

COMPUTATIONAL STABILITY ANALYSIS OF LOTKA-VOLTERRA SYSTEMS

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This paper concerns the computational stability analysis of locally stable Lotka-Volterra (LV) systems by searching for appropriate Lyapunov functions in a general quadratic form composed of higher order monomial terms. The Lyapunov conditions are ensured through the solution of linear matrix inequalities. The stability region is estimated by determining the level set of the Lyapunov function within a suitable convex domain. The paper includes interesting computation results and discussion on the stability regions of higher (3,4) dimensional LV models as well as on the monomial selection for constructing the Lyapunov functions. Finally, the stability region is estimated of an uncertain 2D LV system with an uncertain interior locally stable equilibrium point.

Keywords: nonlinear systems, Lotka-Volterra models, stability analysis, linear matrix inequalities, Lyapunov function

Introduction

Approximating the domain of attraction (DOA) is often a fundamental task in the analysis and control of nonlinear systems. The stability properties of dynamical systems are most often studied using Lyapunov functions. Therefore, an extensive literature exists on the computational construction of Lyapunov functions [1].

Due to their advantageous properties and the availability of efficient numerical solvers, the use of linear matrix inequalities (LMI) and semidefinite programming (SDP) techniques became popular in the field of system and control theory. Important results were announced in [2, 3] in the context of linear uncertain systems, their stability analysis and control synthesis.

Recently, an optimization based method for DOA estimation was published in [4], where the authors use Finsler's lemma and affine parameter dependent LMIs to compute rational Lyapunov functions for a wide class of locally asymptotically stable nonlinear systems. Based on these results an improved method was published in [5, 6], where the transformation of the model to the form required for optimization is done automatically using the linear fraction transformation (LFT) and further automatic model simplification steps, which results in the dimension reduction of the optimization task. As the dimensions of the problem are reduced the method is capable to handle more complex models. For example, in [6]

the DOA is successfully estimated for a 3-dimensional rational uncertain system (bioreactor model with an applied proportional and integral substrate feedback law).

The dynamical descriptive power of Lotka-Volterra systems is so extensive that LV models "have the status of canonical format" within the class of smooth nonlinear dynamical systems [7]. Besides modeling biological/ecological environments, they are widely used in other scientific fields like neural nets [8] or in economy, where the Goodwin-Lotka-Volterra models are applied for modeling the predator-pray mechanism of the technological substitution [9, 10]. In order to model the correlation between the employment rate and the wage share of the working population, the authors of [11] used a stochastic extension of the Goodwin-model.

Some important results that makes LV models even more attractive are the existing techniques to represent a general nonlinear system as a multidimensional LV model [12]. The analysis of stability and behaviour of LV systems has an extended literature [13]. In [14], Plank has shown that n-dimensional LV systems are Hamiltonian if they fulfill certain algebraic properties [14, Theorem 3.1]. He demonstrated that, when using an appropriate Poisson structure, one can obtain the system's Hamiltonian function, which is a key object in determining the system's DOA. Furthermore, Hamiltonian function candidates were defined in [14, Section IV.]. It is important to note that LV systems are kinetic in the sense that they can be formally described as chemical reaction networks

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with mass action kinetics.

In this paper, the local stability properties of LV models are analysed by applying the underlying method presented in [6]. The stability region is estimated on a few locally stable 2-, 3-, and 4-dimensional benchmark LV systems. The DOA of an uncertain 2D system is also estimated by two concentric regions (one contained by the other).

The main motivation of our work was to evaluate the applicability of the approach [4] on a general polynomial system class having low-degree monomials, and to study the limits of the method as the number of dimensions of the state space increase.

Background

In this section, we present the basic notions and results on which our computational method is based.

System class, Lyapunov functions, and the domain of attraction.

Our computational method can handle general nonlinear systems of the form

$$\begin{aligned} \dot{x}(t) &= f(x(t), \delta(t)) \\ x(t) &\in \mathbb{R}^n, x_0 \in \mathcal{X}, \delta(t) \in \mathcal{D}, \dot{\delta}(t) \in \tilde{\mathcal{D}} \end{aligned} \quad (1)$$

where $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{D}, \tilde{\mathcal{D}} \subset \mathbb{R}^d$ are given polytopes, x is the state vector function with its initial condition $x_0 = x(0)$, and δ is a smooth, bounded vector function of uncertain parameters with bounded time derivative. The applied method is presented in detail in technical report [15]. From now on, the time arguments of x and δ will be suppressed as it is commonly done in the literature. We assume that function $f : \mathbb{R}^n \times \mathcal{D} \mapsto \mathbb{R}^n$ of Eq. (1) is a well defined smooth *rational mapping*, with the property $f(0, \delta) = 0$ for all $\delta \in \mathcal{D}$. Additionally, we assume that $x^* = 0 \in \mathbb{R}^n$ is a locally asymptotically stable equilibrium point of (1) for all $\delta \in \mathcal{D}$. The set of all initial conditions, from which the solutions converge to x^* is called the *domain of attraction* (DOA). Furthermore, we assume that function $f(x, \delta)$ can be written in a so-called quasi-LPV form $f(x, \delta) = \mathcal{A}(x, \delta)x$.

