

New asymptotic expansion for the Gamma function

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Abstract. Using a series transformation, the Stirling-De Moivre asymptotic series approximation to the Gamma function is converted into a new one with better convergence properties. The new formula is being compared with those of Stirling, Laplace, and Ramanujan for real arguments greater than 0.5 and turns out to be, for equal number of “correction” terms, numerically superior to all of them. As a side benefit, a closed-form approximation has turned up during the analysis which is about as good as 3rd order Stirling’s (maximum relative error smaller than $1e - 10$ for real arguments greater or equal to 24).

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1. Introduction

1.1. The main result. In this paper we present a new asymptotic expansion for the Gamma function for real arguments greater or equal than one. We give an explicit formula for the coefficients in the series and estimate the error. After the proof we compare our new formula with some classical results. The numerical comparison shows that for equal number of “correction” terms the new formula is the most accurate and it is highly recommended computing the Gamma function for large real arguments.

Theorem 1.1. *Let $x \geq 1$, then for every $n \geq 1$, the following expression holds:*

$$\Gamma(x) = \left(\frac{x}{e} \left(\sum_{k=0}^{n-1} \frac{G_k}{x^{2k}} + R_n(x) \right) \right)^x \sqrt{\frac{2\pi}{x}}, \quad (1.1)$$

where $R_n(x) = \mathcal{O}\left(\frac{1}{x^{2n}}\right)$ and the G_k coefficients are given by

$$G_k = \sum \prod_{r=1}^k \frac{1}{m_r!} \left(\frac{B_{2r}}{2r(2r-1)} \right)^{m_r}, \quad G_0 = 1. \quad (1.2)$$

Here B_r denotes the r th Bernoulli number [3], and the sum extending over all sequences $m_1, m_2, \dots, m_k$ of non-negative integers such that $\sum_{i=1}^k 2im_i = 2k$. Moreover, if $x \geq n+1$, then

$$|R_n(x)| \leq \left(\frac{(n+1)^2}{2n+1} e + \frac{8e^{1/5}}{n(2n-1)(2\pi)^{2n}} \right) \left(\frac{n}{x} \right)^{2n}. \quad (1.3)$$

2. The proof of the theorem

2.1. Explicit formula for the coefficients. The proof of our theorem is based on the following well-known and more precise version of the Stirling-De Moivre series [2].

Proposition 2.1. *Let $x \geq 1$, then for every $n \geq 1$, the following expression holds:*

$$\log \Gamma(x) = \left(x - \frac{1}{2} \right) \log x - x + \frac{1}{2} \log(2\pi) + \sum_{k=1}^{n-1} \frac{B_{2k}}{2k(2k-1)x^{2k-1}} + S_n(x), \quad (2.1)$$

where the $S_n(x)$ remainder satisfies

$$|S_n(x)| \leq \frac{|B_{2n}|}{2n(2n-1)x^{2n-1}}. \quad (2.2)$$

Proof of expression (1.2). A simple algebraic manipulation of the above expansion gives

$$\frac{1}{x} \log \left(\frac{\Gamma(x)e^x}{x^{x-1/2}\sqrt{2\pi}} \right) = \sum_{k=1}^{n-1} \frac{B_{2k}}{2k(2k-1)x^{2k}} + \frac{S_n(x)}{x}.$$

Taking the exponential of both sides yields

$$\left(\frac{\Gamma(x)e^x}{x^{x-1/2}\sqrt{2\pi}} \right)^{1/x} = \exp \left(\sum_{k=1}^{n-1} \frac{B_{2k}}{2k(2k-1)x^{2k}} \right) \cdot \exp \left(\frac{S_n(x)}{x} \right).$$

By the Maclaurin series of the exponential function we obtain

$$\begin{aligned} & \exp \left(\sum_{k=1}^{n-1} \frac{B_{2k}}{2k(2k-1)x^{2k}} \right) \cdot \exp \left(\frac{S_n(x)}{x} \right) \\ &= \exp \left(\frac{S_n(x)}{x} \right) \cdot \prod_{k=1}^{n-1} \exp \left(\frac{B_{2k}}{2k(2k-1)x^{2k}} \right) \\ &= \exp \left(\frac{S_n(x)}{x} \right) \cdot \prod_{k=1}^{n-1} \sum_{l \geq 0} \frac{1}{l!} \left(\frac{B_{2k}}{2k(2k-1)x^{2k}} \right)^l. \end{aligned}$$

