



A note on the Landau constants

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ABSTRACT

In this paper we establish new asymptotic expansions for the n th Landau constant G_n . An open problem is indicated as well.

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1. Introduction and previous results

Let $f(z) = \sum_{k \geq 0} a_k z^k$ be a holomorphic function on the open unit disk that satisfies $|f(z)| < 1$. In 1913, Landau [2] showed that

$$\left| \sum_{k=0}^n a_k \right| \leq G_n,$$

where

$$G_n = 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 + \cdots + \left(\frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)}\right)^2$$

is the n th Landau constant [6,8]. Moreover, if $T_n(f)$ is a polynomial operator associated to $f(z)$, then its norm is given by $\|T_n\| = G_n$.

The investigation of the asymptotic behavior of G_n was begun by Landau, who proved that

$$G_n \sim \frac{1}{\pi} \log n,$$

then Watson [8] established the following asymptotic formula

$$G_n = \frac{1}{\pi} \log(n+1) + \frac{1}{\pi} (\gamma + 4 \log 2) + o(1).$$

Here $\gamma = 0.577215 \dots$ is the Euler–Mascheroni constant. Recently, Zhao [9] gave a much better approximation as follows

$$G_n = \frac{1}{\pi} \log(n+1) + \frac{1}{\pi} (\gamma + 4 \log 2) - \frac{1}{4\pi(n+1)} + \frac{5}{192\pi(n+1)^2} + \mathcal{O}\left(\frac{1}{(n+1)^3}\right).$$

More recently, Mortici [3] extended this asymptotic formula to

$$G_n = \frac{1}{\pi} \log(n+1) + \frac{1}{\pi} (\gamma + 4 \log 2) - \frac{1}{4\pi(n+1)} + \frac{5}{192\pi(n+1)^2} + \frac{3}{128\pi(n+1)^3} - \frac{341}{122880\pi(n+1)^4} + \mathcal{O}\left(\frac{1}{(n+1)^5}\right).$$

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In this paper we will show that the asymptotic formulas mentioned above are the first few terms of a complete asymptotic expansion for G_n in terms of $1/(n+1)$.

2. The main results

Theorem 2.1. *The Landau constants G_n have the following asymptotic expansion*

$$G_n \sim \frac{1}{\pi} \log \left(n + \frac{1}{2} \right) + \frac{1}{\pi} (\gamma + 4 \log 2) - \sum_{k \geq 1} \frac{\alpha_k}{(n + \frac{1}{2})^k} \quad (2.1)$$

as $n \rightarrow +\infty$, where

$$\alpha_k = \frac{B_k}{\pi k} + \sum_{j=1}^k (-1)^{k+j} \binom{2j}{j}^2 \frac{(j-1)! S(k, j)}{2^{4j} \pi}.$$

Here B_k denotes the k th Bernoulli number and $S(k, j)$ is the Stirling number of the second kind.

The Bernoulli numbers and the Stirling numbers of the second kind can be computed by the formulas

$$B_k = \sum_{i=0}^k \frac{1}{i+1} \sum_{r=0}^i (-1)^r \binom{i}{r} r^k$$

and

$$S(k, j) = \frac{1}{j!} \sum_{i=0}^j (-1)^{j-i} \binom{j}{i} i^k$$

(see, e.g., [5, p. 591 and p. 624]). As a corollary we prove the asymptotic expansion in terms of $1/(n+1)$ that we mentioned in the first section.

Theorem 2.2. *The Landau constants G_n have the following asymptotic expansion*

$$G_n \sim \frac{1}{\pi} \log(n+1) + \frac{1}{\pi} (\gamma + 4 \log 2) - \sum_{k \geq 1} \frac{\beta_k}{(n+1)^k} \quad (2.2)$$

as $n \rightarrow +\infty$, where

$$\beta_k = \frac{1}{2^k \pi k} + \sum_{j=1}^k \frac{\alpha_j}{2^{k-j}} \binom{k-1}{j-1}.$$

The computation of the first few coefficients α_k and β_k leads us to the

Conjecture 2.1. *For every $k \geq 1$ we have $\beta_k = (-1)^k \alpha_k$.*

This would be an amazing relationship between the two asymptotic expansions.