We are looking for an appropriate rational Lyapunov function $V(x, \delta)$ which satisfies the following conditions:

$$\begin{aligned} \underline{\alpha}(\|x\|) &\leq V(x, \delta) \leq \bar{\alpha}(\|x\|), \forall (x, \delta) \in \mathcal{X} \times \mathcal{D} \\ \dot{V}(x, \delta, \dot{\delta}) &\leq -\alpha(\|x\|), \forall (x, \delta, \dot{\delta}) \in \mathcal{X} \times \mathcal{D} \times \tilde{\mathcal{D}} \end{aligned} \quad (2)$$

where $\underline{\alpha}, \bar{\alpha}, \alpha$ are continuous strictly positive functions, and being zero if $\|x\| = 0$. Due to conditions (2) any closed level set of the Lyapunov function contained entirely in $\mathcal{X}$ bounds an invariant region of the state space.

Our objective is to find a Lyapunov function which fulfills conditions (2) and to determine its maximal invariant level set completely inside $\mathcal{X}$.

Dynamical system representation

In [4], a Lyapunov function is computed in the form:

$$V(x, \delta) = \pi_b^T(x, \delta) P \pi_b(x, \delta), \quad \pi_b = \begin{bmatrix} x \\ \pi \end{bmatrix} \quad (3)$$

where $P \in \mathbb{R}^{m \times m}$ is a *constant symmetric* matrix, not necessarily positive definite, and $\pi : \mathbb{R}^n \times \mathcal{D} \mapsto \mathbb{R}^p$ is a mapping, in which each element is a monomial in (x, δ) , or a smooth rational function with monomial numerator. The arguments of π and π_b will be suppressed below.

The application of the linear fractional transformation (LFT) and further algebraic steps are proposed in [5] and [6] to transform the system equation $\dot{x} = f(x, \delta) = \mathcal{A}(x, \delta)x$ into the desired differential-algebraic representation that was introduced in the same references:

$$\begin{aligned} \dot{x} &= \mathcal{A}(x, \delta)x = Ax + B\pi \quad x_0 \in \mathcal{X} \\ 0 &= \mathcal{N}_{\pi_b}(x, \delta)\pi_b \quad \delta \in \mathcal{D}, \dot{\delta} \in \tilde{\mathcal{D}} \end{aligned} \quad (4)$$

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times p}$ are constant matrices, and $\mathcal{N}_{\pi_b}(x, \delta) \in \mathbb{R}^{q \times m}$ is an *affine* matrix function in (x, δ) also called as an “annihilator”. These transformations also result in the dimension reduction of the optimization problem compared to the results presented in [4]. Representation (4) separates the linear part of the system (x) from its nonlinear part (π).

The SDP problem for estimating the DOA

In this section, we present how to construct the semidefinite optimization task (SDP), which will uniquely characterize an appropriate Lyapunov function and its maximal invariant level set ε_α as an estimate of the true domain of attraction. We assume that polytopes $\mathcal{X}, \mathcal{Y}$ and $\mathcal{D}$ are already given.

Using the model representation (4), and having a Lyapunov function candidate of the form (3), we can formulate sufficient LMIs for the Lyapunov conditions (2). According to Finsler’s lemma [4, Lemma 2.1], if there exist real-valued matrices $L_b \in \mathbb{R}^{m \times q}$ and $L_a \in \mathbb{R}^{(n+2p+n^2+np) \times (2q+n^2+nq)}$, s.t.

$$\begin{aligned} \forall (x, \delta) \in \vartheta(\mathcal{X} \times \mathcal{D}) : \\ P + L_b \mathcal{N}_{\pi_b}(x, \delta) + \mathcal{N}_{\pi_b}^T(x, \delta) L_b^T > 0 \end{aligned} \quad (5)$$

$$\begin{aligned} \forall (x, \delta, \dot{\delta}) \in \vartheta(\mathcal{X} \times \mathcal{D} \times \tilde{\mathcal{D}}) : \\ P_a + P_a^T + L_a \mathcal{N}_{\pi_a}(x, \delta, \dot{\delta}) + \mathcal{N}_{\pi_a}^T(x, \delta, \dot{\delta}) L_a^T < 0 \end{aligned} \quad (6)$$

then conditions (2) are met for every $(x, \delta, \dot{\delta}) \in \mathcal{X} \times \mathcal{D} \times \check{\mathcal{D}}$. Variables P_a and $\mathcal{N}_{\pi_a}(x, \delta, \dot{\delta})$ are defined in [15] in Eqs. (3.16) and (3.22), respectively. $\vartheta(\cdot)$ denotes the corner points (vertices) of a given polytope.

Since the Lyapunov conditions are satisfied within $\mathcal{X}$, we are looking for the maximal α -level set

$$\varepsilon_\alpha = \{x \in \mathcal{X} \mid V(x) = \alpha, 1 \leq \alpha\},$$

which lies inside $\mathcal{X}$. That level set will be *invariant*, in the sense that every trajectory entering this region will never leave it.

We define $\alpha \geq 1$ as a free variable of the optimization task, and we constrain ε_α to be inside $\mathcal{X}$. One can observe that by maximizing α we obtain an unbounded feasible solution, as the function $V(x, \delta)$ can be scaled arbitrarily. Therefore, as Fig. 1 illustrates, we define an auxiliary polytope $\mathcal{Y}$ around the locally stable origin inside $\mathcal{X}$, constraining the 1-level set $\varepsilon_1 = \{x \in \mathcal{X} \mid V(x) = 1\}$ to be around $\mathcal{Y}$.