Applying the Cauchy product and collecting the coefficients of x^{-2k} we obtain

$$\exp\left(\frac{S_n(x)}{x}\right) \cdot \prod_{k=1}^{n-1} \sum_{l \geq 0} \frac{1}{l!} \left(\frac{B_{2k}}{2k(2k-1)x^{2k}}\right)^l = \sum_{k=0}^{n-1} \frac{G_k}{x^{2k}} + R_n(x),$$

where the G_k coefficients are given by

$$G_k = \sum \prod_{r=1}^k \frac{1}{m_r!} \left(\frac{B_{2r}}{2r(2r-1)}\right)^{m_r}, \quad G_0 = 1.$$

$R_n(x) = \mathcal{O}\left(\frac{1}{x^{2n}}\right)$, and the sum extending over all sequences $m_1, m_2, \dots, m_k$ of non-negative integers such that $\sum_{i=1}^k 2im_i = 2k$. Thus the first part of the theorem is proved. $\square$

2.2. The estimate of the remainder. We will estimate uniformly in x and n the value of the remainder $R_n(x)$.

Proof. In the first part of the proof we showed that

$$\exp\left(\frac{S_n(x)}{x}\right) \cdot \prod_{k=1}^{n-1} \exp\left(\frac{B_{2k}}{2k(2k-1)x^{2k}}\right) = \sum_{k=0}^{n-1} \frac{G_k}{x^{2k}} + R_n(x). \quad (2.3)$$

First let

$$\prod_{k=1}^{n-1} \exp\left(\frac{B_{2k}}{2k(2k-1)x^{2k}}\right) = \sum_{j \geq 0} \frac{K_{j,n}}{x^j}, \quad (2.4)$$

thus $K_{2j,n} = G_j$ if $0 \leq j \leq n-1$ and $K_{2j-1,n} = 0$ if $j \geq 1$. Let us determine the order of the factor $\exp\left(\frac{S_n(x)}{x}\right)$. Assume that $x \geq n+1$, then

$$\left|\frac{S_n(x)}{x}\right| \leq \frac{|B_{2n}|}{2n(2n-1)x^{2n}} \leq \frac{4}{n(2n-1)x^{2n}} \left(\frac{n}{2\pi}\right)^{2n} < \frac{1}{9} \quad (2.5)$$

by the inequality

$$|B_{2n}| = \frac{2(2n)!}{(2\pi)^{2n}} \zeta(2n) \leq 4 \frac{(2n)!}{(2\pi)^{2n}} \leq 4 \frac{2 \cdot n^{2n}}{(2\pi)^{2n}} = 8 \left(\frac{n}{2\pi}\right)^{2n}, \quad (2.6)$$

where ζ denotes Riemann's Zeta function. On the other hand we have

$$\exp\left(\frac{S_n(x)}{x}\right) = \sum_{k \geq 0} \frac{1}{k!} \left(\frac{S_n(x)}{x}\right)^k = 1 + 2\delta_n \frac{S_n(x)}{x}, \quad 0 < \delta_n < 1. \quad (2.7)$$

Now from the properties of the Stirling series one finds

$$\prod_{k=1}^{n-1} \exp\left(\frac{B_{2k}}{2k(2k-1)x^{2k}}\right) \cdot \exp\left(\frac{S_n(x)}{x}\right) < \exp\left(\frac{1}{12x}\right) \leq \exp\left(\frac{1}{12}\right). \quad (2.8)$$

From (2.5) and (2.8) we find that

$$\begin{aligned} \prod_{k=1}^{n-1} \exp\left(\frac{B_{2k}}{2k(2k-1)x^{2k}}\right) &= \prod_{k=1}^{n-1} \exp\left(\frac{B_{2k}}{2k(2k-1)x^{2k}}\right) \\ &\quad \times \exp\left(\frac{S_n(x)}{x}\right) \exp\left(-\frac{S_n(x)}{x}\right) < \exp\left(\frac{1}{5}\right). \end{aligned} \quad (2.9)$$

