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Combinatorial Properties and Applications of the Laplace-Pólya Integral

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Abstract

The Laplace-Pólya integral defined by $J_n(r) = \frac{1}{\pi} \int_{-\infty}^{\infty} \operatorname{sinc}^n t \cos(rt) dt$ has several applications in mathematics, physics, and engineering. We study this quantity by combinatorial methods; accordingly, our investigation focuses mainly on the integer values of the function. Our main result establishes a lower bound for the ratio $\frac{J_n(r+2)}{J_n(r)}$ which extends and generalizes the previous estimates by Lesieur and Nicolas [50]. Yet, our approach is purely combinatorial and elementary, standing in stark contrast with their technically demanding proof. This result provides a natural counterpart to the upper bound established in our previous work [3]. The methods leading to the current bounds and their possible improvements are also analyzed in detail. We conclude the work by presenting some applications in combinatorics and geometry.

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1 Introduction

In this work, we investigate a special type of improper integrals. Anyone who studies or researches mathematics will probably get acquainted with integrals of the type

$$\int_0^\infty \left(\frac{\sin t}{t}\right)^n dt \quad (1.1)$$

where n is an arbitrary positive integer. It is really surprising how many fields of mathematics these quantities turn up, including classical and modern analysis, geometry, combinatorics and algebra. Our first encounter with them arose from a geometric problem in which a certain inequality regarding (1.1) was needed, see [3, Theorem 1.4]. At first glance, that problem seemed to be of an analytical nature. However, interesting combinatorial properties related to the integrals (1.1) simplified our original arguments. Motivated by this success, we started to study this topic in more detail.

In the next section, we briefly discuss the significance of the cardinal sine function before introducing the integrals we intend to study in a more general form. In this section, we shall also present our main result, Theorem 2.2. We also list some other integrals related to (1.1) in the hope of providing a basis for a prospective survey on these quantities whose story is really diversified.

In Section 3, the proof of our main result is presented along with a brief summary of the method that leads to the estimates given by Theorem 2.2 and its corresponding pair, [3, Theorem 1.4]. We also discuss a general scheme that may lead to improvements in the future.

There are some combinatorial structures that are connected to integrals of the form (1.1). Among them, we highlight the notion of Eulerian numbers that is introduced in Section 4 along with a list of some of their properties. We are specifically interested in their maxima, which were studied intensively with analytical methods by Lesieur and Nicolas [50]. The main goal of Section 4 is to establish some consequences of our results which lead to entirely combinatorial proofs of the main results presented in [50], whose original proofs utilized quite heavy analytical tools.

Thereafter we discuss another major application, this time in convex geometry. As we have mentioned above, the present work stemmed from the study of central hyperplane sections of the cube. In Section 5, we elaborate on this connection. In particular, we note that Bartha, Fodor and González Merino [8] proved that the volume of central main diagonal hyperplane sections of the n -cube form a strictly monotone increasing sequence for $n \geq 3$. Our long-term goal is to provide a fully combinatorial proof of this statement based on the approach presented in Section 3. Theoretical background of this procedure is presented in this section.

As we have already pointed out, integrals involving the cardinal sine function have numerous applications in different fields of mathematics. Therefore, we hope that our present work may serve as a starting point for future research, which may link these fields and reveal further connections and consequences.

2 The cardinal sine function and related integrals

Before investigating the topic of Laplace-Pólya integrals, we first need to introduce the *unnormalized cardinal sine function*¹, defined by

$$\operatorname{sinc} x := \begin{cases} \frac{\sin x}{x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases} \quad (2.1)$$

One of the most well-known limits in calculus,

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1,$$

implies that $\frac{\sin x}{x}$ has a removable singularity at $x = 0$ and so (2.1) is indeed well defined in the sense of continuity. See its graph displayed in Figure 1. Furthermore, this also shows that sinc is analytic everywhere, hence it is an entire function on the complex plane. Although we are mostly interested in its restriction to real values.

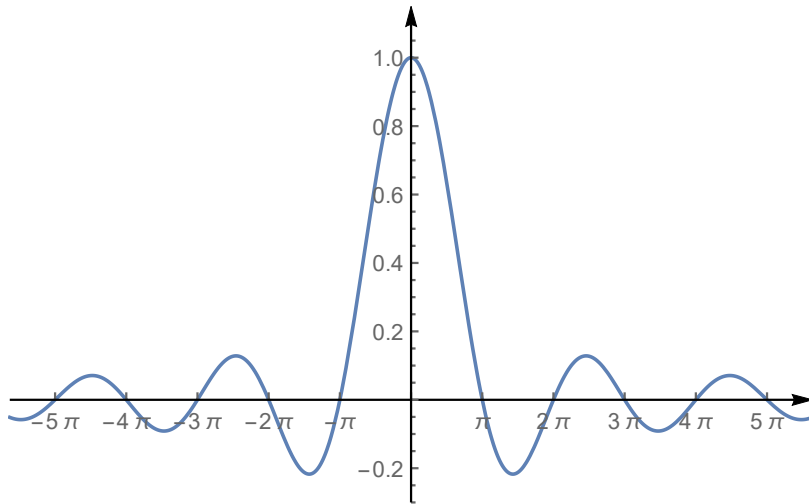


Figure 1: Graph of unnormalized cardinal sine function.

The sinc function was studied first in the 18th century by Euler [30,31], who established some interesting infinite product representations², such as (using the present notation)

$$\operatorname{sinc} x = \prod_{k=1}^{\infty} \left(1 - \left(\frac{x}{\pi k} \right)^2 \right) \quad \text{and} \quad \operatorname{sinc} x = \prod_{k=1}^{\infty} \cos \frac{x}{2^k}.$$

Yet, the definition (2.1) was introduced only in the 20th century, much later than Euler's era: it appeared in the work of Woodward and Davies [73] in the context of

¹Throughout this paper, we will simply refer to this function as *cardinal sine function* or just the sinc function. In this section, we shall give a small remark about the normalized cardinal sine function also.

²These formulae were justified properly later by Weierstraß's factorization theorem.

telecommunication. As they stated: ‘This function occurs so often in Fourier analysis and its applications that it does seem to merit some notation of its own.’

Indeed, many concepts in Fourier analysis are based on the cardinal sine function. For example, the so-called Gibbs phenomenon regarding the convergence of Fourier series can be explained using the integral function of sinc as follows. Consider the Fourier series of a piecewise periodic function with jump discontinuities which converges uniformly on any closed subinterval that does not contain any discontinuities. How does the Fourier series behave on intervals containing points of discontinuities? Obviously, it cannot converge uniformly because the uniform limit of continuous functions also needs to be continuous. The lack of uniformity manifests itself in a particularly dramatic way: as one includes more and more terms, the partial sums overshoot or undershoot the one-sided limits of the function near its discontinuities. Thus, sharp “peaks and valleys” emerge that tend to zero in width, but not in height! See Figure 2, which plots the sign function (periodically) and the 20th partial sum of its Fourier series.

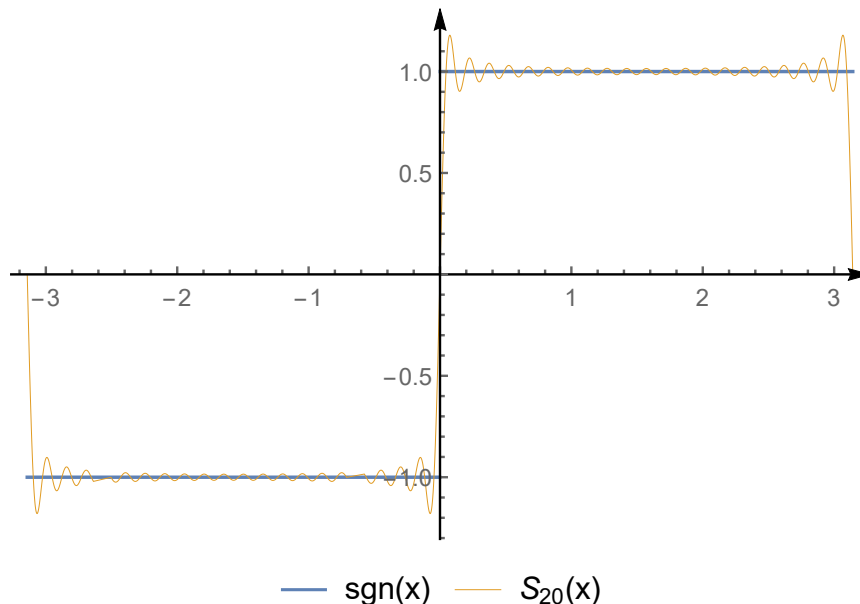


Figure 2: An example of the Gibbs phenomenon.

This surprising observation, known today as the Gibbs phenomenon, is named after Josiah Willard Gibbs [36,37], an American scientist. However, 50 years before his time, an English mathematician, Henry Wilbraham [71] had already notified this behavior. Such eccentricities happen in the history of science.

Quantitatively, the scale of overshoots and undershoots can be expressed with the integral of the cardinal sine function, referred to as the *sine integral*, which is defined by

$$\text{Si}(x) := \int_0^x \text{sinc } t \, dt. \quad (2.2)$$

This function attains its local maxima at the points of odd multiples of π , whereas its local minima are taken at the points of even multiples of π , see Figure 3. However, at none

of these points, or even at any other points³, the integral cannot be expressed explicitly. Only numerical values are known; see [46] for its values in many places. Reasonably, the most famous value is $\text{Si}(\pi)$, which determines the size of overshoots and undershoots of the Fourier series. See for details [60, Subsection 1.2.8] and [69, §5 in Chapter 2].

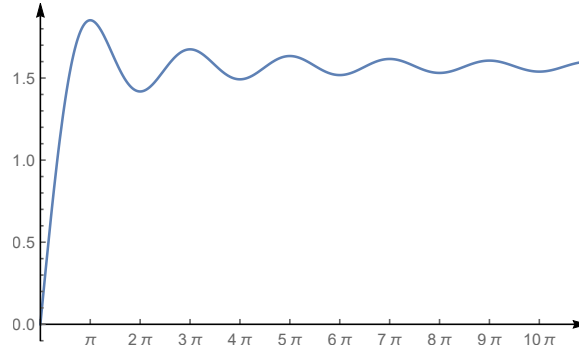


Figure 3: Graph of the sine integral over the interval $[0, 10\pi]$.

In addition to its intrinsic significance, the Gibbs phenomenon is important for (at least) two reasons. First, from a mathematical point of view, it shows precisely how the Fourier series fails to converge uniformly for piecewise smooth functions with jumps. Second, from a physical viewpoint, it explains the phenomenon called “ringing” in electric circuits: a high voltage circuit that is suddenly switched on will throw a spark. For related technical explanations and utilization in circuitry, see [52].

This remark leads us to another important application of Fourier analysis: signal processing. The observation of some event yields certain quantities that depend on time, which are assumed to be measurable. These are simply functions in time, and in practice a signal may represent sound, images, potential fields, current, etc.

One of the simplest signals is the rectangular function, or, by another name, the unit pulse, which is the characteristic function of the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$. In addition to playing a key role in numerous proofs in Fourier analysis, this function is really effective in manipulating signals. On the one hand, it can be used to simplify or cut off some complicated part of a signal via multiplication, while, on the other hand, it works as a building block to model more complex periodic signals.

The Fourier transform of a rectangular pulse is $\text{sinc}(\pi x)$, and vice versa (up to some constant factor). Hence, multiplication with a rectangular pulse in the time domain translates to convolution with sinc in the frequency domain; equivalently, the cardinal sine function works as a filter. See [19, 34] for further details regarding the usage of sinc in signal processing.

In addition to signal processing, there are further applications of the sinc function in approximation theory. Whittaker [70] studied its unique role in the approximation of analytic functions. This theory was further developed by Shannon [66], who studied the general principles of sampling analysis. Through his work, one can see that the functions

³Except at $x = 0$, for which the integral is trivially zero.