According to Finsler's lemma, for every $k = \overline{1, M_{\mathcal{X}}}$ and $l = \overline{1, M_{\mathcal{Y}}}$, if there exist real-valued matrices $L_{c_k}, \bar{L}_{c_l} \in \mathbb{R}^{m \times q}$, $M_{c_k}, \bar{M}_{c_l} \in \mathbb{R}^{(m+1) \times n}$, s.t.

$$Q_k^T P_{c_k}^{(\alpha)}(x, \delta) Q_k \geq 0, \quad \forall (x, \delta) \in \vartheta(\mathcal{F}_k^{(\mathcal{X})} \times \mathcal{D}) \quad (7)$$

$$\bar{Q}_k^T \bar{P}_{c_k}^{(1)}(x, \delta) \bar{Q}_k \leq 0, \quad \forall (x, \delta) \in \vartheta(\mathcal{F}_k^{(\mathcal{Y})} \times \mathcal{D}) \quad (8)$$

than ε_α is inside $\mathcal{X}$ and ε_1 is around $\mathcal{Y}$. Variables $Q_k, \bar{Q}_k, P_{c_k}^{(\alpha)}(x, \delta), \bar{P}_{c_k}^{(1)}(x, \delta)$ are defined in [15] in Eqs. (3.36) and (3.34), $\mathcal{F}_k^{(\mathcal{X})}$ denotes the k th facet of $\mathcal{X}$, furthermore, $M_{\mathcal{X}}$ denotes the number of facets of $\mathcal{X}$.

The obtained LMI conditions (5), (6), (7) and (8) are affine parameter dependent LMIs, which can be computationally handled by checking their feasibility at the corner points of the polytopic region, on which the parameters $x, \delta, \dot{\delta}$ are defined. Depending on the number of the corner points we obtain a given number of parameter *independent* LMI conditions (Table 2).

The SDP task can be summarized as follows. In order to find the maximal invariant level set ε_α of the Lyapunov function, one should maximize α , under the following conditions:

- Eq. (5) $V(x, \delta)$ is positive on $\mathcal{X} \times \mathcal{D}$
- Eq. (6) $\dot{V}(x, \delta, \dot{\delta})$ is negative on $\mathcal{X} \times \mathcal{D} \times \check{\mathcal{D}}$
- Eq. (7) ε_α lies inside $\mathcal{X}$
- Eq. (8) $\mathcal{Y}$ is inside ε_1

The number of LMIs and their dimensions are given in Table 2, furthermore, the free variables of the optimization task are listed in Table 1.

Finding the most appropriate outer polytope

In case of 2D systems, polytope $\mathcal{X}$ is evaluated manually through iterations. When having a higher-dimensional

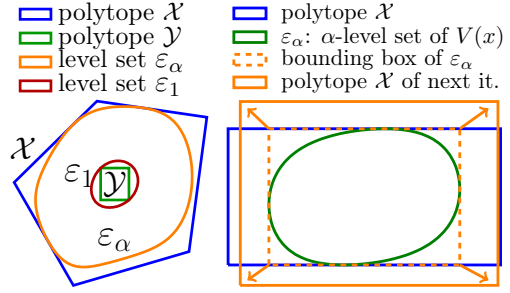


Figure 1: (Left) Illustration of the conditions regarding $\mathcal{X}$ and $\mathcal{Y}$. Inside polytope $\mathcal{X}$ the Lyapunov conditions (2) are required. The α -level set ε_α of the Lyapunov function should lie inside polytope $\mathcal{X}$ (hence it is invariant). The 1-level set ε_1 should be around polytope $\mathcal{Y}$. This condition ensures that the problem has a bounded solution.

Figure 2: (Right) Polytope $\mathcal{X}$ is evaluated through iterations considering the bounding box of the obtained α -level set.

matrix var.	number of (scalar) independent decision variables appearing in the SDP
α	1
P	$\frac{1}{2} m(m+1)$
L_b	mq
L_a	$(n+2p+n^2+np)(2q+n^2+nq)$
L_{c_k} and $\bar{L}_{c_l}$	$(M_{\mathcal{X}} + M_{\mathcal{Y}}) \times mq$
M_{c_k} and $\bar{M}_{c_l}$	$(M_{\mathcal{X}} + M_{\mathcal{Y}}) \times (m+1)n$
The dimensional parameters are the following:	
n	number of state variables (size of x)
p	number of elements in π
m	number of elements in π_b ($n+p$)
q	number of rows in annihilator $\mathcal{N}_{\pi_b}(x, \delta)$
$M_{\mathcal{X}}$	number of corner points of $\mathcal{X}$
$M_{\mathcal{Y}}$	number of corner points of $\mathcal{Y}$
$M_{\mathcal{D}}$	number of corner points of $\mathcal{D}$
$M_{\check{\mathcal{D}}}$	number of corner points of $\check{\mathcal{D}}$

Table 1: Free variables of the optimization task and the number of (scalar) symbolic decision variables they introduce into the optimization task.

Eq. nr.	Dimension of the LMIs	Nr. of LMIs
(5)	m	$M_{\mathcal{X}} \cdot M_{\mathcal{D}}$
(6)	$n+2p+n^2+np$	$M_{\mathcal{X}} \cdot M_{\mathcal{D}} \cdot M_{\check{\mathcal{D}}}$
(7)	$m+1$	$2 M_{\mathcal{X}} \cdot M_{\mathcal{D}}$
(8)	$m+1$	$2 M_{\mathcal{Y}} \cdot M_{\mathcal{D}}$

Table 2: Number and dimensions of parameter independent LMIs of the optimization task

system ($n \geq 3$), we applied the same iterative procedure as it was presented in [6]. This method starts from an initial polytope $\mathcal{X}^{(0)}$, then a new polytope $\mathcal{X}^{(1)}$ is defined by enlarging the axis aligned bounding box of the obtained α -level set $\varepsilon_\alpha^{(0)}$. During the iterations, $\mathcal{X}^{(i)}$ is constrained to be a rectangular polytope. Fig. 2. illustrates the operation of a single iteration step.

