation but run off roughly in parallel with the HF curve in the strongly correlated regime. In Fig. 5.1 the efficiency of SR-PAPT2 is well-manifested, it is able to increase the accuracy of SR-MP2 and provide results that are an order of magnitude closer to FCI unless the system necessitates the consideration of static correlation (above cca. 2.5 Å).

As for the multi-reference methods, a two-electron two-orbital CAS calculation as detailed in Sec. 3.1.2, composed of two determinants can account for the static correlation of the LiH system. This is manifested in Fig. 5.1 in the CAS(2,2) curve staying closer to FCI in the dissociated regime than HF. Overall, the CAS(2,2) curve is observed to be more parallel to the FCI curve than the SR curves. At the same time, CAS(2,2) does not account for sufficient dynamic correlation, as reflected by the significant non-parallelity of the difference curve in Fig. 5.2. Still, it can serve as the reference function for the calculation of perturbative corrections (MR-MP2 and MR-PAPT2) using the pMCPT method as detailed in Sec. 3.2.2.

To illustrate the performance of MR-PAPT with respect to MR-MP and the SR perturbative methods, Fig. 5.2 shows the energy differences from the FCI benchmark of the same dataset as Fig. 5.1. As mentioned before in Sec. 1.3, in order to obtain chemically correct potential curves, a post-HF MR method is required to treat static and dynamic correlation in balance. In other words, besides running close to the zero-point in Fig. 5.2, the more a method’s curve resembles to a straight horizontal line, the better.

Comparing the SR-MP2 curve to the MR-MP2 curve in Fig. 5.2 reveals that the contribution of static correlation allows for a balanced description of the LiH system. MR-MP2 significantly outperforms SR-MP2 throughout the potential curve with ground state energy values of MR-MP2 being consistently only 5 - 10 millihartrees above the FCI results.

As for MR-PAPT, in the weakly correlated regime, the little room left for enhance-

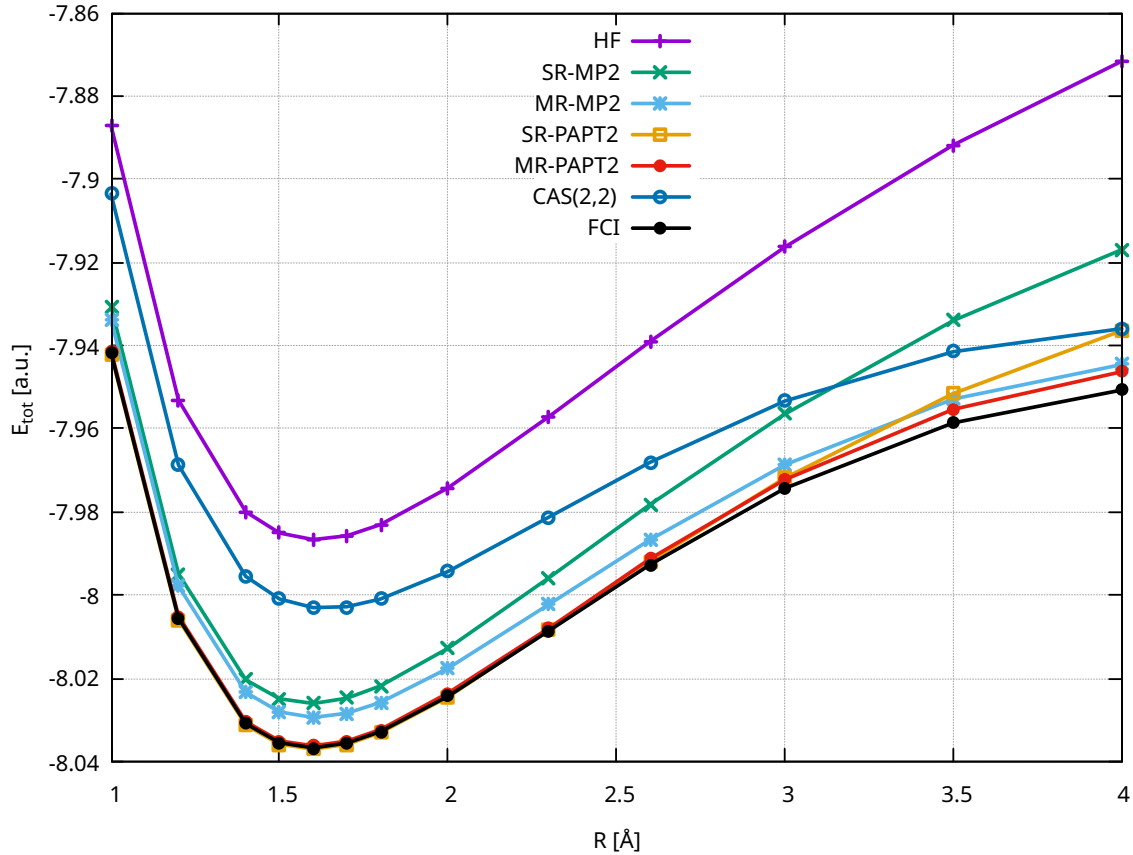


Figure 5.1: Total energies of LiH in cc-pVTZ basis.

ment with respect to SR-PAPT is not exploited well. The MR-PAPT curve is slightly but consistently worse than SR-PAPT at around equilibrium geometry. Implementing the $\tilde{c}_K^{(1)}$ in the projection functions might have a beneficial effect on this result. Comparing with MR-MP2, MR-PAPT2 is observed to bring significant improvement with deviations from FCI under the 5 millihartree lower limit of MR-MP2 in all points, and even under 1 millihartree in some points. However, the deviation from FCI is not balanced, as there is an increasing tendency of the error of MR-PAPT2 as the system exhibits more strongly correlated characteristics.

