

# ASYMPTOTIC EXPANSION FOR $\log n!$ IN TERMS OF THE RECIPROCAL OF A TRIANGULAR NUMBER

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**Abstract.** Ramanujan suggested an expansion for the  $n$ th partial sum of the harmonic series which employs the reciprocal of the  $n$ th triangular number. This has been proved in 2006 by Villarino, who speculated that there might also exist a similar expansion for the logarithm of the factorial. This study shows that such an asymptotic expansion indeed exists and provides formulas for its generic coefficient and for the bounds on its errors.

## 1. Introduction

**1.1. Ramanujan's expansion.** Ramanujan [4], [9] proposed, without proof and without a formula for the general term, the following asymptotic expansion for the partial sum of the harmonic series:

$$(1.1) \quad H_n := \sum_{k=1}^n \frac{1}{k} \sim \frac{1}{2} \log(2m) + \gamma + \frac{1}{12m} - \frac{1}{120m^2} + \frac{1}{630m^3} - \frac{1}{1680m^4} + \cdots,$$

where  $m := \frac{n(n+1)}{2}$  is the  $n$ th triangular number and  $\gamma$  is the Euler–Mascheroni constant. The complete proof of this theorem was given in 2006<sup>1</sup> by M. Villarino [9] who also suggested that there might exist a series expansion for the logarithm of the factorial in terms of  $\frac{1}{m}$ . In this article we prove a formula which represents an affirmative answer to this conjecture.

**1.2. The new asymptotic series.** The following theorem gives a complete asymptotic expansion for the logarithm of the factorial in terms

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of the reciprocal of a triangular number. The proof of the theorem will be given in Section 2.

**THEOREM 1.1.** *Let  $m := \frac{n(n+1)}{2}$ , where  $n$  is a positive integer. Then for every integer  $r \geq 1$  there exist two numbers  $0 < \vartheta_r < 1$  and  $-1 < \kappa_r < 1$  such that*

$$(1.2) \quad \frac{2}{\sqrt{8m+1}} \log \left( \frac{n!}{\sqrt{2\pi}} \right) \\ = \frac{1}{2} \log(2m) - 1 + \sum_{k=1}^r \frac{G_k}{m^k} + \vartheta_r \cdot \frac{G_{r+1}}{m^{r+1}} + \kappa_r \cdot \frac{(-1)^r}{2(r+1) \cdot 8^{r+1} m^{r+1}},$$

where the coefficients  $G_k$  are given by

$$(1.3) \quad G_k = \frac{(-1)^{k-1}}{2k \cdot 8^k} \sum_{j=0}^k \frac{(-1)^j (2^{2j} - 2) B_{2j}}{2j - 1} \binom{k}{j},$$

and  $B_j$  denotes the  $j$ th Bernoulli number (see [1, p. 804]).

Before we prove this, let us first briefly review the history of the Stirling series on which our proof will be based.

**1.3. The history of Stirling's formula.** In 1730, in his *Methodus Differentialis* [8], Stirling presented his results on the summation of logarithms. In particular, he gave a series approximation for  $\log 1 + \log 2 + \cdots + \log n = \log n!$  in the form

$$(1.4) \quad \left(n + \frac{1}{2}\right) \log \left(n + \frac{1}{2}\right) - \left(n + \frac{1}{2}\right) \\ + \frac{1}{2} \log(2\pi) - \frac{1}{24\left(n + \frac{1}{2}\right)} + \frac{7}{2880\left(n + \frac{1}{2}\right)^3} - \cdots.$$

Stirling gave a recurrence for the coefficients in his series, but he was not able to obtain an explicit formula for them. The general term for  $k \geq 1$  is given by

$$(1.5) \quad \frac{(2^{1-2k} - 1) B_{2k}}{2k(2k-1)\left(n + \frac{1}{2}\right)^{2k-1}},$$

where  $B_k$  denotes the  $k$ th Bernoulli number. Having seen Stirling's results, De Moivre in his *Miscellaneis Analyticis Supplementum* discovered a much simpler approximation:

$$(1.6) \quad \left(n + \frac{1}{2}\right) \log n - n + \frac{1}{2} \log(2\pi) + \frac{1}{12n} - \frac{1}{360n^3} + \cdots.$$

This time the formula for the general term is

$$(1.7) \quad \frac{B_{2k}}{2k(2k-1)n^{2k-1}}.$$

De Moivre's series was associated with Stirling's name because the constant  $\frac{1}{2} \log(2\pi)$  was determined by him. Both of the series are divergent for all values of  $n$ , but the partial sums can be made an arbitrarily good approximation for large enough  $n$  (see [5, p. 1]).

