

TRANSITION BETWEEN THE LAGRANGE AND HERMITE–FEJÉR INTERPOLATIONS. I (TRIGONOMETRIC CASE)

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Abstract. We construct a special class of discrete φ -summations to describe the transition between the well known (trigonometric) Lagrange and Hermite–Fejér interpolations. We give general properties, operator norm and convergence order for these cases.

1. Introduction

In one of his papers, Bernstein [1] gives the following class of trigonometric interpolation functions on equidistant point systems with an odd number of nodes, which can be used to describe the transition between the well known (trigonometric) Lagrange and Hermite–Fejér types of interpolation:

$$\frac{1}{(2m+1)(2h+1)} \sum_{k=0}^{2m} \frac{\sin \frac{2m+1}{2}(x-x_k) \sin \frac{2h+1}{2}(x-x_k)}{\sin^2 \frac{x-x_k}{2}} f(x_k),$$

where $2m+1$ is the number of nodes, $0 \leq h \leq m$ and $x_k = \frac{2k\pi}{2m+1}$.

In this paper we study generalizations of these functions using the technique of φ -summation. After some general properties, the precise operator norm and uniform convergence (with order) are investigated.

2. Preliminaries

2.1. Summation of discrete Fourier series. Let $C_{2\pi}$ denote the linear space of complex valued 2π -periodic continuous functions defined on

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$\mathbb{R} = (-\infty, \infty)$. With the maximum norm

$$(2.1) \quad \|f\|_{\infty} := \max_{x \in \mathbb{R}} |f(x)| \quad (f \in C_{2\pi})$$

the $(C_{2\pi}, \|\cdot\|_{\infty})$ space is a Banach space.

Let us fix a natural number $M \in \mathbb{N} = \{1, 2, \dots\}$ and consider the equidistant point system

$$(2.2) \quad X_M := \left\{ x_{k,M} := k \frac{2\pi}{M} : k = 0, 1, \dots, M-1 \right\}.$$

Let $\mathbb{Z}$ denote the set of integers and define the discrete trigonometric Fourier coefficients of $f \in C_{2\pi}$ with respect to the point system (2.2) by

$$(2.3) \quad \hat{f}_M(j) := \frac{1}{M} \sum_{k=0}^{M-1} f(x_{k,M}) e^{-ijx_{k,M}} \quad (j \in \mathbb{Z}).$$

The n -th discrete Fourier partial sum of the function $f \in C_{2\pi}$ corresponding to the point system X_M can be defined as

$$(2.4) \quad (S_{n,M}f)(x) := \sum_{j=-n}^n \hat{f}_M(j) e^{ijx}.$$

Now we define summation of discrete Fourier series. Fix the real even function φ and the number $M \in \mathbb{N}$. The n -th discrete φ -sums with respect to the point system (2.2) of the function $f \in C_{2\pi}$ are defined by

$$(2.5) \quad (S_{n,M}^{\varphi}f)(x) := \sum_{j=-n}^n \varphi\left(\frac{j}{n}\right) \hat{f}_M(j) e^{ijx} \quad (x \in \mathbb{R}, f \in C_{2\pi}, n \in \mathbb{N}).$$

Thus for every function φ we have a two-parameter operator family.

Since φ is an even function thus for every $f \in C_{2\pi}$ and all $n, M \in \mathbb{N}$ we have

$$(2.6) \quad (S_{n,M}^{\varphi}f)(x) = \frac{2}{M} \sum_{k=0}^{M-1} f(x_{k,M}) D_n^{\varphi}(x - x_{k,M}),$$

where

$$(2.7) \quad D_n^{\varphi}(t) := \frac{1}{2} + \sum_{j=1}^n \varphi\left(\frac{j}{n}\right) \cos jt \quad (t \in \mathbb{R}).$$

Let $\mathcal{T}_n$ ($n \in \mathbb{N}$) be the linear space of all complex valued trigonometric polynomials of degree not exceeding n :

$$\mathcal{T}_n := \text{span} \{e^{ijx} : -n \leq j \leq n\}.$$

Then

$$S_{n,M}^\varphi : (C_{2\pi}, \|\cdot\|_\infty) \rightarrow (\mathcal{T}_n, \|\cdot\|_\infty)$$

is a bounded linear operator for all fixed $n, M \in \mathbb{N}$.