This feature of the pMCPT-based adaptation of PAPT is most likely to be caused by the enhanced pivot dependence. The pMCPT method itself is not invariant to the choice of the pivot. In case of MR-PAPT, the PAPT equations are formulated utilizing the concept of the pivot determinant when constructing the parameters that define the zero-order problem. In particular, the restricted summations of the projection functions in Eqs. (4.7), (4.8) and (4.9) classify the MOs to be either occupied or virtual with respect to the pivot determinant.

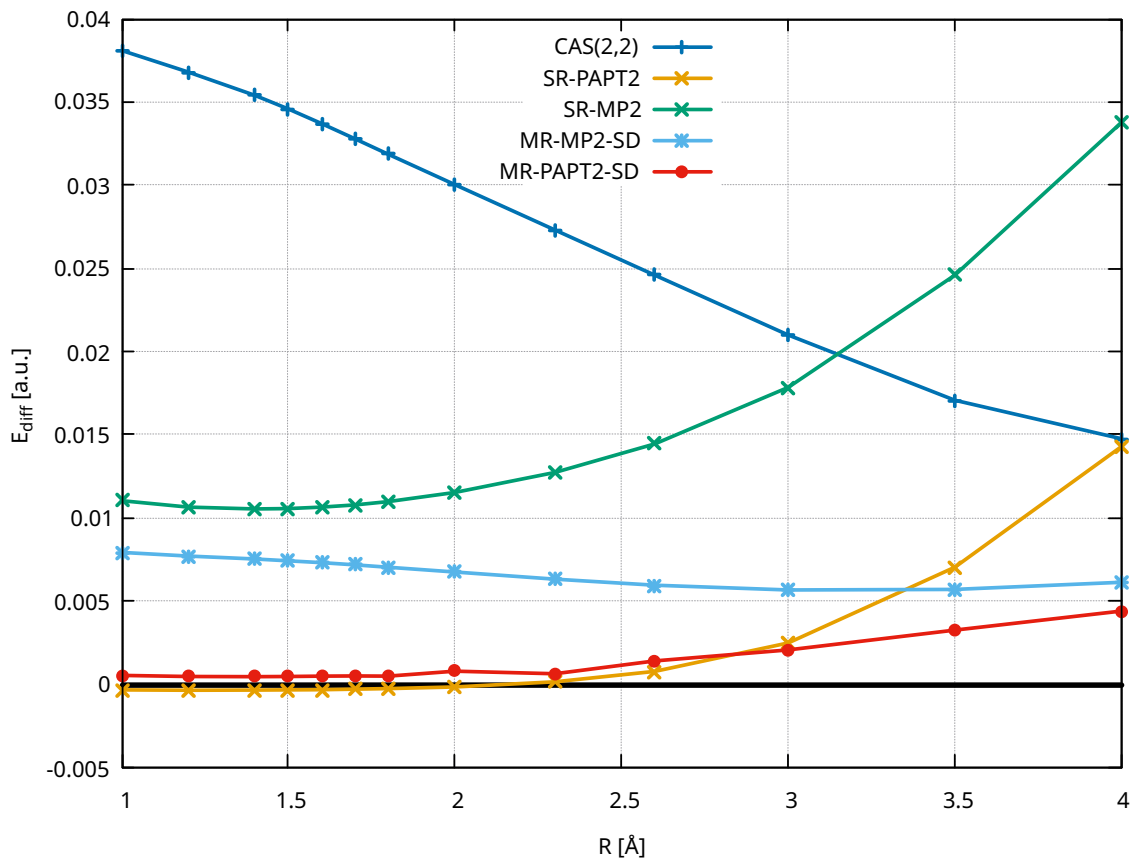


Figure 5.2: Energies differences from Full-CI of LiH in cc-pVTZ basis.

5.2 Size-consistency

An important feature, not necessarily reflected by quantum chemical methods of electronic structure is that extensive quantities should scale properly when extending the system. Terms size-consistency and size-extensivity [48], although being two distinct concepts, are often used interchangeably to describe that characteristic. Let us refer to the strict additive separability of the energy of an $A + B$ supersystem containing two non-interacting systems A and B , i.e.

$$E(A, B) = E(A) + E(B) \quad (5.3)$$

as size-consistency [49].

Theoretically, RSPT can provide size-consistent results with proper choice of zero-order and careful computational implementation [50]. Accordingly, in the single-reference approach, both MP and PAPT are considered to be size-consistent. However, the pMCPT method, that is used to formulate the MR extension of MP and PAPT in this thesis, was shown to be slightly size-inconsistent [51].

To assess whether MR-PAPT formulated in the MCPT framework improves or deteriorates on inconsistency, tests on two distant, non-interacting LiH molecules was conducted. Using a correlation-consistent basis set, computational limitations allowed for frozen core calculations for the dimer and the monomer in double-zeta (cc-pVDZ) basis as presented in Table 5.1. Calculations without freezing the core MOs were performed for the same system in Pople’s 6-31G double-zeta basis set, presented in Table 5.2.

