

COMPUTATIONAL STABILITY ANALYSIS OF LOTKA-VOLTERRA SYSTEMS

P. POLCZ¹ AND G. SZEDERKÉNYI¹

¹Pázány Péter Catholic University Faculty of Information Technology and Bionics, 1083 Budapest, Práter u. 50/A., Hungary

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Introduction

Approximating the domain of attraction (DOA) is often a fundamental task in the analysis and control of nonlinear systems. The stability properties of dynamical systems are most often studied using Lyapunov functions. Therefore, an extensive literature exists on the computational construction of Lyapunov functions [1].

Due to their advantageous properties and the availability of efficient numerical solvers, the use of linear matrix inequalities (LMI) and semidefinite programming (SDP) techniques became popular in the field of system control theory. Important results were announced in [2, 3] in the context of linear uncertain systems, their stability analysis and control synthesis.

Recently, an optimization based method for DOA estimation was published in [4], where the authors use Finsler's lemma and affine parameter dependent LMIs to compute rational Lyapunov functions for a wide class of locally asymptotically stable nonlinear systems. Based on these results an improved method was published in [5, 6], where the transformation of the model to the form required for optimization is done automatically using the linear fraction transformation (LFT) and further automatic model simplification steps, which results in the dimension reduction of the optimization task. As the dimensions of the problem are reduced the method is capable to solve more complex systems. For example, in [6] the DOA is estimated for a 3-dimensional rational uncertain system (bioreactor model with an applied proportional and integral substrate feedback law).

The dynamical descriptive power of Lotka-Volterra systems is so extensive that LV models "have the status of canonical format" within the class of smooth nonlinear dynamical systems [7]. Besides modeling biological/ecological environments, they are widely used in other scientific fields like neural nets [8] or in economy, where the Goodwin-Lotka-Volterra models are applied for modeling the predator-pray mechanism of the

technological substitution [9, 10]. In order to model the correlation between the employment rate and the wage share of the working population, the authors of [11] used a stochastic extension of the Goodwin-model. Some important results that makes LV models even more attractive are the existing techniques to represent a general nonlinear system with a multidimensional LV model [12]. The analysis of stability and behaviour of LV systems has an extended literature [13]. In [14], Plank has shown that n -dimensional LV systems are Hamiltonian if they fulfill some algebraic properties [14, Theorem 3.1]. He demonstrated that, when using an appropriate Poisson structure, one can obtain the system's Hamiltonian function, which is a key object in determining the system's DOA. Furthermore, Hamiltonian function candidates were defined in [14, Section IV.].

In this paper, the local stability properties of LV models are analysed by applying the underlying method presented in [6]. The stability region is estimated on a few locally stable 2-, 3-, and 4-dimensional benchmark LV systems. The DOA of an uncertain 2D system is also determined. The applied method is presented in detail in technical report [15]. The main motivation of our work was to evaluate the applicability of the approach [4] on a general polynomial system class having low-degree monomials, and to study the limits of the method as the number of dimensions of the state space increase.

System class, dynamical system representation

Our computational methods can handle general nonlinear systems of the form

$$\begin{aligned}\dot{x}(t) &= f(x(t), \delta(t)) \\ x(t) &\in \mathbb{R}^n, x_0 \in \mathcal{X}, \delta(t) \in \mathcal{D}, \dot{\delta}(t) \in \tilde{\mathcal{D}}\end{aligned}\tag{1}$$

where $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{D}, \tilde{\mathcal{D}} \subset \mathbb{R}^d$ are given polytopes, x is the state vector function with its initial condition $x_0 = x(0)$, and δ is a smooth, bounded vector function of

uncertain parameters with bounded time derivative. From now on, the time arguments of x and δ will be suppressed. We assume that function $f : \mathbb{R}^n \times \mathcal{D} \mapsto \mathbb{R}^n$ of Eq. (1) is a well defined smooth *rational mapping*, with the property $f(0, \delta) = 0$ for all $\delta \in \mathcal{D}$. In other words, $x^* = 0 \in \mathbb{R}^n$ is a locally asymptotically stable equilibrium point of (1) for all $\delta \in \mathcal{D}$.

We are looking for an appropriate rational Lyapunov function $V(x, \delta)$ which satisfies the following conditions:

$$\begin{aligned} v_l(x) &\leq V(x, \delta) \leq v_u(x), \quad \forall (x, \delta) \in \mathcal{X} \times \mathcal{D} \\ \dot{V}(x, \delta, \dot{\delta}) &\leq -v_d(x), \quad \forall (x, \delta, \dot{\delta}) \in \mathcal{X} \times \mathcal{D} \times \check{\mathcal{D}} \end{aligned} \quad (2)$$

where v_l, v_u, v_d are continuous positive functions on $\mathcal{X}$. Due to conditions (2) any closed level set of the Lyapunov function contained entirely in $\mathcal{X}$ bounds an invariant region of the state space. Our objective is to find a Lyapunov function which fulfills conditions (2) and to determine its maximal invariant level set completely inside $\mathcal{X}$.

