



Approximations for the higher order coefficients in an asymptotic expansion for the Gamma function

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ABSTRACT

We examine the asymptotic behavior as $n \rightarrow +\infty$ of the coefficients G_n appearing in an asymptotic expansion for the Gamma function.

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1. Introduction

The author proved in a recent paper [1] the following asymptotic expansion for the Gamma function

$$\left(\frac{\Gamma(x)}{\left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}}} \right)^{1/x} \sim \sum_{n \geq 0} \frac{G_n}{x^{2n}} \quad (1.1)$$

holds as $x \rightarrow +\infty$. It was shown that the coefficients G_n satisfy the recurrence

$$G_0 = 1, \quad G_n = \frac{1}{2n} \sum_{k=0}^{n-1} \frac{B_{2k+2} G_{n-k-1}}{2k+1}, \quad n \geq 1. \quad (1.2)$$

Here B_k denotes the k th Bernoulli number. In this paper we will prove that (1.1) holds also in the sector $|\arg z| \leq \pi - \delta$, $0 < \delta \leq \pi$ as $z \rightarrow \infty$. Then we will focus attention on the growth of the coefficients G_n as n becomes large. The calculation of higher terms by the recurrence relation can be exceedingly laborious. However, in the theory of asymptotic expansions, it is often not necessary to know the exact value of the late terms. Several times accurate numerical approximations of them are enough for practical purposes.

Through the paper we denote the Stirling numbers of the second kind by $S(m, k)$ and define them by the generating function

$$\frac{x^k}{(1-x)(1-2x)\cdots(1-kx)} = \sum_{m \geq k} S(m, k) x^m \quad (1.3)$$

for every nonnegative integer k . It is known that they can be given explicitly by

$$S(m, k) = \frac{1}{k!} \sum_{j=0}^k (-1)^j \binom{k}{j} (k-j)^m.$$

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We also need the Stirling coefficients γ_n which are given by the following asymptotic expansion:

$$m! \sim \left(\frac{m}{e}\right)^m \sqrt{2\pi m} \sum_{n \geq 0} \frac{\gamma_n}{m^n} \quad (1.4)$$

as $m \rightarrow +\infty$. An exact explicit expression for these coefficients was unknown for a long time. The author proved in a recent paper [2] that

$$\gamma_n = \frac{\Gamma\left(3n + \frac{3}{2}\right)}{\sqrt{\pi}} \sum_{k=0}^{2n} \frac{2^{n+k+1}}{(2n+2k+1)(2n-k)!} \sum_{j=0}^k \frac{(-1)^{k-j} S(2n+2k-j, k-j)}{j!(2n+2k-j)!}.$$

Now we are in a position to formulate our main results.

Theorem 1.1. For every positive integer N we have

$$\Gamma(z) = \left(\frac{z}{e}\right)^z \sqrt{\frac{2\pi}{z}} \left(\sum_{n=0}^{N-1} \frac{G_n}{z^{2n}}\right)^z \left(1 + \mathcal{O}\left(\frac{1}{z^{2N-1}}\right)\right) \quad (1.5)$$

as $z \rightarrow \infty$ in the sector $|\arg z| \leq \pi - \delta$, $0 < \delta \leq \pi$. Here the square root and the logarithm in $\exp[z \log(z \sum_{n \geq 0} G_n z^{-2n})]$ take their principal values. The coefficients G_n are given by the recurrence (1.2). Moreover, for every integer $r > 0$

$$G_n = \sum_{k=0}^{r-1} G_k \frac{B_{2n-2k}}{(2n-2k)(2n-2k-1)} + \mathcal{O}\left(\frac{B_{2n-2r}}{(2n-2r)(2n-2r-1)}\right) \quad (1.6)$$

as n tends to infinity.

As a corollary of the expansion (1.6) we will derive a complete asymptotic series for G_n in terms of $1/(n-1)$.