Conditions of uniform convergence and other properties in this general case were investigated in [5,6], while a summary on many discrete linear interpolatory operators was given in [3].

2.2. Definition of φ_α functions. Our aim is to investigate interpolation methods which form some kind of transition between the (trigonometric) Lagrange and Hermite–Fejér interpolations with a special class of summation functions.

DEFINITION 2.1. For $\alpha \in [0, 1)$ let φ_α be the continuous function which is 1 on the interval $[-\alpha, \alpha]$, 0 on $\mathbb{R} \setminus [-1, 1]$ and is linear on $[-1, -\alpha]$ and $[\alpha, 1]$, i.e. on $x \in [0, 1]$

$$\varphi_\alpha(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq \alpha \\ \frac{1-x}{1-\alpha}, & \text{if } \alpha < x \leq 1. \end{cases}$$

The function φ_α (see Fig. 1) is called the *Fejér summation function* if $\alpha = 0$, and a *de la Vallée Poussin-type summation function* if $0 < \alpha < 1$.

In the following, we investigate properties of the operators $S_{n,M}^{\varphi_\alpha}$. Similar types of operators were investigated (among others) in [2].

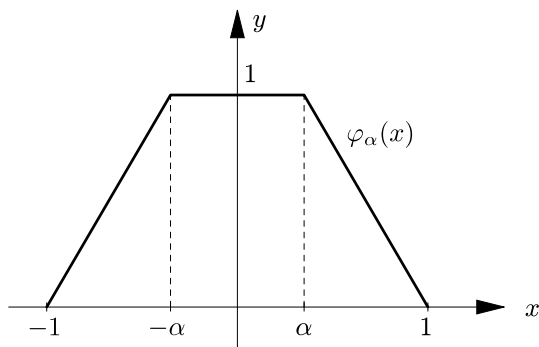


Fig. 1

Note that the term $j = n$ of the sum (2.7) is zero for every φ_α -summation, so the degree of the polynomial $S_{n,M}^{\varphi_\alpha} f$ ($f \in C_{2\pi}$) is at most $n - 1$: $\deg S_{n,M}^{\varphi_\alpha} f \leq n - 1$.

3. Results

The operator $S_{n,M}^{\varphi_\alpha}$ has the following projection property.

THEOREM 3.1. *Let $\alpha \in [0, 1)$ be an arbitrary number and suppose that $n \leq M$. If*

$$0 \leq s \leq \min \{n\alpha, M - n\},$$

then for every trigonometric polynomial $T \in \mathcal{T}_s$ the identity

$$(S_{n,M}^{\varphi_\alpha} T)(x) = T(x) \quad (x \in \mathbb{R})$$

holds.

We will give a necessary and sufficient condition for $S_{n,M}^{\varphi_\alpha} f$ to interpolate the function f .

THEOREM 3.2. *Let $\alpha \in [0, 1)$. $S_{n,M}^{\varphi_\alpha} f$ interpolates $f \in C_{2\pi}$ on the point system X_M (i.e. $(S_{n,M}^{\varphi_\alpha} f)(x_{k,M}) = f(x_{k,M})$, $k = 0, 1, \dots, M - 1$) if and only if one of these conditions holds:*

- (a) $\alpha = \frac{M}{n} - 1$;
- (b) $M = 2n - 1$ and $\alpha > 1 - \frac{1}{n}$.

REMARK 3.1. For arbitrary parameters M , n , α_1 satisfying condition (b), there exists $\alpha_2 \in [0, 1)$ such that

$$S_{n,M}^{\varphi_{\alpha_1}} f = S_{n,M}^{\varphi_{\alpha_2}} f \quad (f \in C_{2\pi})$$

and condition (a) holds for the parameters M , n , α_2 .

Indeed, suppose that $M = 2n - 1$ and $\alpha_1 > 1 - \frac{1}{n}$, and let $\alpha_2 = \frac{M}{n} - 1 = 1 - \frac{1}{n}$. It is clear that condition (a) holds for the values M , n , α_2 , and one can easily see that in (2.5) we have

$$\varphi_{\alpha_1} \left(\frac{j}{n} \right) = \varphi_{\alpha_2} \left(\frac{j}{n} \right) = \begin{cases} 1, & \text{if } j = -n + 1, -n + 2, \dots, n - 1, \\ 0, & \text{if } j = -n, n. \end{cases}$$

Consequently $S_{n,M}^{\varphi_{\alpha_1}} f = S_{n,M}^{\varphi_{\alpha_2}} f$ for every $f \in C_{2\pi}$.