Method	Monomer	Dimer	Error
	E_{mon}	E_{dim}	$ E_{\text{dim}} - 2E_{\text{mon}} $
	[a.u.]	[a.u.]	[a.u.]
CAS	-7.99221261	-15.98442524	$1.9 \cdot 10^{-8}$
MR-MP2	-8.00063141	-16.00072131	$5.4 \cdot 10^{-4}$
MR-PAPT2	-8.00354392	-16.00634667	$7.4 \cdot 10^{-4}$

Table 5.1: Size-consistency error test results on LiH monomer and non-interacting dimer in cc-pVDZ basis with frozen core calculations. Bond length of the monomer units is 2.00 Å.

Method	Monomer	Dimer	Error
	E_{mon}	E_{dim}	$ E_{\text{dim}} - 2E_{\text{mon}} $
	[a.u.]	[a.u.]	[a.u.]
CAS	-7.9892728	-15.9785459	$3.1 \cdot 10^{-7}$
MR-MP2	-7.9910346	-15.9819499	$1.2 \cdot 10^{-4}$
MR-PAPT2	-7.9914723	-15.9827846	$1.6 \cdot 10^{-4}$

Table 5.2: The same as Table 5.1 in 6-31G basis.

Inconsistency values for the CAS reference in Tables 5.1 and 5.2 reflect the numerical precision of the optimization procedure. This method is size-consistent with proper choice of active electrons and orbitals. In this case a CAS(2,2) model was adopted for the monomer and a CAS(4,4) was accordingly specified for the non-interacting dimer.

Results of both size-consistency tests are in agreement with the conclusion that the inconsistency of the MCPT framework is an order of magnitude smaller than the second-order correction and it is not significantly enlarged by the MR adaptation of PAPT.

5.3 Geometry points of berillium-hydride

As a modest size test case of covalent bond cleavage and formation, nine standardized geometry points (A-I) were devised by Purvis and Bartlett [52]. The points describe different configurations along the reaction path where the linear BeH_2 molecule is distorted into a H_2 molecule and distant Be atom. This is considered to be a challenging inherent MR problem of medium-level correlation (based on the Kapuy lecture held by Gustavo Scuseria in 2023 at Eötvös Loránd University, Budapest). In this test, the Be atom lays in the origin and the two H atoms are placed symmetrically to the z-axis with coordinates in atomic unit $(0, \pm 2.54, 0)$, $(0, \pm 2.08, 1.0)$, $(0, \pm 1.62, 2.0)$, $(0, \pm 1.39, 2.5)$, $(0, \pm 1.275, 2.75)$, $(0, \pm 1.16, 3.0)$, $(0, \pm 0.93, 3.5)$, $(0, \pm 0.70, 4.0)$, and $(0, \pm 0.70, 6.0)$, respectively (see Fig. 5.3). The process can also be considered as the perpendicular insertion of a Be atom into a H_2 molecule starting from point I, and proceeding towards point A.