$\{\text{sinc}(\pi n)\}_{n \in \mathbb{Z}}$ form an orthogonal base in the space consisting of square-integrable functions whose Fourier transforms are supported on $\left[-\frac{1}{2}, \frac{1}{2}\right]$. This makes it possible to decompose functions as a series of cardinal sine functions with sample values being coefficients, which is the content of Shannon's formula (see [66, Part III] and [34, Lesson 38] for details). This leads to significant applications throughout numerical analysis, including numerical integration, and the solution of differential and integral equations.

In addition to the applications in analysis described above, the cardinal sine function also appeared in early mathematics. It served as a tool for approximating π by geometric or stochastic means; see [35] for such examples. Without doubt, it is still relevant in probability theory and modern geometry, as we shall see in Section 5.

Although the importance of (2.1) increased in the 1950s due to its applications in signal processing and approximation theory, the function had previously appeared in other concepts. Namely, some interesting properties related to its integral were investigated back in the 19th century. Interpreting the integral function (2.2) was not coincidental at all: the cardinal sine function does not have an elementary indefinite integral. This property can be shown directly via Liouville's theorem from the 1830s, which places an important restriction on antiderivatives that can be expressed by elementary functions. We direct the reader to [63, Chapter III] for details concerning Liouville's work on differential algebra. See also a method for expressing the indefinite integral of sinc with nonelementary, hypergeometric functions in Nijimbere's article [57]⁴.

However, as $x \rightarrow \infty$, the improper Riemann integral⁵ of the cardinal sine function actually exists and can be evaluated explicitly. This is called *Dirichlet integral*⁶:

$$\int_{-\infty}^{\infty} \text{sinc } t \, dt = \pi. \quad (2.3)$$

At this point, we need to mention an alternative definition of (2.1). Dirichlet integral indicates the reason for which in most applications the function sinc is defined by multiplying its argument by π . In this way, its integral becomes 1:

$$\int_{-\infty}^{\infty} \text{sinc}(\pi t) \, dt = 1.$$

This is the so-called *normalized cardinal sine function*. Referring back to the short introduction written on the function sinc and the applications mentioned above, we remark that using this alternative definition instead of (2.1) is more expedient in applied mathematics. Yet, instead of using this concept in the note, we will rather simply normalize the integral itself.

⁴There seems to be a typo in this article, in the 23rd row: 'Liouville 1938's Theorem'. Liouville lived in the 19th century, so it is very unlikely that there exists a theorem named after him from 1938.

⁵We need to consider the improper Riemann integral, as the sinc function is not Lebesgue integrable over the real line.

⁶This terminology is quite ambiguous, since there exist various integrals named after Dirichlet. Such an example is the integral of the Dirichlet kernel multiplied by some appropriate function. However, it seems reasonable to insist on this name. Based on the extensive bibliography in [12], this integral might have appeared first in Dirichlet's lectures from 1850s. Although he did not prepare notes, after his death in 1859 his lectures were compiled and published by several of his former students. See, for instance, Arendt's book [28] from 1904, which closely follows Dirichlet's lectures.

The number of methods to prove the fascinating result (2.3) is literally unlimited. In 1909, Hardy [43] collected seven methods based on the articles [12, 55] and complemented them with [20, §172 in Appendix III, Ex. 1]. Furthermore, he examined these proofs with respect to their ‘artificiality, complexity, etc.’ and gave them marks according to these aspects. The proof obtaining the least marks was regarded as the simplest and best. He noted that ‘Practically all methods of evaluating any definite integral depend ultimately upon the inversion of two or more operations of procedure to a limit. [...] Such inversions, then, constitute what we may call the standard difficulty of the problem, and I shall base my system of marking primarily upon them.’ It is worth to mention here three main methods among the others.

1. One can investigate the limit of the Laplace transform⁷ of sinc at zero. This needs to be executed with great care because of several limit changes. Actually, Dirichlet himself used this method as noticed by Berry [12]; see [28, §92]. (Due to theoretical difficulties, Hardy marked this solution for 40 points.)
2. The range of integration is divided into intervals $[k\pi, (k+1)\pi]$ and the integration is reduced to that of a certain series. (This method got the lowest score with 28 points, so according to Hardy this was the most favorable one.)
3. Another standard way is applying Cauchy’s integral theorem from complex analysis; see [40, §46] for precise details. (The classic solution got 42 points, while Hardy gave 34 points for Berry’s modified version [12].)

It is natural to study the generalization of (2.3), namely integrals of the form

$$\int_{-\infty}^{\infty} \text{sinc}^n t \, dt, \quad (2.4)$$

where n is an arbitrary natural number. For small n ’s, these can be easily determined with similar approaches listed below the case of (2.3), see [39, Formulae 3.821.9, 3.827.4, 3.827.7, 3.827.11 and 3.827.15]. For higher values of n , these methods may work as well, and so one can gain the result as a sum⁸.

For further studies, we need to investigate the integrals of the form (2.4) in a more general way, just as Laplace [49] did in the 19th century. Later, inspired by questions in geometry, Pólya [61] also arrived at the same concepts. In honor of them, we introduce the *Laplace-Pólya integral*:

$$J_n(r) := \frac{1}{\pi} \int_{-\infty}^{\infty} \text{sinc}^n t \cdot \cos(rt) \, dt \quad (2.5)$$

where $n \geq 1$ is an integer, and $r \in \mathbb{R}$. Laplace [49, §42 in Chapter III] showed that this

⁷Evaluating the Laplace transform explicitly can be performed directly by differentiating under the integral sign.

⁸Visit the web page <https://math.stackexchange.com/questions/307510/a-sine-integral-int-0-infty-left-frac-sin-x-x-right^n-mathrm{d}x> (accessed 2 April 2024) for such solutions from Stack Exchange users extending the methods mentioned above along with other integrals including the sinc function.

integral can be evaluated explicitly as follows:

$$J_n(r) = \frac{1}{2^{n-1}(n-1)!} \sum_{i=0}^{\lfloor \frac{n+r}{2} \rfloor} (-1)^i \binom{n}{i} (n+r-2i)^{n-1}. \quad (2.6)$$

Since Laplace's time, justifying equation (2.6) has become a classic exercise in older textbooks, just like [20, Appendix III Example 22] and [72, Problem 1928]. Moreover, it can be found in several famous integral tables, such as [9, Formula 1.6 (11)] and [39, Formula 3.836.5].

Neither formula (2.5), nor (2.6) indicates that the Laplace-Pólya integral becomes zero if $|r| \geq n$, yet this surprising fact is indeed valid. This is implied by the following stochastic approach, which immediately shows that the closed support of the Laplace-Pólya integral is exactly the interval $[-n, n]$. For $n \geq 2$, or $n = 1$ and $r \neq \pm 1$, the integral can also be interpreted as the density function of the following Irwin-Hall distribution:

$$J_n(r) = 2 \cdot f_{\sum_{i=1}^n X_i}(r) \quad (2.7)$$

where $X_1 \dots, X_n$ are independent random variables uniformly distributed on $[-1, 1]$, and $f_X(\cdot)$ denotes the probability density function of a continuous random variable X . By definition, the probability density function of the sum of random variables is given by the convolution of their probability density functions. Based on the properties of convolution, this argument shows that the closed support of (2.7) is indeed $[-n, n]$.

Now we have already presented two ways to evaluate the Laplace-Pólya integral. The first method (2.6) is useful for small n 's, but becomes prohibitively cumbersome, even for $r = 0$, as n increases rapidly. The second method (2.7) begins by describing a "clumsy", piecewise polynomial function⁹, whose computation also becomes a ponderous task for larger n 's. In the second half of the 20th century, the applications in telecommunications listed above raised the need to calculate this type of integrals more effectively.

Evaluation of the integral (2.5) is made possible by determining its power series expansion. Establishing the asymptotic expansion began with Silberstein's article [67] during the Second World War. Unfortunately, his work contained several errors that had been addressed in a series of articles [21, 38, 41, 59]. Finally, the correct coefficients up to order 10 were published by Medhurst and Roberts [53]. For central values, the asymptotic expansion up to order 3 reads as

$$J_n(0) = \sqrt{\frac{6}{\pi n}} \left(1 - \frac{3}{20n} - \frac{13}{1120n^2} + \frac{27}{3200n^3} + O\left(\frac{1}{n^4}\right) \right). \quad (2.8)$$

Even finer estimates were proven in [47, 58, 64]. In addition to asymptotic expansions, other numerical calculations have also been utilized during these decades. For instance, Harumi, Katsura and Wrench [44] calculated the central values up to $n = 30$. Medhurst and Roberts [53] supplemented this table up to $n = 100$. (These values are included

⁹It is worth to mention that in numerical calculations we used formula (2.7), which provided the answer as an exact fraction.

in Table 1 up to $n = 8$.) In addition, Butler [22] and Fettis [33]¹⁰ presented methods to evaluate numerically diverse values of the Laplace-Pólya integral easily, based on Poisson's summation formula.

In addition to providing numerical approximations of the Laplace-Pólya integral, the need for calculating exact values of it also arose. Medhurst and Roberts [53] had already presented some recurrence relations in vertical manner¹¹ for even and odd values of r , separately. Probably the most powerful recursion formula involving this integral (as we shall see in Section 3) was proven by Thompson [68]:

$$J_n(r) = \frac{n+r}{2(n-1)} J_{n-1}(r+1) + \frac{n-r}{2(n-1)} J_{n-1}(r-1) \quad (2.9)$$

for $n \geq 3$. Although this formula is valid for all $r \in \mathbb{R}$, we are going to use it for integer values *throughout this work*, unless otherwise stated. Since $J_n(r)$ is even in r , we specifically derive that

$$J_n(0) = \frac{n}{n-1} J_{n-1}(1) \quad (2.10)$$

for $n \geq 3$. Combined with (2.9) this provides a simple way to calculate the exact central values $J_n(0)$ for small n 's; see Table 1.

n	2	3	4	5	6	7	8
$J_n(0)$	1	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{115}{192}$	$\frac{11}{20}$	$\frac{5887}{11520}$	$\frac{151}{315}$

Table 1: Central values $J_n(0)$ for $n = 2, \dots, 8$.

In this work, we intend to analyze the integrals of the form (2.4), more accurately the sequence $(J_n(0))_{n=1}^{\infty}$. By the recurrence relations (2.9) and (2.10), this requires investigation of the Laplace-Pólya integral for integer values. In our previous work, we estimated the decay of the two-step ratio of the Laplace-Pólya integral from above.

Theorem 2.1 ([3, Theorem 1.4.]). *Let $n \geq 4$ and r be integers satisfying $-1 \leq r \leq n-2$. Then*

$$\frac{J_n(r+2)}{J_n(r)} \leq d_{n,r} \quad (2.11)$$

where

$$d_{n,r} = \frac{(n-r)}{(n+r+2)} \cdot \frac{(n-r-2)(n-r+2)}{(n+r)(n+r+4)}. \quad (2.12)$$

Now we present a corresponding lower bound for the same ratio.

¹⁰In [33, Formula (7)] the factor $\frac{2}{\pi}$ in the second row should be $\frac{2}{n}$ and should appear on the right hand side of the equality sign.

¹¹This means that determining the central values $J_n(0)$ enables to calculate non-central values with the aid of these recursive equations.

Theorem 2.2. Let $n \geq 4$ and r be integers satisfying $-1 \leq r \leq \left\lceil \frac{4n-8}{7} \right\rceil$. Then

$$c_{n,r} \leq \frac{J_n(r+2)}{J_n(r)} \quad (2.13)$$

where

$$c_{n,r} = \frac{(n-r)^2}{(n+r+2)^2} \cdot \frac{(4n-7r-8)(4n+3r+6)}{(4n+7r+6)(4n-3r)}. \quad (2.14)$$

There are major differences between the two theorems, which will be discussed in Section 3, along with the proof of Theorem 2.2. We also provide some insight into the method leading to the above inequalities and discuss further improvements.