Now we show how we can obtain the (trigonometric) Lagrange and Hermite-Fejér interpolation polynomials as special cases.

THEOREM 3.3. (a) *The (trigonometric) Lagrange interpolation, odd case: If*

$$M = 2m + 1, \quad n = m + 1 = \left\lceil \frac{M}{2} \right\rceil + 1 \quad \text{and} \quad \alpha = \frac{M - n}{n} = 1 - \frac{1}{n}$$

then $S_{n,M}^{\varphi_\alpha} f$ is the uniquely determined trigonometric polynomial of degree $\leq \left\lceil \frac{M}{2} \right\rceil = m = n - 1$ which interpolates the function $f \in C_{2\pi}$ at the points X_M .

(b) *The (trigonometric) Lagrange interpolation, even case: If*

$$M = 2m, \quad n = m + 1 = \frac{M}{2} + 1 \quad \text{and} \quad \alpha = \frac{M - n}{n} = 1 - \frac{2}{n}$$

then $S_{n,M}^{\varphi_\alpha} f$ is a (not uniquely determined) trigonometric polynomial of degree $\leq \frac{M}{2} = m = n - 1$ which interpolates the function $f \in C_{2\pi}$ at the points X_M .

(c) *The (trigonometric) Hermite–Fejér interpolation: Fix an arbitrary natural number M . Let $n = M$, $\alpha = 0$ and $f \in C_{2\pi}$. Then $S_{n,M}^{\varphi_\alpha} f$ is the uniquely determined trigonometric polynomial of degree $\leq (n - 1)$ which satisfies the following Hermite–Fejér interpolation properties:*

$$(S_{n,M}^{\varphi_\alpha} f)(x_{k,M}) = f(x_{k,M}), \quad (S_{n,M}^{\varphi_\alpha} f)'(x_{k,M}) = 0 \quad (k = 0, 1, \dots, M - 1).$$

Next, we give a two-sided estimation of the operator norm

$$\begin{aligned} (3.1) \quad \|S_{n,M}^{\varphi_\alpha}\| &= \sup \{ \|S_{n,M}^{\varphi_\alpha} f\|_\infty : \|f\|_\infty = 1 \} \\ &= \max_{x \in [0, 2\pi)} \left\{ \frac{2}{M} \sum_{k=0}^{M-1} |D_n^{\varphi_\alpha}(x - x_{k,M})| \right\}. \end{aligned}$$

We will prove the following theorem concerning the norm of the operator $S_{n,M}^{\varphi_\alpha}$.

THEOREM 3.4. *To any interpolatory operator $S_{n,M}^{\varphi_\alpha}$ there exist positive constants c_1, c_2 independent of n and M such that the inequalities*

$$\frac{2}{\pi} \log N + c_1 \leq \|S_{n,M}^{\varphi_\alpha}\| \leq \frac{2}{\pi} \log N + c_2$$

hold for every n, M , where $N = \frac{M}{2n-M}$.

Let us denote the error of the n -th degree best approximation of $f \in C_{2\pi}$ by

$$E_n(f) := \inf_{T \in \mathcal{T}_n} \|f - T\|_\infty.$$

Choose the index sequences $(n_k, k \in \mathbb{N})$ and $(M_k, k \in \mathbb{N})$ arbitrarily. We shall investigate the convergence of the operator sequence

$$(3.2) \quad S_{n_k, M_k}^{\varphi_\alpha} : (C_{2\pi}, \|\cdot\|_\infty) \rightarrow (C_{2\pi}, \|\cdot\|_\infty) \quad (k \in \mathbb{N}).$$

Using the estimations above, we have (cf. [1, p. 150]) the following result.

THEOREM 3.5. *Let $f \in C_{2\pi}$. Consider the trigonometric polynomial sequence $S_{n_k, M_k}^{\varphi_{\alpha_k}} f$ ($k \in \mathbb{N}$). Suppose that*

- (i) $M_k \rightarrow +\infty$ ($k \rightarrow +\infty$), and let
- (ii) $\alpha_k = \frac{M_k - n_k}{n_k}$, $N_k = \frac{M_k}{2n_k - M_k}$ ($k \in \mathbb{N}$).