Theorem 1.2. The coefficients G_n have the asymptotic expansion

$$G_n \sim (-1)^{n-1} \left(\frac{n-1}{\pi e}\right)^{2n-2} \sqrt{\frac{n-1}{\pi^3}} \sum_{k \geq 0} \frac{g_k}{(n-1)^k} \quad (1.7)$$

as $n \rightarrow +\infty$, where $g_0 = 1$, $g_1 = \gamma_1/2 = 1/24$ and

$$g_k = \frac{\gamma_k}{2^k} + \sum_{j=2}^k \sum_{m=1}^{\lfloor j/2 \rfloor} (-1)^m 2^{2m-k} \pi^{2m} G_m \gamma_{k-j} S(j-1, 2m-1)$$

for $k \geq 2$.

The following closed-form approximation is a consequence of (1.5).

Corollary 1.1. The Gamma function has the asymptotic approximation

$$\Gamma(z) = \left(\frac{z}{e}\right)^z \sqrt{\frac{2\pi}{z}} \left(1 + \frac{1}{12z^2 - \frac{1}{10}}\right)^z \left(1 + \mathcal{O}\left(\frac{1}{z^5}\right)\right)$$

which holds as $z \rightarrow \infty$ in the sector $|\arg z| \leq \pi - \delta$, $0 < \delta \leq \pi$.

2. The proof of the theorems

To prove (1.5) we shall use the following theorem of Love [3].

Theorem 2.1. Let $S = \{z : \alpha \leq \arg z \leq \beta\}$, where $-\pi \leq \alpha < \beta \leq \pi$, be a sector in the complex plane. If

$$g(z) \sim \sum_{n \geq 1} \frac{g_n}{z^n}$$

as $z \rightarrow \infty$ in S and

$$h(w) = \sum_{n \geq 1} h_n w^n$$

in a neighborhood of $w = 0$, then

$$h(g(z)) \sim \sum_{n \geq 1} \frac{p_n}{z^n}$$

as $z \rightarrow \infty$ in S , where p_n is the coefficient of z^{-n} in the formal expansion of

$$\sum_{n \geq 1} h_n \left(\sum_{k \geq 1} \frac{g_k}{z^k} \right)^n.$$

Proof of formula (1.5). It is well known that

$$\log \Gamma(z) \sim \left(z - \frac{1}{2} \right) \log z - z + \frac{1}{2} \log(2\pi) + \sum_{n \geq 1} \frac{B_{2n}}{2n(2n-1)z^{2n-1}}$$

valid as $z \rightarrow \infty$ in the sector $|\arg z| \leq \pi - \delta$, $0 < \delta \leq \pi$ [4, p. 140]. Here and through the proof $\log$ denotes the principal value of the logarithm. Simple algebraic manipulation gives

$$\frac{1}{z} \log \frac{\Gamma(z)}{\left(\frac{z}{e}\right)^z \sqrt{\frac{2\pi}{z}}} \sim \sum_{n \geq 1} \frac{B_{2n}}{2n(2n-1)z^{2n}}.$$

Now we apply Theorem 2.1 with $S = \{z : -\pi + \delta \leq \arg z \leq \pi - \delta\}$ and

$$g(z) = \frac{1}{z} \log \frac{\Gamma(z)}{\left(\frac{z}{e}\right)^z \sqrt{\frac{2\pi}{z}}}, \quad h(w) = e^w - 1.$$

We find that

$$\left(\frac{\Gamma(z)}{\left(\frac{z}{e}\right)^z \sqrt{\frac{2\pi}{z}}} \right)^{1/z} - 1 \sim \sum_{n \geq 1} \frac{p_n}{z^n}.$$

