

Computation of integrated subtraction terms numerically

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Abstract

We report on a numerical representation of the integrated subtraction terms of the NNLO subtraction scheme defined in Refs. [1, 2, 3, 4]. The integrated approximate cross sections themselves can be written as products of insertion operators (in colour space) times the Born, or the one-loop cross section. The insertion operator is constructed from the numerical representation of the integrated subtraction terms. We give selected results for the integrated doubly-collinear subtraction term.

1. INTRODUCTION

We consider the NNLO correction to a generic m -jet observable,

$$\sigma^{\text{NNLO}} = \int_{m+2} d\sigma_{m+2}^{\text{RR}} J_{m+2} + \int_{m+1} d\sigma_{m+1}^{\text{RV}} J_{m+1} + \int_m d\sigma_m^{\text{VV}} J_m. \quad (1)$$

The three contributions on the right hand side are separately divergent in $d = 4$ dimensions, but their sum is finite for IR safe observables. To obtain the finite NNLO correction, we first continue analytically all integrals to $d = 4 - 2\epsilon$ dimensions and then rewrite Eqn. (1) as

$$\sigma^{\text{NNLO}} = \int_{m+2} d\sigma_{m+2}^{\text{NNLO}} + \int_{m+1} d\sigma_{m+1}^{\text{NNLO}} + \int_m d\sigma_m^{\text{NNLO}}, \quad (2)$$

that is a sum of three integrals where the integrands,

$$d\sigma_{m+2}^{\text{NNLO}} = \left\{ d\sigma_{m+2}^{\text{RR}} J_{m+2} - d\sigma_{m+2}^{\text{RR},A_2} J_m - \left[d\sigma_{m+2}^{\text{RR},A_1} J_{m+1} - d\sigma_{m+2}^{\text{RR},A_{12}} J_m \right] \right\}_{\epsilon=0}, \quad (3)$$

$$d\sigma_{m+1}^{\text{NNLO}} = \left\{ \left[d\sigma_{m+1}^{\text{RV}} + \int_1 d\sigma_{m+2}^{\text{RR},A_1} \right] J_{m+1} - \left[d\sigma_{m+1}^{\text{RV},A_1} + \left(\int_1 d\sigma_{m+2}^{\text{RR},A_1} \right)^{A_1} \right] J_m \right\}_{\epsilon=0}, \quad (4)$$

and

$$d\sigma_m^{\text{NNLO}} = \left\{ d\sigma_m^{\text{VV}} + \int_2 \left[d\sigma_{m+2}^{\text{RR},A_2} - d\sigma_{m+2}^{\text{RR},A_{12}} \right] + \int_1 \left[d\sigma_{m+1}^{\text{RV},A_1} + \left(\int_1 d\sigma_{m+2}^{\text{RR},A_1} \right)^{A_1} \right] \right\}_{\epsilon=0} J_m, \quad (5)$$

are integrable in four dimensions by construction. The approximate cross sections $d\sigma_{m+2}^{\text{RR},A_2}$ and $d\sigma_{m+2}^{\text{RR},A_1}$ regularise the doubly- and singly-unresolved limits of the real-emission contribution, $d\sigma_{m+2}^{\text{RR}}$ respectively. The double subtraction due to the overlap of these two terms is compensated by $d\sigma_{m+2}^{\text{RR},A_{12}}$. These terms are given explicitly in Ref. [3]. Finally, $d\sigma_{m+1}^{\text{RV},A_1}$ and $\left(\int_1 d\sigma_{m+2}^{\text{RR},A_1} \right)^{A_1}$ regularise the singly-unresolved limits of $d\sigma_{m+1}^{\text{RV}}$ and $\int_1 d\sigma_{m+2}^{\text{RR},A_1}$ respectively. They are given explicitly in Ref. [4].