Lotka-Volterra systems

The n -dimensional LV equation has the form

$$\dot{\bar{x}} = \text{diag}(\bar{x})(A\bar{x} + b), \quad A \in \mathbb{R}^{n \times n}, \quad b \in \mathbb{R}^n, \quad (9)$$

where $\text{diag}(a)$, $a \in \mathbb{R}^n$ denotes an $n \times n$ square diagonal matrix with the $x_1, \dots, x_n$ on the main diagonal. We translate the system into its, by assumption, unique interior equilibrium point $\bar{x}^* = -A^{-1}b$ by introducing the centered state vector $x = \bar{x} - \bar{x}^*$. Then, we obtain an autonomous nonlinear system of the form $\dot{x} = \mathcal{A}(x)x$, where the matrix function $\mathcal{A}$ can be expressed as: $\mathcal{A}(x) = \text{diag}(x + \bar{x}^*)A$. Applying the LFT and the further algebraic model transformation steps, we obtain a model in representation (4), where the entries of $\pi(x)$ are second order monomials of the state variables.

Numerical results

In this section, we illustrate the applicability of the approach presented above through different locally stable Lotka-Volterra models. The results presented in this section have been computed in the Matlab environment. For symbolic computations, we used Matlab's built-in Symbolic Math Toolbox based on MuPAD. For linear fractional transformations (LFT), we used the Enhanced LFR-toolbox [16, 17]. To model and solve semidefinite optimization (SDP) problems we used Mosek solver with YALMIP [18].

2D Lotka-Volterra system

We consider a locally stable LV system with the model matrix

$$A = \begin{bmatrix} -2 & -3 \\ 1.4 & 1 \end{bmatrix},$$

and having a unique interior equilibrium point $\bar{x}^* = [1 \ 1]^T$. In Fig. 3 we can see some trajectories of the *centered* system from different initial conditions. The red trajectories converge to the equilibrium point, the blue ones do not tend to the equilibrium point.

After solving the corresponding SDP, the obtained Lyapunov function is $V(x) = \pi_b^T P \pi_b$ (Fig. 4), where:

$$\pi = [x_1^2 \ x_1 x_2 \ x_2^2]^T, \quad \pi_b = \begin{bmatrix} x \\ \pi \end{bmatrix} \quad (10)$$

$$P = \begin{bmatrix} 87.39 & 63.83 & -3.45 & 0 & 0 \\ 63.83 & 165.26 & 67.24 & 100.85 & 28.76 \\ -3.45 & 67.24 & 7.63 & 0.15 & 0 \\ 0 & 100.85 & 0.15 & 38.78 & 39.97 \\ 0 & 28.76 & 0 & 39.97 & 15.53 \end{bmatrix}$$

In Fig. 3, we can see the shape of the *manually chosen* polytope $\mathcal{X}$, and the obtained invariant α -level set of the Lyapunov function, which is considered as the estimated DOA (filled orange region). We can also observe that the areas where the Lyapunov function's time derivative is positive (green region) are completely outside of $\mathcal{X}$. Fig. 5 illustrates how the value of the Lyapunov function decreases along the trajectories.

3D Lotka-Volterra system

We have chosen a locally stable 3D LV system with the model matrix

$$A = \begin{bmatrix} 0.06 & 0.21 & 0.83 \\ -2.47 & -2.10 & -3.64 \\ 0.06 & 0.47 & -0.45 \end{bmatrix} \quad (11)$$

Similarly to the 2D example, the equilibrium point is set to $\bar{x}^* = [1 \ 1 \ 1]^T$. The monomial set appearing in π_b contains the state variables and every possible 2^{nd} order monomial:

$$\pi^T = [x_1^2 \ x_1 x_2 \ x_1 x_3 \ x_2^2 \ x_2 x_3 \ x_3^2] \quad (12)$$

Matrix P is a 9×9 symmetric matrix. Fig. 7 illustrates the invariant α -level set of the obtained Lyapunov function (3D red volume). In Fig. 6, one can see the cross-sections of the 3D invariant region from three different viewpoints. Some trajectories of the system are shown in red (stable) and blue (unstable).

4D Lotka-Volterra system

We applied the method to a locally stable 4D LV system having a unique interior equilibrium point

$$\bar{x}^* = [1 \ 1 \ 1 \ 1]^T. \quad (13)$$

The model matrix of the LV system was chosen to be:

$$A = \begin{bmatrix} -4.7126 & -1.5833 & 1.5346 & 2.323 \\ -9.2461 & -3.1634 & 2.8648 & 2.7796 \\ -14.258 & -4.5477 & 3.6104 & 4.6238 \\ -3.1687 & -0.8016 & 1.2287 & 1.2656 \end{bmatrix} \quad (14)$$