In particular, for $r = 0$, we obtain the following bounds.

Corollary 2.3. For $n \geq 4$ we have

$$\frac{n(n-2)}{(n+2)^2} \leq \frac{J_n(2)}{J_n(0)} \leq \frac{n-2}{n+4}. \quad (2.15)$$

Based on (2.9), this transforms into the following estimate for central values:

$$\frac{n}{n+1} \leq \frac{J_{n+2}(0)}{J_n(0)} \leq \frac{(n+2)(n^2+2n-2)}{n(n+1)(n+4)}. \quad (2.16)$$

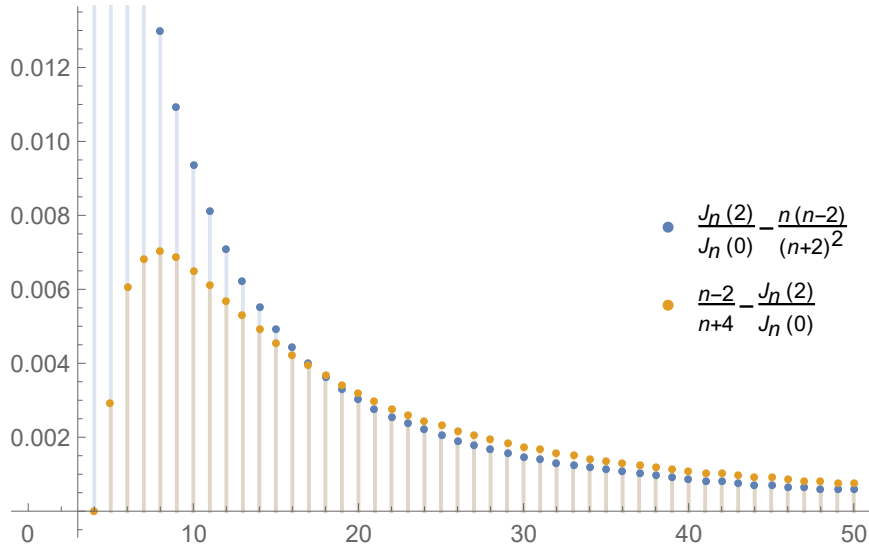


Figure 4: Difference of the central ratio and the proven estimates for $n = 4, \dots, 50$.

According to the eminent article of Lesieur and Nicolas [50], the upper bound in (2.16) may be improved to $\frac{n+1}{n+2}$, which they proved for even values of n with analytical tools. (This will be discussed in more detail in Section 4.) We believe that our combinatorial methods could be extended to achieve this stronger bound, which actually requires $\frac{n(n^2-2)}{(n+2)^3}$ to be proven to be an upper bound in (2.15), see Figure 5.

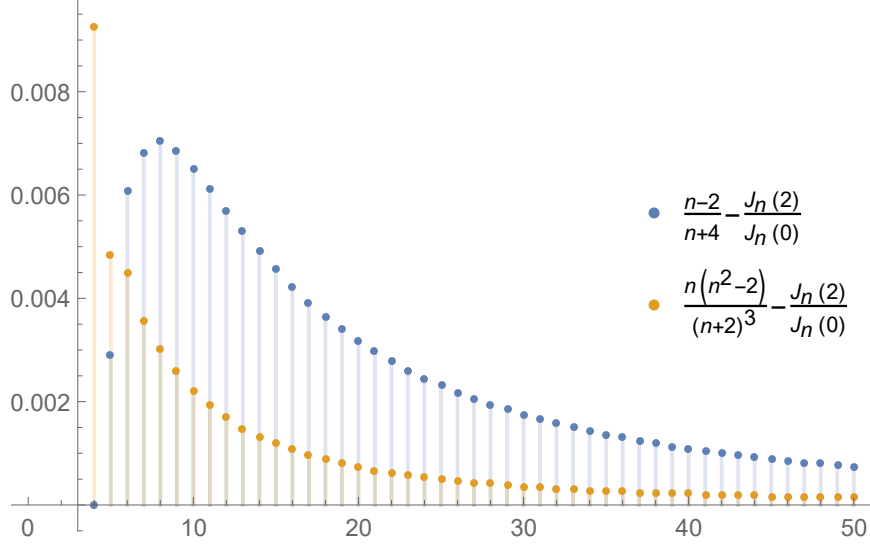


Figure 5: Difference of the central ratio and the two upper bounds (the proven one and the suspected improvement) for $n = 4, \dots, 50$.

Before moving on to the next section on the proof of Theorem 2.2, we present some related integrals involving the cardinal sine function, which might be investigated with our methods in the future.

First, a natural generalization of (2.5) is considering arbitrary coefficients in the argument of sinc functions. Suppose that we have $\mathbf{a} = (a_1 \dots, a_n) \in \mathbb{R}^n$, then we define

$$J_{\mathbf{a}}(r) := \frac{1}{\pi} \int_{-\infty}^{\infty} \prod_{i=1}^n \text{sinc}(a_i t) \cdot \cos(rt) dt. \quad (2.17)$$

In this general form, these integrals will take a key part in Section 5 in the manner of Pólya's article [61]. Now we consider only the case where $r = 0$ and specifically $\mathbf{p}_n = \left(\frac{1}{2i-1}\right)_{i=1}^n$, that is,

$$J_{\mathbf{p}_n}(0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \prod_{i=1}^n \text{sinc}\left(\frac{t}{2i-1}\right) dt. \quad (2.18)$$

For $n = 1 \dots, 6$, these integrals are equal to 1, while for $n \geq 7$, this nice pattern breaks down:

$$J_{\mathbf{p}_7}(0) = \frac{467807924713440738696537864469}{467807924720320453655260875000} \approx 0.9999999998529371865.$$

As stated in [6, p. 221]: ‘When this was first discovered by a researcher, using a well-known computer algebra package, both he and the software vendor concluded there was a “bug” in the software. Not so!’ Borwein and Borwein [15] resolved this anomaly by establishing important inequalities and exact formulae regarding integrals of the type (2.17) with Fourier transform techniques. As an honor to their work, integrals (2.18) are now known as *Borwein integrals* in the literature, and their article [15] inspired many authors to

investigate this phenomenon in more detail. For example, some articles [4, 65] focused on providing a nice graphical explanation through the convolution of rectangular pulses. Meanwhile, further concrete values of integrals of the type (2.17) (with small modifications in some cases) and similar anomalies are discussed in the papers [2, 7, 10, 11, 15]¹² and in [6, p. 255, Problem 10 (A sincing feeling)].

Returning to the original definition of (2.1), it allows us to handle another generalization of (2.5). Consider the integrals

$$J_{m,n}(r) = \frac{2}{\pi} \int_0^\infty \frac{\sin^m t}{t^n} \cos(rt) dt \quad (2.19)$$

where m, n are either both odd or both even positive integers with $m \geq n > 2$. Here we needed to reduce the domain of integration to the real half-line, as otherwise the integral would become zero in some specific cases. The integrals $J_{m,n}(0)$ may be expressed with explicit formulae or with recursion equations, similar to (2.6) and (2.9), respectively. For details of the calculations, see [29, §1023–1034 in Chapter XXVI]. For further history and recent developments in the study of integrals of the type (2.19), see [1].

We may also consider several other generalizations of (2.5). In such a possible issue the classical sine function is replaced with the generalized trigonometric sine function; see [54] where the eigenfunction of a particular non-homogeneous eigenvalue problem is considered as a generalization of the sine function. Another subject concerns more complicated arguments in the terms of sinc. See [18] and [52, Example 7.36 on p. 125] for such examples. However, in most cases, these are quite difficult to handle.

Lastly, we take a few words about multivariate sinc functions and related integrals. Similarly to the one-dimensional case, multivariate sinc functions also occur in several fields of mathematics and physics, specifically optics.

Suppose that a parallel wave of light¹³ is cast on an opaque screen with a tiny aperture. If the wavelength and the size of the aperture are comparable, then the light “bends” around the edges of the aperture and spreads out. This phenomenon is called diffraction. Now, let an image plane gather all the light emerging from the aperture on which diffracted light creates a pattern with alternating dark and bright regions. This is called a diffraction pattern; see Figure 6. There are two types of diffraction depending on whether the light source and the image plane are relatively close to each other or if their distance is sufficiently large. They are called Fresnel and Fraunhofer diffraction, respectively (the latter one is easier to describe with mathematical tools).

We consider the simplest aperture: a rectangle. Suppose that we have a rectangular aperture of width $a > 0$ in direction x and height $b > 0$ in direction y , centered at the origin. Then the intensity of light at the point (x, y) is expressed with a bivariate function involving sinc: $I(x, y) = a^2 b^2 \operatorname{sinc}^2(ax) \operatorname{sinc}^2(by)$. This function precisely portrays the diffraction pattern; see Figure 6.

¹²It is worth to mention that in the second paper [7, Theorem 2.] provides a method to evaluate $J_n(0)$ as a series involving only the sinc function.

¹³Naturally, we may consider other type of waves, but our argument remains pertinent, if we insist on light waves.

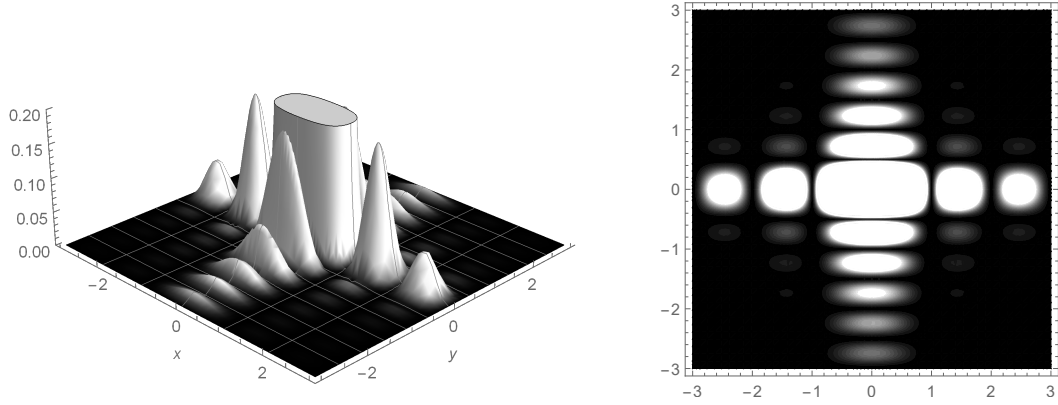


Figure 6: Graph of intensity distribution for a rectangular aperture with parameters $a = 1$ and $b = 2$ along with the predicted diffraction pattern on the image plane.

Similarly to the one-dimensional integrals described previously in this section, the physical interpretation of bivariate sinc functions suggests that it may be useful to consider their integral. Indeed, integrating the intensity $I(x, y)$ along the rectangle corresponding to the aperture gives us the amount of energy radiated through the aperture per unit time. For a more detailed explanation of Fraunhofer diffraction and its characterization with Fourier analytical tools involving sinc function, see [69, Sections 7.4 and 7.5].

In a mathematical point of view, restricting ourselves to the two-dimensional case motivated by a physical problem is not so practical. Hence, let us consider integrals of multivariable sinc functions. In this topic, we highlight the article of Borwein *et al.* [17], in which they considered multivariate integrals of the form

$$\Sigma_{m,n}(S) := \int_{\mathbb{R}^m} \prod_{i=1}^{m+n} \text{sinc}\langle s_i, y \rangle dy \quad (2.20)$$

where $S = [s_1, \dots, s_{m+n}] \in \mathbb{R}^{m \times (m+n)}$, $\langle \cdot, \cdot \rangle$ indicates the Euclidean scalar product in $\mathbb{R}^m$ and $n \geq m \geq 1$. We need to note that integrals of the type (2.20) are not convergent for every S , unlike in the one-dimensional case (2.17). In order for (2.20) to exist, a necessary and sufficient condition for S is to contain two completely disjoint $m \times m$ submatrices that are nonsingular. It is actually possible to provide an explicit formula for such integrals, as the authors deduced in [17, Theorem 4]. This formula can be viewed as a high dimensional generalization of (2.6).