The construction of each approximate cross section in Eqns. (3–5) is based on the known and universal IR limits of tree level and one-loop squared matrix elements, and proceeds in two steps. First, the IR factorisation formulae are written in such a way that their complicated overlap structure can be disentangled (“matching of limits”) [1, 5]. Second, we define “extensions” of the formulae, so that they

are unambiguously defined away from the strict IR limits [2, 3, 4]. These extensions are defined by the use of various momentum mappings that map a set of $m+1$ or $m+2$ momenta into a set of m momenta,

$$\{p\}_{m+1} \longrightarrow \{\tilde{p}\}_m \quad \text{and} \quad \{p\}_{m+2} \longrightarrow \{\tilde{p}\}_m, \quad (6)$$

such that (i) the delicate structure of cancellations among the matched limit formulae in various limits is respected (ii) exact momentum conservation is implemented, and (iii) the original $m+1$ or $m+2$ particle phase space factorises exactly into the product of an m particle phase space and a one- or two-particle phase space measure,

$$d\phi_{m+r}(\{p\}_{m+r}; Q) = d\phi_m(\{\tilde{p}\}_m; Q) [dp_{r,m}], \quad r = 1, 2. \quad (7)$$

To finish the definition of the scheme, one must compute once and for all the one- and two-particle integrals, denoted formally as $\int_1$ and $\int_2$, appearing in Eqns. (4–5).

In general the integrated subtraction terms are integrals of extensions over the whole phase space of combinations of the QCD splitting functions and squared soft currents. In this proceedings we discuss two examples: (i) the singly-collinear subtractions $\mathcal{C}_{ir}^{(\ell,0)}$ and (ii) the doubly-collinear subtractions $\mathcal{C}_{ir,js}^{(0,0)}$, which are part of $d\sigma_{m+2}^{\text{RR},A_1}$ and $d\sigma_{m+2}^{\text{RR},A_2}$ in Eqn. (3), respectively. The precise definitions of these terms can be found in Ref. [3]. The meaning of the superscript is irrelevant for our present purpose (also explained in Ref. [3]).

Denoting a generic subtraction term by $\mathcal{X}^{(\ell,k)}$ (such as $\mathcal{C}_{ir}^{(\ell,0)}$) the integrated counterterms can be written in the following general form:

$$\int_r \mathcal{X}^{(\ell,k)} = \left[\frac{\alpha_s}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2} \right)^\epsilon \right]^{r+\ell} N_X(\epsilon) X^{(\ell)}(x, \dots) \text{Re} \langle \mathcal{M}_m^{(0)}(\{\tilde{p}\}) | \mathbf{T}_i \cdot \mathbf{T}_j \dots | \mathcal{M}_m^{(k)}(\{\tilde{p}\}) \rangle, \quad (8)$$

where $S_\epsilon = (4\pi)^\epsilon / \Gamma(1-\epsilon)$, and $X^{(\ell)}(x, \dots)$ represents a function that depends on kinematical invariants of the factorized m -parton phase space. It results in the integration of the subtraction term $\mathcal{X}^{(\ell,k)}$ over the factorized phase spaces $[dp_{r,m}]$ in Eqn. (7). In a NNLO computation the possible cases are $r+\ell+k=1$ with $\ell+k=0$ or 1 , and $r=2$ with $\ell+k=0$. We use the colour- and spin-state notation of Ref. [6], when the amplitude for a scattering process involving m final-state momenta, $|\mathcal{M}_m^{(k)}\rangle$, is an abstract vector in colour and spin space; k denotes the number of loops. Colour interactions at QCD vertices are represented by associating colour charges $\mathbf{T}_i$ with the emission of a gluon from each parton i . There are $2r$ such colour charges. Then the functions $X^{(\ell)}$ are dimensionless in colour-space. For certain subtraction terms, universal, possibly ϵ -dependent numerical factors, $N_X(\epsilon)$ appear naturally, which can be factored out. Our purpose is to compute all functions $X^{(\ell)}$, which we discuss next.

2. INTEGRATING THE COUNTERTERMS

The actual computation of the integrated counterterms leads to a large number of multi-dimensional integrals. The ultimate goal is to find the analytical form of the coefficients of a Laurent expansion (in ϵ) of these integrals, which turns out to be a rather a tedious job. In order to compute these coefficients as efficiently as possible, we have explored several methods.