In [4], a Lyapunov function was searched in the form:

$$V(x, \delta) = \pi_b^T(x, \delta) P \pi_b(x, \delta), \quad \pi_b = \begin{bmatrix} x \\ \pi \end{bmatrix} \quad (3)$$

where $P \in \mathbb{R}^{m \times m}$ is a *constant symmetric* matrix, not necessarily positive definite, and $\pi : \mathbb{R}^n \times \mathcal{D} \mapsto \mathbb{R}^p$ is a mapping, in which each element is a monomial in (x, δ) , or a smooth rational function with monomial numerator. The arguments of π and π_b will be suppressed below.

Using LFT and further algebraic simplification steps the system equation $\dot{x} = f(x, \delta) = \mathcal{A}(x, \delta)x$ can be transformed to the differential-algebraic representation that was introduced in [6]:

$$\begin{aligned} \dot{x} &= f(x, \delta)x = Ax + B\pi \quad x_0 \in \mathcal{X} \\ 0 &= \mathcal{N}_{\pi_b}(x, \delta)\pi_b \quad \delta \in \mathcal{D}, \dot{\delta} \in \check{\mathcal{D}} \end{aligned} \quad (4)$$

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times p}$ are constant matrices, and $\mathcal{N}_{\pi_b}(x, \delta) \in \mathbb{R}^{q \times m}$ is an *affine* matrix function in (x, δ) also called as an “annihilator”.

The SDP problem for estimating the DOA

In this section, we present how to construct the semidefinite optimization task (SDP), which will uniquely characterize an appropriate Lyapunov function and its maximal invariant level set ε_α as an estimate of the true domain of attraction. We assume that polytopes $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{D}$ are already given.

Using the model representation (4), and having a Lyapunov function candidate of the form (3), we can formulate sufficient LMIs for the Lyapunov conditions (2). According to Finsler’s lemma [4, Lemma 2.1], if there exist real-valued matrices $L_b \in \mathbb{R}^{m \times q}$ and $L_a \in$

$$\mathbb{R}^{(n+2p+n^2+np) \times (2q+n^2+nq)}, \text{ s.t.}$$

$$\begin{aligned} \forall (x, \delta) &\in \vartheta(\mathcal{X} \times \mathcal{D}) : \\ P + L_b \mathcal{N}_{\pi_b}(x, \delta) + \mathcal{N}_{\pi_b}^T(x, \delta) L_b^T &> 0 \end{aligned} \quad (5)$$

$$\begin{aligned} \forall (x, \delta, \dot{\delta}) &\in \vartheta(\mathcal{X} \times \mathcal{D} \times \check{\mathcal{D}}) : \\ P_a + P_a^T + L_a \mathcal{N}_{\pi_a}(x, \delta, \dot{\delta}) + \mathcal{N}_{\pi_a}^T(x, \delta, \dot{\delta}) L_a^T &< 0 \end{aligned} \quad (6)$$

then conditions (2) are met for every $(x, \delta, \dot{\delta}) \in \mathcal{X} \times \mathcal{D} \times \check{\mathcal{D}}$. Variables P_a and $\mathcal{N}_{\pi_a}(x, \delta, \dot{\delta})$ are define in [15] in Eqs. (3.16) and (3.22), respectively. $\vartheta(\cdot)$ denotes the corner points (vertices) of a given polytope.

Since the Lyapunov conditions are satisfied within $\mathcal{X}$, we are looking for the maximal α -level set $\varepsilon_\alpha = \{x \in \mathcal{X} \mid V(x) = \alpha, 1 \leq \alpha\}$, which lies inside $\mathcal{X}$. That level set will be invariant, in the sense that every trajectory entering this region will never leave it.

We define $\alpha \geq 1$ as a free variable of the optimization task, and we constrain ε_α to be inside $\mathcal{X}$. One can observe that by maximizing α we obtain an unbounded feasible solution, as the function $V(x, \delta)$ can be scaled arbitrarily. Therefore, we define an auxiliary polytope $\mathcal{Y}$ around the locally stable origin inside $\mathcal{X}$, constraining the 1-level set $\varepsilon_1 = \{x \in \mathcal{X} \mid V(x) = 1\}$ to be around $\mathcal{Y}$.