Then $S_{n_k, M_k}^{\varphi_{\alpha_k}} f$ interpolates f at the points X_{M_k} for every $k \in \mathbb{N}$, and

$$\|S_{n_k, M_k}^{\varphi_{\alpha_k}} f - f\|_\infty \leq \left\{ \frac{2}{\pi} \log N_k + O(1) \right\} \cdot E_{M_k - n_k}(f) \quad (f \in C_{2\pi}).$$

COROLLARY 3.1. *Let $f \in C_{2\pi}$. Consider the polynomial sequence $S_{n_k, M_k}^{\varphi_{\alpha_k}} f$ ($k \in \mathbb{N}$). Suppose that conditions (i) and (ii) of Theorem 3.5 hold, and*

(iii) the sequence

$$N_k = \frac{M_k}{2n_k - M_k} \quad (k \in \mathbb{N})$$

is bounded.

Then the sequence $S_{n_k, M_k}^{\varphi_{\alpha_k}} f$ ($k \in \mathbb{N}$) converges uniformly to f .

By Theorem 3.5 the order of convergence is near the best approximation.

For a sequence of the (trigonometric) Lagrange interpolation polynomials, if $M_k \rightarrow +\infty$ ($k \rightarrow +\infty$) then the sequence

$$N_k = \frac{M_k}{2n_k - M_k} = \begin{cases} M_k, & \text{if } M_k \text{ is odd} \\ \frac{M_k}{2}, & \text{if } M_k \text{ is even} \end{cases}$$

is not bounded so we do not have uniform convergence.

A sequence of the (trigonometric) Hermite-Fejér polynomials satisfies conditions (i)–(iii) and therefore uniformly converges for any $f \in C_{2\pi}$. Moreover for any fixed $0 \leq \alpha < 1$ if conditions (i) and (ii) hold for a sequence $S_{n_k, M_k}^{\varphi_\alpha} f$ then

$$N_k = \frac{M_k}{2n_k - M_k} = \frac{1 + \alpha}{1 - \alpha}$$

is constant (i.e. bounded), so $S_{n_k, M_k}^{\varphi_\alpha} f$ uniformly converges to f .

4. Proofs

4.1. Proof of Theorem 3.1. Let us consider an arbitrary polynomial

$$T(x) = \sum_{l=-s}^s c_l e^{ilx} \quad (x \in \mathbb{R}).$$

Then

$$(S_{n,M}^{\varphi_\alpha} T)(x) = \sum_{j=-n}^n \varphi_\alpha\left(\frac{j}{n}\right) \hat{T}_M(j) e^{ijx},$$

where

$$\hat{T}_M(j) = \frac{1}{M} \sum_{l=-s}^s c_l \sum_{k=0}^{M-1} e^{ilx_{k,M}} e^{-ijx_{k,M}}.$$

We have $s \leq M - n$ and $s \leq n\alpha < n$, which means that

$$s - (-s) < M - n + n = M,$$

so in $\hat{T}_M(j)$ the inner sum is 1 if $l = j$ and 0 otherwise. Hence

$$(S_{n,M}^{\varphi_\alpha} T)(x) = \sum_{j=-s}^s \varphi_\alpha\left(\frac{j}{n}\right) \cdot c_j e^{ijx}.$$

Since $s \leq n\alpha \Leftrightarrow \frac{s}{n} \leq \alpha$ then $\varphi_\alpha\left(\frac{j}{n}\right) = 1$ for every $-s \leq j \leq s$. $\square$

4.2. Proof of Theorem 3.2. First we recall the following statement (cf. [5, Lemma A, p. 137]):

LEMMA 4.1. *Suppose that $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is an even function supported in $[-1, 1]$, $\varphi(0) = 1$ and $M \geq n$. The polynomial $S_{n,M}^\varphi f$ interpolates the function f at the points X_M if and only if*

$$(4.1) \quad \varphi\left(\frac{j}{n}\right) + \varphi\left(\frac{M-j}{n}\right) = 1 \quad (j = 1, 2, \dots, M-1).$$

This lemma visually states that for $j = 1, 2, \dots, M-1$ the points $\left(\frac{j}{n}, \varphi\left(\frac{j}{n}\right)\right)$ from the graph of φ are positioned symmetrically to the point $\left(\frac{M}{2n}, \frac{1}{2}\right)$ on the interval $\left[0, \frac{M}{n}\right]$. This property is demonstrated in Fig. 2.