On the positive real line both this and (1.1) hold. Thus, from the uniqueness theorem on asymptotic series it follows that $p_{2n} = G_n$ and $p_{2n+1} = 0$ for all $n \geq 0$, hence

$$\left(\frac{\Gamma(z)}{\left(\frac{z}{e}\right)^z \sqrt{\frac{2\pi}{z}}} \right)^{1/z} - 1 \sim \sum_{n \geq 1} \frac{G_n}{z^{2n}}$$

as $z \rightarrow \infty$ in the sector $|\arg z| \leq \pi - \delta$, $0 < \delta \leq \pi$. By the definition of an asymptotic expansion, for every positive integer N , there exist positive numbers C_N and $M_{N,1}$, such that

$$\left(\frac{\Gamma(z)}{\left(\frac{z}{e}\right)^z \sqrt{\frac{2\pi}{z}}} \right)^{1/z} = \sum_{n=0}^{N-1} \frac{G_n}{z^{2n}} + \frac{\varphi_N(z)}{z^{2N}} \quad (2.1)$$

where $|\varphi_N(z)| \leq C_N$ for $|z| > M_{N,1}$ and $|\arg z| \leq \pi - \delta$, $0 < \delta \leq \pi$. Especially

$$\left(\frac{\Gamma(z)}{\left(\frac{z}{e}\right)^z \sqrt{\frac{2\pi}{z}}} \right)^{1/z} = 1 + \mathcal{O}\left(\frac{1}{z^2}\right)$$

hence

$$0 < \left| \sum_{n=0}^{N-1} \frac{G_n}{z^{2n}} + \frac{\varphi_N(z)}{z^{2N}} \right| < 2$$

when $|z| > M_{N,2} > 0$ and $|\arg z| \leq \pi - \delta$, $0 < \delta \leq \pi$. Thus, (2.1) can be written as

$$\Gamma(z) = \left(\frac{z}{e}\right)^z \sqrt{\frac{2\pi}{z}} \left(\sum_{n=0}^{N-1} \frac{G_n}{z^{2n}} \right)^z \left(1 + \frac{\varphi_N(z)}{z^{2N} \sum_{n=0}^{N-1} G_n z^{-2n}} \right)^z \quad (2.2)$$

when $|z| > M_{N,3} := \max\{M_{N,1}, M_{N,2}\}$ and $|\arg z| \leq \pi - \delta$, $0 < \delta \leq \pi$. Since for every positive integer N , $\lim_{z \rightarrow \infty} \sum_{n=0}^{N-1} G_n z^{-2n} = 1$, we conclude that

$$\left| \frac{\varphi_N(z)}{z^{2N} \sum_{n=0}^{N-1} G_n z^{-2n}} \right| \leq \frac{2C_N}{|z|^{2N}} < \frac{1}{2} \quad (2.3)$$

as $|z| > M_{N,4} > 0$ and $|\arg z| \leq \pi - \delta$, $0 < \delta \leq \pi$. The inequality $|\log(1+z)| \leq -\log(1-|z|) < 2|z|$ ($|z| < 1/2$) and (2.3) yield

$$\begin{aligned} \left| \left(1 + \frac{\varphi_N(z)}{z^{2N} \sum_{n=0}^{N-1} G_n z^{-2n}} \right)^z \right| &\leq \exp \left(-|z| \log \left(1 - \left| \frac{\varphi_N(z)}{z^{2N} \sum_{n=0}^{N-1} G_n z^{-2n}} \right| \right) \right) \\ &< \exp \left(|z| \frac{4C_N}{|z|^{2N}} \right) = \exp \left(\frac{4C_N}{|z|^{2N-1}} \right) \end{aligned}$$

for $|z| > M_{N,4} > 0$ and $|\arg z| \leq \pi - \delta$, $0 < \delta \leq \pi$. Since $\exp(x) < 1 + 2x$ for $0 \leq x < 1 < 1.2564 \dots$, by (2.3) we have

$$\exp \left(\frac{4C_N}{|z|^{2N-1}} \right) < 1 + \frac{8C_N}{|z|^{2N-1}}.$$

Thus,

$$\left(1 + \frac{\varphi_N(z)}{z^{2N} \sum_{n=0}^{N-1} G_n z^{-2n}} \right)^z = 1 + \mathcal{O} \left(\frac{1}{z^{2N-1}} \right)$$

as $z \rightarrow \infty$ in the sector $|\arg z| \leq \pi - \delta$, $0 < \delta \leq \pi$. From this and (2.2) the formula (1.5) follows. $\square$

To prove (1.6) we shall use the following theorem of Bender about formal power series [5, Theorem 2] (quoted in [6, Theorem 7.3]).