The formula (2.20) has a very inspiring geometric interpretation as follows. Consider a centered polytope obtained by cutting the unit cube in $\mathbb{R}^n$ by m slabs. Then, its volume can be determined with the integral (2.20). See some concrete examples at the end of [17, Section 7].

In this section, in addition to presenting our main result, Theorem 2.2, our intention was to offer a convenient collection of applications of the sinc function along with several integrals involving it. This task is also justified by the fact that – as far as I am aware – there exists no survey on this topic. The above compilation may serve as a basis for such a work in the near future.

3 Estimating the two-step ratio of the Laplace-Pólya integral

3.1 General form of the estimates

Recurrence (2.9) provides us an approach to prove Theorems 2.1 and 2.2 by induction. This amounts to justifying certain inequalities involving the bounds (2.12) and (2.14), respectively. For the sake of simplicity, in this subsection we just refer to them as “induction inequalities” (which differ only in reversing the inequality sign). To see their exact form, we direct the reader to [3, Section 3.] and (3.4).

Our intention is to seek estimates given by rational functions of the form $\frac{P(n, r)}{Q(n, r)}$ with P and Q being polynomials, since proving the validity of the induction inequality simply amounts to comparing polynomials in this case.

First, we need to determine the degree of the polynomials P and Q . Numerical simulations show that for a fixed n , the cubic and quartic roots of $\frac{J_n(r+2)}{J_n(r)}$ are nearly linear functions of r in the domain of investigation; see Figure 7. Hence, we set out to search P and Q of degree 3 and 4 for the upper and lower bounds, respectively.

In order to simplify the induction inequality, it is favorable to have a factor of the form $\frac{n-r}{n+r+2}$ in the estimates. This term appears with different exponents in our upper and lower bounds.

According to the recursion (2.9),

$$\begin{aligned} J_{n+2}(r) &= \frac{n+r+2}{2(n+1)} J_{n+1}(r+1) + \frac{n-r+2}{2(n+1)} J_{n+1}(r-1) = \\ &= \frac{n+r+2}{2(n+1)} \left(\frac{n+r+2}{2n} J_n(r+2) + \frac{n-r}{2n} J_n(r) \right) + \\ &+ \frac{n-r+2}{2(n+1)} \left(\frac{n-r}{2n} J_n(r) + \frac{n-r+2}{2n} J_n(r-2) \right) = \\ &= \frac{(n+r+2)^2}{4n(n+1)} J_n(r+2) + \frac{(n-r)(n+2)}{2n(n+1)} J_n(r) + \frac{(n-r+2)^2}{4n(n+1)} J_n(r-2), \end{aligned}$$

hence

$$\frac{J_{n+2}(r)}{J_n(r)} = \frac{1}{4n(n+1)} \left((n+r+2)^2 \frac{J_n(r+2)}{J_n(r)} + 2(n-r)(n+2) + (n-r+2)^2 \frac{J_n(r-2)}{J_n(r)} \right).$$

This also directs us towards incorporating the factor $\frac{n-r}{n+r+2}$ in the estimates.

Since we are going to prove the validity of the estimates by recursion, it is critical to guarantee their correctness at the endpoints of the domain in question, that is, $r = -1$ and $r = n - 2$. At these values of r , we obtain

$$\frac{J_n(1)}{J_n(-1)} = 1 \quad \text{and} \quad \frac{J_n(n)}{J_n(n-2)} = 0,$$

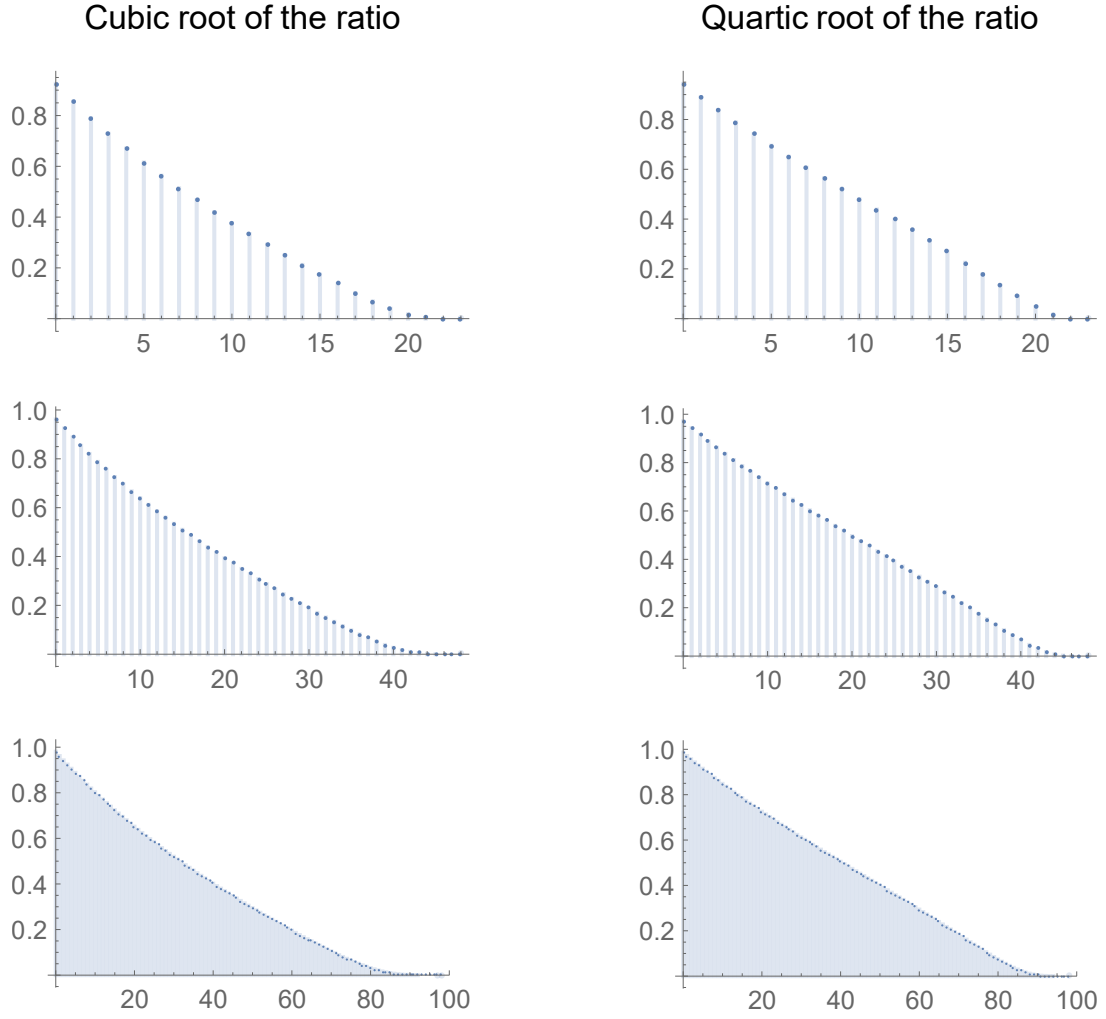


Figure 7: Cubic and quartic root of the ratio $\frac{J_n(r+2)}{J_n(r)}$ as sequences in r , for fixed $n = 25, 50, 100$.

and we require the bounds to be sharp here. The other main criterion for the bounds to satisfy is to extend the estimates for $r = 0$ and n even, proven by Lesieur and Nicolas [50] (see remark after Corollary 2.3) which read as

$$\frac{n(n-2)}{(n+2)^2} < \frac{J_n(2)}{J_n(0)} < \frac{n(n^2-2)}{(n+2)^3}. \quad (3.1)$$

Our lower bound given by Theorem 2.2 meets this requirement, while the upper bound of Theorem 2.1 is slightly weaker than the one above (although it holds in a larger domain); see Figure 5.

Based on the above guidelines, we decided to seek the bounds of the form

$$\frac{(n-r)^2}{(n+r+2)^2} \cdot \frac{(n+ar+b)(n+er+f)}{(n+gr+h)(n+ir+j)} \quad (3.2)$$

for the lower estimate and

$$\frac{(n-r)}{(n+r+2)} \cdot \frac{(n+Ar+B)(n+Er+F)}{(n+Gr+H)(n+Ir+J)}$$

for the upper estimate, where roman letters indicate real parameters. Finding the latter bound was accomplished in our previous article [3]. From now on, we focus on finding a suitable lower bound for $\frac{J_n(r+2)}{J_n(r)}$ of the form (3.2).

3.2 Establishing the lower bound

Before we turn to the proof of Theorem 2.2, let us add some comments on comparing our lower and upper bounds. Remarkably, there is a major difference between them, regarding the validity region of r .

In Theorem 2.1, the upper bound (2.12) holds in the entire region $-1 \leq r \leq n-2$. Meanwhile, we could only provide a *non-negative* lower bound only on a smaller set, namely for $-1 \leq r \leq \left\lceil \frac{4n-8}{7} \right\rceil$. Figure 7 illustrates the reason for this anomaly graphically: linear approximations seem to cross the x -axis before the right end of the region.

Hence, we suspect that one cannot give a lower bound in the form of some rational function similar to (2.14), which is valid in the whole region $-1 \leq r \leq n-2$. This shortcoming might be fixed by introducing an exponential factor; however, this would greatly complicate the calculations according to Section 3.1. For our present purpose, a lower bound in the form of (2.14) is suitable. However, sufficient care is needed to validate the estimate at the right end of the region.

Proof of Theorem 2.2. Suppose that r is an integer that satisfies $r \geq -1$. We will proceed by induction on n , with $n=4$ being the base case.

The values of $J_4(r)$ for $r = -1, \dots, 4$ can be calculated using the formula (2.6). Based on these, the $n=4$ case of (2.13) is easy to check, see Table 2.

r	-1	0	1	2
$\frac{J_4(r+2)}{J_4(r)}$	1	$\frac{1}{4}$	$\frac{1}{23}$	0
$c_{4,r}$	1	$\frac{2}{9}$	$\frac{225}{18473}$	$-\frac{7}{240}$

Table 2: Two sides of the inequality (2.13) for $n=4$ and $r = -1, \dots, 2$.

Suppose now that for some $n \geq 4$, (2.13) holds for each $-1 \leq r \leq \left\lceil \frac{4n-8}{7} \right\rceil$. We need to show that

$$c_{n+1,r} J_{n+1}(r) \leq J_{n+1}(r+2) \quad (3.3)$$

for each r with $-1 \leq r \leq \left\lceil \frac{4n-4}{7} \right\rceil$.

For $r = -1$, equality holds above, because $c_{n,-1} = \frac{J_n(1)}{J_n(-1)} = 1$ for each $n \geq 4$. If $r = \left\lceil \frac{4n-4}{7} \right\rceil$, then $c_{n+1,r} \leq 0$, and so (2.13) holds in this case as well.

