

The method of double chains for largest families with excluded subposets

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Abstract

For a given finite poset P , $La(n, P)$ denotes the largest size of a family $\mathcal{F}$ of subsets of $[n]$ not containing P as a weak subposet. We exactly determine $La(n, P)$ for infinitely many P posets. These posets are built from seven base posets using two operations. For arbitrary posets, an upper bound is given for $La(n, P)$ depending on $|P|$ and the size of the longest chain in P . To prove these theorems we introduce a new method, counting the intersections of $\mathcal{F}$ with double chains, rather than chains.

1 Introduction

Let $[n] = \{1, 2, \dots, n\}$ be a finite set. We investigate families $\mathcal{F}$ of subsets of $[n]$ avoiding certain configurations of inclusion.

Definition Let P be a finite poset, and $\mathcal{F}$ be a family of subsets of $[n]$. We say that P is contained in $\mathcal{F}$ if there is an injective mapping $f : P \rightarrow \mathcal{F}$ satisfying $a <_P b \Rightarrow f(a) \subset f(b)$ for all $a, b \in P$. $\mathcal{F}$ is called P -free if P is not contained in it.

Let $La(n, P) = \{\max |\mathcal{F}| \mid \mathcal{F} \text{ contains no } P\}$

Note that we do not want to find P as an induced subposet, so the subsets of $\mathcal{F}$ can satisfy more inclusions than the elements of the poset P .

We are interested in determining $La(n, P)$ for as many posets as possible. The first theorem of this kind was proved by Sperner. Later it was generalized by Erdős.

Theorem 1.1 (Sperner) [1] *Let $\mathcal{F}$ be a family of subsets of $[n]$, with no member of $\mathcal{F}$ being the subset of another one. Then*

$$|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor} \quad (1)$$

Theorem 1.2 (Erdős) [2] *Let $\mathcal{F}$ be a family of subsets of $[n]$, with no $k+1$ members of $\mathcal{F}$ satisfying $A_1 \subset A_2 \subset \dots \subset A_{k+1}$ ($k \leq n$). Then $|\mathcal{F}|$ is at most the sum of the k biggest binomial coefficients belonging to n . The bound is sharp, since it can be achieved by choosing all subsets F with $\lfloor \frac{n-k+1}{2} \rfloor \leq |F| \leq \lfloor \frac{n+k-1}{2} \rfloor$.*

Since choosing all the subsets with certain sizes near $n/2$ is the maximal family for many excluded posets, we use the following notation.

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Notation $\Sigma(n, m) = \sum_{i=\lfloor \frac{n-m+1}{2} \rfloor}^{\lfloor \frac{n+m-1}{2} \rfloor} \binom{n}{i}$ denotes the sum of the m largest binomial coefficients belonging to n .

Now we can reformulate Theorem 1.2. Let P_{k+1} be the path poset with $k+1$ elements. Then

$$La(n, P_{k+1}) = \Sigma(n, k) \quad (2)$$

We give here a proof of Theorem 1.2 to illustrate the chain method introduced by Lubell [3].

Proof (Theorem 1.2) A chain is $n+1$ subsets of $[n]$ satisfying $L_0 \subset L_1 \subset L_2 \subset \dots \subset L_n$ and $|L_i| = i$ for all $i = 0, 1, 2, \dots, n$. The number of chains is $n!$. We use double counting for the pairs (C, F) where C is a chain, $F \in C$ and $F \in \mathcal{F}$.

The number of chains going through some subset $F \in \mathcal{F}$ is $|F|!(n - |F|)!$. So the number of pairs is

$$\sum_{F \in \mathcal{F}} |F|!(n - |F|)!$$

One chain can contain at most k elements of $\mathcal{F}$, otherwise a P_{k+1} poset would be formed. So the number of pairs is at most $k \cdot n!$. It implies

$$\sum_{F \in \mathcal{F}} |F|!(n - |F|)! \leq k \cdot n! \quad (3)$$

$$\sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \leq k \quad (4)$$

Fixing $|\mathcal{F}|$, the left side takes its minimum when we choose the subsets with sizes as near to $n/2$ as possible. Choosing all $\Sigma(n, k)$ subsets with sizes $\lfloor \frac{n-k+1}{2} \rfloor \leq |F| \leq \lfloor \frac{n+k-1}{2} \rfloor$, we have equality. So we have

$$La(n, P_{k+1}) = \Sigma(n, k) \quad \blacksquare \quad (5)$$

$La(n, P)$ is determined asymptotically for many posets, but its exact value is known for very few P . (See [4] and [5])