First, it is possible to extend the method of integration-by-parts identities and solving of differential equations, developed for computing multi-loop Feynman integrals [7, 8], to the relevant phase space integrations [9]. This method yields ϵ -expansions with fully analytical coefficients, with the final results being expressed in terms of two-dimensional harmonic polylogarithms (after a suitable basis extension, see Ref. [9] for details). This approach was used successfully to compute a class of singly-unresolved integrals [9].

Second, the phase space integrals that arise can be computed via the method of Mellin–Barnes (MB) representations [10, 11, 12]. Here we obtain the ϵ -expansion coefficients in terms of complex

contour integrals over Γ -functions. Performing these integrals by the use of the residue theorem, a representation in terms of harmonic sums is obtained. In many cases, the sums can be evaluated in a closed form, yielding an analytical result. In some instances however, we find multi-dimensional MB integrals that are very difficult to compute fully analytically. Nevertheless, in these situations a direct numerical evaluation of the appropriate MB representations provides a fast and reliable way to obtain final results with small numerical uncertainties. We stress that for phenomenological applications, this is all that is required, since the numerical uncertainty of the complete computation is dominated by the phase space integrations. We have used the MB method to compute all singly-unresolved integrals [13], and all two-particle integrals appearing in $\int_2 d\sigma_{m+2}^{RR,A_{12}}$ as well [14].

Finally, the method of iterated sector decomposition [15] can also be used to calculate the integrals we encounter [16]. Sector decomposition produces a representation of the ϵ -expansion where the coefficients are given in terms of (mostly quite cumbersome) finite integrals over the unit hypercube. The analytical evaluation of these integrals is not feasible except for the simplest cases. Nevertheless, this method is simple to implement and can be automated to a large extent. In fact there are several computer programs that use various implementations of sector decomposition to provide numerical values of coefficients of the powers of ϵ in the Laurent expansion of dimensionally regulated integrals [17, 18, 19]. We found the program `SecDec` powerful and flexible to generate sufficiently precise values of our integrated subtraction terms.

Choosing the Cuhre integrator implemented in `SecDec`, we can easily reach 10^{-7} relative precision for the integration. Such precision is sufficient for our purposes: (i) to demonstrate the cancellation of the ϵ poles numerically, and (ii) to compute the finite integrals in Eqns. (4) and (5). As the numerical uncertainty of the second item is limited more by the Monte Carlo integration over the $m + 1$ and m particle phase spaces, for item (ii) much lower (not better than 10^{-3}) precision is sufficient. This looser requirement on the precision for the $O(1)$ terms and the fact that the integrated subtraction terms are smooth functions of their parameters, with logarithmic behaviour for asymptotically small values of the parameters, makes possible that we find sufficient approximations to the integrated subtraction terms.

3. APPROXIMATE INTEGRATED SUBTRACTION TERMS

The computation of the integrated subtraction terms at any given values of the kinematical parameters, as required in the Monte Carlo integration over the phase space, is not feasible. In order to demonstrate the cancellation of the ϵ poles numerically we can choose several randomly selected phase space points and evaluate the necessary integrals with high precision. The cancellation cannot depend on the particular phase space point. In the case of the finite remainders, in order to compute the phase space integrals in Eqns. (4) and (5), we are able to find sufficiently precise approximations to the integrated subtraction terms using a procedure that can be automated to high degree. The latter point is also important as there are several hundred integrals to compute. In the following, we outline our procedure for two cases: (i) an example with integrals depending on one kinematical parameter, ($\int_1 \mathcal{X}^{(\ell,k)} = \int_1 \mathcal{C}_{ir}^{(\ell,0)}$) and (ii) another example with integrals depending on two kinematical parameters, ($\int_2 \mathcal{X}^{(\ell,k)} = \int_2 \mathcal{C}_{ir,js}^{(0,0)}$).