Starting the proof of the theorem, first, let us assume (a). In this case it is enough to show that the graph of φ_α on the interval $\left(0, \frac{M}{n}\right)$ is symmetrical

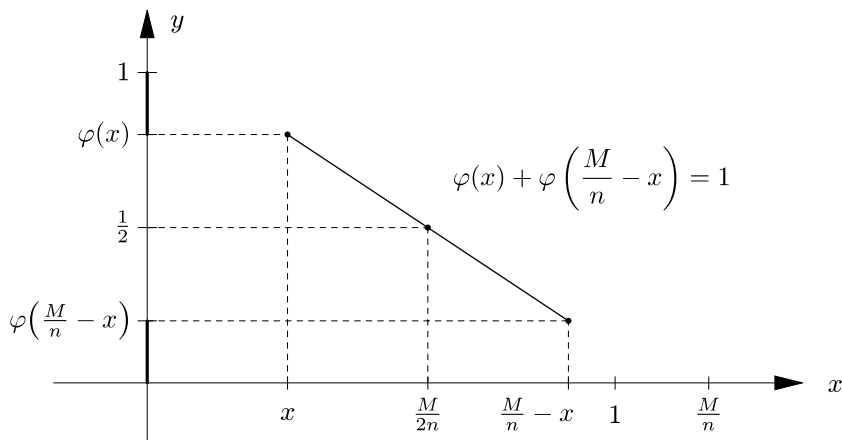


Fig. 2

to the point $(\frac{M}{2n}, \frac{1}{2})$. Namely in this case

$$\varphi_{\alpha}(x) + \varphi_{\alpha}\left(\frac{M}{n} - x\right) = 1$$

holds and considering Lemma 4.1 the polynomial $S_{n,M}^{\varphi_{\alpha}} f$ interpolates f .

The graph of φ_{α} is symmetrical to the point $(\frac{M}{2n}, \frac{1}{2})$ because the center points of the intervals $(0, \frac{M}{n})$ and $(\alpha, 1)$ are the same, and

$$\varphi_{\alpha}\left(\frac{\alpha+1}{2}\right) = \frac{1 - \frac{\alpha+1}{2}}{1 - \alpha} = \frac{1}{2}.$$

Now let us assume that condition (b) holds. From $\alpha \geq 1 - \frac{1}{n}$ we have that $\varphi_{\alpha}(\frac{n-1}{n}) = 1$, and also we can see that

$$\varphi_{\alpha}\left(\frac{M - (n-1)}{n}\right) = \varphi_{\alpha}\left(\frac{(2n-1) - (n-1)}{n}\right) = \varphi_{\alpha}(1) = 0.$$

This means that (4.1) holds for $j = n-1$ and $j = n$, as well. It also implies that for every $j < n-1$ we have $\varphi_{\alpha}(\frac{j}{n}) = 1$, and $\varphi_{\alpha}(\frac{M-j}{n}) = 0$, so (4.1) holds for every $j \leq n-1$. From this we get that it is true for every $j \geq n$, and by using Lemma 4.1 we proved that $S_{n,M}^{\varphi_{\alpha}} f$ interpolates f .

Conversely let us assume that $S_{n,M}^{\varphi_{\alpha}} f$ interpolates $f \in C_{2\pi}$, that means condition (4.1) holds. We consider two cases:

(i) If $\alpha < 1 - \frac{1}{n}$, then we have $\frac{n-1}{n} \in (\alpha, 1)$. Using (4.1) for $j = n$, and considering $\varphi_{\alpha}(1) = 0$, we get $\varphi_{\alpha}(\frac{M-n}{n}) = 1$, whence $\frac{M-n}{n} \leq \alpha$.

Now using (4.1) for $j = n - 1$ we can write

$$\begin{aligned}\varphi_\alpha\left(\frac{M-n}{n}\right) - \varphi_\alpha\left(\frac{M-n+1}{n}\right) &= 1 - \varphi_\alpha\left(\frac{M-n+1}{n}\right) \\ &= \varphi_\alpha\left(\frac{n-1}{n}\right) = \varphi_\alpha\left(\frac{n-1}{n}\right) - \varphi_\alpha(1),\end{aligned}$$

and considering that φ_α is linear on $[\alpha, 1]$, this can only happen if $\alpha = \frac{M-n}{n}$, that means condition (a) holds.