2 The method of double chains

The main purpose of the present paper is to exactly determine $La(n, P)$ for some posets P . Our main tool is a modification of the the chain method, double chains are used rather than chains.

Definition Let $C : L_0 \subset L_1 \subset L_2 \subset \dots \subset L_n$ be a chain. The double chain assigned to C is a set $D = \{L_0, L_1, \dots, L_n, M_1, M_2, \dots, M_{n-1}\}$, where $M_i = L_{i-1} \cup (L_{i+1} \setminus L_i)$.

Note that $|M_i| = |L_i| = i$,
 $i < j \Rightarrow L_i \subset L_j$, $L_i \subset M_j$, $M_i \subset L_j$ and $i + 1 < j \Rightarrow M_i \subset M_j$.

$\{L_0, L_1, \dots, L_n\}$ is called the primary line of D and $\{M_1, M_2, \dots, M_{n-1}\}$ is the secondary line.

$\mathcal{D}$ denotes the set of all $n!$ double chains.

Lemma 2.1 Let $\mathcal{F}$ be a family of subsets of $[n]$ ($n \geq 2$), and let m be a positive real number. Assume that

$$\sum_{D \in \mathcal{D}} |\mathcal{F} \cap D| \leq 2m \cdot n! \quad (6)$$

Then

$$|\mathcal{F}| \leq m \binom{n}{\lfloor n/2 \rfloor} \quad (7)$$

If m is an integer and $m \leq n - 1$, we have the following better bound:

$$|\mathcal{F}| \leq \Sigma(n, m) \quad (8)$$

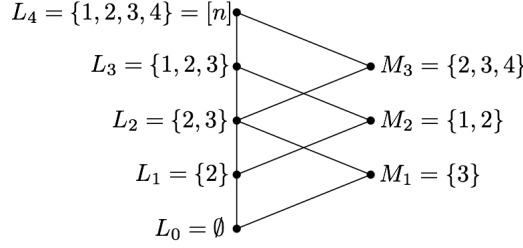


Figure 1: The double chain assigned to the chain $\emptyset \subset \{2\} \subset \{2, 3\} \subset \{1, 2, 3\} \subset \{1, 2, 3, 4\}$.

Proof First we count how many double chains contains a given subset $F \subset [n]$. $\emptyset$ and $[n]$ are contained in all $n!$ double chains. Now let $F \notin \{\emptyset, [n]\}$. F is contained in the primary line of $|F|!(n - |F|)!$ double chains. Now count the double chains containing F in the secondary line. Letting $F = M_{|F|}$, we have $|F| \cdot (n - |F|)$ possibilities to choose $L_{|F|}$, since we have to replace one element of $M_{|F|}$ with a new one. $M_{|F|}$ and $L_{|F|}$ already define $L_{|F|-1}$ and $L_{|F|+1}$. We have $(|F| - 1)!$ and $(n - |F| - 1)!$ possibilities for the first and last part of the primary line, so the number of double chains containing F in the secondary line is $|F|(n - |F|)(|F| - 1)!(n - |F| - 1)! = |F|!(n - |F|)!$. It gives a total of $2|F|!(n - |F|)!$ double chains containing F .

