

Quasi-linear analysis of high frequency waves driven by electron beams

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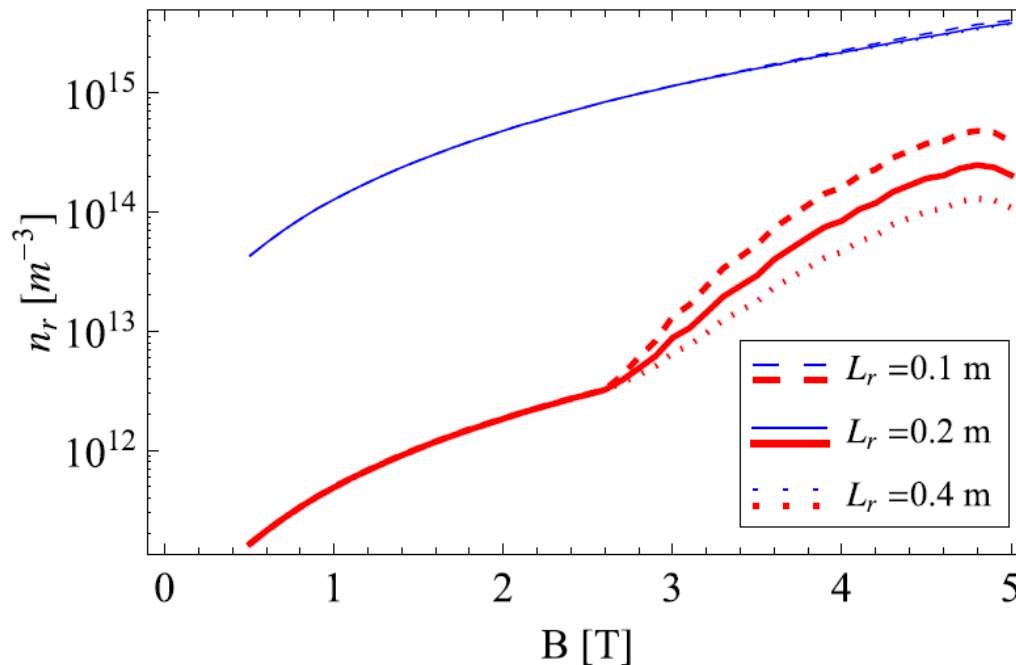
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Motivation

- **Magnetic field threshold** for runaway production at ~ 2.2 T
- Recent **results on EXEL wave** suggest destabilization by a moderate number of runaway electrons in near-critical electric field.

See previous talk by Anna Kómár:



$$E/E_c = 1.3$$

$$n_e = 5 \cdot 10^{19} \text{ m}^{-3}$$

$$T = 20 \text{ eV}$$

$$T = 1000 \text{ eV}$$

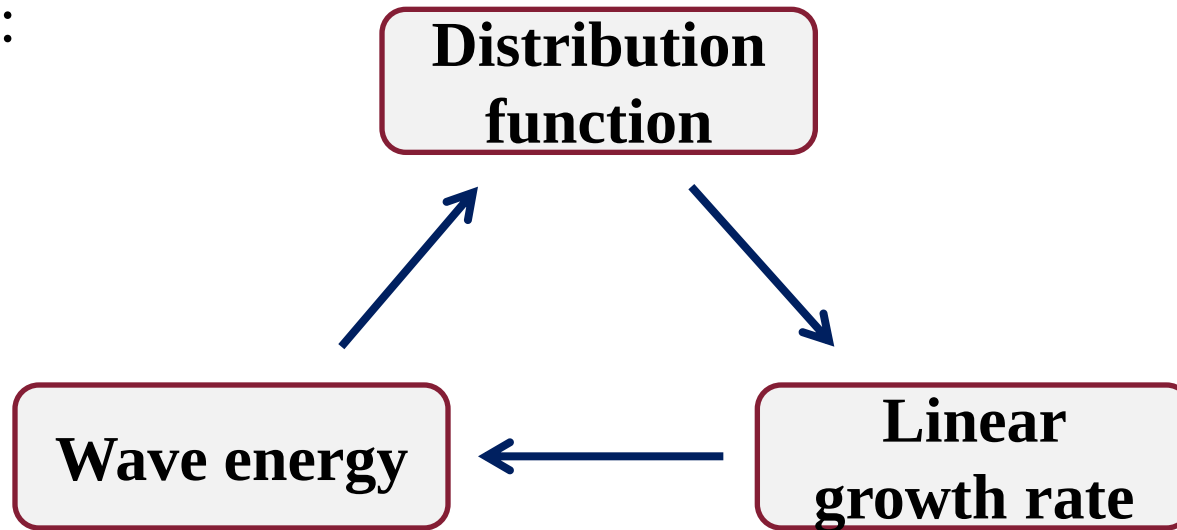
- We speculate similar low thresholds for other runaway distributions.