A particular case of (2.2) is

$$G_n = \frac{1}{\pi} \log(n+1) + \frac{1}{\pi} (\gamma + 4 \log 2) - \frac{1}{4\pi(n+1)} + \frac{5}{192\pi(n+1)^2} + \frac{3}{128\pi(n+1)^3} + \mathcal{O}\left(\frac{1}{(n+1)^4}\right).$$

Since

$$\begin{aligned} \frac{1}{\pi} \log(n+1) &= \frac{1}{\pi} \log \left(n + \frac{3}{4} \right) - \frac{1}{\pi} \log \left(1 - \frac{1}{4(n+1)} \right) \\ &= \frac{1}{\pi} \log \left(n + \frac{3}{4} \right) + \frac{1}{4\pi(n+1)} + \frac{1}{32\pi(n+1)^2} + \frac{1}{192\pi(n+1)^3} + \mathcal{O}\left(\frac{1}{(n+1)^4}\right) \\ &= \frac{1}{\pi} \log \left(n + \frac{3}{4} \right) + \frac{1}{4\pi(n+1)} + \frac{1}{32\pi(n+\frac{3}{4})^2} - \frac{1}{96\pi(n+\frac{3}{4})^3} + \mathcal{O}\left(\frac{1}{(n+\frac{3}{4})^4}\right), \\ \frac{5}{192\pi(n+1)^2} &= \frac{5}{192\pi(n+\frac{3}{4})^2} - \frac{5}{384\pi(n+\frac{3}{4})^3} + \mathcal{O}\left(\frac{1}{(n+\frac{3}{4})^4}\right) \end{aligned}$$

and

$$\frac{3}{128\pi(n+1)^3} = \frac{3}{128\pi(n+\frac{3}{4})^3} + \mathcal{O}\left(\frac{1}{(n+\frac{3}{4})^4}\right),$$

we find that

$$G_n = \frac{1}{\pi} \log\left(n + \frac{3}{4}\right) + \frac{1}{\pi}(\gamma + 4 \log 2) + \frac{11}{192\pi(n+\frac{3}{4})^2} + \mathcal{O}\left(\frac{1}{(n+\frac{3}{4})^4}\right).$$

Thus, we obtain an asymptotic formula with a fourth order error term. This suggests an asymptotic series for G_n in terms of $1/(n+3/4)$. This can be derived in a similar way as we will prove (2.2) from (2.1). This time we get

$$G_n \sim \frac{1}{\pi} \log\left(n + \frac{3}{4}\right) + \frac{1}{\pi}(\gamma + 4 \log 2) + \frac{11}{192\pi(n+\frac{3}{4})^2} - \frac{1541}{122880\pi(n+\frac{3}{4})^4} + \frac{63433}{8257536\pi(n+\frac{3}{4})^6} - \frac{9199901}{1006632960\pi(n+\frac{3}{4})^8} \\ + \frac{317959723}{17716740096\pi(n+\frac{3}{4})^{10}} - \dots$$

At least, the first few odd indexed coefficients are zero. It seems that we obtain an alternating even type asymptotic series for the G_n 's. We propose the following open problem.

Open problem 2.1. The Landau constants G_n have the following asymptotic expansion

$$G_n \sim \frac{1}{\pi} \log\left(n + \frac{3}{4}\right) + \frac{1}{\pi}(\gamma + 4 \log 2) - \sum_{k \geq 1} \frac{(-1)^k \gamma_k}{\pi(n+\frac{3}{4})^{2k}}$$

as $n \rightarrow +\infty$, where the coefficients γ_k are positive rational numbers.

3. The proofs of the theorems

Our proofs will be based on the following theorem of Cvijović and Klinowski [1].