Theorem 2.2. Suppose that

$$\begin{aligned} A(x) &= \sum_{n \geq 1} a_n x^n, \quad F(x, y) = \sum_{n, k \geq 0} f_{nk} x^n y^k, \\ B(x) &= \sum_{n \geq 0} b_n x^n = F(x, A(x)), \quad D(x) = \sum_{n \geq 0} d_n x^n = \left. \frac{\partial F(x, y)}{\partial y} \right|_{y=A(x)}. \end{aligned}$$

Assume that $F(x, y)$ is analytic in x and y in a neighborhood of $(0, 0)$, $a_n \neq 0$ and

(i)

$$a_{n-1} = o(a_n) \quad \text{as } n \rightarrow +\infty,$$

(ii)

$$\sum_{k=r}^{n-r} |a_k a_{n-k}| = \mathcal{O}(a_{n-r}) \quad \text{for some } r > 0 \text{ as } n \rightarrow +\infty.$$

Then

$$b_n = \sum_{k=0}^{r-1} d_k a_{n-k} + \mathcal{O}(a_{n-r}) \quad \text{as } n \rightarrow +\infty.$$

Proof of formula (1.6). As we saw in the previous proof the formal expansion

$$\exp \left(\sum_{n \geq 1} \frac{B_{2n}}{2n(2n-1)z^{2n}} \right) \sim \sum_{n \geq 0} \frac{G_n}{z^{2n}}$$

holds in the sector $|\arg z| \leq \pi - \delta$, $0 < \delta \leq \pi$ as $z \rightarrow \infty$. From this we have the following formal generating function

$$\exp \left(\sum_{n \geq 1} \frac{B_{2n}}{2n(2n-1)} x^n \right) = \sum_{n \geq 0} G_n x^n.$$

We wish to apply [Theorem 2.2](#) with

$$A(x) = \sum_{n \geq 1} \frac{B_{2n}}{2n(2n-1)} x^n, \quad F(x, y) = e^y.$$

It follows that

$$B(x) = D(x) = \sum_{n \geq 0} G_n x^n.$$

From the representation

$$a_n = \frac{B_{2n}}{2n(2n-1)} = (-1)^{n-1} \frac{2(2n-2)!}{(2\pi)^{2n}} \zeta(2n)$$

it is clear that condition (i) of [Theorem 2.2](#) holds. On the other hand

$$\frac{2(2n-2)!}{(2\pi)^{2n}} < |a_n| < \frac{4(2n-2)!}{(2\pi)^{2n}},$$

thus (ii) holds for every integer $r > 0$ since for $n > 2r$

$$\begin{aligned} \sum_{k=r}^{n-r} |a_k a_{n-k}| &< \sum_{k=r}^{n-r} \frac{4(2k-2)!}{(2\pi)^{2k}} \frac{4(2n-2k-2)!}{(2\pi)^{2n-2k}} \\ &= \frac{2(2n-2r-2)!}{(2\pi)^{2n-2r}} \sum_{k=r}^{n-r} \frac{4(2k-2)!}{(2\pi)^{2k}} \frac{4(2n-2k-2)!}{(2\pi)^{2n-2k}} \frac{(2\pi)^{2n-2r}}{2(2n-2r-2)!} \\ &< |a_{n-r}| \frac{8}{(2\pi)^{2r} (2n-2r-2)!} \sum_{k=r}^{n-r} (2k-2)! (2n-2k-2)! \\ &< |a_{n-r}| \frac{8}{(2\pi)^{2r} (2n-2r-2)!} (2(2r-2)! (2n-2r-2)! + (n-2r-1)(2r)!(2n-2r-4)!) \\ &< |a_{n-r}| \frac{8}{(2\pi)^{2r}} (2(2r-2)! + (2r)!). \end{aligned}$$