Outline

1. The quasi-linear treatment (structure and conditions)
2. Example of whistler wave instability
[Fulop POP 2006, Pokol PPCF 2008, Fulop POP 2009]
 1. Model and solution
 2. Limitations and difficulties
 3. Recent developments
3. Quasi-linear analysis of the EXEL wave
 1. Distribution function – linear growth rate
 2. Quasi-linear equations
 3. Numerical solution
 4. Possible outcomes, further studies implied → Work to be done!
4. Summary

The quasi-linear treatment

Structure:



Conditions:

1. **Wave amplitude small enough** for the linear approximation to hold.
2. **Incoherent, wide spectrum waves** to avoid particle trapping.
3. **Small growth and damping rates** compared to the real part of wave frequency.

Example of whistler wave instability

Model runaway distribution

- Pure secondary generation with avalanche mechanism (Rosenbluth and Putvinski, NF 37):

$$\frac{dn_r}{dt} = \frac{n_r(E - 1)}{c_Z \tau}$$

$\tau = 4\pi\epsilon_0^2 m_{e0}^2 c^3 / n_e e^4 \ln \Lambda$ is the collision time
 $E = e|E_{||}| \tau / m_{e0} c$ the normalized parallel electric field
 $c_Z = \sqrt{3(Z + 5) / \pi \ln \Lambda}$

- Kinetic equation for relativistic electrons in homogenous magnetic field with conditions $E \gg 1$ and $p_{\perp} \ll p_{||} \simeq p$

$$\tau \frac{\partial f}{\partial t} + (E - 1) \frac{\partial f}{\partial p_{||}} = \frac{1 + Z}{2} \frac{1}{p_{\perp}} \frac{\partial}{\partial p_{\perp}} p_{\perp} \frac{\partial f}{\partial p_{\perp}}$$

$p = \gamma v / c$ is the normalized momentum
 $\gamma = \sqrt{1 + p^2}$ the Lorentz factor

- Distribution function of runaway electrons:

$$f(p_{||}, p_{\perp}, t) = \frac{C}{p_{||}} \exp \left(\frac{(E - 1)t / \tau - p_{||}}{c_Z} - \frac{\alpha p_{\perp}^2}{2p_{||}} \right) \quad \alpha = (E - 1) / (1 + Z)$$

Example of whistler wave instability

Whistler wave dispersion

- Homogenous plasma wave dispersion in electromagnetic approximation for coordinates $\mathbf{B} = (0, 0, B)$, $\mathbf{k} = (k_{\perp}, 0, k_{\parallel})$ and $E_{\parallel} = 0$

$$\left(\epsilon_{11} - \frac{k_{\parallel}^2 c^2}{\omega^2} \right) \left(\epsilon_{22} - \frac{k_{\perp}^2 c^2}{\omega^2} \right) + \epsilon_{12}^2 = 0$$

- The dielectric tensor is:

$$\epsilon(\mathbf{k}, \omega) = \mathbf{I} + \chi_i(\mathbf{k}, \omega) + \chi_e(\mathbf{k}, \omega) + \chi_r(\mathbf{k}, \omega)$$

$\mathbf{I}$ is the dyadic unit

χ_s is the susceptibility tensor

- For cold plasma, $|k| \gg |k_{\parallel}|$, $\omega_{ci} \ll \omega \ll \omega_{ce}$, $kc \gg \omega_{pi}$ and $k_{\parallel}^2 c^2 / \omega_{pi}^2 \gg 1$ we get the whistler wave dispersion:

$$\omega_0 = k k_{\parallel} v_A^2 / \omega_{ci}$$

Example of whistler wave instability

Linear growth rate

- Instability growth rate for small runaway perturbation

$$\omega = \omega_0 + \delta\omega \quad \gamma_i = \text{Im}\delta\omega \quad \frac{\gamma_i}{\omega_0} = -\frac{k^2 v_A^2}{2\omega_{pi}^2} \text{Im}\chi_{11}^r$$