Note that Ramanujan [7] also gave an approximation for  $\log n!$  in the form

$$(1.8) \quad \log n! \approx n \log n - n + \frac{1}{2} \log \pi + \frac{1}{6} \log \left( 8n^3 + 4n^2 + n + \frac{1}{30} \right).$$

This formula has been studied in detail in [3] and [6].

## 2. Proof of the new asymptotic expansion

The proof of our theorem is based on the following more precise version of the Stirling series (see [2]).

**THEOREM 2.1.** *Let  $n$  be a positive integer. Then for every integer  $r \geq 1$  there exists a number  $0 < \rho_r < 1$  for which the following expression holds:*

$$(2.1) \quad \begin{aligned} & \frac{1}{\left(n + \frac{1}{2}\right)} \log \left( \frac{n!}{\sqrt{2\pi}} \right) \\ &= \log \left( n + \frac{1}{2} \right) - 1 + \sum_{k=1}^r \frac{S_k}{\left(n + \frac{1}{2}\right)^{2k}} + \rho_r \cdot \frac{S_{r+1}}{\left(n + \frac{1}{2}\right)^{2r+2}}, \end{aligned}$$

where the coefficients  $S_k$  are given by

$$(2.2) \quad S_k = \frac{(2^{1-2k} - 1)B_{2k}}{2k(2k-1)}. \quad \square$$

**PROOF OF THEOREM 1.1.** First, let us modify the term  $\log \left( n + \frac{1}{2} \right)$ :

$$\begin{aligned} \log \left( n + \frac{1}{2} \right) &= \frac{1}{2} \log \left( n + \frac{1}{2} \right)^2 = \frac{1}{2} \log \left( 2m + \frac{1}{4} \right) \\ &= \frac{1}{2} \log \left( 2m \left( 1 + \frac{1}{8m} \right) \right) = \frac{1}{2} \log(2m) + \frac{1}{2} \log \left( 1 + \frac{1}{8m} \right) \end{aligned}$$

$$= \frac{1}{2} \log(2m) + \frac{1}{2} \sum_{i \geq 1} \frac{(-1)^{i-1}}{i \cdot 8^i m^i} = \frac{1}{2} \log(2m) + \sum_{i=1}^r \frac{(-1)^{i-1}}{2i \cdot 8^i m^i} + \delta_r,$$

where

$$\delta_r := \sum_{i \geq r+1} \frac{(-1)^{i-1}}{2i \cdot 8^i m^i}.$$

To obtain the series expansion in terms of  $\frac{1}{m}$ , we apply the binomial theorem:

$$\begin{aligned} \sum_{k=1}^r \frac{S_k}{\left(n + \frac{1}{2}\right)^{2k}} &= \sum_{k=1}^r \frac{S_k}{\left(2m + \frac{1}{4}\right)^k} = \sum_{k=1}^r \frac{S_k}{(2m)^k} \left(1 + \frac{1}{8m}\right)^{-k} \\ &= \sum_{k=1}^r \frac{S_k}{(2m)^k} \sum_{l \geq 0} \binom{-k}{l} \frac{1}{8^l m^l} = \sum_{k=1}^r \frac{S_k}{2^k} \sum_{l \geq 0} \binom{k+l-1}{l} \frac{(-1)^l}{8^l m^{k+l}}. \end{aligned}$$

Replacing every  $k$  by  $k-l$ , and every  $l$  by  $k-j$  we obtain

$$\sum_{k=1}^r \frac{S_k}{\left(n + \frac{1}{2}\right)^{2k}} = \sum_{k=1}^r \left\{ \sum_{j=1}^k \frac{S_j}{2^j} \binom{k-1}{k-j} \frac{(-1)^{k-j}}{8^{k-j}} \right\} \frac{1}{m^k} + \varepsilon_r,$$

where

$$\varepsilon_r := \sum_{k=1}^r \frac{S_k}{2^k} \sum_{l \geq r-k+1} \binom{k+l-1}{l} \frac{(-1)^l}{8^l m^{k+l}}.$$