From (2.7) and (2.9) we have

$$\begin{aligned} &\prod_{k=1}^{n-1} \exp\left(\frac{B_{2k}}{2k(2k-1)x^{2k}}\right) \cdot \exp\left(\frac{S_n(x)}{x}\right) \\ &= \prod_{k=1}^{n-1} \exp\left(\frac{B_{2k}}{2k(2k-1)x^{2k}}\right) \left(1 + 2\delta_n \frac{S_n(x)}{x}\right) \\ &= \prod_{k=1}^{n-1} \exp\left(\frac{B_{2k}}{2k(2k-1)x^{2k}}\right) + 2\eta_n e^{1/5} \frac{S_n(x)}{x} \end{aligned} \quad (2.10)$$

for $-1 \leq \eta_n \leq 1$. Comparing (2.3), (2.4), and (2.10) we find the following expression for our remainder

$$R_n(x) = \sum_{j \geq 2n-1} \frac{K_{j,n}}{x^j} + 2\eta_n e^{1/5} \frac{S_n(x)}{x}, \quad x \geq n+1$$

and

$$|R_n(x)| \leq \left| \sum_{j \geq 2n-1} \frac{K_{j,n}}{x^j} \right| + 2e^{1/5} \left| \frac{S_n(x)}{x} \right| \leq \sum_{j \geq 2n-1} \frac{|K_{j,n}|}{x^j} + \frac{e^{1/5} |B_{2n}|}{n(2n-1)x^{2n}}. \quad (2.11)$$

We are going to estimate the value of the sum on the right hand side of (2.11). Let

$$f(z) := \prod_{k=1}^{n-1} \exp\left(\frac{B_{2k}}{2k(2k-1)} z^{2k}\right) = \sum_{j \geq 0} K_{j,n} z^j.$$

Hence the $K_{j,n}$ coefficients are the coefficients in the Taylor expansion of f . Thus for every $w > 0$ we have

$$K_{j,n} = \frac{1}{2\pi i} \int_{|v|=w} \frac{f(v)}{v^{j+1}} dv.$$

From this

$$|K_{j,n}| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{w |f(we^{i\varphi})| d\varphi}{w^{j+1}} \leq \max_{0 \leq \varphi \leq 2\pi} \frac{|f(we^{i\varphi})|}{w^j}. \quad (2.12)$$

Moreover,

$$|f(we^{i\varphi})| \leq \prod_{k=1}^{n-1} \exp\left(\frac{|B_{2k}|}{2k(2k-1)} w^{2k}\right) = \exp\left(\sum_{k=1}^{n-1} \frac{|B_{2k}|}{2k(2k-1)} w^{2k}\right).$$

Let $w = 1/n$, from (2.6)

$$\begin{aligned} \sum_{k=1}^{n-1} \frac{|B_{2k}|}{2k(2k-1)} w^{2k} &\leq \sum_{k=1}^{n-1} \frac{4}{k(2k-1)} \left(\frac{kw}{2\pi}\right)^{2k} \\ &\leq \sum_{k=1}^{n-1} \frac{4}{k(2k-1)} \left(\frac{1}{2\pi}\right)^{2k} < 1. \end{aligned}$$

Hence $|f(we^{i\varphi})| \leq e$ valid when $w = 1/n$ and by (2.12) we finally have