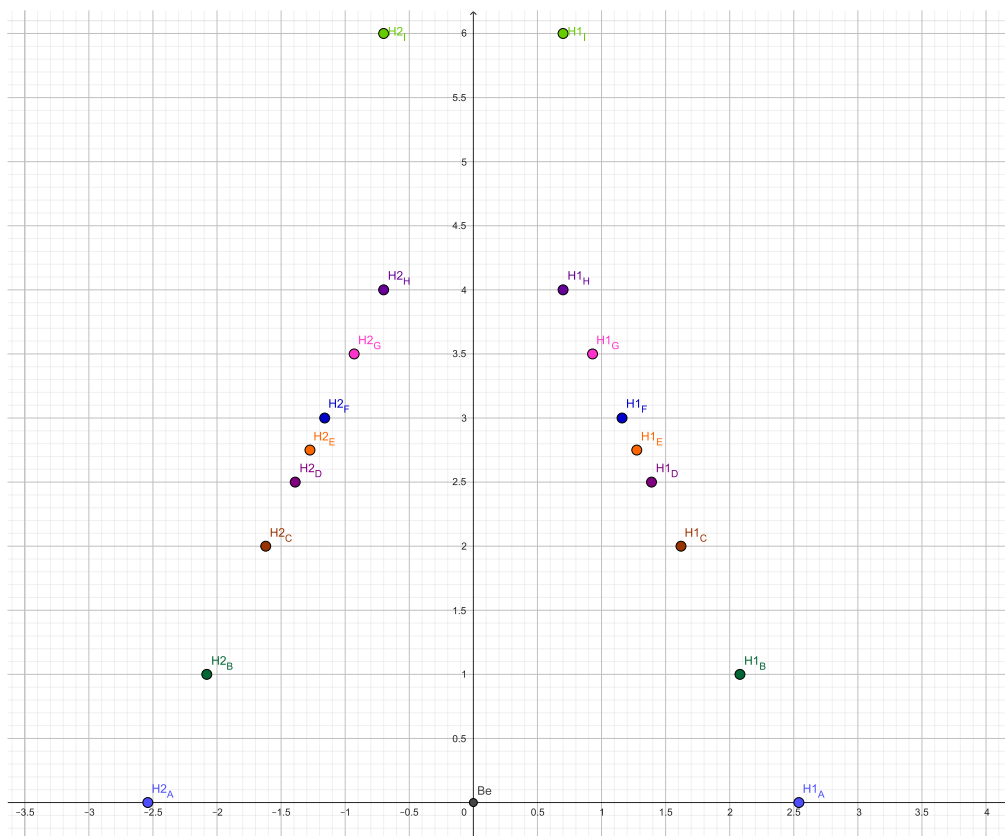


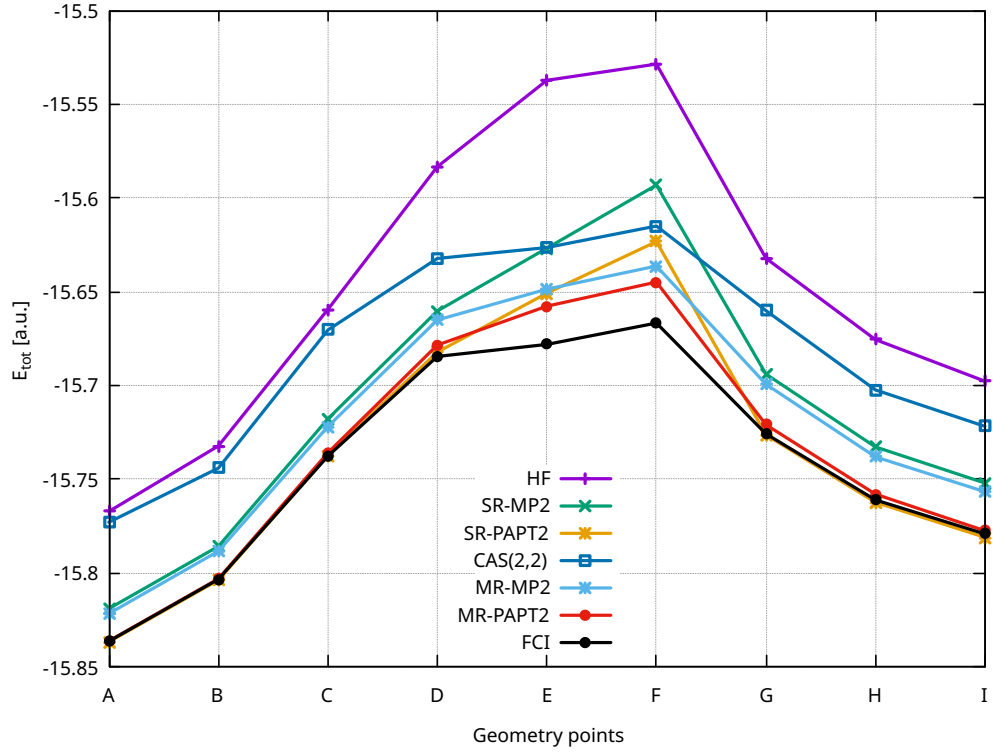
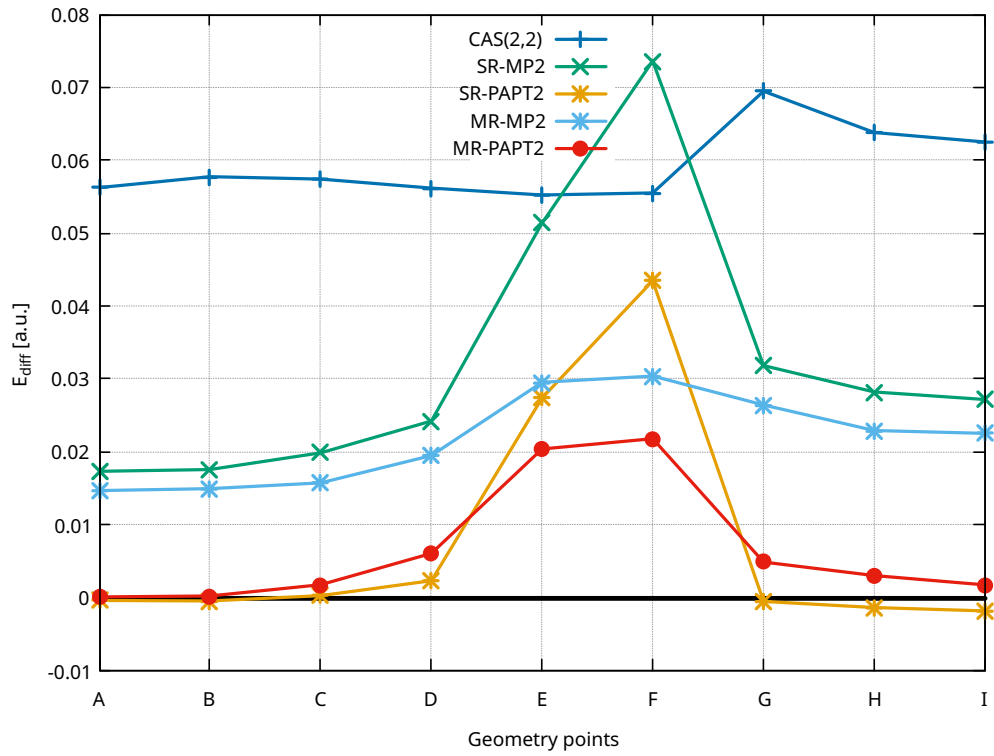
Figure 5.3: Geometry points of BeH_2 .