Now, collecting all the partial results

$$\begin{aligned} \frac{1}{\left(n + \frac{1}{2}\right)} \log\left(\frac{n!}{\sqrt{2\pi}}\right) &= \frac{1}{2} \log(2m) + \sum_{i=1}^r \frac{(-1)^{i-1}}{2i \cdot 8^i m^i} - 1 \\ &+ \sum_{k=1}^r \left\{ \sum_{j=1}^k \frac{S_j}{2^j} \binom{k-1}{k-j} \frac{(-1)^{k-j}}{8^{k-j}} \right\} \frac{1}{m^k} + \delta_r + \varepsilon_r + \rho_r \cdot \frac{S_{r+1}}{\left(n + \frac{1}{2}\right)^{2r+2}} \\ &= \frac{1}{2} \log(2m) - 1 + \sum_{k=1}^r \left\{ \frac{(-1)^{k-1}}{2k \cdot 8^k} + \sum_{j=1}^k \frac{S_j}{2^j} \binom{k-1}{k-j} \frac{(-1)^{k-j}}{8^{k-j}} \right\} \frac{1}{m^k} \\ &+ \delta_r + \varepsilon_r + \rho_r \cdot \frac{S_{r+1}}{\left(n + \frac{1}{2}\right)^{2r+2}}. \end{aligned}$$

We have thus derived a series expansion in terms of  $\frac{1}{m}$ , but the general term looks different from the one in (1.3). However, inserting (2.2) into our general coefficient, we get

$$\begin{aligned}
 G_k &= \frac{(-1)^{k-1}}{2k \cdot 8^k} + \sum_{j=1}^k \frac{S_j}{2^j} \binom{k-1}{k-j} \frac{(-1)^{k-j}}{8^{k-j}} \\
 &= (-1)^{k-1} \left\{ \frac{1}{2k \cdot 8^k} + \sum_{j=1}^k \frac{(2^{1-2j} - 1) B_{2j}}{2^j \cdot 2j(2j-1)} \binom{k-1}{k-j} \frac{(-1)^{1-j}}{8^{k-j}} \right\} \\
 &= (-1)^{k-1} \left\{ \frac{1}{2k \cdot 8^k} + \sum_{j=1}^k \frac{(2^{1-2j} - 1) B_{2j}}{2^j(2j-1)} \frac{1}{2k} \binom{k}{j} \frac{(-1)^{1-j}}{8^{k-j}} \right\} \\
 &= \frac{(-1)^{k-1}}{2k \cdot 8^k} \left\{ 1 + \sum_{j=1}^k \frac{(2^{1-2j} - 1) B_{2j}}{2^j(2j-1)} \binom{k}{j} \frac{(-1)^{1-j}}{8^{-j}} \right\} \\
 &= \frac{(-1)^{k-1}}{2k \cdot 8^k} \left\{ 1 + \sum_{j=1}^k \frac{(-1)^j (2^{2j} - 2) B_{2j}}{2j-1} \binom{k}{j} \right\} \\
 &= \frac{(-1)^{k-1}}{2k \cdot 8^k} \sum_{j=0}^k \frac{(-1)^j (2^{2j} - 2) B_{2j}}{2j-1} \binom{k}{j},
 \end{aligned}$$

which is the desired form of  $G_k$ .

Since  $(n + \frac{1}{2})^{-1} = \frac{2}{\sqrt{8m+1}}$ , we now have

$$\begin{aligned}
 &\frac{2}{\sqrt{8m+1}} \log \left( \frac{n!}{\sqrt{2\pi}} \right) = \frac{1}{2} \log(2m) - 1 \\
 &+ \sum_{k=1}^r \left\{ \frac{(-1)^{k-1}}{2k \cdot 8^k} \sum_{j=0}^k \frac{(-1)^j (2^{2j} - 2) B_{2j}}{2j-1} \binom{k}{j} \right\} \frac{1}{m^k} \\
 &+ \delta_r + \varepsilon_r + \rho_r \cdot \frac{S_{r+1}}{(n + \frac{1}{2})^{2r+2}}.
 \end{aligned}$$