According to Finsler’s lemma, for every $k = \overline{1, M_{\mathcal{X}}}$ and $l = \overline{1, M_{\mathcal{Y}}}$, if there exist real-valued matrices $L_{c_k}, \bar{L}_{c_l} \in \mathbb{R}^{m \times q}, M_{c_k}, \bar{M}_{c_l} \in \mathbb{R}^{(m+1) \times n}$, s.t.

$$Q_k^T P_{c_k}^{(\alpha)}(x, \delta) Q_k \geq 0, \quad \forall (x, \delta) \in \vartheta(\mathcal{F}_k^{(\mathcal{X})} \times \mathcal{D}) \quad (7)$$

$$\bar{Q}_k^T \bar{P}_{c_k}^{(1)}(x, \delta) \bar{Q}_k \leq 0, \quad \forall (x, \delta) \in \vartheta(\mathcal{F}_k^{(\mathcal{Y})} \times \mathcal{D}) \quad (8)$$

than ε_α is inside $\mathcal{X}$ and ε_1 is around $\mathcal{Y}$. Variables $Q_k, \bar{Q}_k, P_{c_k}^{(\alpha)}(x, \delta), \bar{P}_{c_k}^{(1)}(x, \delta)$ are defined in [15] in Eqs. (3.36) and (3.34), $\mathcal{F}_k^{(\mathcal{X})}$ denotes the k th facet of $\mathcal{X}$, furthermore, $M_{\mathcal{X}}$ denotes the number of facets of $\mathcal{X}$.

The SDP task can be summarized as follows. In order to find the maximal invariant level set ε_α of the Lyapunov function, one should maximize α , under the following conditions:

- Eq. (5) $V(x, \delta)$ is positive on $\mathcal{X} \times \mathcal{D}$
- Eq. (6) $\dot{V}(x, \delta, \dot{\delta})$ is negative on $\mathcal{X} \times \mathcal{D} \times \check{\mathcal{D}}$
- Eq. (7) ε_α lies inside $\mathcal{X}$
- Eq. (8) $\mathcal{Y}$ is inside ε_1

The free variables of the optimization task are $\alpha, P, L_b, L_a, L_{c_k}, \bar{L}_{c_l}, M_{c_k}$ and $\bar{M}_{c_l}$, for $k = \overline{1, M_{\mathcal{X}}}$ and $l = \overline{1, M_{\mathcal{Y}}}$.

Finding the most appropriate outer polytope

In case of a 2D systems, polytope $\mathcal{X}$ is evaluated manually through iterations. When having a higher-dimensional system ($n \geq 3$), we applied the same iterative procedure as it was presented in [6].

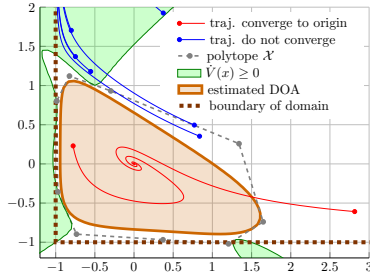


Figure 1

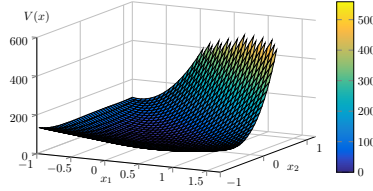


Figure 2: The obtained Lyapunov function

Lotka-Volterra systems

The n -dimensional LV equation has the form

$$\dot{\bar{x}}_i = \bar{x}_i(A_i\bar{x} + b_i), \quad A \in \mathbb{R}^{n \times n}, \quad b \in \mathbb{R}^n, \quad (9)$$

where A_i denotes the i th row of matrix A . We translate the system into its, by assumption, unique interior equilibrium point $x^* = -A^{-1}b$ by introducing $\bar{x} = x + x^*$:

$$\dot{x}_i = (x_i + x^*)A_i x \quad (10)$$

The LFT produced second order monomials of x in vector π .

Numerical results on a 2D LV system

After translating to its unique interior equilibrium point, the equation of the centered system is:

$$\dot{x}_i = (x_i + x_i^*)A_i x, \quad A = \begin{bmatrix} -2 & -3 \\ 1.4 & 1 \end{bmatrix}, \quad x^* = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

A_i denotes the i th row of model matrix A . In Fig. 1, we can see the shape of the *manually chosen* polytope $\mathcal{X}$, and the obtained invariant α level set of the Lyapunov function (Fig. 2). We can also observe that the regions where the Lyapunov function's time derivative is positive (green region) are completely outside of $\mathcal{X}$.

Summary and further results

In this work, we successfully computed the Lyapunov function of LV systems by using the improved optimization-based method presented in [6, 15]. In each case, we gave an invariant region as the estimated DOA.

The conference presentation and the final paper will contain already existing interesting computation results and discussion on the stability regions of higher (3,4) dimensional LV models as well as on the monomial selection for constructing the Lyapunov functions.

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