In order to compute $\int_1 \mathcal{C}_{ir}^{(\ell,0)}$, we have to integrate the azimuthally averaged Altarelli-Parisi splitting functions $P_{f_i f_r}^{(\ell)}(z_{i,r}, z_{r,i}; \epsilon)$ in $4 - 2\epsilon$ dimensions for the splitting process $f_{ir} \rightarrow f_i + f_r$, with z_i being the momentum fraction of parton f_i . It was discussed in Ref. [13] that the corresponding functions $\mathcal{C}_{ir}^{(\ell)}$ can be expressed as combinations of the integrals (we changed the notation from $\mathcal{I}$ to $\mathcal{I}_C$)

$$\mathcal{I}_C(x; \epsilon, \alpha_0, d_0, \kappa, k, \delta, g_I^{(\pm)}) = \frac{16\pi^2}{S_\epsilon} Q^{2\epsilon} \int_1 [dp_{1,m+1}^{(ir)}] \frac{z_r^{k+\delta\epsilon}}{s_{ir}^{1+\kappa\epsilon}} g_I^{(\pm)}(z_r) f(\alpha_0, \alpha_{ir}, d(m, \epsilon)). \quad (9)$$

In terms of explicit integration variables these collinear integrals have the general form [13]

$$\mathcal{I}_C(x; \epsilon, \alpha_0, d_0; \kappa, k, \delta, g_I^{(\pm)}) = x \int_0^{\alpha_0} d\alpha \alpha^{-1-(1+\kappa)\epsilon} (1-\alpha)^{2d_0-1} [\alpha + (1-\alpha)x]^{-1-(1+\kappa)\epsilon} \\ \times \int_0^1 dv [v(1-v)]^{-\epsilon} \left(\frac{\alpha + (1-\alpha)xv}{2\alpha + (1-\alpha)x} \right)^{k+\delta\epsilon} g_I^{(\pm)} \left(\frac{\alpha + (1-\alpha)xv}{2\alpha + (1-\alpha)x} \right). \quad (10)$$

The necessary functions $g_I^{(\pm)}$ are listed in Ref. [13], where analytic results of these integrals for $\alpha_0 = 1$ and $d_0 = 3$ are also presented.

Our present goal is to provide sufficiently precise numerical approximations to the functions $\mathcal{I}_C(x)$ in a simple way. The motivation is that often it is difficult to perform the analytic computation with arbitrary values of the parameters. For instance, the derivation with $\alpha_0 = 1$ is rather different from a derivation with $\alpha_0 < 1$. Also, the choice for d_0 is to some extent arbitrary, and a new choice requires a completely new analytic computation. Thus, for the sake of flexibility we propose a fully numerical approach here.

First we used the program `SecDec`, modified such that it can compute the value of the integral at multiple values of the parameter x in a single run. For simplicity, we call the $\mathcal{O}(1)$ terms of the integral ‘measurements’. Then, inspired by the analytic results in Ref. [13], we fitted these measurements by combinations of logarithms and polynomials in x of the form

$$\mathcal{F}_C(x; \kappa = 0, k, \delta = 0, g_I^{(\pm)} = 1) = \sum_{n=0}^{n_{\max}} P_n^{(m)}(x, k) \log^n(x), \quad P_n^{(m)}(x, k) = \sum_{n=0}^m a_n^{(k)} x^n \quad (11)$$

where the upper limit $n_{\max}$ is determined by the power $-n_{\max}$ of the leading pole in the Laurent-expansion (in ϵ) of the integral. As for the degree of the polynomials we tried several simple choices ($m = 1, 2, 3$). We found that splitting the region of the parameter space into an asymptotic ($0 < x \leq 10^{-4}$) and a non-asymptotic ($10^{-4} < x \leq 1$) region, we could provide a fit with $m = 2$ that approximates the analytic result within relative difference few times 10^{-4} . The loss of relative precision is associated with phase space points where the function changes sign, and its numerical value is close to zero (around $x = 0.2$).

In Fig. 1 we show the approximate function $\mathcal{F}_C(x; \alpha_0, d_0, 0, -1, 0, 1)$ together with the ‘measurements’, which coincide with the known exact analytic result to at least six digit accuracy. We find very good agreement, which is characterized by the ratio of the two values in the lower panels. In Fig. 1b we show the approximate function for $\alpha_0 = 0.1$ and $d_0 = 3 - 3\epsilon$ together with the corresponding ‘measurements’. In this case the analytic results are not available.