Let $t = |\mathcal{F} \cap \{\emptyset, [n]\}|$. Double counting the pairs (D, F) where $D \in \mathcal{D}$, $F \in D$ and $F \in \mathcal{F}$ we have

$$t \cdot n! + \sum_{F \in \mathcal{F} \setminus \{\emptyset, [n]\}} 2|F|!(n - |F|)! \leq 2m \cdot n! \quad (9)$$

$$t \cdot \frac{1}{2} + \sum_{F \in \mathcal{F} \setminus \{\emptyset, [n]\}} \frac{1}{\binom{n}{|F|}} \leq m \quad (10)$$

Since $\binom{n}{\lfloor n/2 \rfloor}$ is the biggest binomial coefficient, and $\binom{n}{\lfloor n/2 \rfloor} \geq 2$ we have

$$\frac{|\mathcal{F}|}{\binom{n}{\lfloor n/2 \rfloor}} \leq m \quad (11)$$

It proves (7). If m is an integer, and $m \leq n - 1$, considering $|\mathcal{F}|$ fixed, the left side of (10) is minimal when we choose subsets with sizes as near to $n/2$ as possible. Choosing all $\Sigma(n, m)$ subsets with such sizes, we have equality in (10). It implies $|\mathcal{F}| \leq \Sigma(n, m)$, so (8) is proved. ■

Definition The infinite double chain is an infinite poset with elements L_i , $i \in \mathbb{Z}$ and M_i , $i \in \mathbb{Z}$. The defining relations between the elements are

$$i < j \Rightarrow L_i \subset L_j, L_i \subset M_j, M_i \subset L_j$$

Note that the poset formed by the elements of any double chain with the inclusion as relation is a subposet of the infinite double chain.

Lemma 2.2 Let m be an integer or half of an integer and P be a finite poset. Assume that any subset of size $2m + 1$ of the infinite double chain contains P as a (not necessarily induced) subposet. Let $\mathcal{F}$ be a family of subsets of $[n]$ such that $\mathcal{F}$ does not contain P . Then

$$|\mathcal{F}| \leq m \binom{n}{\lfloor n/2 \rfloor} \quad (12)$$

If m is an integer and $m \leq n - 1$ we have the following better bound:

$$|\mathcal{F}| \leq \Sigma(n, m) \quad (13)$$

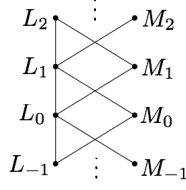


Figure 2: The infinite double chain.

Proof Since the poset formed by the elements of any double chains is a subposet of the infinite double chain, $|\mathcal{F} \cap D| \leq 2m$ for all double chains D . There are $n!$ double chains, so

$$\sum_{D \in \mathcal{D}} |\mathcal{F} \cap D| \leq 2m \cdot n! \quad (14)$$

holds. Now we can use Lemma 2.1 and finish the proof. $\blacksquare$

3 An upper estimate for arbitrary posets

Definition The size of the longest chain in a finite poset P is the largest integer $L(P)$ such that for some $a_1, a_2, \dots, a_{L(P)} \in P$, $a_1 <_P a_2 <_P \dots <_P a_{L(P)}$ holds.

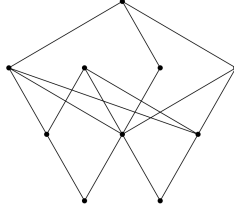


Figure 3: A poset with $|P| = 10$ elements and longest chain of length $L(P) = 4$.

Theorem 3.1 Let P be a finite poset and let $\mathcal{F}$ be a P -free family of subsets of $[n]$. Then

$$|\mathcal{F}| \leq \left(\frac{|P| + L(P)}{2} - 1 \right) \binom{n}{\lfloor n/2 \rfloor} \quad (15)$$

If $\frac{|P| + L(P)}{2} - 1$ is an integer and $\frac{|P| + L(P)}{2} \leq n$ we have the following better bound:

$$|\mathcal{F}| \leq \Sigma \left(n, \frac{|P| + L(P)}{2} - 1 \right) \quad (16)$$

Proof We want to use Lemma 2.2 with $m = \frac{|P| + L(P)}{2} - 1$. So the only thing we have to prove is the following lemma.

Lemma 3.2 Let P be a finite poset. Then any subset S of size $|P| + L(P) - 1$ of the infinite double chain contains P as a (not necessarily induced) subposet.