Assessment of the performance of MR-PAPT, on the geometry points A-I of the BeH_2 molecule in cc-pVDZ basis set is presented in Figs. 5.4 and 5.5. Total energies are displayed in Fig. 5.4 while Fig. 5.5 depicts difference from FCI.

As can be observed in Fig. 5.4, the single reference approach gradually breaks down as the distortion takes place. The reason behind this phenomenon is that two determinants become comparable in weight in the FCI expansion. From this perspective, the middle points E and F are the most challenging, exhibiting strongly correlated characteristics illustrated by the increasing distance between the benchmark FCI curve and the SR curves (HF, SR-MP2, SR-PAPT2).

Similarly as observed in Sec. 5.1, a two-electron two-orbital CAS calculation can correct the shape of the SR curves with respect to FCI thanks to the description of static correlation. The fact that CAS(2,2) is off from FCI by cca. 100 millihartree reflects the lack of dynamical correlation.

Further evaluation is assisted by Fig. 5.5 showing that for the weakly correlated points (A-D and G-I) the performance of SR-PAPT2 is superior to MR-PAPT2 in both running close to FCI with errors under 5 millihartree and exhibiting balanced description, i.e. being more horizontal than SR-MP2. As for the MR methods, similar conclusions can be drawn than in case of the LiH potential curve detailed in Sec. 5.1: MR-PAPT2 accounts for a significantly greater part of the total correlation with only slightly being inferior to SR-PAPT in the weakly correlated regime. In the strongly correlated points, MR-PAPT2 outperforms MR-MP2 but its overall balance is significantly worse which can be attributed to the enhanced pivot-dependence.


 Figure 5.4: Total energies of BeH₂ in cc-pVDZ basis.

 Figure 5.5: Energies differences from Full-CI of BeH₂ in cc-pVDZ basis.

6. Conclusion and Future Directions

As it was previously observed [3], single-reference PAPT is a viable candidate for multi-reference extension. This statement is further supported by the results presented in Ch. 5 that illustrate the promising properties of SR-PAPT. In particular, SR-PAPT accounts for a significantly greater part of the dynamic correlation than the Møller-Plesset partitioning deteriorating only in strongly correlated scenarios, a direct consequence of the limitation of the single-reference approach itself.

To develop a multi-reference version of PAPT, the philosophy of SR-PAPT is found to be straightforwardly extendable in the projected Multiconfigurational PT framework for an arbitrary multiconfigurational reference function. The working formulae of MR-PAPT presented in Ch. 4 are devised in a way that allows to preserve the most prominent features of SR-PAPT in the MR approach, e.g. the general superiority to the MP partitioning and the vanishing of the third-order energy ensured by self-consistency.

The computational cost when optimally coded, is expected to scale in accordance with the SR version, with the sixth power of the number of basis functions also introducing a linear increase with the number of determinants in the CI expansion of the reference function. Preliminary tests performed with a parallelized working code indicate the mitigation of the ill-effect of SR-PAPT in strongly correlated scenarios, capitalizing on the enhanced description of static correlation by the reference function while maintaining the advance from MR-MP in accounting for the dynamic correlation. Additionally, the inherent slight size-inconsistency is not observed to be enlarged.

On the other hand, the parallelity of MR-PAPT with FCI compared to MR-MP significantly deteriorates in the strongly correlated scenarios. This can be attributed to the enhanced pivot dependence of the MCPT framework caused by utilizing the pivot determinant when obtaining the PAPT parameters for the zero-order problem. This may be rectified by completely abandoning the concept of the pivot. While a suitable variant of MCPT has been worked out recently [53], generalization of the PAPT equations for this case is less straightforward. Further considerations along this line are among our future plans.

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Appendix

A Reciprocal vectors in the pMCPT formulae

The overlap of the projected determinants in pMCPT method yields

$$S_{0K} = \langle 0|K' \rangle = \underbrace{\langle 0|K \rangle}_0 - c_K \langle 0|\Phi \rangle = - \sum_L c_K c_L \langle 0|L \rangle = 0 \quad \forall K \neq 0, \quad (1)$$

but

$$S_{KL} = \langle K'|L' \rangle = \underbrace{\langle K|L \rangle}_{\delta_{KL}} - c_L \underbrace{\langle K|\Phi \rangle}_{c_K} - c_K \underbrace{\langle \Phi|L \rangle}_{c_L} + c_K c_L \underbrace{\langle \Phi|\Phi \rangle}_1 = \delta_{KL} - c_K c_L \quad \forall K, L \neq 0. \quad (2)$$