Building on the experience gained in studying the one-parameter case, we worked out a similar strategy for the integrated subtraction term $\int_2 \mathcal{C}_{ir,js}^{(0,0)}$. The corresponding functions $C_{ir,js}^{(0,0)}$ can be expressed as combination of the integrals

$$\mathcal{I}_{2C}(x_i, x_j; \epsilon, \alpha_0, d_0; k, l) = x_i x_j \int_0^1 d\alpha \int_0^1 d\beta \Theta(\alpha_0 - \alpha - \beta) \\ \times (1 - \alpha - \beta)^{2d_0-2(1-\epsilon)} \alpha^{-1-\epsilon} \beta^{-1-\epsilon} (\alpha + (1 - \alpha - \beta)x_i)^{-1-\epsilon} (\beta + (1 - \alpha - \beta)x_j)^{-1-\epsilon} \\ \times \int_0^1 dv v^{-\epsilon} (1-v)^{-\epsilon} \int_0^1 du u^{-\epsilon} (1-u)^{-\epsilon} \left(\frac{\alpha + (1 - \alpha - \beta)x_i v}{2\alpha + (1 - \alpha - \beta)x_i} \right)^k \left(\frac{\beta + (1 - \alpha - \beta)x_j u}{2\beta + (1 - \alpha - \beta)x_j} \right)^l. \quad (12)$$

We again run `SecDec` with $\alpha_0 = 0.1$ and $d_0 = 3 - 3\epsilon$ at several hundred different values of the kinematic parameters to obtain the ‘measurements’. To reach 10^{-7} relative precision for all such ‘measurements’

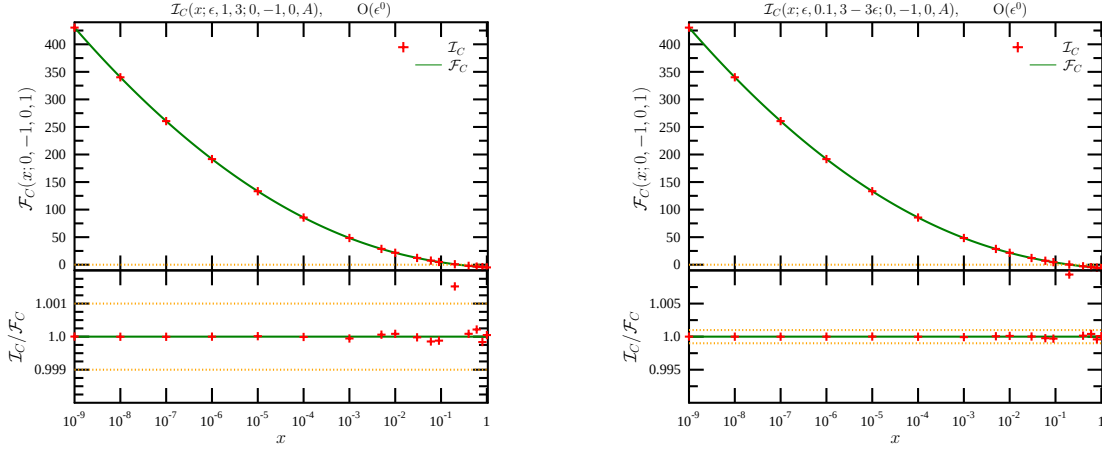


Fig. 1: The fitted function $\mathcal{F}_C(x; 0, -1, 0, 1)$ compared to the integral $\mathcal{I}_C(x; 0, -1, 0, 1)$ at a) $\alpha_0 = 1$ and $d_0 = 3$, b) $\alpha_0 = 0.1$ and $d_0 = 3 - 3\epsilon$.

takes several hours on a single CPU. Then we fitted these ‘measurements’ with the function

$$\mathcal{F}_{2C}(x_i, x_j; k, l) = \sum_{n_i=0}^{n_{\max}} \sum_{n_j=0}^{n_{\max}-n_i} P_{n_i}^{(m)}(x_i, k, l) P_{n_j}^{(m)}(x_j, k, l) \log^{n_i}(x_i) \log^{n_j}(x_j). \quad (13)$$

We divide the parameter space $0 < x_i, x_j \leq 1$ into four regions: (i) $0 < x_i, x_j \leq 10^{-4}$, (ii) $0 < x_i \leq 10^{-2}$ and $10^{-4} < x_j \leq 1$, (iii) $0 < x_j \leq 10^{-2}$ and $10^{-4} < x_i \leq 1$, (iv) $10^{-2} < x_i, x_j \leq 1$. Using $m = 2$, we are able to fit the original function $\mathcal{I}_{2C}$ to per mille precision almost everywhere. The ratio of the fitted function $\mathcal{F}_{2C}$ to the numerical evaluation of $\mathcal{I}_{2C}$ is shown in Fig. 2 together with the fitted function $\mathcal{F}_{2C}$ itself.