Proof We prove the lemma using induction on $L(P)$. When $L(P) = 1$, we have a subset of size $|P|$ in the infinite double chain. We can choose them all, we get the poset P , since there are no relations between its elements. Assume that we already proved the lemma for all posets with longest chain of size $l - 1$, and prove it for a poset P with $L(P) = l$.

Arrange the elements of the infinite double chain as follows:

$$\dots L_{-1}, M_{-1}, L_0, M_0, L_1, M_1, L_2, M_2 \dots$$

Assume that P has k minimal elements, and choose the k first elements of S for them according to the above arrangement. Note that all remaining elements of S , except for at most one, are greater in the infinite double chain than all the k elements we just chose. If there is such an exception, delete that element from S . Now we have at least $|P| + L(P) - k - 2$ elements of S left, all greater than the k we chose for the minimal elements. Denote the set of these elements by S' .

Let P' be the poset obtained by P after deleting its minimal elements. It has $|P'| = |P| - k$ elements and a longest chain of size $L(P') = L(P) - 1$. By the inductive hypothesis P' is formed by some elements of S' , since $|S'| \geq |P| + L(P) - k - 2 = |P'| + L(P') - 1$. Considering these elements together with the first k , they form P as a weak subposet in S . ■

Remark The previously known upper bound for maximal families not containing a general P as weak subposet was $\Sigma(n, |P| - 1)$. We can get it from Theorem 1.2 since P is a subposet of the path poset $P_{|P|}$. The new upper bound, $\Sigma\left(n, \frac{|P| + L(P)}{2} - 1\right)$ is better since $L(P) \leq |P|$, and equality occurs only when P is a path poset.

4 Exact results

In this section we will describe infinitely many posets for which Theorem 3.1 provides a sharp bound.

Definition For a finite poset P , $e(P)$ is the maximal m such that the family formed by all subsets of $[n]$ of size $k, k + 1, \dots, k + m - 1$ is P -free for all n and k .

We will prove that $La(n, P) = \Sigma(n, e(P))$ if n is large enough for infinitely many P , verifying the following conjecture for these posets.

Conjecture [6] For all finite poset P

$$La(n, P) = e(P) \binom{n}{\lfloor n/2 \rfloor} (1 + O(1/n)) \quad (17)$$

In [6] Bukh proved the conjecture for all posets whose Hasse-diagram is a tree.

Notation

$$b(P) = \frac{|P| + L(P)}{2} - 1, \text{ The bound used in Theorem 3.1} \quad (18)$$

Lemma 4.1 Assume that $e(P) = b(P)$ for a finite poset P and $n \geq b(P) + 1$. Then

$$La(n, P) = \Sigma(n, e(P)) = \Sigma(n, b(P)) \quad (19)$$

Proof The family of subsets of size $\lfloor \frac{n - e(P) + 1}{2} \rfloor \leq |F| \leq \lfloor \frac{n + e(P) - 1}{2} \rfloor$ has $\Sigma(n, e(P))$ elements and is P -free by the definition of $e(P)$. On the other hand, Theorem 3.1 states that a P -free family has at most $\Sigma(n, b(P))$ elements. ■

Now we show some posets satisfying $e(P) = b(P)$.

Definition (See figure 4).

E is the poset with one element.

The elements of the following posets are divided into levels so that a is greater than b in the poset if and only if a is in a higher level than b .

B is the butterfly poset, a poset with 2 elements on each level.

D_3 is the 3-diamond poset, a poset with respectively 1, 3 and 1 element on its levels.

Q is a poset with respectively 2, 3 and 2 elements on its levels.

R is a poset with respectively 1, 4, 4 and 1 element on its levels.

S is a poset with respectively 1, 4 and 2 elements on its levels.

S' is a poset with respectively 2, 4 and 1 element on its levels.

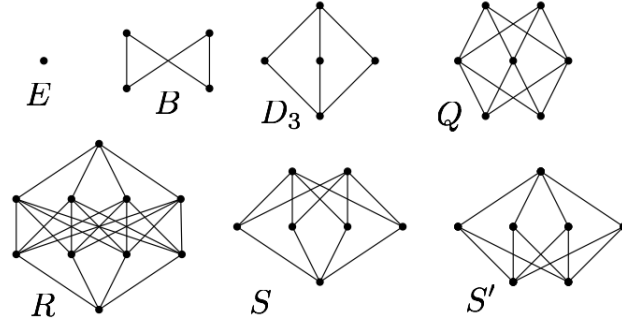


Figure 4: 7 small posets satisfying $e(P) = b(P)$.