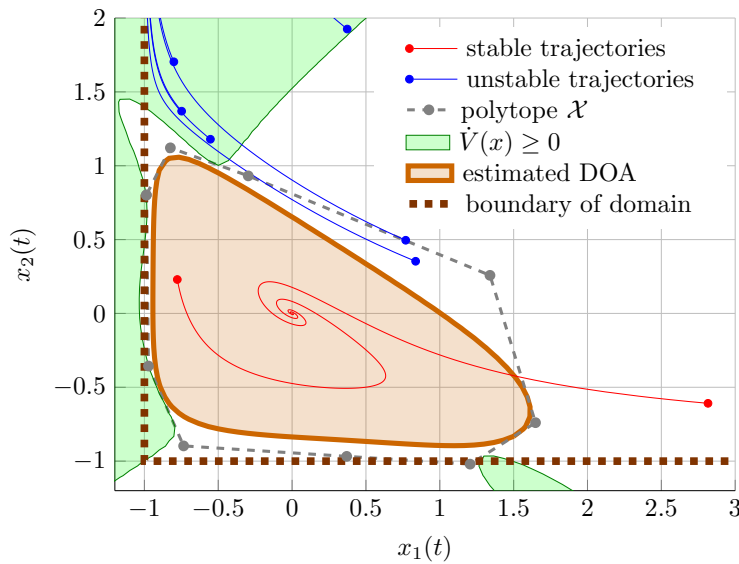


Figure 3: Phase diagram of the 2D locally stable system. Red trajectories converge to the origin, the blue ones do not converge to the origin. The estimated DOA and the used polytope $\mathcal{X}$ are illustrated by the orange filled region and by the gray dashed polytope, respectively. The green area highlights the subset of the state space where the Lyapunov function features a positive time-derivative.

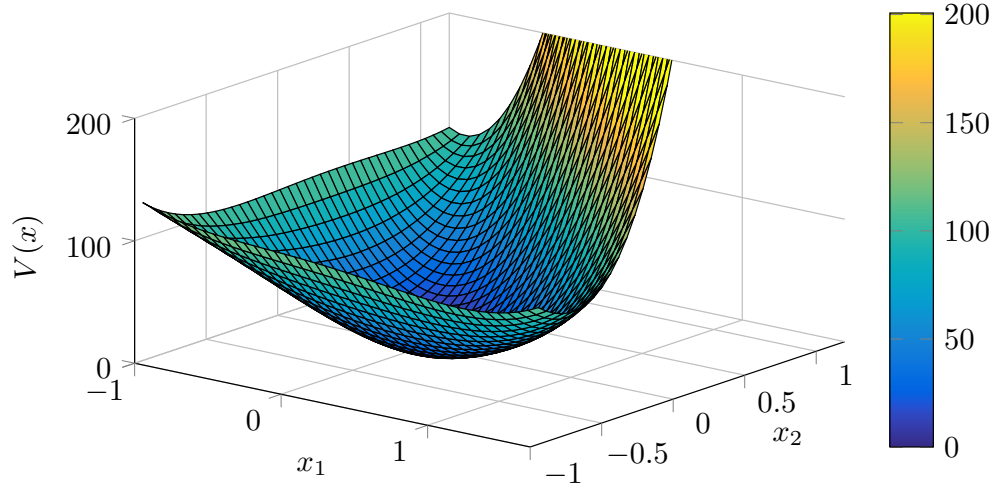


Figure 4: The obtained Lyapunov function in case of the 2D LV system

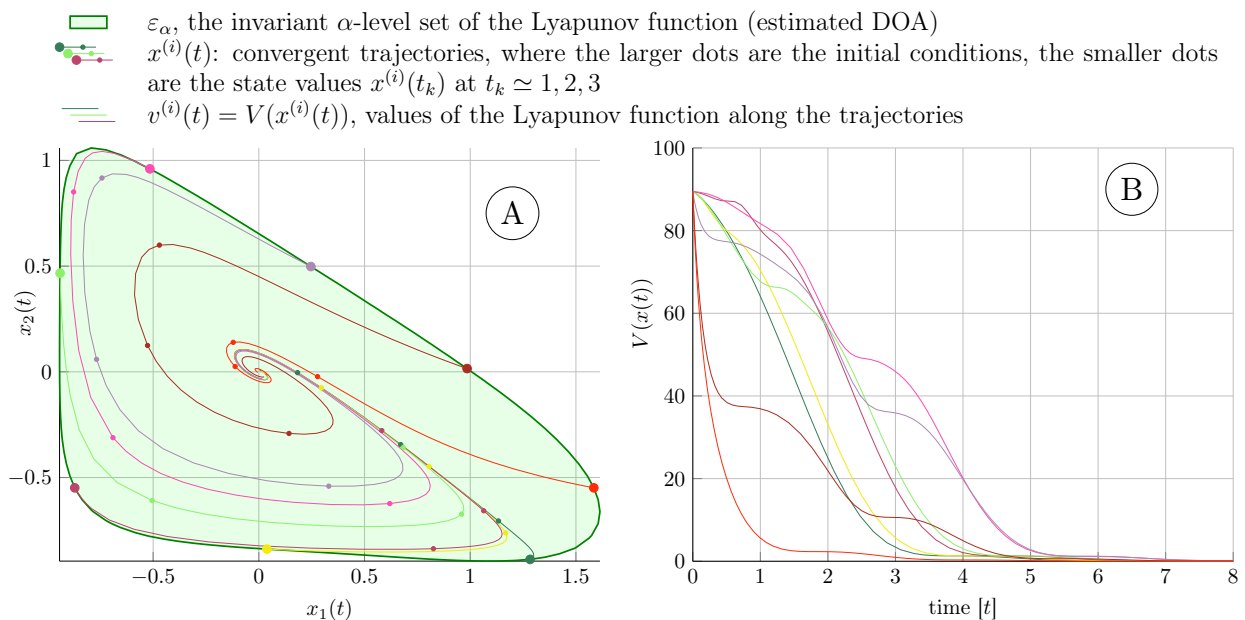


Figure 5: In Panel A, the obtained invariant level set is shown, furthermore some trajectories $x^{(i)}(t)$ of the system can be seen in different colors with initial conditions $x_0^{(i)}$ close to the boundary of the estimated DOA. The values of the Lyapunov function along the trajectories ($v^{(i)}(t)$) are presented in Panel B, respectively.