The choice of

$$\langle \widetilde{K}' | = \langle K | - \frac{c_K}{c_0} \langle 0 |, \quad (3)$$

that corresponds to

$$\begin{aligned} \langle \widetilde{K}' | &= \sum_{J \neq 0} (S^{-1})_{JK} \langle J' | \\ &= \langle K' | - \frac{c_K}{c_0^2} \sum_{J \neq 0} c_J \langle J' | \end{aligned} \quad (4)$$

being the reciprocal vectors used in the working formulae of the pMCPT method can, therefore, be justified by

$$\begin{aligned} \langle \widetilde{K}' | L' \rangle &= \left(\langle K | - \frac{c_K}{c_0} \langle 0 | \right) \left(|L \rangle - c_L |\Phi \rangle \right) \\ &= \underbrace{\langle K | L \rangle}_{\delta_{KL}} - \frac{c_K}{c_0} \underbrace{\langle 0 | L \rangle}_0 - \underbrace{\langle K | \Phi \rangle}_{c_K} c_L + \frac{c_K c_L}{c_0} \underbrace{\langle 0 | \Phi \rangle}_{c_0} \\ &= \delta_{KL} - c_K c_L + c_K c_L = \delta_{KL} \end{aligned} \quad (5)$$

the fact that it ensures orthonormality.

B Intermediars in the SR-PAPT working code

Computation time is reduced, if in the cycles constructing the normal-ordered version of the

$$Y_{kl}^{\Lambda} = \langle \tilde{\Theta}_{kl} | \hat{H} | \Psi^{(1)} \rangle \quad (1)$$

$\mathbf{Y}$ inhomogenous vector (considering kl as a hyperindex) to

$$Y_{kl}^{\Lambda_N} = \langle \tilde{\Theta}_{kl} | [\hat{H} - \langle 0 | \hat{H} | 0 \rangle] | \Psi^{(1)} \rangle \quad (2)$$

of the single-reference PAPT equations, suitable intermediars are found, that reduce the overall $(n_{\text{basis}})^7$ cost to $(n_{\text{basis}})^6$.

As for the inhomogenous part with occupied indices

$$\begin{aligned} Y_{ab}^{\Lambda_N} = & -8 \sum_{c,d,p,q}' F_{dc} c_{pq}^{ac} \bar{c}_{pq}^{bd} - 4 \sum_{c,d,p,q}' F_{db} c_{pq}^{ac} \bar{c}_{pq}^{dc} - 4 \sum_{c,d,p,q}' F_{da} c_{pq}^{bc} \bar{c}_{pq}^{dc} \\ & + 4 \sum_{c,p,q,r}' F_{pr} (c_{rq}^{cb} \bar{c}_{qp}^{ac} + c_{qr}^{cb} \bar{c}_{pq}^{ac} + c_{rq}^{ca} \bar{c}_{qp}^{bc} + c_{qr}^{ca} \bar{c}_{pq}^{bc}) \\ & + 4 \sum_{c,d,p,q,r}' [pd|cr] (\bar{c}_{qp}^{ac} \bar{c}_{qr}^{bd} + \bar{c}_{qp}^{bc} \bar{c}_{qr}^{ad}) \\ & - 4 \sum_{c,d,p,q,r}' [pd|rc] (c_{qp}^{ac} \bar{c}_{qr}^{bd} + c_{pq}^{ac} \bar{c}_{rq}^{bd} + c_{qp}^{bc} \bar{c}_{qr}^{ad} + c_{pq}^{bc} \bar{c}_{rq}^{ad}) \\ & + 4 \sum_{c,d,p,q,r}' [pd|br] \bar{c}_{pq}^{ac} \bar{c}_{qr}^{cd} - 4 \sum_{c,d,p,q,r}' [pd|rb] (c_{pq}^{ac} \bar{c}_{qr}^{cd} + c_{qp}^{ac} \bar{c}_{rq}^{cd}) \\ & + 4 \sum_{c,d,p,q,r}' [pd|ar] \bar{c}_{pq}^{bc} \bar{c}_{qr}^{cd} - 4 \sum_{c,d,p,q,r}' [pd|ra] (c_{pq}^{bc} \bar{c}_{qr}^{cd} + c_{qp}^{bc} \bar{c}_{rq}^{cd}) \\ & + 4 \sum_{d,e,c,p,q}' [de||bc] c_{pq}^{de} c_{pq}^{ac} + 4 \sum_{d,e,c,p,q}' [de||ac] c_{pq}^{de} c_{pq}^{bc} + 8 \sum_{c,p,q,r,s}' [pq||rs] c_{rs}^{bc} c_{pq}^{ac}, \quad (3) \end{aligned}$$

the following form can be introduced:

$$\begin{aligned} Y_{ab}^{\Lambda_N} = & \sum_{c,p,q}' \left[-\text{IM1}(b, c, p, q) c_{pq}^{ac} \right. \\ & - \left[\text{IM2}(a, c, p, q) c_{pq}^{bc} + \text{IM2}(b, c, p, q) c_{pq}^{ac} \right] \\ & + \left[\text{IM3}(a, c, p, q) \bar{c}_{pq}^{bc} + \text{IM3}(b, c, p, q) \bar{c}_{pq}^{ac} \right] \\ & \left. + \left[\text{IM4}(a, c, p, q) \bar{c}_{qp}^{bc} + \text{IM4}(b, c, p, q) \bar{c}_{qp}^{ac} \right] \right] \end{aligned}$$