Assume that $0 \leq r \leq \left\lceil \frac{4n-4}{7} \right\rceil - 2 = \left\lceil \frac{4n-18}{7} \right\rceil$. By (2.9) we have

$$\begin{aligned} J_{n+1}(r+2) &= \frac{n+r+3}{2n} J_n(r+3) + \frac{n-r-1}{2n} J_n(r+1), \\ J_{n+1}(r) &= \frac{n+r+1}{2n} J_n(r+1) + \frac{n-r+1}{2n} J_n(r-1). \end{aligned}$$

Applying the induction hypothesis for the pairs $(n, r+1)$ and $(n, r-1)$ shows that

$$\begin{aligned} c_{n,r+1} J_n(r+1) &\leq J_n(r+3) \\ J_n(r-1) &\leq \frac{1}{c_{n,r-1}} J_n(r+1). \end{aligned}$$

(Remember that $c_{n,r-1} > 0$ in this region.) Thus, (3.3) follows from the inequality

$$c_{n+1,r} \left(\frac{n+r+1}{2n} + \frac{n-r+1}{2n} \cdot \frac{1}{c_{n,r-1}} \right) \leq \frac{n+r+3}{2n} c_{n,r+1} + \frac{n-r-1}{2n}.$$

Multiplying both sides by $2n$ and rearranging all terms to one side leads to the inequality¹⁴

$$0 \leq (n+r+3)c_{n,r+1} + (n-r-1) - c_{n+1,r} \left((n+r+1) + (n-r+1) \frac{1}{c_{n,r-1}} \right). \quad (3.4)$$

Using (2.14) the right-hand side can be combined to a single fraction with the numerator

$$\begin{aligned} &8(r+1) \left(128n^6 + 128n^5(41r^2 + 82r + 36) + 8n^4(1807r^2 + 3614r + 1585) - \right. \\ &-4n^3(1018r^4 + 4072r^3 + 5441r^2 + 2738r + 45) - 4n^2(958r^4 + 3832r^3 + 9243r^2 + 10822r + 3697) + \\ &\left. + 3n(r+1)^2(455r^4 + 1820r^3 + 2299r^2 + 958r - 1476) - 9(r+1)^2(91r^4 + 364r^3 - 17r^2 - 762r - 220) \right), \end{aligned} \quad (3.5)$$

and denominator

$$(4n-7r-1)(4n-3r-3)(4n-3r+4)(n+r+3)^2(4n+3r+3)(4n+7r+10)(4n+7r+13). \quad (3.6)$$

Clearly, (3.6) is positive if $r \leq \left\lceil \frac{4n-18}{7} \right\rceil$. Thus, it suffices to prove the non-negativity of (3.5). To that end, introduce the following bivariate polynomial

$$p_n(s) = \alpha_n s^3 + \beta_n s^2 + \gamma_n s + \delta_n,$$

¹⁴This is the induction inequality in the case of lower bound, mentioned earlier in Subsection 3.1.

where the coefficients depending on n are

$$\begin{aligned}\alpha_n &= 1365n - 819, \\ \beta_n &= -4072n^3 - 3832n^2 - 1293n + 5067, \\ \gamma_n &= 5248n^5 + 14456n^4 + 2668n^3 - 13980n^2 - 4500n - 2268, \\ \delta_n &= 128n^6 - 640n^5 - 1776n^4 + 1224n^3 + 3024n^2.\end{aligned}$$

Then, (3.5) reads as

$$8(r+1)p_n((r+1)^2).$$

Thus we need to show that $p_n(s) \geq 0$ for any $1 \leq s \leq \left\lceil \frac{4n-11}{7} \right\rceil^2$ (indeed, this will hold for any value of s at least 1, and the sign of (3.6) solely determines the validity of inequality (3.4)). By direct substitution, we have

$$p_n(1) = 128n^6 + 4608n^5 + 12680n^4 - 180n^3 - 14788n^2 - 4428n + 1980,$$

which is clearly non-negative for all $n \geq 4$. Notice that if we show that p_n is monotone increasing in s , then our proof will be complete. Since we supposed that r is an integer, s needs to be an integer as well. For a moment, forget this requirement and investigate p_n for all real values $s \geq 1$. With this extension, we may calculate the sign of p'_n :

$$p'_n(s) = 3\alpha_n s^2 + 2\beta_n s + \gamma_n,$$

where $3\alpha_n > 0$ for all $n \geq 4$. The discriminant of the above quadratic polynomial is given by

$$\begin{aligned}4\beta_n^2 - 12\alpha_n\gamma_n &= -19637504n^6 - 60380704n^5 + 199229392n^4 + \\ &+ 129789120n^3 - 21331996n^2 - 59489208n + 80408052.\end{aligned}$$

By comparing the coefficients, the above quantity is immediately seen to be strictly negative for $n \geq 4$, which implies that $p'_n(s) > 0$ for all $n \geq 4$ and $s \geq 1$. $\square$

3.3 The general recursive scheme

The key part of Theorem 2.2 was justifying the inequality (3.4). Similarly, in order to prove Theorem 2.1, one has to check the inequality

$$0 \leq d_{n+1,r-1} \left((n+r) + \frac{n-r+2}{d_{n,r-2}} \right) - (n-r) - (n+r+2)d_{n,r}$$

for the given bound (2.12), see [3, Section 3]. This is analogous to (3.4), but with a reverse inequality sign. It is an interesting question what happens if we choose a completely new sequence, which reduces the “gap” in the induction inequalities, leading to a very strong approximation in both cases. Moreover, at the beginning we can ensure simple estimates in both cases based on (3.1). Notice that all is given to introduce a recursive sequence for the bounds:

$$\begin{aligned}\mathfrak{d}_{n,-1} &:= 1, \quad \mathfrak{d}_{n,0} := \frac{n(n^2 - \zeta)}{(n+2)^3}, \\ \mathfrak{d}_{n,r} &:= \frac{1}{n+r+2} \left(\mathfrak{d}_{n+1,r-1} \left((n+r) + \frac{n-r+2}{\mathfrak{d}_{n,r-2}} \right) - (n-r) \right) \quad \text{for } r \geq 1.\end{aligned}\tag{3.7}$$

Here, ζ is an arbitrary parameter. As we encountered in Section 3.1, based on (3.1) we expect $\mathfrak{d}_{n,r}$ to be a lower bound for $\zeta = 4$ and an upper bound for $\zeta = 2$.

Not surprisingly, the proof of the estimates provided by a recurrence of type (3.7) is now immediate as long as we can justify the positivity of the sequence. This proves to be critical, since the term $\mathfrak{d}_{n,r-2}$ appears in the denominator in (3.7). Actually it would suffice to show the inequality

$$\mathfrak{d}_{n,r} \left((n+r) + \frac{n-r}{\mathfrak{d}_{n-1,r-1}} \right) > n-r-2 \quad (3.8)$$

for all $n \geq 4$ and $r \geq 0$. However, our calculations indicate that this may prove to be a very difficult task. It would be natural to use induction; however, our efforts in this direction were unsuccessful due to the reciprocal relationship in the definition (3.7).

We suspect that only an accurate examination of the recurrence relation (3.7) may lead to inequality (3.8). This investigation surely includes finding or describing the minimum of the sequence (3.7). However, this is also problematic. Based on numerical calculations, we found a strange monotonicity behavior of the sequence (3.7): there is a certain “moving minimum plateau” of the sequence. More precisely, for any initial value ζ , we can observe that the minimum shifts to the right in n as r increases; see Figure 8. Therefore, one

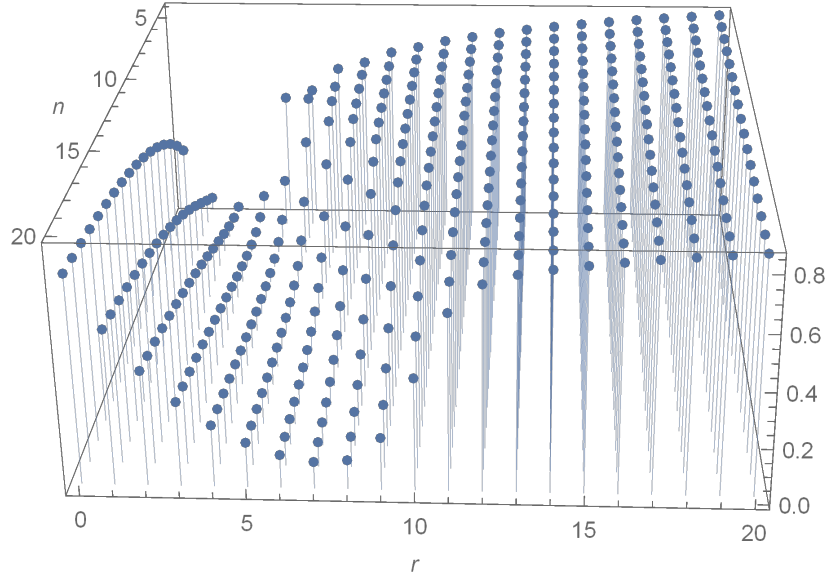


Figure 8: The sequence $\mathfrak{d}_{n,r}$ for $n = 4, \dots, 20$, $r = 0, \dots, 20$ and $\zeta = 2$.

actually needs to characterize this “plateau” in order to estimate the recurrence $\mathfrak{d}_{n,r}$ from below and so validate the inequality (3.8).

We also need to mention the importance of the initial parameter ζ . For example, if $\zeta = 2$, then the recurrence defined by (3.7) is positive on all values (see Figure 8), and so is a valid upper bound of the ratio $\frac{J_n(r+2)}{J_n(r)}$. Meanwhile, if we consider $\zeta = 4$, then the sequence $\mathfrak{d}_{n,r}$ also attains negative values. Hence, we suspect that this recursive approach works only for establishing upper bounds.

In addition to the difficulties listed above about handling the sequence $\mathbf{d}_{n,r}$, there are certainly some advantages. We mention that more precise upper estimates for the central ratio $\frac{J_n(2)}{J_n(0)}$ may also be within reach using the recursive approach. Our numerical calculations suggest that the smallest ζ , which seems to be an upper bound of the form $\frac{n(n^2 - \zeta)}{(n+2)^3}$, is roughly 2.33, see Figure 9.

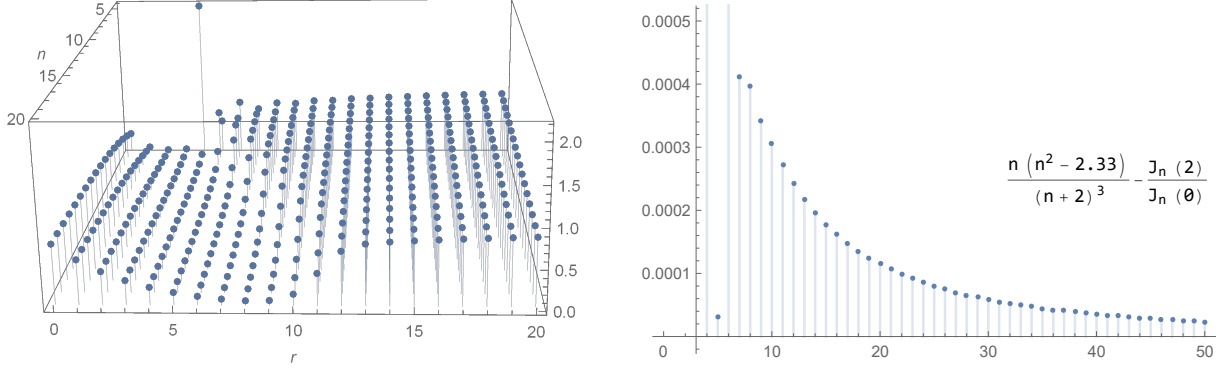


Figure 9: The sequence $\mathbf{d}_{n,r}$ for $n = 4, \dots, 20$, $r = 0, \dots, 20$ and $\zeta = 2.33$, beside the difference of the initial value $\mathbf{d}_{n,0}$ and the ratio $\frac{J_n(2)}{J_n(0)}$.