(ii) If $\frac{n-1}{n} \leq \alpha$, then we only have to show that $M = 2n - 1$. Consider (4.1) for $j = n - 1$. We get

$$\varphi_\alpha\left(\frac{M-(n-1)}{n}\right) = 0,$$

because in this case $\varphi_\alpha\left(\frac{n-1}{n}\right) = 1$. It can only happen if $\frac{M-(n-1)}{n} \geq 1$, so $M \geq 2n - 1$.

Observe that we have

$$\varphi_\alpha\left(\frac{M-n}{n}\right) = 1$$

as well, because $\varphi_\alpha\left(\frac{n}{n}\right) = 0$. This means that $\frac{M-n}{n} < \frac{n}{n}$, so $M < 2n$. Putting these together we have $M = 2n - 1$. $\square$

4.3. Proof of Theorem 3.3. (a) The degree of the polynomial $S_{n,M}^{\varphi_\alpha} f$ in this case is $n - 1 = \left[\frac{M}{2}\right]$, and it interpolates f at the points X_M because condition (b) of Theorem 3.2 holds.

(b) The degree of the polynomial $S_{n,M}^{\varphi_\alpha} f$ in this case is $n - 1 = \frac{M}{2} = \left[\frac{M}{2}\right]$, because M is an even number. Also we have

$$\alpha = \frac{n-2}{n} = \frac{2(n-1)-n}{n} = \frac{M-n}{n},$$

so condition (a) of Theorem 3.2 holds, meaning that $S_{n,M}^{\varphi_\alpha} f$ interpolates f .

(c) It was shown (cf. [5, Theorem 1, p. 142]) that $S_{n,M}^{\varphi} f$ is the Hermite–Fejér interpolation polynomial if φ is the Fejér summation function. In our case φ_α is the Fejér summation function if $\alpha = 0$. From Theorem 3.2 condition (b) cannot hold, so necessarily $\alpha = \frac{M-n}{n} = 0$, meaning $M = n$. $\square$

4.4. Proof of Theorem 3.4. $S_{n,M}^{\varphi_\alpha}$ has the interpolation property, so considering Remark 3.1 we can assume that

$$(4.2) \quad \alpha = \frac{M}{n} - 1 = \frac{M-n}{n}$$

and $\lceil \frac{M}{2} \rceil < n \leq M$.

Let $H := 2n - M$. Observe that M and H have the same parity and $1 \leq H \leq M$.

Let $f \in C_{2\pi}$. First we give two different forms of the polynomial

$$\begin{aligned} (S_{n,M}^{\varphi_\alpha} f)(x) &= \frac{2}{M} \sum_{k=0}^{M-1} f(x_{k,M}) D_n^{\varphi_\alpha}(x - x_{k,M}) \\ &= \frac{2}{M} \sum_{k=0}^{M-1} f(x_{k,M}) \left\{ \frac{1}{2} + \sum_{j=1}^n \varphi_\alpha \left(\frac{j}{n} \right) \cos j(x - x_{k,M}) \right\}. \end{aligned}$$

LEMMA 4.2. *The polynomial $S_{n,M}^{\varphi_\alpha} f$ has the form*

$$(S_{n,M}^{\varphi_\alpha} f)(x) = \frac{1}{MH} \sum_{k=0}^{M-1} \frac{\sin \frac{M}{2}(x - x_{k,M}) \sin \frac{H}{2}(x - x_{k,M})}{\sin^2 \frac{x - x_{k,M}}{2}} f(x_{k,M}).$$

PROOF. Via induction one can easily prove that

$$(4.3) \quad \frac{\sin hx}{\sin x} = \begin{cases} 2 \cos x + 2 \cos 3x + \dots + 2 \cos (h-1)x, & h \text{ even,} \\ 1 + 2 \cos 2x + \dots + 2 \cos (h-1)x, & h \text{ odd.} \end{cases}$$

Considering (2.6) we have to show that

$$(4.4) \quad \frac{\sin \frac{M}{2}(x - x_{k,M}) \sin \frac{H}{2}(x - x_{k,M})}{2H \sin^2 \frac{x - x_{k,M}}{2}} = D_n^{\varphi_\alpha}(x - x_{k,M}).$$