- The ultra-relativistic $\gamma \simeq p_{\parallel}$ resonance condition:

$$p_{\parallel} = p_{res} \equiv -n\omega_{ce}/(k_{\parallel}c - \omega_0)$$

- Growth rate with the runaway susceptibility expanded:

$$\frac{\gamma_i}{\omega_0} = \frac{\pi^2 \omega_{pr}^2}{\omega_{pi}^2} \frac{k^2 v_A^2}{\omega_0^2} \frac{\omega_{ce}^2}{k_{\perp}^2 c^2} \frac{1}{1-y} \int_0^{\infty} dp_{\perp} \sum_{n=-\infty}^{\infty} n^2 J_n^2 \left(\frac{n\omega_{ce}}{k_{\parallel}c p_{\parallel}} \frac{\partial f_r}{\partial p_{\perp}} + \frac{p_{\perp}}{p_{\parallel}} \frac{\partial f_r}{\partial p_{\parallel}} \right) \Bigg|_{p_{\parallel}=p_{res}}$$

$$y = \omega_0/k_{\parallel}c$$

Example of whistler wave instability

Linear growth rate

- Substituting $n = -1$ to get the dominant anomalous Doppler resonance, and doing the algebra, we arrive to the analytical expression:

$$\gamma_i(\omega_0, k, k_{\parallel}) = \frac{\pi}{4c_Z} \frac{\omega_{pr}^2}{\omega_{pi}^2} \frac{k^2 v_A^2}{\omega_0} \exp \left[\frac{-\omega_{ce}}{(k_{\parallel} c - \omega_0) c_Z} \right]$$

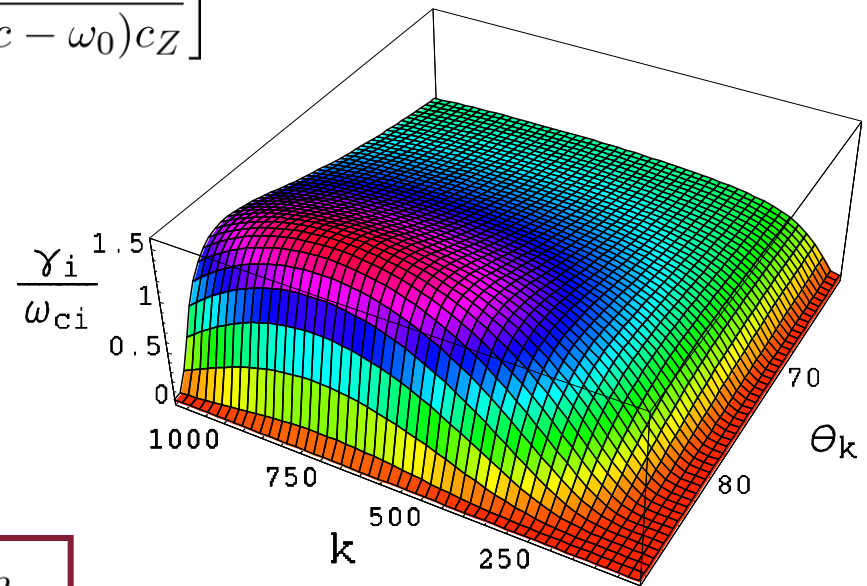
- This has a **maximum inside the range of validity:**

$$k v_A / \omega_{pi} = 1/2$$

$$k_{\parallel} c = 2\omega_{ce} / c_Z$$

$$\omega_0 = \omega_{ce} / c_Z$$

$$\gamma_i^{\max} = \frac{\pi}{16} \frac{\omega_{pr}^2}{\omega_{ce}} \exp(-1) = 1.3 \cdot 10^{-9} \frac{n_r}{B_T}$$



Example of whistler wave instability

Wave stability

- Damping dominated by **electron-ion collisions** [Brambilla POP 2]:

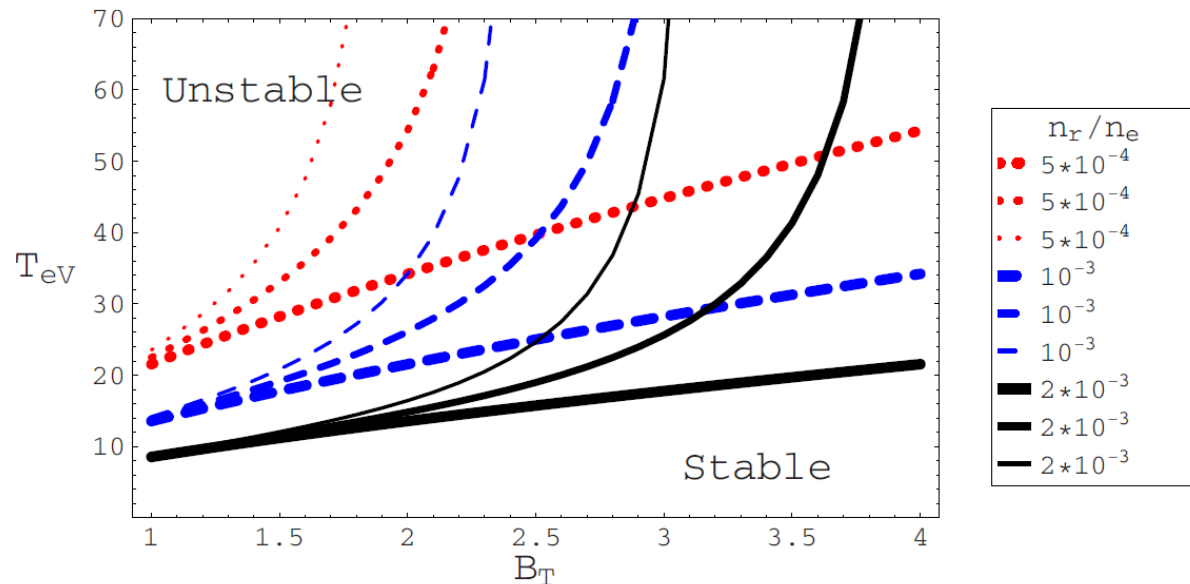
$$\gamma_d \approx 1.5 \tau_{ei}^{-1}$$

- Additional convective damping:

$$\gamma_v \equiv (\partial \omega / \partial k_{\perp}) / (4 L_r) = v_A^2 k_{\parallel} / 4 \omega_{ci} L_r$$

- $\gamma_i - \gamma_d - \gamma_v > 0$ gives the **stability threshold**:

$$\frac{n_r}{n_e} > \frac{Z^2 B_T}{20 T_{eV}^{3/2}} + \frac{B_T^3}{90 c Z n_{19}^2 L_r}$$



Example of whistler wave instability

Quasi-linear equations

1. Time evolution of runaway distribution function

[Stix: Waves in plasmas]:

$$\frac{\partial f}{\partial t} = \frac{\pi e^2}{m_{e0}^2 c^2} \int d^3 k \hat{\Pi} \frac{|E_k|^2}{4} \frac{(k(1-A) + k_{\parallel})^2}{(k^2 + k_{\parallel}^2)} \frac{p_{\perp}^2}{\omega^2} \delta(\omega + \Omega - k_{\parallel} p_{\parallel} c / \gamma) \hat{\Pi} f.$$

where $A = \frac{k_{\parallel} p_{\parallel} c}{\omega_{ce}} \frac{k_{\perp}^2 c^2}{\omega_{pe}^2}$ and $\hat{\Pi} = \frac{\omega - k_{\parallel} p_{\parallel} c / \gamma}{p_{\perp}} \frac{\partial}{\partial p_{\perp}} + \frac{k_{\parallel} c}{\gamma} \frac{\partial}{\partial p_{\parallel}}$

2. Time evolution of the wave energy:

$$\frac{dW_k}{dt} = 2\gamma_k(t)W_k \qquad W_k(t) = \frac{\epsilon_0}{2}|E_k(t)|^2$$

3. Linear growth rate from perturbation analysis on the modified distribution.

Example of whistler wave instability

Semi-analytic solution

- **Diffusion** for $\gamma \simeq p_{\parallel}$, $k_{\parallel} \ll k_{\perp}$ and $\alpha \gg k_{\parallel}c/\omega_{ce}$ goes purely **in the perpendicular direction**:

$$\frac{\partial f}{\partial \tau} = \frac{1}{p_{\perp}} \frac{\partial}{\partial p_{\perp}} p_{\perp} \frac{\partial f}{\partial p_{\perp}} \quad \tau(p_{\parallel}, t) = \int_0^t dt' \tilde{D}(p_{\parallel}, t')$$

- With the diffusion constant:

$$\tilde{D}(p_{\parallel}, t) = \frac{\pi^2 e^2 \omega_{ce}^2 \omega_{ci}^2}{\epsilon_0 m_{e0}^2 c^2 v_A^4} \frac{1}{p_{\parallel}} \int_{k_{\parallel min}}^{k_{\parallel max}} dk_{\parallel} \frac{W_k(k_{\parallel}, t)}{k_{\parallel}^2 (k_{\parallel} p_{\parallel} c - \omega_{ce})} \frac{(k_{\perp r} (1 - A) + k_{\parallel})^2}{k_{\perp r}^2},$$

where integration is along the line of resonance:

$$k_{\perp r} = \frac{\omega_{ci}}{v_A^2} \left(c - \frac{\omega_{ce}}{p_{\parallel} k_{\parallel}} \right)$$