Lemma 4.2 For all $P \in \{E, B, D_3, Q, R, S, S'\}$, $e(P) = b(P)$ holds.

Proof $b(P)$ is an integer for all the above posets. Assume that $e(P) \geq b(P) + 1$. Then for $n \geq b(P) + 1$ there would be a P -free family $\mathcal{F}$ of subsets of $[n]$ with $|\mathcal{F}| = \Sigma(n, b(P) + 1) > \Sigma(n, b(P))$, contradicting Theorem 3.1. So $e(P) \leq b(P)$. We will show that for every poset $P \in \{E, B, D_3, Q, R, S, S'\}$ and integers n, k the family formed by all subsets of $[n]$ of size $k, k + 1, \dots, k + b(P) - 1$ is P -free. It gives us $e(P) \geq b(P)$, and completes the proof.

The statement is trivial for $P = E$ since $b(E) = 0$.

$b(B) = 2$. The set of all subsets with k and $k + 1$ elements is B -free since two subsets of size $k + 1$ can not have two different common subsets of size k .

$b(D_3) = 3$. The set of all subsets with $k, k + 1$ and $k + 2$ elements is D_3 -free since for two subsets A, B , $|B| - |A| \leq 2$ there are at most two subsets F satisfying $A \subset F \subset B$.

$b(Q) = 4$. Assume that Q is formed by 7 subsets of size $k, k + 1, k + 2$ or $k + 3$. There are at least 4 subsets in the lower 2 or the upper 2 levels. They should form a B poset, that is not possible.

$b(R) = 6$. Assume that R is formed by 10 subsets of size $k, k + 1, \dots, k + 5$. Let A be the least, and B be the greatest subset. Let U be the union of the 5 smaller subsets. At least 3 subsets in the second level are different from U , and contained in it. Similarly, at least 3 subsets of the third level are different from U , and contain it. Since D_3 is not formed by subsets of size $m, m + 1$ and $m + 2$, $|A| + 6 \leq |U| + 3 \leq |B|$, a contradiction.

$b(S) = 4$. Assume that S is formed by 7 subsets of size $k, k + 1, k + 2$ or $k + 3$. Let V be the intersection of the two elements of the top level, then $|V| \leq k + 2$. V contains all elements of the middle level, and is different from at least 3 of them. These 3 elements together with the least element and V form a D_3 from subsets of size $k, k + 1$ and $k + 2$, and it is a contradiction.

A family is S' -free if and only if the family of the complements of its elements is S -free. It gives $e(S') = e(S) \geq b(S) = b(S')$. ■

We define two ways of building posets from smaller ones, keeping the property $e(P) = b(P)$.

Definition Let P_1, P_2 posets. $P_1 \oplus P_2$ is the poset obtained by P_1 and P_2 adding the relations $a < b$ for all $a \in P_1, b \in P_2$.

Assume that P_1 has a greatest element and P_2 has a least element. $P_1 \otimes P_2$ is the poset obtained by identifying the greatest element of P_1 with the least element of P_2 .

Lemma 4.3 $e(P_1 \oplus P_2) \geq e(P_1) + e(P_2) + 1$. If $P_1 \otimes P_2$ is defined, then $e(P_1 \otimes P_2) \geq e(P_1) + e(P_2)$.