Theorem 3.1. The Landau constants G_n have the following expansion

$$G_n = \frac{1}{\pi} \psi\left(n + \frac{3}{2}\right) + \frac{1}{\pi}(\gamma + 4 \log 2) - \sum_{k \geq 1} \binom{2k}{k}^2 \frac{(k-1)!}{2^{4k} \pi} \frac{1}{(n+\frac{3}{2})((n+\frac{3}{2})+1) \cdots ((n+\frac{3}{2})+k-1)}.$$

Here $\psi = \Gamma'/\Gamma$ denotes the digamma function.

Proof of theorem 2.1. It is well known that

$$\psi(x+1) \sim \log x - \sum_{k \geq 1} \frac{B_k}{k} \frac{1}{x^k}$$

as $x \rightarrow +\infty$ (see, e.g., [4, p. 295]). From this, it follows that

$$\frac{1}{\pi} \psi\left(n + \frac{3}{2}\right) = \frac{1}{\pi} \psi\left(\left(n + \frac{1}{2}\right) + 1\right) \sim \frac{1}{\pi} \log\left(n + \frac{1}{2}\right) - \sum_{k \geq 1} \frac{B_k}{\pi k} \frac{1}{(n+\frac{1}{2})^k} \quad (3.1)$$

as $n \rightarrow +\infty$. The Stirling numbers $S(m, k)$ of the second kind satisfy the identity

$$\frac{x^k}{(1-x)(1-2x) \cdots (1-kx)} = \sum_{m \geq 0} S(m, k) x^m,$$

where the series converges for $|x| < 1$ (see, e.g., [7, p. 34]). This gives

$$\frac{1}{(y+1)(y+2) \cdots (y+k)} = \sum_{m \geq 0} (-1)^{k+m} S(m, k) \frac{1}{y^m},$$

when $|y| > 1$. Hence

$$\frac{1}{(n+\frac{3}{2})((n+\frac{3}{2})+1) \cdots ((n+\frac{3}{2})+k-1)} = \frac{1}{((n+\frac{1}{2})+1)((n+\frac{1}{2})+2) \cdots ((n+\frac{1}{2})+k)} = \sum_{m \geq 0} (-1)^{k+m} S(m, k) \frac{1}{(n+\frac{1}{2})^m}$$

This yields the asymptotic expansion

$$\begin{aligned} & \sum_{k \geq 1} \binom{2k}{k}^2 \frac{(k-1)!}{2^{4k} \pi} \frac{1}{(n + \frac{3}{2})((n + \frac{3}{2}) + 1) \cdots ((n + \frac{3}{2}) + k - 1)} \\ &= \sum_{k \geq 1} \binom{2k}{k}^2 \frac{(k-1)!}{2^{4k} \pi} \sum_{m \geq 0} (-1)^{k+m} S(m, k) \frac{1}{(n + \frac{1}{2})^m} \sim \sum_{m \geq 1} \left(\sum_{k=1}^m (-1)^{k+m} \binom{2k}{k}^2 \frac{(k-1)! S(m, k)}{2^{4k} \pi} \right) \frac{1}{(n + \frac{1}{2})^m} \end{aligned}$$

as $n \rightarrow +\infty$. Putting this and (3.1) together completes the proof. $\square$

Proof of theorem 2.2. Series transformation gives

$$\sum_{k \geq 1} \frac{\alpha_k}{(n + \frac{1}{2})^k} = \sum_{k \geq 1} \frac{\alpha_k}{((n + 1) - \frac{1}{2})^k} = \sum_{k \geq 1} \left(\sum_{j=1}^k \frac{\alpha_j}{2^{k-j}} \binom{k-1}{j-1} \right) \frac{1}{(n + 1)^k}$$

and

$$\frac{1}{\pi} \log \left(n + \frac{1}{2} \right) = \frac{1}{\pi} \log \left((n + 1) - \frac{1}{2} \right) = \frac{1}{\pi} \log(n + 1) + \frac{1}{\pi} \log \left(1 - \frac{1}{2(n + 1)} \right) = \frac{1}{\pi} \log(n + 1) - \sum_{k \geq 1} \frac{1}{2^k \pi (n + 1)^k}.$$

Putting this forms into (2.1) yields (2.2). $\square$

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