Since $F(x, y) = e^y$ is analytic in x and y , it follows that

$$G_n = \sum_{k=0}^{r-1} G_k \frac{B_{2n-2k}}{(2n-2k)(2n-2k-1)} + \mathcal{O} \left(\frac{B_{2n-2r}}{(2n-2r)(2n-2r-1)} \right)$$

for every integer $r > 0$ as $n \rightarrow +\infty$. $\square$

Proof of Theorem 1.2. Using

$$\frac{B_{2n}}{2n(2n-1)} = (-1)^{n-1} \frac{2(2n-2)!}{(2\pi)^{2n}} \zeta(2n)$$

in (1.6) we find that

$$G_n = (-1)^{n-1} \frac{2(2n-2)!}{(2\pi)^{2n}} \zeta(2n) \left(\sum_{k=0}^{r-1} (-1)^k (2\pi)^{2k} G_k \frac{\zeta(2n-2k)}{\zeta(2n)} \frac{(2n-2k-2)!}{(2n-2)!} + \mathcal{O} \left(\frac{(2n-2r-2)!}{(2n-2)!} \right) \right).$$

Since

$$\frac{\zeta(2n-2k)}{\zeta(2n)} = 1 + \mathcal{O} \left(\frac{1}{4^{n-k}} \right)$$

as $n \rightarrow +\infty$, it follows that

$$G_n = (-1)^{n-1} \frac{2(2n-2)!}{(2\pi)^{2n}} \zeta(2n) \left(\sum_{k=0}^{r-1} (-1)^k (2\pi)^{2k} G_k \frac{(2n-2k-2)!}{(2n-2)!} + \mathcal{O}\left(\frac{(2n-2r-2)!}{(2n-2)!}\right) \right)$$

for every integer $r > 0$ as $n \rightarrow +\infty$. This can be written as

$$G_n = (-1)^{n-1} \frac{2(2n-2)!}{(2\pi)^{2n}} \zeta(2n) \left(\sum_{k=0}^{r-1} \frac{(-1)^k (2\pi)^{2k} G_k}{(2n-2)_{2k}} + \mathcal{O}\left(\frac{1}{(2n-2)_{2r}}\right) \right)$$

where

$$(x)_m = \begin{cases} x(x-1) \cdots (x-m+1), & m \geq 1, \\ 1, & m = 0 \end{cases}$$

is the falling factorial. This leads us to the formal asymptotic series

$$G_n \sim (-1)^{n-1} \frac{2(2n-2)!}{(2\pi)^{2n}} \zeta(2n) \sum_{k \geq 0} \frac{(-1)^k (2\pi)^{2k} G_k}{(2n-2)_{2k}} \quad (2.4)$$

which holds as $n \rightarrow +\infty$. By the Stirling formula (1.4) we have

$$(-1)^{n-1} \frac{2(2n-2)!}{(2\pi)^{2n}} \zeta(2n) \sim (-1)^{n-1} \left(\frac{n-1}{\pi e} \right)^{2n-2} \sqrt{\frac{n-1}{\pi^3}} \sum_{k \geq 0} \frac{\gamma_k}{2^k} \frac{1}{(n-1)^k}. \quad (2.5)$$