$$|K_{j,n}| \leq en^j.$$

Substituting this into (2.11) and noting that $K_{2j-1,n} = 0$ we obtain

$$\begin{aligned} |R_n(x)| &\leq \sum_{j \geq 2n-1} \frac{|K_{j,n}|}{x^j} + \frac{e^{1/5} |B_{2n}|}{n(2n-1)x^{2n}} = \sum_{i \geq n} \frac{|K_{2i,n}|}{x^{2i}} + \frac{e^{1/5} |B_{2n}|}{n(2n-1)x^{2n}} \\ &\leq \sum_{i \geq n} \frac{en^{2i}}{x^{2i}} + \frac{e^{1/5} |B_{2n}|}{n(2n-1)x^{2n}} = e \left(\frac{n}{x}\right)^{2n} \frac{1}{1 - (n/x)^2} + \frac{e^{1/5} |B_{2n}|}{n(2n-1)x^{2n}} \\ &\leq \frac{(n+1)^2}{2n+1} e \left(\frac{n}{x}\right)^{2n} + \frac{8e^{1/5}}{n(2n-1)x^{2n}} \left(\frac{n}{2\pi}\right)^{2n} \\ &= \left(\frac{(n+1)^2}{2n+1} e + \frac{8e^{1/5}}{n(2n-1)(2\pi)^{2n}}\right) \left(\frac{n}{x}\right)^{2n}, \end{aligned}$$

which completes the proof. $\square$

3. Numerical properties

3.1. The computation of the coefficients. From (1.2) we can compute the numerical values of G_k . The Bernoulli numbers start

$$B_j = 1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, -\frac{1}{30}, \dots (j \geq 0).$$

So the first few values of the G_k coefficients are

$$\begin{aligned} G_0 &= 1, \\ G_1 &= \frac{1}{2}B_2 = \frac{1}{12}, \\ G_2 &= \frac{1}{12}B_4 + \frac{1}{8}B_2^2 = \frac{1}{1440}, \\ G_3 &= \frac{1}{30}B_6 + \frac{1}{24}B_2B_4 + \frac{1}{48}B_2^3 = \frac{239}{362880}, \\ G_4 &= \frac{1}{56}B_8 + \frac{1}{60}B_2B_6 + \frac{1}{288}B_4^2 + \frac{1}{96}B_4B_2^2 + \frac{1}{384}B_2^4 = -\frac{46409}{87091200}. \end{aligned}$$

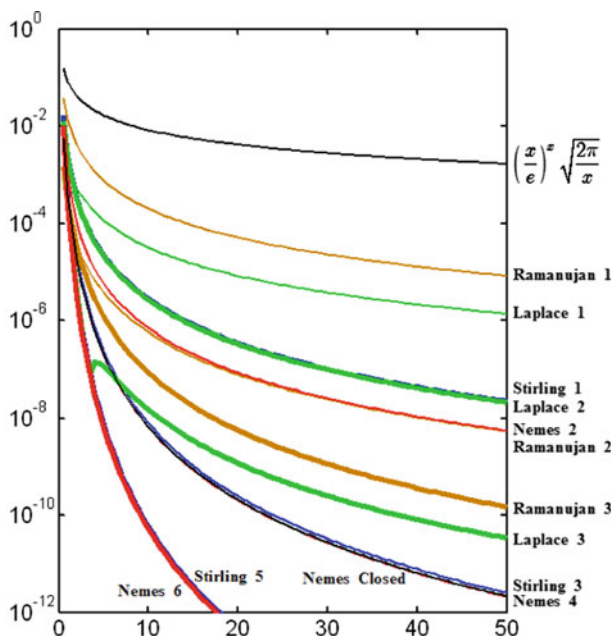


FIGURE 1. Relative errors of Gamma function approximations

We compare the following approximation formulas [4, 5]

$$\left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}} \exp\left(\frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \cdots\right) \quad (\text{Stirling}), \quad (3.4)$$

$$\left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}} \left(1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} - \cdots\right) \quad (\text{Laplace}), \quad (3.5)$$

$$\left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}} \sqrt[6]{\left(1 + \frac{1}{2x} + \frac{1}{8x^2} + \frac{1}{240x^3} - \cdots\right)} \quad (\text{Ramanujan}), \quad (3.6)$$

$$\left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}} \left(1 + \frac{1}{12x^2} + \frac{1}{1440x^4} + \frac{239}{362880x^6} - \cdots\right)^x \quad (\text{Nemes}). \quad (3.7)$$

The second expression is sometimes incorrectly called Stirling's formula (see [1]).