It remains to show that for  $r \geq 1$  the error term has the properties specified in the theorem. To estimate the error, we use the fact that a convergent alternating series whose terms decrease in absolute value monotonically to

zero evaluates to any of its partial sums and a remainder comprised between zero and the first neglected term. It follows that

$$v_r := \rho_r \cdot \frac{S_{r+1}}{\left(n + \frac{1}{2}\right)^{2r+2}} = \rho_r \cdot \frac{S_{r+1}}{(2m)^{r+1}} \left(1 + \frac{1}{8m}\right)^{-(r+1)} = \sigma_r \cdot \frac{S_{r+1}}{(2m)^{r+1}},$$

$$\delta_r = \sum_{i \geq r+1} \frac{(-1)^{i-1}}{2i \cdot 8^i m^i} = \tau_r \cdot \frac{(-1)^r}{2(r+1) \cdot 8^{r+1} m^{r+1}},$$

$$\begin{aligned} \varepsilon_r &= \sum_{k=1}^r \frac{S_k}{2^k} \sum_{l \geq r-k+1} \binom{k+l-1}{l} \frac{(-1)^l}{8^l m^{k+l}} \\ &= \sum_{k=1}^r \frac{S_k}{2^k} \left\{ \omega_k \cdot \binom{r}{r-k+1} \frac{(-1)^{r-k+1}}{8^{r-k+1}} \right\} \frac{1}{m^{r+1}} \\ &= \Omega_r \cdot \sum_{k=1}^r \frac{S_k}{2^k} \left\{ \binom{r}{r-k+1} \frac{(-1)^{r-k+1}}{8^{r-k+1}} \right\} \frac{1}{m^{r+1}}, \end{aligned}$$

where  $0 < \sigma_r, \tau_r, \omega_k, \Omega_r < 1$  for  $1 \leq k \leq r$  and  $r \geq 1$ . Moreover,  $0 < \Omega_r < 1$  because for a fixed  $r$ , all the terms with the weights  $\omega_k$  have the same sign. Since  $v_r$  and  $\varepsilon_r$  have always the same sign (positive for odd  $r$  and negative for even  $r$ ), we have

$$\begin{aligned} v_r + \varepsilon_r &= \sigma_r \cdot \frac{S_{r+1}}{(2m)^{r+1}} + \Omega_r \cdot \sum_{k=1}^r \frac{S_k}{2^k} \left\{ \binom{r}{r-k+1} \frac{(-1)^{r-k+1}}{8^{r-k+1}} \right\} \frac{1}{m^{r+1}} \\ &= \vartheta_r \cdot \sum_{k=1}^{r+1} \frac{S_k}{2^k} \left\{ \binom{r}{r-k+1} \frac{(-1)^{r-k+1}}{8^{r-k+1}} \right\} \frac{1}{m^{r+1}}, \end{aligned}$$

where  $0 < \vartheta_r < 1$ . But  $\delta_r$  has always the opposite sign so that we can set  $\tau_r = \vartheta_r + \kappa_r$ , where  $-1 < \kappa_r < 1$ . Finally

$$\begin{aligned} &\vartheta_r \cdot \left( \frac{(-1)^r}{2(r+1) \cdot 8^{r+1} m^{r+1}} + \sum_{k=1}^{r+1} \frac{S_k}{2^k} \left\{ \binom{r}{r-k+1} \frac{(-1)^{r-k+1}}{8^{r-k+1}} \right\} \frac{1}{m^{r+1}} \right) \\ &= \vartheta_r \cdot \frac{(-1)^r}{2(r+1) \cdot 8^{r+1} m^{r+1}} \left( 1 + \sum_{k=1}^{r+1} \frac{(2^{1-2k} - 1) B_{2k}}{2^k \cdot (2k-1)} \left\{ \binom{r+1}{k} \frac{(-1)^{-k+1}}{8^{-k}} \right\} \right) \end{aligned}$$

$$\begin{aligned}
&= \vartheta_r \cdot \frac{(-1)^r}{2(r+1) \cdot 8^{r+1} m^{r+1}} \left( 1 + \sum_{k=1}^{r+1} \frac{(-1)^k (2^{2k} - 2) B_{2k}}{(2k-1)} \binom{r+1}{k} \right) \\
&= \vartheta_r \cdot \frac{(-1)^r}{2(r+1) \cdot 8^{r+1} m^{r+1}} \sum_{k=0}^{r+1} \frac{(-1)^k (2^{2k} - 2) B_{2k}}{(2k-1)} \binom{r+1}{k} \\
&= \vartheta_r \cdot \frac{G_{r+1}}{m^{r+1}}. \quad \square
\end{aligned}$$