$$\begin{aligned}
& + \left[\text{IM5}(a, c, p, q) \bar{c}_{qp}^{bc} + \text{IM5}(b, c, p, q) \bar{c}_{qp}^{ac} \right] \\
& - \left[\text{IM6}(a, c, p, q) c_{qp}^{bc} + \text{IM6}(b, c, p, q) c_{qp}^{ac} \right] \\
& - \left[\text{IM7}(a, c, p, q) c_{pq}^{bc} + \text{IM7}(b, c, p, q) c_{pq}^{ac} \right] \\
& + \left[\text{IM8}(a, c, p, q) \bar{c}_{pq}^{bc} + \text{IM8}(b, c, p, q) \bar{c}_{pq}^{ac} \right] \\
& - \left[\text{IM9}(a, c, p, q) c_{pq}^{bc} + \text{IM9}(b, c, p, q) c_{pq}^{ac} \right] \\
& - \left[\text{IM10}(a, c, p, q) c_{qp}^{bc} + \text{IM10}(b, c, p, q) c_{qp}^{ac} \right] \\
& + \left[\text{IM11}(a, c, p, q) c_{pq}^{bc} + \text{IM11}(b, c, p, q) c_{pq}^{ac} \right] \\
& + \text{IM12}(b, c, p, q) c_{pq}^{ac} \Big], \tag{4}
\end{aligned}$$

where the intermediers are

$$\begin{aligned}
\text{IM1}(b, c, p, q) &= 8 \sum_d' F_{dc} \bar{c}_{pq}^{bd} \\
\text{IM2}(a, c, p, q) &= 4 \sum_d' F_{da} \bar{c}_{pq}^{dc} \\
\text{IM3}(a, c, p, q) &= 4 \sum_r' F_{pr} c_{rq}^{ac} \\
\text{IM4}(a, c, p, q) &= 4 \sum_r' F_{pr} c_{qr}^{ac} \\
\text{IM5}(a, c, p, q) &= 4 \sum_{d,r}' [pd|cr] \bar{c}_{qr}^{ad} \\
\text{IM6}(a, c, p, q) &= 4 \sum_{d,r}' [pd|rc] \bar{c}_{qr}^{ad} \\
\text{IM7}(a, c, p, q) &= 4 \sum_{d,r}' [pd|rc] \bar{c}_{rq}^{ad} \\
\text{IM8}(a, c, p, q) &= 4 \sum_{d,r}' [pd|ar] \bar{c}_{qr}^{cd} \\
\text{IM9}(a, c, p, q) &= 4 \sum_{d,r}' [pd|ra] \bar{c}_{qr}^{cd} \\
\text{IM10}(a, c, p, q) &= 4 \sum_{d,r}' [pd|ra] \bar{c}_{rq}^{cd}
\end{aligned}$$

$$\begin{aligned}
 \text{IM11}(a, c, p, q) &= 4 \sum'_{d,e} [de||ac] c_{pq}^{de} \\
 \text{IM12}(a, c, p, q) &= 8 \sum'_{r,s} [pq||rs] c_{rs}^{ac},
 \end{aligned} \tag{5}$$

respectively.

The same can be applied to the

$$\begin{aligned}
 Y_{pq}^{\Lambda_N} &= +8 \sum'_{a,b,r,s} F_{rs} c_{pr}^{ab} \bar{c}_{qs}^{ab} + 4 \sum'_{a,b,r,s} F_{qs} c_{pr}^{ab} \bar{c}_{sr}^{ab} + 4 \sum'_{a,b,r,s} F_{ps} c_{qr}^{ab} \bar{c}_{sr}^{ab} \\
 &\quad - 4 \sum'_{a,b,c,r} F_{ca} \left(c_{pr}^{ba} \bar{c}_{rq}^{cb} + c_{pr}^{ab} \bar{c}_{rq}^{bc} + c_{qr}^{ba} \bar{c}_{rp}^{cb} + c_{qr}^{ab} \bar{c}_{rp}^{bc} \right) \\
 &\quad + 4 \sum'_{a,b,c,r,s} [rc|as] \left(\bar{c}_{qs}^{bc} \bar{c}_{pr}^{ba} + \bar{c}_{ps}^{bc} \bar{c}_{qr}^{ba} \right) \\
 &\quad - 4 \sum'_{a,b,c,r,s} [rc|sa] \left(c_{pr}^{ba} \bar{c}_{qs}^{bc} + c_{rp}^{ba} \bar{c}_{sq}^{bc} + c_{qr}^{ba} \bar{c}_{ps}^{bc} + c_{rq}^{ba} \bar{c}_{sp}^{bc} \right) \\
 &\quad + 4 \sum'_{a,b,c,r,s} [qc|bs] \bar{c}_{rp}^{ab} \bar{c}_{rs}^{ac} - 4 \sum'_{a,b,c,r,s} [qc|sb] \left(c_{pr}^{ab} \bar{c}_{sr}^{ac} + c_{rp}^{ab} \bar{c}_{rs}^{ac} \right) \\
 &\quad + 4 \sum'_{a,b,c,r,s} [pc|bs] \bar{c}_{rq}^{ab} \bar{c}_{rs}^{ac} - 4 \sum'_{a,b,c,r,s} [pc|sb] \left(c_{qr}^{ab} \bar{c}_{sr}^{ac} + c_{rq}^{ab} \bar{c}_{rs}^{ac} \right) \\
 &\quad + 4 \sum'_{a,b,r,s,t} [qr||st] c_{st}^{ab} c_{pr}^{ab} + 4 \sum'_{a,b,r,s,t} [pr||st] c_{st}^{ab} c_{qr}^{ab} + 8 \sum'_{a,b,c,d,r} [cd||ba] c_{qr}^{cd} c_{pr}^{ba}
 \end{aligned} \tag{6}$$