Figure 1 shows the relative error of these formulas to the Gamma function. The plotted quantity is $|\log(a(x)/\Gamma(x))| \approx |1 - a(x)/\Gamma(x)|$ ($a(x)$ denotes the corresponding approximation). The first computed values are for $x = 0.5$. Thick traces indicate $a(x) > \Gamma(x)$, thin ones $a(x) < \Gamma(x)$. Color codes for various families: blue—Stirling, green—Laplace, brown—Ramanujan, red—Nemes. The number after a name indicates the power of $(1/x)$ in the last kept term of the expansion. The following curves overlap: Laplace 2 overlaps Stirling 1, Ramanujan 2 overlaps Nemes 2 for large x , Nemes 4 overlaps Nemes Closed, Nemes 6 overlaps Stirling 5. For Nemes Closed formula, see Section 4.1.

Conclusion. From the graph in Fig. 1, we see that the first and the third Laplace approximations outperform the corresponding Ramanujan formulas, however these expansions contain the same number of terms. Ramanujan 2 gives better approximation than Laplace 2 and Nemes 2, though the latter is nearly identical for larger values of x . The graph also shows that, for equal number of “correction” terms, the Nemes formulas are always better than any other (e.g., Nemes 4 is better than Stirling 3, Ramanujan 2 and Laplace 2). It seems for numerical computation the most useful are the Stirling and Nemes formulas. These expansions use only the half of the powers of the variable contrary to the Laplace and Ramanujan series. The behavior of the closed formula is very interesting, it gives approximately the same value as Nemes 4 and a better one than Stirling 3, even though it contains only one “correction” term.

4. Corollaries

4.1. Closed approximation. The following closed approximation is a consequence of (3.1).

Corollary 4.1. *The following holds as $x \rightarrow \infty$*

$$\Gamma(x) \sim \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}} \left(1 + \frac{1}{12x^2 - \frac{1}{10}}\right)^x. \quad (4.1)$$

Proof. If $|t| > 1/10$

$$1 + \frac{1}{t - \frac{1}{10}} = 1 + \frac{1}{t} + \frac{1}{10t^2} + \frac{1}{100t^3} + \cdots.$$

Let $t = 12x^2 > 1$, then

$$1 + \frac{1}{12x^2 - \frac{1}{10}} = 1 + \frac{1}{12x^2} + \frac{1}{1440x^4} + \frac{1}{172800x^6} + \cdots,$$

from which the approximation is reasonable (compare it with (3.1)). $\square$

4.2. Asymptotic expansion of $\sqrt[n]{n!}$. It is well known that

$$\sqrt[n]{n!} \sim \frac{n}{e}. \quad (4.2)$$

By our new formula we can easily deduce a complete asymptotic expansion for $\sqrt[n]{n!}$.

Corollary 4.2. *As $n \rightarrow \infty$*

$$\sqrt[n]{n!} \sim \frac{n}{e} \sum_{k \geq 0} \frac{P_k(\log(2\pi n))}{n^k} \quad (4.3)$$

where P_k is a polynomial in degree k and for every real number x we have

$$P_k(x) = \sum_{j=0}^k \frac{V_{k-j}}{2^j j!} x^j, \quad (4.4)$$

and $V_{2i+1} = 0$, $V_{2i} = G_i$ for $i \geq 0$.

Proof. For positive integer n we have $n\Gamma(n) = n!$, thus by our formula

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \left(\sum_{k \geq 0} \frac{G_k}{n^{2k}}\right)^n$$

or

$$\sqrt[n]{n!} \sim \frac{n}{e} \sqrt[2n]{2\pi n} \sum_{k \geq 0} \frac{G_k}{n^{2k}}.$$

It is well-known that

$$\sqrt[2n]{t} = \exp\left(\frac{1}{2n} \log t\right) = \sum_{j \geq 0} \frac{1}{j!} \left(\frac{\log t}{2n}\right)^j,$$

hence by setting $t = 2\pi n$ and applying the definition of V_i we find

$$\sqrt[n]{n!} \sim \frac{n}{e} \sum_{j \geq 0} \frac{1}{j!} \left(\frac{\log(2\pi n)}{2n}\right)^j \sum_{k \geq 0} \frac{V_k}{n^k}.$$

The Cauchy product of the series gives the desired form. $\square$

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