CONJECTURE 2.2. *Numerical computations suggest that for  $r \geq 2$  we can use  $\kappa_r = 0$  in (1.2).*

### 3. Numerical comparisons

We will compare in this section the numerical performance of the classical asymptotic formulas to the factorial function with our formula for large values. We compare the following approximation formulas:

$$\begin{aligned}
&\left(\frac{k}{e}\right)^k \sqrt{2\pi} \exp\left(-\frac{1}{24k} + \frac{7}{2880k^3} - \frac{31}{40320k^5} + \cdots\right) \quad (\text{Stirling}), \\
&\left(\frac{n}{e}\right)^n \sqrt{2\pi n} \exp\left(\frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \cdots\right) \quad (\text{De Moivre}), \\
&\left(\frac{\sqrt{2m}}{e}\right)^k \sqrt{2\pi} \exp\left(k\left(\frac{1}{24m} - \frac{1}{1440m^2} - \frac{1}{4032m^3} + \cdots\right)\right) \quad (\text{New}),
\end{aligned}$$

where  $2k = 2n + 1$  and  $2m = n(n + 1)$ .

The following table displays the number of exact decimal digits (edd) of the formulas for some values of  $n$ . Exact decimal digits are defined as follows:

$$(3.1) \quad \text{edd}(n) = -\log_{10} \left| 1 - \frac{\text{approximation}(n)}{n!} \right|.$$

In the table below the  $(i)$ -th entry ( $i = 0, 1, 2, \dots$ ) in a line starting with “name” is the edd of the given approximation using the first  $i$  terms in the asymptotic expansion. The “−” sign indicates that the approximation is smaller and the “+” sign (not displayed) indicates that the approximation is larger than the true value.

Our formula is almost as good as the earlier ones of Stirling or De Moivre. With  $> 0$  terms Stirling’s formula is numerically on par with De Moivre’s approximation but has even larger rational coefficients. If we have to do only asymptotic analysis, De Moivre’s formula is highly recommended.

| Formula   | $n$   | (0)  | (1)   | (2)   | (3)   | (4)   |
|-----------|-------|------|-------|-------|-------|-------|
| Stirling  | 100   | 3.4  | −8.6  | 13.1  | −17.2 | 21.1  |
| De Moivre | 100   | −3.1 | 8.6   | −13.1 | 17.2  | −21.1 |
| New       | 100   | −3.1 | 8.6   | 12.7  | −16.8 | 20.7  |
| Stirling  | 1000  | 4.4  | −11.6 | 18.1  | −24.2 | 30.1  |
| De Moivre | 1000  | −4.1 | 11.6  | −18.1 | 24.2  | −30.1 |
| New       | 1000  | −4.1 | 11.6  | 17.7  | −23.7 | 29.7  |
| Stirling  | 10000 | 5.4  | −14.6 | 23.1  | −31.2 | 39.1  |
| De Moivre | 10000 | −5.1 | 14.6  | −23.1 | 31.2  | −39.1 |
| New       | 10000 | −5.1 | 14.6  | 22.7  | −30.7 | 38.7  |

Table 1: The number of exact decimal digits of the formulas for some values of  $n$ 

### Table of coefficients

Table 2 gives the first ten coefficients of the asymptotic series.

|       |                      |          |                                  |
|-------|----------------------|----------|----------------------------------|
| $G_1$ | $\frac{1}{24}$       | $G_6$    | $\frac{7619}{138378240}$         |
| $G_2$ | $-\frac{1}{1440}$    | $G_7$    | $-\frac{7439}{92252160}$         |
| $G_3$ | $-\frac{1}{4032}$    | $G_8$    | $\frac{2302207}{13442457600}$    |
| $G_4$ | $\frac{1}{8960}$     | $G_9$    | $-\frac{39982913}{81265766400}$  |
| $G_5$ | $-\frac{23}{380160}$ | $G_{10}$ | $\frac{242180131}{132433100800}$ |

Table 2: The first ten  $G_k$  coefficients

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