CONCLUSIONS

We have worked out a numerical procedure for providing simple approximations of the integrated subtraction terms of the NNLO subtraction scheme defined in Refs. [1, 2, 3, 4]. We use the publicly available program `SecDec` to compute the coefficients of the Laurent expansion of the necessary integrals to high numerical precision. We found that the integrals that depend on one or two kinematical invariants can be approximated with simple combinations of polynomials and logarithms. The precision of these approximations is usually at per mille or better.

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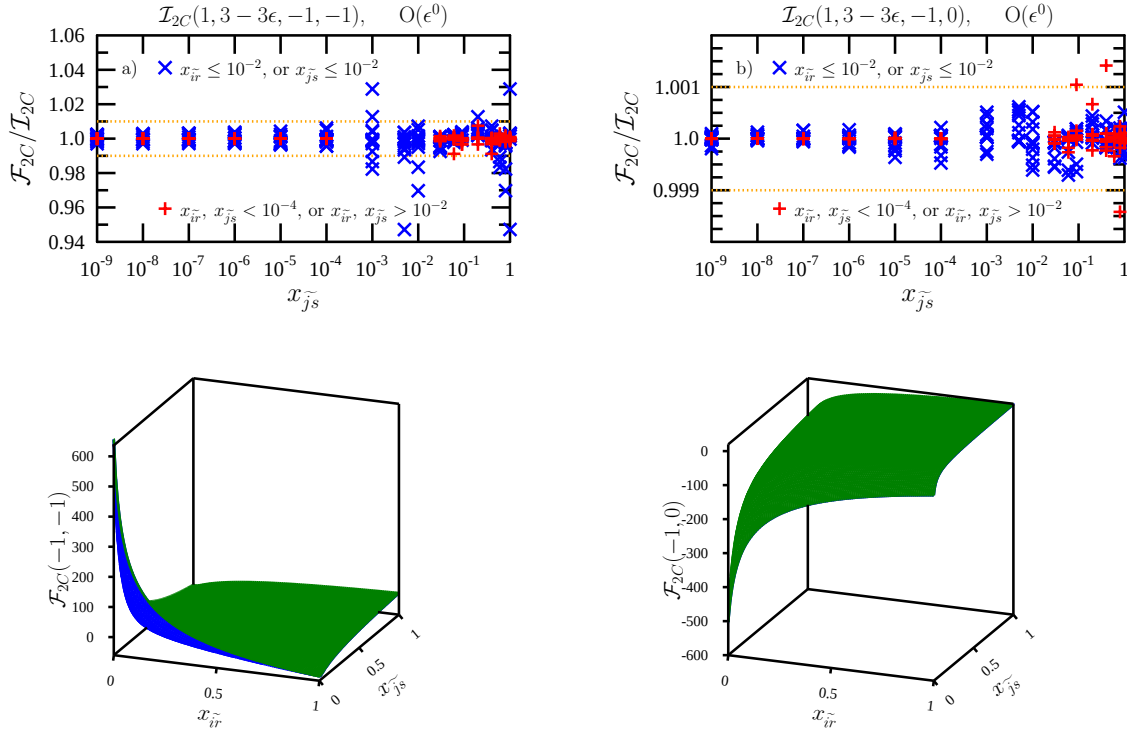


Fig. 2: The ratio of the fitted function $\mathcal{F}_{2C}(x_i, x_j; -1, l)$ to the integral $\mathcal{I}_{2C}(x_i, x_j; -1, l)$. a) $l = -1$ b) $l = 0$ Also shown the fitted function $\mathcal{F}_{2C}(x_i, x_j; -1, -l)$.

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