From the generating function of the Stirling numbers (1.3) one can easily verify that

$$\frac{1}{(2n-2)_{2k}} = \sum_{m \geq 2k} \frac{S(m-1, 2k-1)}{2^m} \frac{1}{(n-1)^m}$$

for $k \geq 1$. Hence

$$\begin{aligned} \sum_{k \geq 0} \frac{(-1)^k (2\pi)^{2k} G_k}{(2n-2)_{2k}} &= 1 + \sum_{k \geq 1} (-1)^k (2\pi)^{2k} G_k \sum_{m \geq 2k} \frac{S(m-1, 2k-1)}{2^m} \frac{1}{(n-1)^m} \\ &\sim 1 + \sum_{m \geq 2} \left[\sum_{k=1}^{\lfloor m/2 \rfloor} \frac{(-1)^k (2\pi)^{2k} G_k S(m-1, 2k-1)}{2^m} \right] \frac{1}{(n-1)^m} \end{aligned}$$

as n tends to infinity. By plugging this and (2.5) into (2.4) and performing the product of the asymptotic series we finally have

$$G_n \sim (-1)^{n-1} \left(\frac{n-1}{\pi e} \right)^{2n-2} \sqrt{\frac{n-1}{\pi^3}} \sum_{k \geq 0} \frac{g_k}{(n-1)^k}$$

as $n \rightarrow +\infty$, where $g_0 = 1$, $g_1 = \gamma_1/2 = 1/24$ and

$$g_k = \frac{\gamma_k}{2^k} + \sum_{j=2}^k \sum_{m=1}^{\lfloor j/2 \rfloor} (-1)^m 2^{2m-k} \pi^{2m} G_m \gamma_{k-j} S(j-1, 2m-1)$$

for $k \geq 2$. $\square$

Proof of Corollary 1.1. Since

$$\sum_{n=0}^2 \frac{G_n}{z^{2n}} = 1 + \frac{1}{12z^2} + \frac{1}{1440z^4},$$

by formula (1.5) we have

$$\Gamma(z) = \left(\frac{z}{e} \right)^z \sqrt{\frac{2\pi}{z}} \left(1 + \frac{1}{12z^2} + \frac{1}{1440z^4} \right)^z \left(1 + \mathcal{O}\left(\frac{1}{z^5}\right) \right) \quad (2.6)$$

as $z \rightarrow \infty$ in the sector $|\arg z| \leq \pi - \delta$, $0 < \delta \leq \pi$. On the other hand for $|120z^2| > 1$

$$1 + \frac{1}{12z^2 - \frac{1}{10}} = 1 + \frac{1}{12z^2} \frac{1}{1 - \frac{1}{120z^2}} = 1 + \frac{1}{12z^2} + \frac{1}{1440z^4} + \frac{1}{172800z^6 - 1440z^2}.$$

Table 1
Approximations for G_{40} with various r , using (3.1).

Exact numerical value of G_{40}	$-3.16528798320637051517856337854 \cdot 10^{51}$
Value of r	5
Approximation for G_{40}	$-3.16528798320642434418777493911 \cdot 10^{51}$
Error	$5.382900921156057 \cdot 10^{37}$
Value of r	10
Approximation for G_{40}	$-3.16528798320637051518803217341 \cdot 10^{51}$
Error	$9.46879487 \cdot 10^{30}$
Value of r	15
Approximation for G_{40}	$-3.16528798320637051517856716638 \cdot 10^{51}$
Error	$3.78784 \cdot 10^{27}$

Since for $|z| > 1$

$$\left| \frac{1}{172800z^6 - 1440z^2} \right| < \frac{1}{171360 |z|^6}$$

we obtain

$$\begin{aligned} \left(1 + \frac{1}{12z^2 - \frac{1}{10}} \right)^z &= \left(1 + \frac{1}{12z^2} + \frac{1}{1440z^4} + \psi(z) \right)^z \\ &= \left(1 + \frac{1}{12z^2} + \frac{1}{1440z^4} \right)^z \left(1 + \frac{\psi(z)}{1 + \frac{1}{12z^2} + \frac{1}{1440z^4}} \right)^z \end{aligned}$$