Due to the fact that model matrix A is a “full” matrix (i.e., there are no zero entries in it), every 2nd order monomial will appear in the equation of the system. As a consequence, the LFT will produce a model (4), in which π will contain every possible 2nd order monomial. This means that the number of monomials (including the state variables) is $m_b = (n^2 + 3n)/2$ in case of a “full” model matrix.

In the following table, we have summarized the number and sizes of the LMIs in the case of n -dimensional LV systems with a “full” model matrix:

Size	Number	Description
$m_b \times m_b$	2^n	positivity of $V(x)$
$m_a \times m_a$	2^n	negativity of $\dot{V}(x)$
$m_b \times m_b$	$n \cdot 2^n$	$\varepsilon_\alpha \in \mathcal{J}nt(\mathcal{X})$
$m_b \times m_b$	$n \cdot 2^n$	$\mathcal{Y} \in \mathcal{J}nt(\varepsilon_\alpha)$

where m_b is the size of the Lyapunov matrix P , $m_a = \frac{1}{2}(n^3 + 5n^2 + 4n)$ is the size of matrix $P_a(P)$, which appears in the LMI ensuring the negativity of the time derivative of the Lyapunov function. The exponential factor (2^n) comes from the rectangular shape of polytope $\mathcal{X}$ having 2^n corner points. On the other hand, the feasibility of LMI conditions (7) and (8) should be checked in each corner point of every facet of the polytope, furthermore an n -dimensional rectangular polytope has $2n$ facet with 2^{n-1} corner points. As we can see, a rectangular polytope introduces an exponential increase on the problems dimension, however, n -dimensional intervals can be easily handled compared to arbitrarily shaped polytopes defined by (hyper-) triangle meshes. In Figure 8, we illustrate the cross sections of the obtained invariant domain along the different pair of axes.

2D uncertain LV system with an uncertain equilibrium point

In this section, we present the same 2-dimensional system that appeared in Section with the same model parameters but with an additional uncertain term $K\delta$, where $K = [1 \ 1]^T$. We assume, that $\delta \in \mathcal{D} = [-0.1, 0.1]$ is an unknown *constant* parameter ($\dot{\delta} = 0$). The equation of the uncertain model is the following:

$$\dot{\bar{x}} = \text{diag}(\bar{x}) \left(A\bar{x} + K\delta + b \right) \quad (15)$$

The equilibrium point of the uncertain system is an affine function of the uncertain parameter δ :

$$\bar{x}^*(\delta) = A^{-1} \left(-K\delta - b \right) = -A^{-1}K\delta + \bar{x}_0^*, \quad (16)$$

where $\bar{x}_0^*$ is the equilibrium point of the system presented in Section (i.e. when $\delta = 0$). We translate the equation (15) into the following form:

$$\dot{\bar{x}} = \text{diag}(\bar{x}) A \left(\bar{x} - \bar{x}^*(\delta) \right) \quad (17)$$

The centered model of the system around the uncertain equilibrium point $\bar{x}^*(\delta)$ can be calculated if we introduce the new state vector $x = \bar{x} - \bar{x}^*(\delta)$. The equation of the centered system is the following:

$$\dot{x} = \text{diag}(x + \bar{x}^*(\delta)) Ax \quad (18)$$

If the Lyapunov function depends on δ , especially when having an uncertain equilibrium point, it is not straightforward to determine an invariant region by considering the α -level set of the Lyapunov function. However, it is possible to compute a region $\mathcal{J}$, which is “*invariant with respect to*” a larger region $\mathcal{U}$ ($\mathcal{J} \subset \mathcal{U}$), in the sense that every trajectory with an initial condition from $\mathcal{J}$ will not leave $\mathcal{U}$.

In order to compute $\mathcal{J}$ and $\mathcal{U}$, we determine the α -level set $\bar{\varepsilon}_\alpha(\delta)$ of the obtained Lyapunov function $V(x, \delta)$ in the coordinates system of the *original* system and the corresponding equilibrium point for every $\delta \in \mathcal{D}$. The overlining in the notation $\bar{\varepsilon}_\alpha(\delta)$ means that this level set is translated into the original coordinates system. As Fig. 9 illustrates, we have computed the intersection $\mathcal{J}$ and the union $\mathcal{U}$ of the α -level sets obtained for different values of δ . We can state that every trajectory starting from region $\mathcal{J}$ will not leave region $\mathcal{U}$.

Conclusion

In this work, we successfully computed the Lyapunov function of n -dimensional LV systems ($n = 2, 3, 4$) by using the improved optimization-based method presented in [6, 15]. In case of each deterministic system, we gave an invariant region as the estimated DOA. In case of an uncertain LV system with an uncertain interior equilibrium point, we gave two regions ($\mathcal{J} \subset \mathcal{U}$), which describes the stability region of the system.