4 Connection with Eulerian numbers

A combinatorial interpretation of the Laplace-Pólya integral originates from the connection with Eulerian numbers of the first kind $A(m, l)$ (which will simply be referred to as Eulerian numbers). The classic definition of $A(m, l)$ is the number of permutations of the set $\{1, \dots, m\}$ in which exactly $l - 1$ elements are greater than the previous element; see Table 3. This fact immediately shows a symmetry property of Eulerian numbers:

$$A(m, l) = A(m, m - l + 1). \quad (4.1)$$

Eulerian numbers can also be defined recursively [24, pp. 240-243.]: for integers $m, l \geq 0$ by

$$\begin{aligned} A(0, 0) &:= 1, \quad A(m, 0) := 0, \text{ ha } m > 0, \\ A(m, l) &= (m - l + 1)A(m - 1, l - 1) + lA(m - 1, l). \end{aligned} \quad (4.2)$$

Moreover they may be evaluated explicitly with formula

$$A(m, l) = \sum_{i=0}^l (-1)^i \binom{m+1}{i} (l-i)^m. \quad (4.3)$$

$m \backslash l$	1	2	3	4	5	6	7	8	9	10
1	1									
2	1	1								
3	1	4	1							
4	1	11	11	1						
5	1	26	66	26	1					
6	1	57	302	302	57	1				
7	1	120	1191	2416	1191	120	1			
8	1	247	4293	15619	15619	4293	247	1		
9	1	502	14608	88234	156190	88234	14608	502	1	
10	1	1013	47840	455192	1310354	1310354	455192	47840	1013	1

Table 3: Eulerian numbers up to $m = 10$.

see [14, Theorem 1.11], which presents a direct, combinatorial proof¹⁵ based on the principle of inclusion and exclusion. Further properties of the Eulerian numbers can be found in [14, Section 1.1] and [24, Section 6.5].

Eulerian numbers have been subject to study in numerous fields of mathematics since Euler's time. These numbers occur mostly in combinatorics and discrete mathematics, but also appear in analysis. Not surprisingly, these numbers were introduced by Euler in his magnificent work [32, §173], which laid the basis for differential calculus. He actually studied the bivariate generating function of Eulerian numbers, which could be written as

$$\sum_{m=0}^{\infty} A_m(u) \frac{v^m}{m!} = \frac{1-v}{v - e^{u(v-1)}}. \quad (4.4)$$

where the polynomials

$$A_m(u) = \sum_{l=0}^m A(m, l) u^l \quad (4.5)$$

are actually the ordinary generating functions of Eulerian numbers in the second variable. These are called *Eulerian polynomials*.

Eulerian numbers may be studied by purely combinatorial tools, e.g. by their generating function. Moreover, they also provide an alternative approach with respect to several combinatorial structures. One such example is the Costas arrays. In an algebraic concept, these are permutation matrices that have the additional property that all index vectors between pairs of ones are distinct. This property is called the Costas condition. It seems an extremely difficult task to find Costas arrays for large (at least 30) dimensions, even using computers. Thus, reducing the time of enumeration has become a real demand due

¹⁵Most of the proofs of formula (4.3) uses generating functions, manipulations of double sums of binomial coefficients or inversion formulae.

to applications in radar engineering. This requires a detailed study of the structural properties of Costas arrays. One way to examine them is through permutations with different numbers of ascends or descends. This is the point where Eulerian numbers come into view: upper bounds on permutations considered may be given with Eulerian numbers. See [25–27] for details.

One aspect that complicates the study and application of Eulerian numbers is their certain asymmetry; see (4.1) or the generating function (4.4). There are techniques to symmetrize Eulerian numbers. One way was performed by Carlitz [23], who introduced the *symmetric Eulerian numbers* $\bar{A}(m, l)$, defined by

$$\bar{A}(m, l) = A(m + l + 1, l + 1). \quad (4.6)$$

The table of these is obtained from Table 3 by sliding all columns upward, see Table 4.

$m \backslash l$	0	1	2	3	4	5
0	1	1	1	1	1	1
1	1	4	11	26	57	120
2	1	11	66	302	1191	4293
3	1	26	302	2416	15619	88234
4	1	57	1191	15619	156190	1310354
5	1	120	4293	88234	1310354	15724248

Table 4: Symmetric Eulerian numbers up to $m = 5$ and $l = 5$.

Another way is to slide all the rows leftward so that maximal values appear below each other; see Table 5. This could be done by introducing the numbers

$$\hat{A}(m, l) = \begin{cases} A\left(m - 1, \frac{m + l}{2}\right), & \text{if } m + l \text{ is even,} \\ 0, & \text{if } m + l \text{ is odd.} \end{cases} \quad (4.7)$$

These numbers have actually a convenient combinatorial meaning in addition to Eulerian numbers, as follows. Consider star polygons with m vertices in the plane. Two of such polygons are said to be *combinatorially equivalent* if the indices of their vertices determine the same permutation relative to the positive orientation of their convex hull. If two star polygons are combinatorially equivalent, then the sum of angles is equal; see [13, Theorem 4]. Thus, it is worth discussing the set of nonequivalent star polygons with m vertices and sum of angles $k\pi$. The cardinality of this set is given by $\hat{A}(m, l)$, see [51]¹⁶. A notable physical interpretation of $\hat{A}(m, l)$ is also mentioned in [51, Section 3], in which integrals of the form (2.4) appear and foreshadow our next argument. In fact, based on (2.6) and (4.3), [51, Theorem 2.6] asserts that we have

$$\hat{A}(m, l) = (m - 1)! J_m(l),$$

¹⁶In the paper’s conception, $(m)_l = \hat{A}(m, l)$.

$m \backslash l$	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
2							1						
3						1	0	1					
4					1	0	4	0	1				
5				1	0	11	0	11	0	1			
6			1	0	26	0	66	0	26	0	1		
7		1	0	57	0	302	0	302	0	57	0	1	
8	1	0	120	0	1191	0	2416	0	1191	0	120	0	1

Table 5: $\hat{A}(m, l)$ numbers up to $m = 8$.

and so (4.7) implies that

$$(m-1)!J_m(l) = A\left(m-1, \frac{m+l}{2}\right) \quad (4.8)$$

provided that $m+l$ is even. Our results for the Laplace-Pólya integral lead to estimates for Eulerian numbers via the connection

$$A(m, l) = m!J_{m+1}(2l-m-1). \quad (4.9)$$

Based on this equity, we derive that

$$\frac{A(m, l+1)}{A(m, l)} = \frac{J_{m+1}(2l-m+1)}{J_{m+1}(2l-m-1)}.$$

This allows us to establish lower and upper estimates on the ratio of consecutive Eulerian numbers as consequences of Theorems 2.1 and 2.2.

Corollary 4.1. *For integers $m \geq 3$ and $\frac{m}{2} \leq l \leq \frac{1}{2} \left\lceil \frac{11m+3}{7} \right\rceil$,*

$$c_{m+1, 2l-m-1} \leq \frac{A(m, l+1)}{A(m, l)} \leq d_{m+1, 2l-m-1}, \quad (4.10)$$

where $c_{n,r}$ and $d_{n,r}$ are defined by formulae (2.14) and (2.12) respectively.

Notice that we have stated this corollary on the restricted region specified in Theorem 2.2. However, the inequalities (4.10) are valid on the whole region, with the lower bound becoming negative and therefore irrelevant for $l > \frac{1}{2} \left\lfloor \frac{11m+3}{7} \right\rfloor$. Also note that Theorem 4.1 was stated only for $l \geq \frac{m}{2}$. Obviously, according to (4.1), Corollary 4.3 could be rephrased with similar bounds in the first half of the row¹⁷.

¹⁷We actually presented the upper estimate in [3, Proposition 3.1] this way. But now we prefer to use the form of (4.10) since it nicely parallels our results regarding the ratio of Laplace-Pólya integral.

Many aspects of these numbers may be investigated. In the remainder of this section, we focus on the maxima of Eulerian numbers in a fixed row, which were studied extensively by Lesieur and Nicolas [50] in 1992. Based on their work, we introduce the notation

$$M_m := \max_{l=1, \dots, m} A(m, l) = A\left(m, \left\lfloor \frac{m}{2} \right\rfloor + 1\right). \quad (4.11)$$

Using formula (4.9), this can also be expressed with the Laplace-Pólya integral:

$$M_m = m! J_{m+1}\left(2 \left\lfloor \frac{m}{2} \right\rfloor - m + 1\right) = \begin{cases} m! J_{m+1}(1), & \text{if } m \text{ is even,} \\ m! J_{m+1}(0), & \text{if } m \text{ is odd.} \end{cases} \quad (4.12)$$

Note that M_m is a “peak” on the line if m is odd and a “plateau” on the line if m is even; see Tables 3 and 5. In order to simplify further studies, we also define the function

$$f(m) := \frac{M_m}{m!} = \begin{cases} J_{m+1}(1), & \text{if } m \text{ is even,} \\ J_{m+1}(0), & \text{if } m \text{ is odd,} \end{cases} \quad (4.13)$$

just like Lesieur and Nicolas [50, Subsection 3.5] did. With tedious analytical estimates based on power series expansions, they showed that

Theorem 4.2 ([50, Theorem 2]). *We have, for all $p \geq 1$,*

$$\frac{2p}{2p+1} < \frac{f(2p+1)}{f(2p-1)} < \frac{2p+1}{2p+2}. \quad (4.14)$$

This indicates that $f(2p+1)$ is decreasing, which means that $f(2p)$ is also decreasing; see [50, Theorem 3 (i) and (ii)]. Our main intention is to provide similar estimates with the combinatorial argument discussed in Section 3. Based on the recursion (2.9), we have

$$J_{n+2}(0) = \frac{n+2}{n+1} J_{n+1}(1) = \frac{(n+2)^2}{2n(n+1)} J_n(2) + \frac{n+2}{2(n+1)} J_n(0).$$

After rearranging,

$$J_n(2) = \frac{2n(n+1)}{(n+2)^2} J_{n+2}(0) - \frac{n}{n+2} J_n(0).$$

Thus, we may express the two-step ratio of the Laplace-Pólya integral as

$$\frac{J_n(2)}{J_n(0)} = \frac{2n(n+1)}{(n+2)^2} \cdot \frac{J_{n+2}(0)}{J_n(0)} - \frac{n}{n+2}.$$

Applying Corollary 2.3, we obtain the following bounds.

$$\frac{(n+2)^2}{2n(n+1)} \cdot \left(\frac{n(n-2)}{(n+2)^2} + \frac{n}{n+2} \right) \leq \frac{J_{n+2}(0)}{J_n(0)} \leq \frac{(n+2)^2}{2n(n+1)} \cdot \left(\frac{n-2}{n+4} + \frac{n}{n+2} \right)$$

After simplification, we gain (2.16) from Corollary 2.3:

$$\frac{n}{n+1} \leq \frac{J_{n+2}(0)}{J_n(0)} \leq \frac{(n+2)(n^2+2n-2)}{n(n+1)(n+4)}.$$

Based on (4.13), we deduce the following estimates on the ratio of sequence $f(m)$ for odd values.

Corollary 4.3. *For every $p \geq 2$, we have*

$$\frac{2p}{2p+1} \leq \frac{f(2p+1)}{f(2p-1)} \leq \frac{(p+1)(2p^2+2p-1)}{p(p+2)(2p+1)} \quad (4.15)$$

Hence, we managed to give a fully combinatorial proof for the lower bound in Theorem 4.2. Ideally, if $\frac{n(n^2-2)}{(n+2)^3}$ was the upper bound in Corollary 2.3 (and this indeed seems to be an upper bound; see Figure 5), it would also imply the upper estimate in (4.14).