Since

$$\alpha = \frac{M-n}{n} = \frac{\frac{M-(2n-M)}{2}}{n} = \frac{\frac{M-H}{2}}{n}$$

and

$$\frac{1 - \frac{j}{n}}{1 - \alpha} = \frac{1 - \frac{j}{n}}{1 - \left(\frac{M-H}{2n}\right)} = 1 - \frac{j - \frac{M-H}{2}}{H},$$

thus the used values of φ_α are

$$(4.5) \quad \varphi_\alpha \left(\frac{j}{n} \right) = \begin{cases} 1, & j = 1, 2, \dots, \frac{M-H}{2}; \\ 1 - \frac{j - \frac{M-H}{2}}{H}, & j = \frac{M-H}{2} + 1, \dots, n. \end{cases}$$

First, let us assume that $M = 2m + 1$ and $H = 2h + 1$, so $n = m + h + 1$. The following identity was showed by S. N. Bernstein in [1, p. 147]:

$$(4.6) \quad \frac{1}{2(2h+1)} \frac{\sin \frac{2m+1}{2}x \sin \frac{2h+1}{2}x}{\sin^2 \frac{x}{2}} = \frac{1}{2(2h+1)} \left[(2h+1) \right. \\ \left. + 2(2h+1) \sum_{j=1}^{m-h} \cos jx + 2 \sum_{j=m-h+1}^{m+h} (m+h+1-j) \cos jx \right].$$

Observe that (4.6) completes the proof of this case as the coefficients of cosines are the needed $\varphi_\alpha\left(\frac{j}{m+h+1}\right)$ values.

Now consider the case $M = 2m$ and $H = 2h$ (cf. [1, p. 151]). This means $n = m + h$. By (2.6) now we have to show that

$$(4.7) \quad D_n^{\varphi_\alpha}(x) = \frac{1}{2} + \sum_{j=1}^n \varphi_\alpha\left(\frac{j}{n}\right) \cos jx = \frac{\sin mx \sin hx}{4h \sin^2 \frac{x}{2}} = \frac{1}{h} \cdot \frac{\sin mx}{2 \tan \frac{x}{2}} \cdot \frac{\sin hx}{\sin x},$$

where the middle fraction is (cf. [7, p. 50])

$$\frac{\sin mx}{2 \tan \frac{x}{2}} = \frac{1}{2} + \sum_{j=1}^m \cos jx - \frac{1}{2} \cos mx.$$

For the last fraction we shall use (4.3). We consider two cases:

(i) If $h = 2l + 1$ then the right side of (4.7) becomes

$$\frac{1}{2l+1} \cdot \left(\frac{1}{2} + \sum_{j=1}^m \cos jx - \frac{1}{2} \cos mx \right) \cdot \left(1 + 2 \sum_{j=1}^l \cos 2jx \right).$$

We multiply the second and third terms and use the trigonometric identity for $\cos px \cdot \cos qx$ ($p, q \in \mathbb{N}$). The result becomes

$$\frac{1}{2l+1} \left[\frac{2l+1}{2} + (2l+1) \sum_{j=1}^{m-2l-1} \cos jx + \sum_{j=m-2l}^{m+2l} \frac{m+2l+1-j}{2} \cos jx \right].$$

Here if we write $l = \frac{h-1}{2}$, we get the wanted $\varphi_\alpha\left(\frac{j}{m+h}\right)$ coefficients for the cosines.

(ii) If $h = 2l$ then the right side of (4.7) becomes

$$\frac{1}{2l} \cdot \left(\frac{1}{2} + \sum_{j=1}^m \cos jx - \frac{1}{2} \cos mx \right) \cdot \left(2 \sum_{j=1}^l \cos (2j-1)x \right).$$

Again, we multiply the second and third terms and use the trigonometric identity. The result becomes

$$\frac{1}{2l} \left[\frac{2l}{2} + 2l \sum_{j=1}^{m-2l} \cos jx + \sum_{j=m-2l+1}^{m+2l-1} \frac{m+2l-j}{2} \cos jx \right].$$

Here if we write $l = \frac{h}{2}$, we get the wanted $\varphi_\alpha\left(\frac{j}{m+h}\right)$ coefficients for the cosines.

Cases (i) and (ii) together yield (4.7), so the proof of the lemma is complete. $\square$

Also from the form (2.6) by using the trigonometric identity

$$\cos j(x - x_{k,M}) = \cos jx \cos jx_{k,M} + \sin jx \sin jx_{k,M},$$

one can get the following simple form of $S_{n,M}^{\varphi_\alpha} f$ (cf. [1, p. 147]).