SYMBOLS

$\dot{x}$ denotes the time-derivative of function $x(t)$

L^T denotes the transpose of a matrix L

$\vartheta(\cdot)$ denotes the corner points of a polytope

$\text{diag}(x)$ denotes the diagonal matrix $\begin{bmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{bmatrix}$

$x = x(t) \in \mathbb{R}^n$ represents the state variables of a dynamical system

$\delta \in \mathbb{R}^d$ is a vector of uncertain parameters

$\pi \in \mathbb{R}^p$ is a vector valued function, which represents the set of monomial and rational nonlinear terms to be considered in the Lyapunov function

$\pi_b \in \mathbb{R}^{m=n+p}$ is an auxiliary variable denoting $\begin{bmatrix} x \\ \pi \end{bmatrix}$.

$V(x)$ denotes the Lyapunov function

$\mathcal{X} \subset \mathbb{R}^n$, $\mathcal{D}, \tilde{\mathcal{D}} \subset \mathbb{R}^d$ are bounded polytopes

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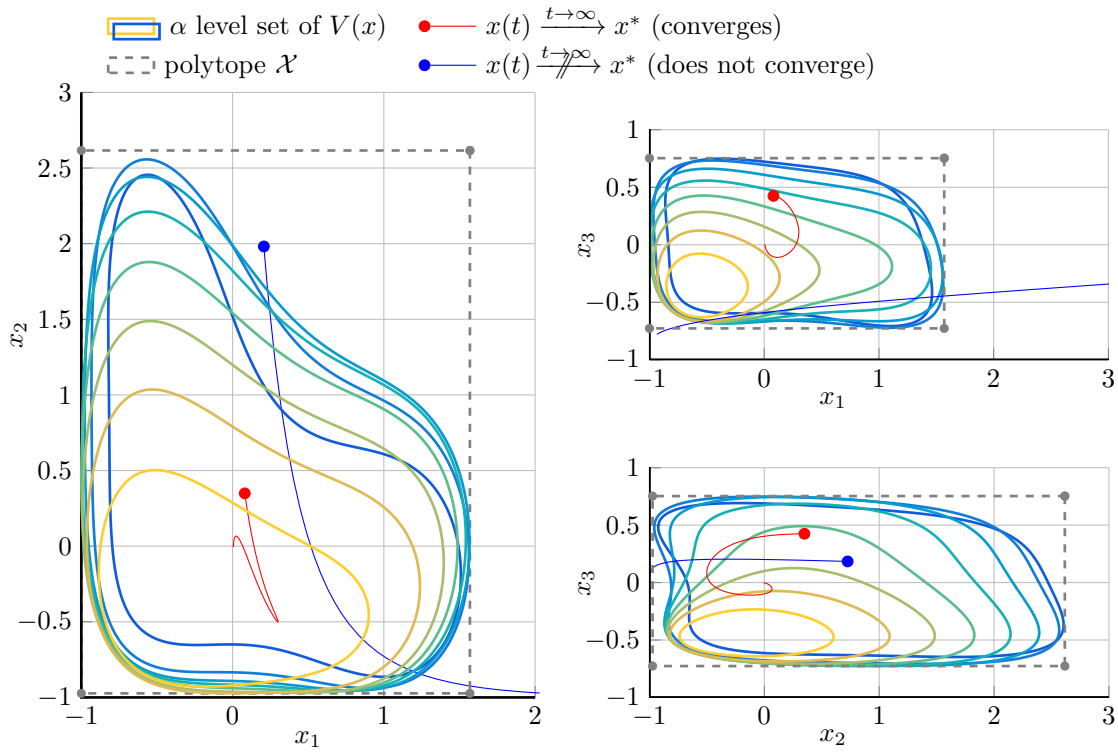


Figure 6: Cross-sectional view of the 3D estimate along the major planes $(x_i x_j)$ and using different values for x_k , where (i, j, k) is a permutation of $(1, 2, 3)$

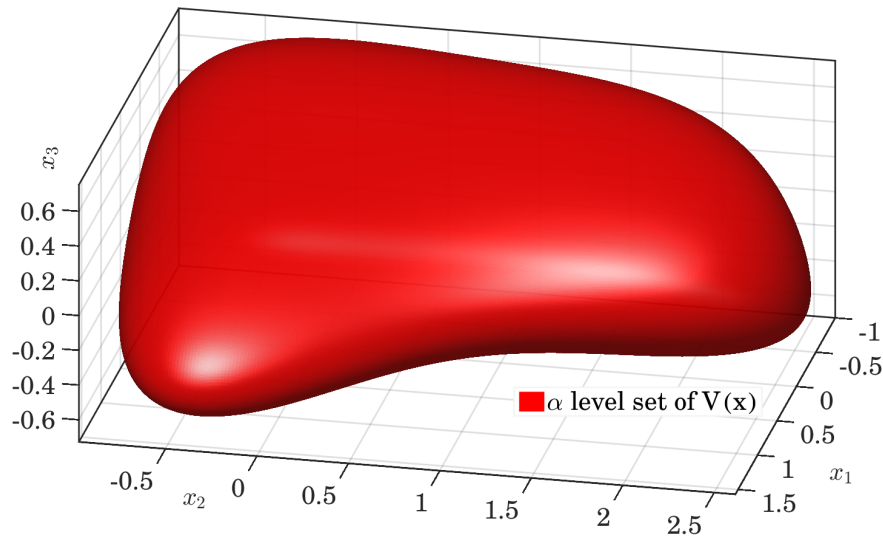


Figure 7: Estimated DOA of the locally stable 3D Lotka-Volterra system (red mesh) and the corresponding final polytope $\mathcal{X}$ (bounding box)