Example of whistler wave instability

Semi-analytic solution

- **Analytical solution** with the unperturbed distribution as initial condition:

$$f(p_{\perp}, p_{\parallel}, t) = \frac{n_{r0}\alpha}{2\pi c_Z \phi(p_{\parallel}, t)} \exp \left(\frac{(E - 1)t/\tau_c - p_{\parallel}}{c_Z} - \frac{\alpha p_{\perp}^2}{2\phi(p_{\parallel}, t)} \right)$$

$$\phi(p_{\parallel}, t) = 2\alpha\tau(p_{\parallel}, t) + p_{\parallel}$$

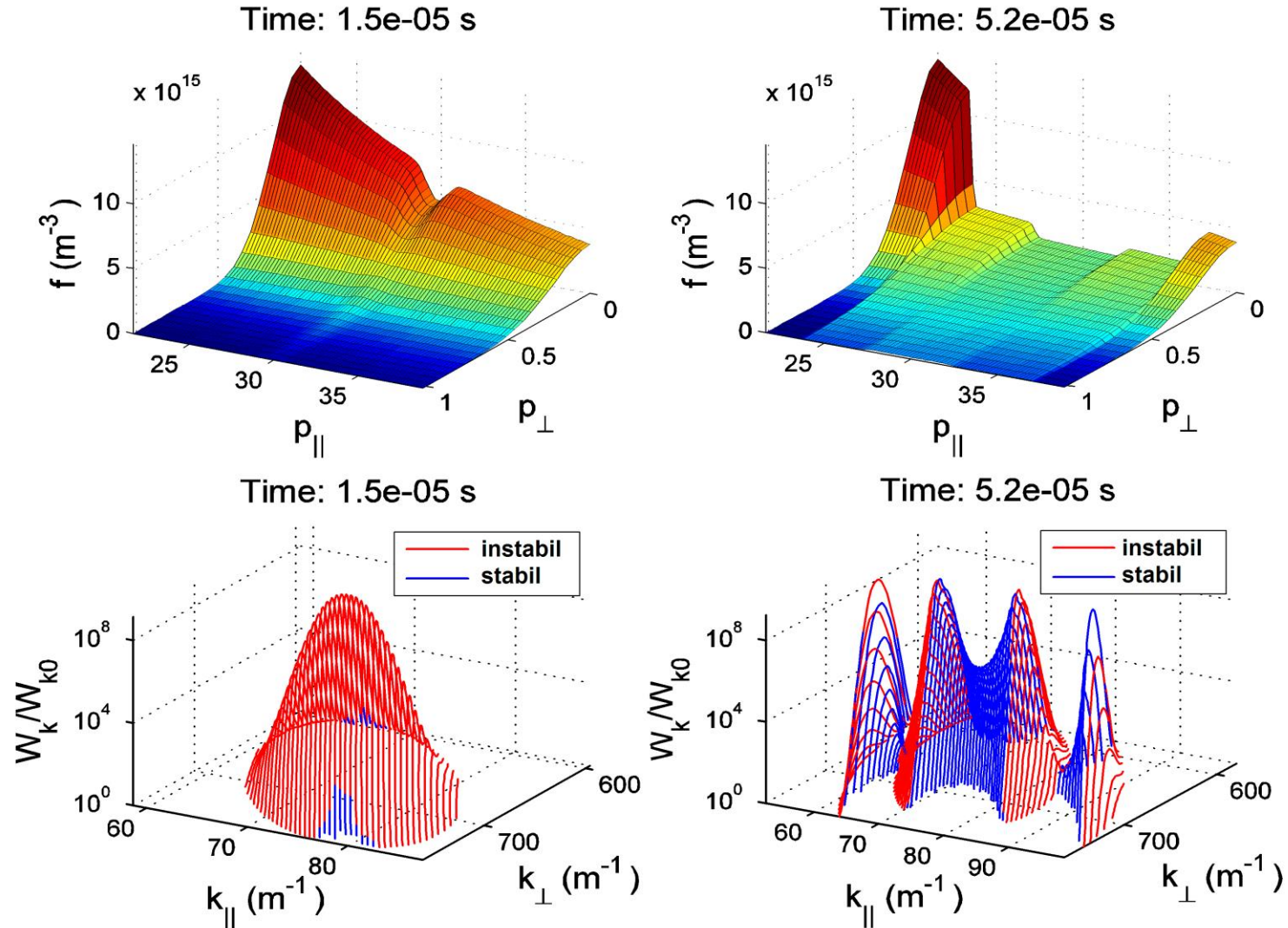
- The linear **growth rate** decreases with decreasing velocity space anisotropy:

$$\gamma_k(k, k_{\parallel}, t) = \frac{\pi}{4c_Z} \frac{\omega_{pr}^2}{\omega_{pi}^2} \frac{k^2 v_A^2}{\omega_0} \frac{I_1(\zeta)}{\zeta} e \left[-\zeta - \frac{\omega_{ce}}{(k_{\parallel} c - \omega_0) c_Z} \right] - \gamma_d - \gamma_v$$

$$\zeta = k_{\perp}^2 c^2 \phi(\omega_{ce}/(k_{\parallel} c - \omega_0), t) / \omega_{ce}^2 \alpha$$

Example of whistler wave instability

Semi-analytic solution



Example of whistler wave instability

Limitations - difficulties

- **Analytical distribution** with constant electric field (needed for semi-analytical solution)
- **Ultra-relativistic condition** (limiting isotropization that can be modelled)
- **Assumptions narrowing down the region of validity in wave parameters** (only close to most unstable waves can be taken)
- Question is if **pitch angle scattering causes losses?** (not modelled)
- Effect of continued **runaway production modelled by increasing amplitude** of the distribution (not correct, gives false dynamics)
- **Marginal stability** even for strong runaway beams

Example of whistler wave instability

Recent developments

- **Analytical distribution** with constant electric field (needed for semi-analytical solution) → **CODE/LUKE**
- **Ultra-relativistic condition** (limiting isotropization that can be modelled) → **general resonance condition**
- **Assumptions narrowing down the region of validity in wave parameters** (only close-to most unstable waves can be taken) → **more general wave dispersion, EXEL wave**
- Question is if **pitch angle scattering causes losses?** (not modelled) → **coupling to CODE/LUKE**
- Effect of continued **runaway production modelled by increasing amplitude** of the distribution (not correct, gives false dynamics) → **coupling to CODE/LUKE**
- **Marginal stability** even for strong runaway beams → **EXEL**

Quasi-linear analysis of the EXEL wave

Distribution function – linear growth rate

- **Distribution** to be taken from CODE or LUKE
- Calculating for EXEL wave → closed form solution of

$$\begin{aligned} \omega^8 - \omega^6 (2k^2 c^2 + \omega_{ce}^2 + 3\omega_{pe}^2) + \omega^4 [k^4 c^4 + 2k^2 c^2 (\omega_{ce}^2 + 2\omega_{pe}^2) + \omega_{pe}^2 (\omega_{ce}^2 + 3\omega_{pe}^2)] \\ - \omega^2 [k^4 c^4 (\omega_{ce}^2 + \omega_{pe}^2) + k^2 c^2 \omega_{pe}^2 (3/2\omega_{ce}^2 + 2\omega_{pe}^2 + 1/2\omega_{ce}^2 \cos 2\theta) + \omega_{pe}^6] = \\ -1/2 k^4 c^4 \omega_{ce}^2 \omega_{pe}^2 (1 + \cos 2\theta) \end{aligned}$$

- Growth rate has to be calculated numerically [Anna's talk]

$$\begin{aligned} \frac{\gamma_l}{\omega^6 (\omega^2 - \omega_{ce}^2)} = \Im \{ -\chi_{11} (\epsilon_{11} - k^2 c^2 / \omega^2) (\epsilon_{33} - k_{\perp}^2 c^2 / \omega^2) - 2\chi_{12} \epsilon_{12} (\epsilon_{33} - k_{\perp}^2 c^2 / \omega^2) - \\ \chi_{22} [(\epsilon_{11} - k_{\parallel}^2 c^2 / \omega^2) (\epsilon_{33} - k_{\perp}^2 c^2 / \omega^2) - k_{\parallel}^2 k_{\perp}^2 c^4 / \omega^4] - \\ \chi_{33} [\epsilon_{11}^2 - \epsilon_{11} (k^2 c^2 / \omega^2 + k_{\parallel}^2 c^2 / \omega^2) + k^2 k_{\parallel}^2 c^4 / \omega^4 + \epsilon_{12}^2] \} / P(\omega, k, \theta). \end{aligned}$$