Proof In order to find a P_1 , we need at least $e(P_1) + 1$ levels, for a P_2 , we need at least $e(P_2) + 1$ levels. It follows from the properties of $\oplus$ that the lowest level of P_2 is above the highest level of P_1 in any occurrence of $P_1 \oplus P_2$, which thus needs at least $e(P_1) + 1 + e(P_2) + 1$ levels. In the case of $P_1 \otimes P_2$, the same reasoning applies, noting that highest level of P_1 and the lowest level of P_2 coincide. ■

Lemma 4.4 Assume that P_1 and P_2 are finite posets such that $e(P_1) = b(P_1)$ and $e(P_2) = b(P_2)$. Then

$$e(P_1 \oplus P_2) = b(P_1 \oplus P_2) \quad (20)$$

Assume that P_1 has a greatest element and P_2 has a least element. Then

$$e(P_1 \otimes P_2) = b(P_1 \otimes P_2) \quad (21)$$

Proof Note that $|P_1 \oplus P_2| = |P_1| + |P_2|$, $L(P_1 \oplus P_2) = L(P_1) + L(P_2)$, and $e(P_1 \oplus P_2) \geq e(P_1) + e(P_2) + 1$. Similarly, $|P_1 \otimes P_2| = |P_1| + |P_2| - 1$, $L(P_1 \otimes P_2) = L(P_1) + L(P_2) - 1$, and $e(P_1 \otimes P_2) \geq e(P_1) + e(P_2)$.

From the above equations and (18) we have

$$e(P_1 \oplus P_2) \geq e(P_1) + e(P_2) + 1 = b(P_1) + b(P_2) + 1 = b(P_1 \oplus P_2) \quad (22)$$

and

$$e(P_1 \otimes P_2) \geq e(P_1) + e(P_2) = b(P_1) + b(P_2) = b(P_1 \otimes P_2) \quad (23)$$

if P_1 has a greatest element and P_2 has a least element. We have already seen that $e(P) \leq b(P)$ always holds. ■

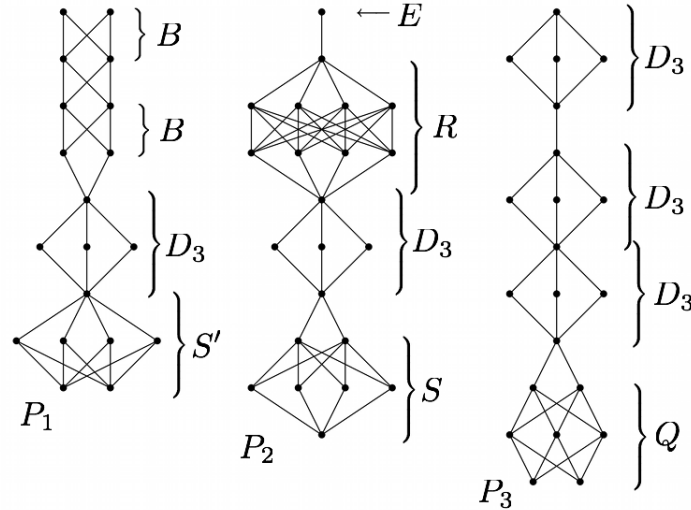


Figure 5: Posets built from E, B, D_3, Q, R, S and S' using $\oplus$ and $\otimes$. $P_1 = S' \otimes D_3 \oplus B \oplus B$, $P_2 = S \oplus D_3 \otimes R \oplus E$ and $P_3 = Q \oplus D_3 \otimes D_3 \oplus D_3$.

The following theorem summarizes our results.

Theorem 4.5 *Let P be a finite poset built from the posets E, B, D_3, Q, R, S and S' using the operations $\oplus$ and $\otimes$. (See figure 5 for examples.) For $n \geq b(P) + 1$*

$$La(n, P) = \Sigma(n, b(P)) = \Sigma(n, e(P)) \quad (24)$$

Proof From Lemma 4.2 and Lemma 4.4 we have $e(P) = b(P)$. Then Lemma 4.1 proves the theorem. ■

Remark Theorem 4.5 is the generalization of the theorem of Erdős (Theorem 1.2), and the following two results.

Theorem 4.6 (De Bonis, Katona, Swanepoel) [7] *for $n \geq 3$*

$$La(n, B) = \Sigma(n, 2) \quad (25)$$

Theorem 4.7 (Griggs, Li, Lu) (Special case of Theorem 2.5 in [8]) *for $n \geq 2$*

$$La(n, D_3) = \Sigma(n, 3) \quad (26)$$

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