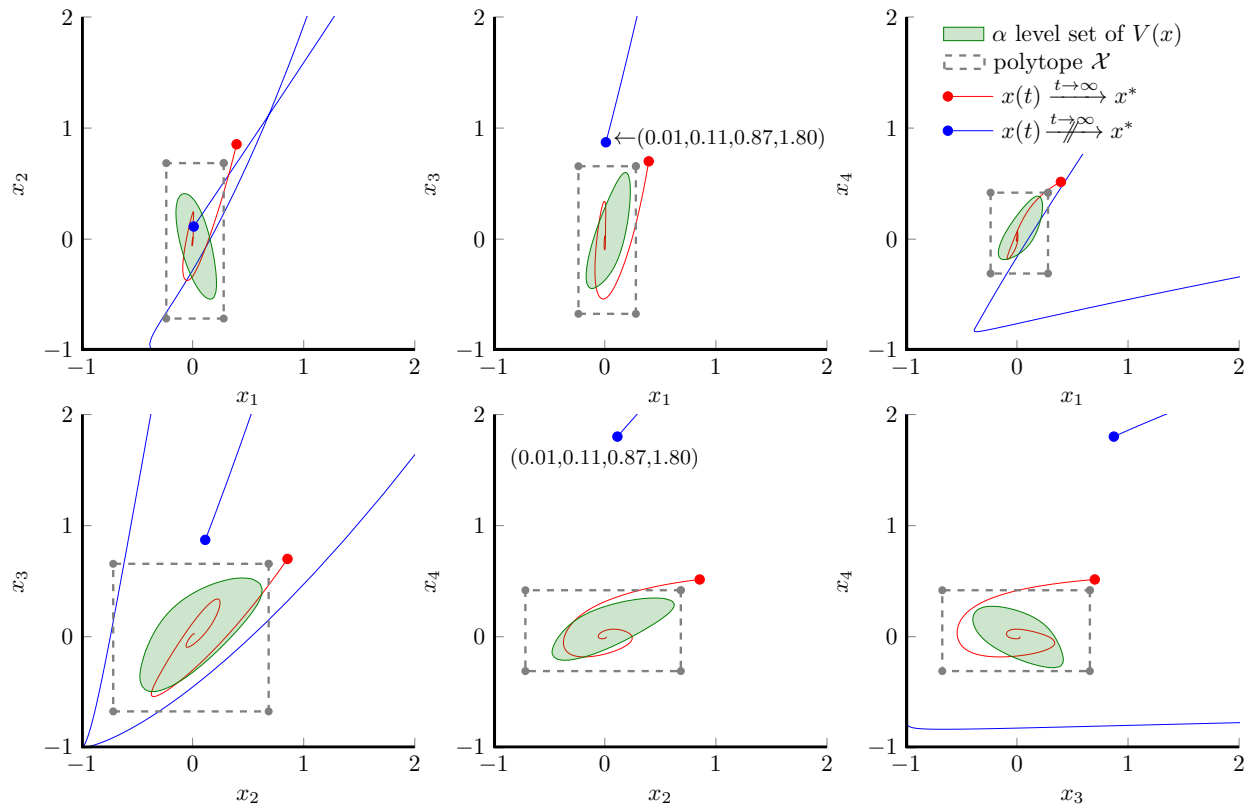


Figure 8: Cross sectional view of the DOA estimate of the 4D LV system.

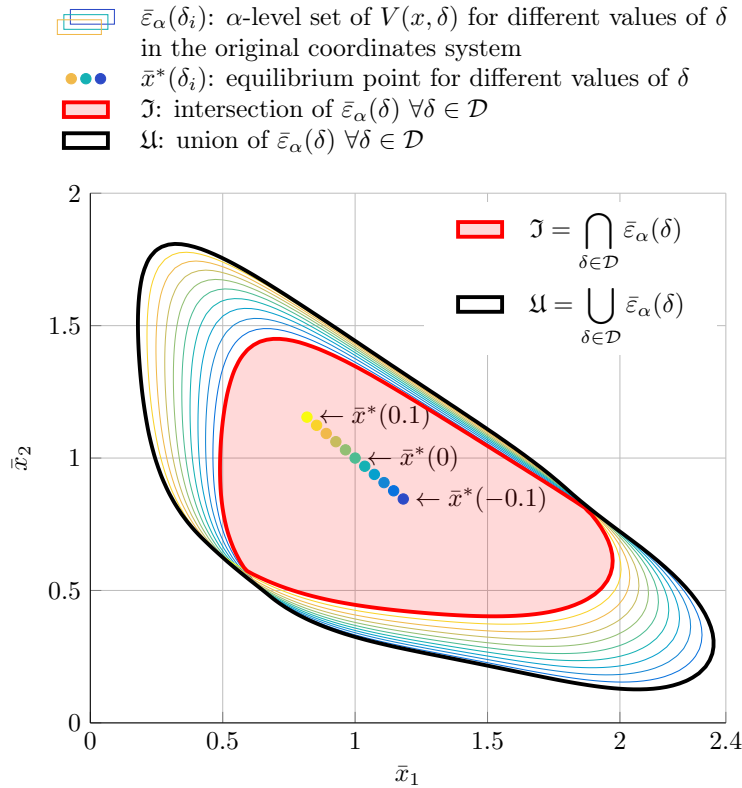


Figure 9: $\bar{\varepsilon}_\alpha(\delta_i)$ of the Lyapunov function in the original coordinates system for some values δ_i of the uncertain parameter and the corresponding equilibrium point $x^*(\delta_i)$ having the same color. The filled red region is the intersection of the level sets for every possible values of δ . The black line shows the contour of the union of the level sets.