$$\begin{aligned} P(\omega, k, \theta) = 8\omega^7 - 6\omega^5 (2k^2 c^2 + \omega_{ce}^2 + 3\omega_{pe}^2) + 4\omega^3 [k^4 c^4 + 2k^2 c^2 (\omega_{ce}^2 + 2\omega_{pe}^2) + \omega_{pe}^2 (\omega_{ce}^2 + 3\omega_{pe}^2)] \\ - 2\omega [k^4 c^4 (\omega_{ce}^2 + \omega_{pe}^2) + k^2 c^2 \omega_{pe}^2 (3/2\omega_{ce}^2 + 2\omega_{pe}^2 + 1/2\omega_{ce}^2 \cos 2\theta) + \omega_{pe}^6] \end{aligned}$$

- **Numerical growth rate** calculation to be repeated in each time step → must be effective → start with ultra-relativistic resonance

Quasi-linear analysis of the EXEL wave

Quasi-linear equations

- At this point we keep the **ultra-relativistic resonance** condition $\rightarrow$ **diffusion in perpendicular direction** with

$$\tilde{D}(p_{\perp}, p_{\parallel}, t) = \frac{\pi^2 e^2 \omega_{ce}^2}{\epsilon_0 m_{e0}^2 c^2} \int d\theta \left[\frac{W_k(t)}{p_{\parallel}^2 \omega_0(k, \theta)^2} P(\omega_0, k, \theta, p_{\parallel}) \frac{k^2 \sin \theta}{\frac{d\omega_0}{dk} - c \cos \theta} \right]_{k=k_{res}}$$

where

$$P(\omega_0, k, \theta, p_{\parallel}) = \frac{\left| 1 - \frac{\omega_{pe}^2 \omega_{ce} / \omega_0}{\omega_0^2 - k^2 c^2 - \omega_{ce}^2 - \omega_{pe}^2 + k^2 c^2 \omega_{ce}^2 / \omega_0^2} - p_{\parallel} \frac{k_{\parallel} k_{\perp}^2 c^3}{\omega_{ce} (\omega_{pe}^2 + k_{\perp}^2 c^2 - \omega_0^2)} \right|^2}{|e_x|^2 + |e_y|^2 + |e_z|^2}$$

with polarization

$$(e_x, e_y, e_z) = \left(1, -i \frac{\omega_{pe}^2 \omega_{ce} / \omega_0}{\omega_0^2 - k^2 c^2 - \omega_{ce}^2 - \omega_{pe}^2 + k^2 c^2 \omega_{ce}^2 / \omega_0^2}, \frac{k_{\parallel} k_{\perp} c^2}{\omega_{pe}^2 + k_{\perp}^2 c^2 - \omega_0^2} \right)$$

and $k=k_{res}$ calculated numerically.

Quasi-linear analysis of the EXEL wave

Numerical solution

- For *ultra-relativistic resonance* condition and *no coupling to electric field or collision terms*

$$\frac{\partial f}{\partial \tau} = \frac{1}{p_{\perp}} \frac{\partial}{\partial p_{\perp}} p_{\perp} \frac{\partial f}{\partial p_{\perp}}$$

can be calculated for each resonant momentum, and interpolated for each time step by calculating

$$\tau(p_{\parallel}, t) = \int_0^t dt' \tilde{D}(p_{\parallel}, t')$$

- For generalized resonance condition and/or for coupled simulation to CODE diffusion has to be solved in each time step.