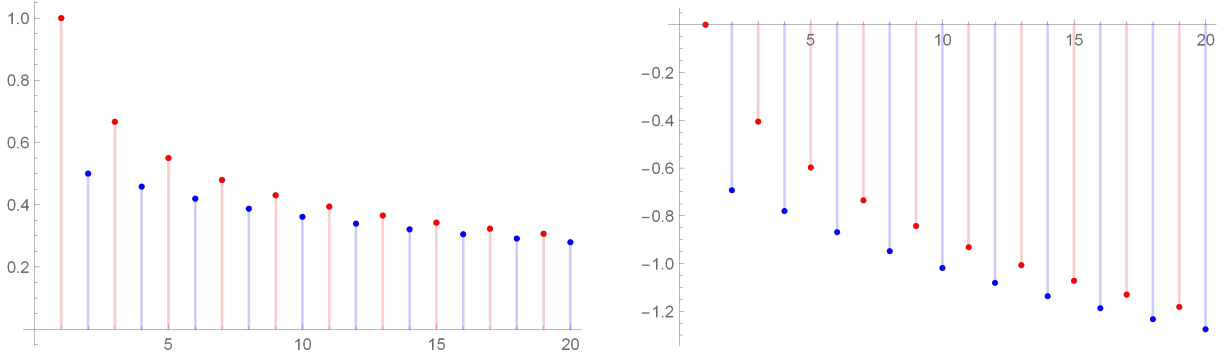


Figure 10: The sequence $f(m)$ and its logarithm up to $m = 20$ marked with different colors for odd and even values of m .

The effectiveness of (4.15) is well illustrated by establishing basic properties of the sequence $(f(2p-1))_{p=1}^{\infty}$, such as convexity and log-convexity in addition to monotonicity; see Figure 10. The latter is actually immediate from (4.15); we need to check the former two. This amounts to showing that

$$\frac{f(2p+3)}{f(2p+1)} + \frac{f(2p-1)}{f(2p+1)} \geq 2 \quad \text{and} \quad \frac{f(2p+3)}{f(2p+1)} \cdot \frac{f(2p-1)}{f(2p+1)} \geq 1$$

respectively. Applying the bounds (4.15) leads to

$$\begin{aligned} \frac{1}{2} \left(\frac{f(2p+3)}{f(2p+1)} + \frac{f(2p-1)}{f(2p+1)} \right) &\geq \frac{1}{2} \left(\frac{2p+2}{2p+3} + \frac{p(p+2)(2p+1)}{(p+1)(2p^2+2p-1)} \right) = \\ &= \frac{2(p+1)^2(2p^2+2p-1) + p(p+2)(2p+1)(2p+3)}{(p+1)(2p+3)(2p^2+2p-1)} = \\ &= \frac{8p^4 + 28p^3 + 29p^2 + 6p - 2}{8p^4 + 28p^3 + 28p^2 + 2p - 6} > 1, \end{aligned}$$

meanwhile

$$\begin{aligned} \frac{f(2p+3)}{f(2p+1)} \cdot \frac{f(2p-1)}{f(2p+1)} &\geq \frac{2p+2}{2p+3} \cdot \frac{p(p+2)(2p+1)}{(p+1)(2p^2+2p-1)} = \frac{2p(p+2)(2p+1)}{(2p+3)(2p^2+2p-1)} = \\ &= \frac{4p^3 + 10p^2 + 4p}{4p^3 + 10p^2 + 4p - 3} > 1 \end{aligned}$$

We arrive at the following statement.

Corollary 4.4. *The sequence $(f(2p+1))_{p=2}^{\infty}$ is convex and logarithmically convex.*

In order to extend this result to even values of the sequence $f(m)$, we employ the simple connection between odd and even values provided by (4.13) and (2.10):

$$f(2p) = \frac{2p+1}{2p+2} f(2p+1)$$

for any $p \geq 1$. Thus, we may easily establish the counterpart of Corollary 4.3 for even values.

Corollary 4.5. *For every $p \geq 2$, we have*

$$\frac{2p^2}{2p^2+p-1} \leq \frac{f(2p)}{f(2p-2)} \leq \frac{2p^2+2p-1}{(p+2)(2p-1)}. \quad (4.16)$$

We can apply these bounds to establish convexity of $f(m)$ for even values, just as we did for odd values:

$$\begin{aligned} \frac{1}{2} \left(\frac{f(2p+2)}{f(2p)} + \frac{f(2p-2)}{f(2p)} \right) &\geq \frac{1}{2} \left(\frac{2p^2+4p+2}{2p^2+5p+2} + \frac{(p+2)(2p-1)}{2p^2+2p-1} \right) = \\ &= \frac{8p^4+28p^3+25p^2-4p-6}{8p^4+28p^3+24p^2-2p-4} > 1. \end{aligned}$$

Hence, Corollary 4.5 implies the following statement.

Corollary 4.6. *The sequence $(f(2p))_{p=2}^{\infty}$ is convex.*

Although the sequence $(f(2p))_{p=2}^{\infty}$ seems to be log-convex as illustrated on Figure 10, the bounds (4.16) lead only to a slightly weaker inequality:

$$\frac{f(2p+2)}{f(2p)} \cdot \frac{f(2p-2)}{f(2p)} \geq \frac{2p^2+4p+2}{2p^2+5p+2} \cdot \frac{(p+2)(2p-1)}{2p^2+2p-1} = \frac{4p^3+6p^2-2}{4p^3+6p^2-1} < 1.$$

5 Hyperplane sections of the cube

We already mentioned in Section 1 that our research on the Laplace-Pólya integral originated from a topic in convex geometry, namely hyperplane sections of the cube. Now we discuss this subject in more detail and represent some consequences of our results. Furthermore, we intend to give a theoretical generalization of our methods from Section 3, which would lead to fascinating results, as we shall see in Subsection 5.2.

Let $Q_n = \left[-\frac{1}{2}, \frac{1}{2}\right]$ denote the centered n -dimensional unit cube. For a given nonzero vector $\mathbf{v} \in \mathbb{R}^n$, we introduce the *central section function*

$$\sigma(\mathbf{v}) = \text{Vol}_{n-1}(Q_n \cap \mathbf{v}^\perp) \quad (5.1)$$

where $\mathbf{v}^\perp$ denotes the central hyperplane with normal vector $\mathbf{v}$. We are primarily interested in the volume of the hyperplane sections of Q_n which is orthogonal to its main diagonal. In order to be historically consistent, we denote this by

$$I(n) := \sigma(\mathbf{1}_n), \quad (5.2)$$

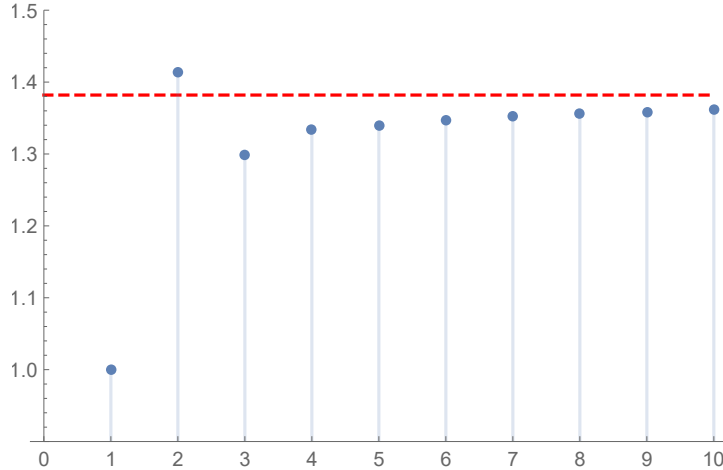


Figure 11: Diagonal hyperplane sections of the cube up to dimension $n = 10$.

where $\mathbf{1}_n$ denotes the n -dimensional vector $(1, \dots, 1)$. These sections prove to be essential for studying the critical points of (5.1). Pournin [62] and the authors of the present work [3] independently showed that the volume of the section perpendicular to the main diagonal of the cube is locally strictly maximal for dimensions of at least 4. It is worth noting that the key step in our proof relied on Corollary 2.3.

Extrema and critical points of $\sigma(\mathbf{v})$ have been intensively studied in the last 50 years. We only summarize here the relevant results; for further details see [56].

It is natural to expect that the sequence $(I(n))_{n=1}^{\infty}$ is bounded. The sharp bound is $1 = I(1)$, as Hadwiger [42] proved in 1972. Hensley [45] gave an alternative proof with a probabilistic argument a few years later. He also provided an upper bound on $I(n)$ and formulated a conjecture about the sharp estimate. This was proven by Ball [5] who showed that $I(n)$ is maximal for $n = 2$, that is, maximal central sections are orthogonal to the main diagonal of a 2-dimensional face of Q_n .

What can we tell about the volume of the other diagonal sections? Hensley [45] described a probabilistic argument, proving that

$$\lim_{n \rightarrow \infty} I(n) = \sqrt{\frac{6}{\pi}},$$

which is slightly less than $I(2) = \sqrt{2}$. König and Koldobsky [48] investigated (5.2) in a more general concept, leading to the fact that $I(n)$ stays below $\sqrt{\frac{6}{\pi}}$ for $n \geq 3$.

The above results complete the characterization regarding boundedness of $(I(n))_{n=1}^{\infty}$. Naturally, the following question arises: What other aspects does the sequence have? Possible properties include monotonicity, convexity, log-convexity, and so on. It had been conjectured that the sequence $(I(n))_{n=1}^{\infty}$ is strictly monotonically increasing for $n \geq 3$. This was shown by Bartha, Fodor, and González Merino:

Theorem 5.1 ([8, Theorem 1]). *The volume $I(n)$ is a strictly monotonically increasing function of n for all $n \geq 3$.*

Based on the asymptotic expansion of Pólya's integral formula, they provided very strong estimates of $I(n)$ with the Laplace method, which allowed to prove monotonicity from some specific n_0 onward. The remaining cases were checked with numerical methods. As the authors noted: 'Using the same ideas as above, one could show the concavity of $I(n)$ for $n \geq 3$.' Yet, the calculations are proven to be too intricate for publication.

The results listed above are all based on the same formula: Pólya [61] proved that the central section function $\sigma(\mathbf{u})$ of a *unit* vector $\mathbf{u} = (u_1, \dots, u_n) \in S^{n-1}$ can be evaluated using the classical integral formula

$$\sigma(\mathbf{u}) = \frac{1}{\pi} \int_{-\infty}^{\infty} \prod_{i=1}^n \text{sinc}(u_i t) dt. \quad (5.3)$$

One can shortly derive an extension for arbitrary non-zero normal vectors via the probabilistic interpretation. For details, see [3, Section 2]. However, we only need the formula for main diagonals, that is,

$$I(n) = \frac{1}{\pi} \int_{-\infty}^{\infty} \text{sinc}^n\left(\frac{t}{\sqrt{n}}\right) dt = \frac{\sqrt{n}}{\pi} \int_{-\infty}^{\infty} \text{sinc}^n t dt = \sqrt{n} J_n(0). \quad (5.4)$$

This relation shows that characterizing the sequence $I(n)$ requires precise estimates of the central values of the Laplace-Pólya integral. In particular, the fact of monotonicity stated in Theorem 5.1 is equivalent to the following inequality:

$$\sqrt{\frac{n}{n+1}} \leq \frac{J_{n+1}(0)}{J_n(0)}. \quad (5.5)$$

Notice that (5.5) is quite similar to (2.16) in Corollary 2.3. However, this time we need to estimate the ratio of consecutive central values of the Laplace-Pólya integral.

Our original motivation was to give a combinatorial proof of (5.5) using the methods presented in Section 3, which would provide a nice and much simpler proof of Theorem 5.1. First, we tried to conclude this fact from the already verified Corollary 2.3. However, this proves to be impossible, as we shall show in the next subsection. Hence, we tried to find another methods; we shall demonstrate some possibilities in the last part of the section.

5.1 Applying the lower bound of the two-step ratio

We now present a hypothetical proof based on Corollary 2.3 and induction by n . Suppose that we would like to prove a lower bound

$$\alpha_n \leq \frac{J_{n+1}(0)}{J_n(0)} \quad (5.6)$$

which verifies monotonicity (5.5), that is we need

$$\alpha_n \geq \sqrt{\frac{n}{n+1}}. \quad (5.7)$$

Suppose that (5.6) holds for some n . Then we need to show that

$$\alpha_{n+1} \leq \frac{J_{n+2}(0)}{J_{n+1}(0)}$$

is also true. According to Corollary 2.3, we have

$$\frac{n}{n+1} \leq \frac{J_{n+2}(0)}{J_n(0)},$$

hence if we could show

$$\alpha_n \alpha_{n+1} \leq \frac{n}{n+1} \quad (5.8)$$

then we would be done. However, since we need (5.7), for (5.8) it would mean that

$$\sqrt{\frac{n}{n+1}} \cdot \sqrt{\frac{n+1}{n+2}} \leq \alpha_n \alpha_{n+1} \leq \frac{n}{n+1}.$$

By equivalent transforms this leads to the inequality

$$n^2 + 2n + 1 \leq n^2 + 2n$$

which is clearly false. Therefore, this approach seems to fail, even though a stronger lower estimate of the two-step ratio might imply the required inequality.