part with virtual indices in a similar manner to obtain:

$$\begin{aligned}
 Y_{pq}^{\Lambda_N} &= \sum'_{a,b,r} \left[+ \text{IM1}(a, b, q, r) c_{pr}^{ab} \right. \\
 &\quad + \left[\text{IM2}(a, b, p, r) c_{qr}^{ab} + \text{IM2}(a, b, q, r) c_{pr}^{ab} \right] \\
 &\quad - \left[\text{IM3}(a, b, p, r) c_{rq}^{ab} + \text{IM3}(a, b, q, r) c_{rp}^{ab} \right] \\
 &\quad - \left[\text{IM4}(a, b, p, r) c_{qr}^{ab} + \text{IM4}(a, b, q, r) c_{pr}^{ab} \right] \\
 &\quad \left. + \left[\text{IM5}(a, b, p, r) \bar{c}_{qr}^{ba} + \text{IM5}(a, b, q, r) \bar{c}_{pr}^{ba} \right] \right]
 \end{aligned}$$

$$\begin{aligned}
 & - \left[\text{IM6}(a, b, p, r) \, c_{qr}^{ba} + \text{IM6}(a, b, q, r) \, c_{pr}^{ba} \right] \\
 & - \left[\text{IM7}(a, b, p, r) \, c_{rq}^{ba} + \text{IM7}(a, b, q, r) \, c_{rp}^{ba} \right] \\
 & + \left[\text{IM8}(a, b, p, r) \, \bar{c}_{rq}^{ab} + \text{IM8}(a, b, q, r) \, \bar{c}_{rp}^{ab} \right] \\
 & - \left[\text{IM9}(a, b, p, r) \, c_{rq}^{ab} + \text{IM9}(a, b, q, r) \, c_{rp}^{ab} \right] \\
 & - \left[\text{IM10}(a, b, p, r) \, c_{qr}^{ab} + \text{IM10}(a, b, q, r) \, c_{pr}^{ab} \right] \\
 & + \left[\text{IM11}(a, b, p, r) \, c_{qr}^{ab} + \text{IM10}(a, b, q, r) \, c_{pr}^{ab} \right] \\
 & + \text{IM12}(a, b, q, r) \, c_{pr}^{ba} \Big], \tag{7}
 \end{aligned}$$

where the intermediers are

$$\text{IM1}(a, b, q, r) = 8 \sum_s F_{rs} \, \bar{c}_{qs}^{ab} \tag{8}$$

$$\text{IM2}(a, b, p, r) = 4 \sum_s F_{ps} \, \bar{c}_{sr}^{ab} \tag{9}$$

$$\text{IM3}(a, b, p, r) = 4 \sum_c F_{ca} \, \bar{c}_{pr}^{bc} \tag{10}$$

$$\text{IM4}(a, b, p, r) = 4 \sum_c F_{ca} \, \bar{c}_{rp}^{bc} \tag{11}$$

$$\text{IM5}(a, b, p, r) = 4 \sum_{c,s} [rc|as] \, \bar{c}_{ps}^{bc} \tag{12}$$

$$\text{IM6}(a, b, p, r) = 4 \sum_{c,s} [rc|sa] \, \bar{c}_{ps}^{bc} \tag{13}$$

$$\text{IM7}(a, b, p, r) = 4 \sum_{c,s} [rc|sa] \, \bar{c}_{sp}^{bc} \tag{14}$$

$$\text{IM8}(a, b, p, r) = 4 \sum_{c,s} [pc|bs] \, \bar{c}_{rs}^{ac} \tag{15}$$

$$\text{IM9}(a, b, p, r) = 4 \sum_{c,s} [pc|sb] \, \bar{c}_{rs}^{ac} \tag{16}$$

$$\text{IM10}(a, b, p, r) = 4 \sum_{c,s} [pc|sb] \, \bar{c}_{sr}^{ac} \tag{17}$$

$$\text{IM11}(a, b, p, r) = 4 \sum_{s,t} [pr||st] \, c_{st}^{ab} \tag{18}$$

$$\text{IM12}(a, b, q, r) = 8 \sum_{c,d}' [cd||ba] c_{qr}^{cd}, \quad (19)$$

correspondingly.

This optimization enabled us to run test in larger basis sets for publication. For comparison, presented are here two such examples.

BeH2 Geometry Point D in cc-pVTZ basis	

Y without intermediers (~Nbasis^7)	
real	3m4.131s
user	3m2.308s
sys	0m1.185s
Y with intermediers (~Nbasis^6)	
real	0m53.093s
user	0m51.697s
sys	0m1.110s

Figure 1: Computational times of single-reference BeH₂ molecule in cc-pVTZ basis.

CH4 Tetrahedral in cc-pVTZ basis	

Y without intermediers (~Nbasis^7)	
real	59m19.372s
user	59m12.227s
sys	0m1.082s
Y with intermediers (~Nbasis^6)	
real	6m53.949s
user	6m52.134s
sys	0m1.086s

Figure 2: Computational times of single-reference CH₄ molecule in cc-pVTZ basis.