where $|\psi(z)| \leq 1/(171360 |z|^6)$ and $|z| > 1$. From

$$\left| \frac{\psi(z)}{1 + \frac{1}{12z^2} + \frac{1}{1440z^4}} \right| < \frac{1440}{1319} \frac{1}{171360 |z|^6} = \frac{1}{156961 |z|^6}$$

it follows that

$$\begin{aligned} \left| \left(1 + \frac{\psi(z)}{1 + \frac{1}{12z^2} + \frac{1}{1440z^4}} \right)^z \right| &\leq \left(1 + \left| \frac{\psi(z)}{1 + \frac{1}{12z^2} + \frac{1}{1440z^4}} \right| \right)^{|z|} \\ &< \left(1 + \frac{1}{156961 |z|^6} \right)^{|z|} < \exp \left(\frac{1}{156961 |z|^5} \right) < 1 + \frac{2}{156961 |z|^5}, \end{aligned}$$

since $\exp(x) < 1 + 2x$ for $0 \leq x < 1 < 1.2564, \dots$. Thus,

$$\left(1 + \frac{1}{12z^2 - \frac{1}{10}} \right)^z = \left(1 + \frac{1}{12z^2} + \frac{1}{1440z^4} \right)^z \left(1 + \mathcal{O} \left(\frac{1}{z^5} \right) \right)$$

as $z \rightarrow \infty$. Comparing this with (2.6) the statement follows. $\square$

3. Numerical examples

From (1.6) we have the following approximation formula

$$G_n \approx \sum_{k=0}^{r-1} G_k \frac{B_{2n-2k}}{(2n-2k)(2n-2k-1)}. \quad (3.1)$$

Table 1 shows the numerical performance of this approximation for $n = 40$ and $r = 5, 10$ and 15 as well. Another family of approximations comes from (1.7)

$$G_n \approx (-1)^{n-1} \left(\frac{n-1}{\pi e} \right)^{2n-2} \sqrt{\frac{n-1}{\pi^3}} \sum_{k=0}^r \frac{g_k}{(n-1)^k}. \quad (3.2)$$

Table 2 shows the numerical performance of this approximation for $n = 40$ and $r = 0, 5$ and 10 as well.

Table 2
Approximations for G_{40} with various r , using (3.2).

Exact numerical value of G_{40}	$-3.16528798320637051517856337854 \cdot 10^{51}$
Value of r	0
Approximation for G_{40}	$-3.16364091760415824540255075458 \cdot 10^{51}$
Error	$-1.64706560221226977601262396 \cdot 10^{48}$
Value of r	5
Approximation for G_{40}	$-3.16528798354896112475362750946 \cdot 10^{51}$
Error	$3.4259060957506413092 \cdot 10^{41}$
Value of r	10
Approximation for G_{40}	$-3.16528798320644784339852009652 \cdot 10^{51}$
Error	$7.732821995671798 \cdot 10^{37}$

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References

- [1] G. Nemes, New asymptotic expansion for the gamma function, Arch. Math. (Basel) 95 (2010) 161–169.
- [2] G. Nemes, On the coefficients of the asymptotic expansion of $n!$, J. Integer Seqs. 13 (2010).
- [3] E.R. Love, Functions of asymptotic expansions, Bull. Austral. Math. Soc. 6 (1972) 307–312.
- [4] F.W.J. Olver, D.W. Lozier, R.F. Boisvert, C.W. Clark (Eds.), NIST Handbook of Mathematical Functions, Cambridge University Press, New York, 2010.
- [5] E.A. Bender, An asymptotic expansion for the coefficients of some formal power series, J. London Math. Soc. 9 (1975) 451–458.
- [6] A.M. Odlyzko, Asymptotic enumeration methods, in: R.L. Graham, M. Grötschel, L. Lovász (Eds.), Handbook of Combinatorics (Vol. II), M.I.T. Press and North-Holland, 1995, pp. 1063–1229.