5.2 Parallel induction

Here we present a second, more favorable approach to prove (5.5). Let us suppose that we have the following bounds

$$a_{n,r} \leq \frac{J_n(r+1)}{J_n(r)} \leq b_{n,r} \quad (5.9)$$

We have to ensure strict estimates for $r = 0$. Inequality (5.5) suggests that we need

$$a_{n,0} \geq \frac{n}{n+1} \cdot \sqrt{\frac{n}{n+1}}.$$

If we choose the lower bound to be the right hand side above, then the general lower bound would contain a square root function. Hence, justifying the induction inequalities would become literally impossible. Therefore, we decided to look for bounds which have a simpler form. Numerical calculations suggested that the estimates

$$a_{n,0} = \frac{n}{n+1} \cdot \frac{20n+11}{20n+21} \quad \text{and} \quad b_{n,0} = \frac{n}{n+1} \cdot \frac{20n+12}{20n+21}$$

would be acceptable. Similarly to two-step decay ratios, we extended the above bounds naturally and so we ran our search of the form

$$a_{n,r} = \frac{n-r}{n+r+1} \cdot \frac{20n+er+11}{20n+fr+21} \quad \text{and} \quad b_{n,r} = \frac{n-r}{n+r+1} \cdot \frac{20n+gr+12}{20n+hr+21}$$

where e, f, g, h have to be chosen in such a way that the recursion works. But how do the recursion inequalities look like this time? Well, we found two forms of these inequalities such that the proof works via recursion. Here we shall present the theoretical background of our parallel inductive proof.

Suppose that we have proved the inequalities

$$a_{n,r} \leq \frac{J_n(r+1)}{J_n(r)} \leq b_{n,r} \quad (5.10)$$

for all r in the region¹⁸ $-1 \leq r \leq n-1$. We need to show that

$$a_{n+1,r} \leq \frac{J_{n+1}(r+1)}{J_{n+1}(r)} \leq b_{n+1,r}$$

for each r with $-1 \leq r \leq n$. Obviously, we need to apply (2.9) in order to use the induction hypothesis (5.10). In the two-step decay ratio, we had a quite trivial choice of how we should perform that. This time it may be performed with different approaches; we present two of them.

First method. Based on (2.9) we have

$$\begin{aligned} \frac{J_n(r+1)}{J_{n+1}(r)} &= \frac{\frac{n+r+2}{2n} J_n(r+2) + \frac{n-r}{2n} J_n(r)}{\frac{n+r+1}{2n} J_n(r+1) + \frac{n-r+1}{2n} J_n(r-1)} = \\ &= \frac{(n+r+2)J_n(r+2) + (n-r)J_n(r)}{(n+r+1)J_n(r+1) + (n-r+1)J_n(r-1)} \geq \\ &\geq \frac{(n+r+2)a_{n,r+1}a_{n,r}J_n(r) + (n-r)J_n(r)}{(n+r+1)J_n(r+1) + (n-r+1)\frac{1}{a_{n,r-1}a_{n,r}}J_n(r+1)} \geq \\ &\geq \frac{(n+r+2)a_{n,r+1}a_{n,r} + (n-r)}{(n+r+1) + (n-r+1) \cdot \frac{1}{a_{n,r-1}a_{n,r}}} \cdot \frac{1}{b_{n,r}} \stackrel{?}{\geq} a_{n+1,r} \end{aligned}$$

and similarly for the upper bound

$$\begin{aligned} \frac{J_n(r+1)}{J_{n+1}(r)} &= \frac{\frac{n+r+2}{2n} J_n(r+2) + \frac{n-r}{2n} J_n(r)}{\frac{n+r+1}{2n} J_n(r+1) + \frac{n-r+1}{2n} J_n(r-1)} = \\ &= \frac{(n+r+2)J_n(r+2) + (n-r)J_n(r)}{(n+r+1)J_n(r+1) + (n-r+1)J_n(r-1)} \leq \\ &\leq \frac{(n+r+2)b_{n,r+1}b_{n,r}J_n(r) + (n-r)J_n(r)}{(n+r+1)J_n(r+1) + (n-r+1)\frac{1}{b_{n,r-1}b_{n,r}}J_n(r+1)} \leq \\ &\leq \frac{(n+r+2)b_{n,r+1}b_{n,r} + (n-r)}{(n+r+1) + (n-r+1) \cdot \frac{1}{b_{n,r-1}b_{n,r}}} \cdot \frac{1}{a_{n,r}} \stackrel{?}{\leq} b_{n+1,r}. \end{aligned}$$

Hence, the induction inequalities take the form

$$\begin{aligned} (n+r+2)a_{n,r+1}a_{n,r} + (n-r) &\geq a_{n+1,r}b_{n,r} \left((n+r+1) + (n-r+1)\frac{1}{a_{n,r-1}a_{n,r}} \right), \\ (n+r+2)b_{n,r+1}b_{n,r} + (n-r) &\leq b_{n+1,r}a_{n,r} \left((n+r+1) + (n-r+1)\frac{1}{b_{n,r-1}b_{n,r}} \right). \end{aligned} \quad (5.11)$$

¹⁸Or at least in a part of this region. It is not excluded that we have to face the same problem as in Subsection 3.2 in the proof, though we can handle it similarly.

Moreover, we have to check estimates at the beginning in order to start the recursion in the r th row. That is, we need for $r = -1$

$$\frac{J_n(0)}{J_n(-1)} \geq \frac{1}{b_{n,0}} \stackrel{?}{\geq} a_{n,-1} \quad \text{and} \quad \frac{J_n(0)}{J_n(-1)} \geq \frac{1}{a_{n,0}} \stackrel{?}{\geq} b_{n,-1}. \quad (5.12)$$

If one finds parameters e, f, g, h such that they satisfy inequalities (5.11) and (5.12), then it will imply (5.5).

Note that in these forms, the estimates for the two-step decay already occur as a product of one-step decay estimates. Based on the previous subsection, this is necessary in fact, because the former ones turn out to be weaker¹⁹. Thus, we suspect that if we can provide the latter ones, then we could strengthen the estimates of two-step decay.

Second method. Based on (2.9) we have

$$\begin{aligned} \frac{J_n(r+1)}{J_{n+1}(r)} &= \frac{\frac{n+r+2}{2n} J_n(r+2) + \frac{n-r}{2n} J_n(r)}{\frac{n+r+1}{2n} J_n(r+1) + \frac{n-r+1}{2n} J_n(r-1)} = \\ &= \frac{(n+r+2)J_n(r+2) + (n-r)J_n(r)}{(n+r+1)J_n(r+1) + (n-r+1)J_n(r-1)} \geq \\ &\geq \frac{(n+r+2)a_{n,r+1}J_n(r+1) + (n-r)\frac{1}{b_{n,r}}J_n(r+1)}{(n+r+1)b_{n,r}J_n(r) + (n-r+1)\frac{1}{a_{n,r-1}}J_n(r)} \geq \\ &\geq \frac{(n+r+2)a_{n,r+1} + (n-r)\frac{1}{b_{n,r}}}{(n+r+1)b_{n,r} + (n-r+1)\frac{1}{a_{n,r-1}}} \stackrel{?}{\geq} a_{n+1,r} \end{aligned}$$

and similarly for the upper bound

$$\begin{aligned} \frac{J_n(r+1)}{J_{n+1}(r)} &= \frac{\frac{n+r+2}{2n} J_n(r+2) + \frac{n-r}{2n} J_n(r)}{\frac{n+r+1}{2n} J_n(r+1) + \frac{n-r+1}{2n} J_n(r-1)} = \\ &= \frac{(n+r+2)J_n(r+2) + (n-r)J_n(r)}{(n+r+1)J_n(r+1) + (n-r+1)J_n(r-1)} \leq \\ &\leq \frac{(n+r+2)b_{n,r+1}J_n(r+1) + (n-r)\frac{1}{a_{n,r}}J_n(r+1)}{(n+r+1)a_{n,r}J_n(r) + (n-r+1)\frac{1}{b_{n,r-1}}J_n(r)} \leq \\ &\leq \frac{(n+r+2)b_{n,r+1} + (n-r)\frac{1}{a_{n,r}}}{(n+r+1)a_{n,r} + (n-r+1)\frac{1}{b_{n,r-1}}} \stackrel{?}{\leq} b_{n+1,r} \end{aligned}$$

Hence, the induction inequalities take the form

$$\begin{aligned} a_{n,r} \left((n+r+2)a_{n,r+1} + (n-r)\frac{1}{b_{n,r}} \right) &\geq a_{n+1,r} \left((n+r+1)b_{n,r} + (n-r+1)\frac{1}{a_{n,r-1}} \right), \\ b_{n,r} \left((n+r+2)b_{n,r+1} + (n-r)\frac{1}{a_{n,r}} \right) &\leq b_{n+1,r} \left((n+r+1)a_{n,r} + (n-r+1)\frac{1}{b_{n,r-1}} \right). \end{aligned} \quad (5.13)$$

¹⁹At least for those forms we proved previously in Theorem 2.1 and 2.2

Moreover, we have to check estimates at the beginning in order to start the recursion in the r th row. These are naturally the same as in the previous method:

$$\frac{J_n(0)}{J_n(-1)} \geq \frac{1}{b_{n,0}} \stackrel{?}{\geq} a_{n,-1} \quad \text{and} \quad \frac{J_n(0)}{J_n(-1)} \geq \frac{1}{a_{n,0}} \stackrel{?}{\geq} b_{n,-1}. \quad (5.14)$$

If one finds parameters e, f, g, h such that they satisfy inequalities (5.13) and (5.14), then it will imply (5.5).

We preferred the second method over the first because it shows more symmetry between the two estimates. However, we could not find yet the appropriate parameters such that either of the recursive inequality pairs is satisfied.

6 Conclusion

The central topic of the work is the Laplace-Pólya integral. As we noted earlier, along with its relatives, it has several applications in mathematics, physics, and engineering. Here, we restricted our investigation to integral values of the Laplace-Pólya integral, which led to interesting and important properties.

The main result of this work gives a lower bound for the two-step decay ratio of the function $J_n(r)$. This nicely complements our result from [3] which provides an upper bound for the same quantity. The bound was previously proven only for even values of n and $r = 0$ by Lesieur and Nicolas [50], whereas we achieved it for every possible pair in the domain under study. Our arguments presented in Section 3 are purely combinatorial, as opposed to the previous proof which is based on analytic methods.

We also discussed the possibility of pushing our method further in order to extend the bound proven by Lesieur and Nicolas [50] to its full generality. These possibilities are discussed in detail in Section 3. We also intend to explore the limitations of the combinatorial method. Some interesting combinatorial consequences based on the connection with Eulerian numbers are also surveyed here, including estimates for the maxima, Costas arrays, or star polygons. We wish to extend and clarify these connections, which might lead to other interesting results in the field of combinatorics.

In Section 5 we mapped the connection to the volume of hyperplane sections, which have been studied for over a century. Our long-term objective is to provide a combinatorial proof for the monotonicity of the volume of central hyperplane sections, which was proven by Bartha, Fodor and González Merino [8] by thorough and precise analytical tools. We describe this alternative approach along with its theoretical background. We believe that refining the criteria for finding the estimates may lead to strong results about the Laplace-Pólya integral.

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