Quasi-linear analysis of the EXEL wave

Possible outcomes, further studies implied

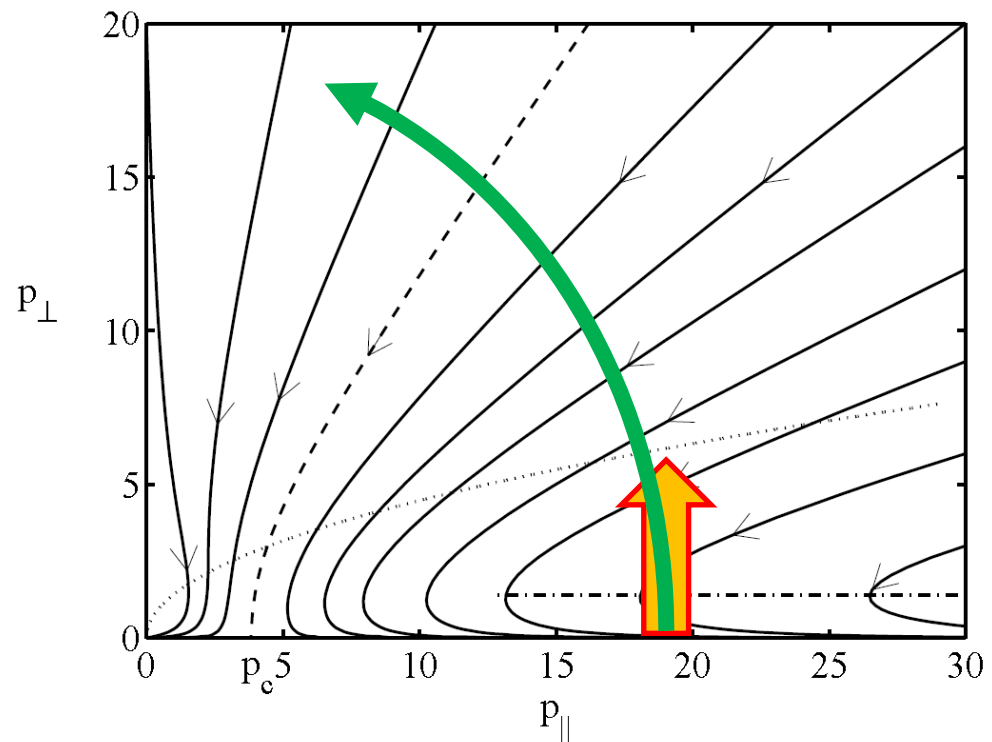
Result of first runs: **diffusion in perpendicular direction for ultra-relativistic resonance**

→ will need to go for **general resonance condition**

→ integration changes in the **runaway susceptibility terms** of the linear growth rate, and **diffusion equation** will have to be generalized:

perpendicular direction

→ curve on the $\mathbf{p}$ plane



Quasi-linear analysis of the EXEL wave

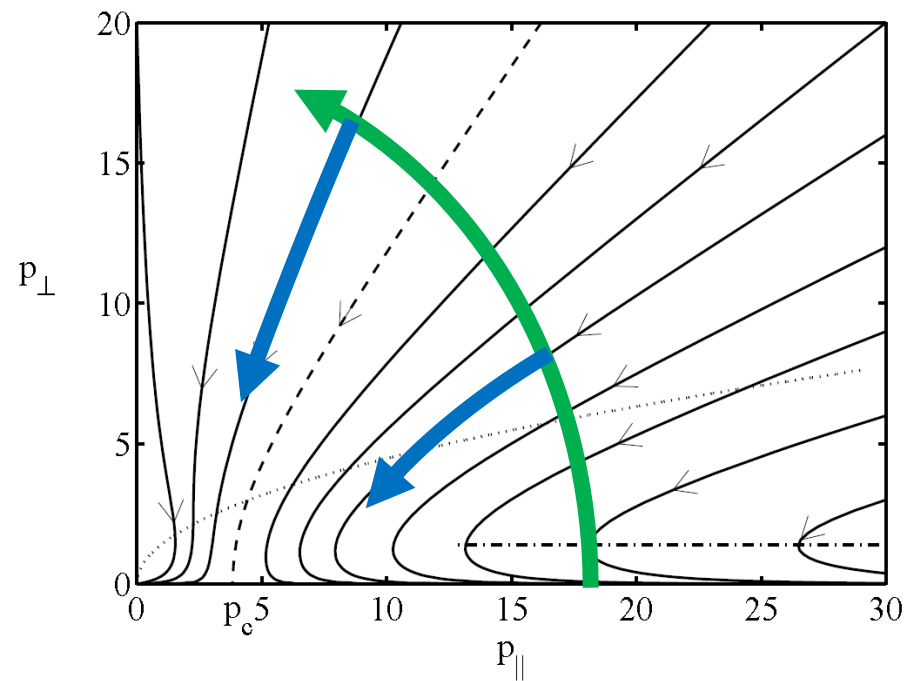
Possible outcomes, further studies implied

- Result: **significant isotropization**

→ will need to look at **loss mechanisms** and effect

on runaway generation

→ couple to CODE/LUKE



- Result: **losses at destabilized wave location** (homogenous approx.) → will need to see global effects with electric field
→ couple to GO

Summary

- *Calculated quasi-linear diffusion for whistler wave back in 2008-2009, but stopped development because of problems with the instability and lack of tools to proceed.*
- *Now we have tools that can help, and found EXEL instability that might be important.*
- *More-or-less finished linear growth rate calculations.*
- *Have equations for quasi-linear diffusion for the ultra-relativistic limit.*
- *Equations are to be implemented.*
- *Depending on the results further steps are foreseen.*