LEMMA 4.3.

$$(4.8) \quad (S_{n,M}^{\varphi_\alpha} f)(x) = \frac{A_0}{2} + \sum_{j=1}^n A_j \cos jx + B_j \sin jx,$$

where

$$A_j = \varphi_\alpha\left(\frac{j}{n}\right) \cdot \frac{2}{M} \sum_{k=0}^{M-1} f(x_{k,M}) \cos jx_{k,M}, \quad (j \leq n);$$

$$B_j = \varphi_\alpha\left(\frac{j}{n}\right) \cdot \frac{2}{M} \sum_{k=0}^{M-1} f(x_{k,M}) \sin jx_{k,M}, \quad (j \leq n).$$

With these lemmas at hand, we can show the following identity (cf. [1, p. 148]).

LEMMA 4.4. *If $M, H \in \mathbb{N}$, $1 \leq H \leq M$ and X_M is the point system defined by (2.2) then*

$$\frac{1}{MH} \sum_{k=0}^{M-1} \frac{\sin^2 \frac{H}{2}(x - x_{k,M})}{\sin^2 \frac{x - x_{k,M}}{2}} = 1 \quad (x \in \mathbb{R}).$$

PROOF. From (4.4) and (4.5) with $M = H$ we obtain that

$$\frac{1}{MH} \sum_{k=0}^{M-1} \frac{\sin^2 \frac{H}{2}(x - x_{k,M})}{\sin^2 \frac{x - x_{k,M}}{2}} = \frac{2}{M} \sum_{k=0}^{M-1} \left[\frac{1}{2} + \sum_{j=1}^{H-1} \frac{H-j}{H} \cos j(x - x_{k,M}) \right].$$

By changing the order of sums this equals

$$(4.9) \quad \frac{2}{M} \left[\left(\sum_{k=0}^{M-1} \frac{1}{2} \right) + \sum_{j=1}^{H-1} \left(\frac{H-j}{H} \cdot \sum_{k=0}^{M-1} \cos j(x - x_{k,M}) \right) \right].$$

Since X_M is an equidistant point system, one can see that the following equation holds:

$$\sum_{k=0}^{M-1} \cos j(x - x_{k,M}) = 0 \quad (x \in \mathbb{R}, j = 1, 2, \dots, M-1).$$

Using this in (4.9) we get our statement. $\square$

Now we prove the following statement regarding the operator norm (cf. [1, pp. 148–150]).

LEMMA 4.5. *Consider the interpolatory operator $S_{n,M}^{\varphi_\alpha}$ and $N = \frac{M}{2n-M}$.*

(a) *If $N \in \mathbb{N}$ then the norm of the operator is*

$$\|S_{n,M}^{\varphi_\alpha}\| = \frac{1}{N} \sum_{\varrho=0}^{N-1} \frac{1}{\sin \frac{1+2\varrho}{2N}\pi} = \frac{1}{N} \left[\frac{1}{\sin \frac{1}{2N}\pi} + \frac{1}{\sin \frac{3}{2N}\pi} + \dots + \frac{1}{\sin \frac{2N-1}{2N}\pi} \right].$$

(b) *If N is not a natural number then the norm can be estimated as*

$$\frac{1}{[N]} \sum_{\varrho=0}^{[N]-1} \frac{1}{\sin \frac{1+2\varrho}{2[N]}\pi} \leq \|S_{n,M}^{\varphi_\alpha}\| \leq \frac{1}{[N+1]} \sum_{\varrho=0}^{[N]} \frac{1}{\sin \frac{1+2\varrho}{2[N+1]}\pi}.$$

PROOF. First let us assume that $N := \frac{M}{H} \in \mathbb{N}$. Using this, the index k in (3.1) can be expressed as $k = \lambda N + \varrho$, where $\lambda = 0, 1, \dots, H-1$ and $\varrho = 0, 1, \dots, N-1$. For the norm of operator $S_{n,M}^{\varphi_\alpha}$ we get

$$\begin{aligned} & \|S_{n,M}^{\varphi_\alpha}\| \\ &= \max_{x \in [0, 2\pi)} \left\{ \frac{1}{MH} \sum_{\varrho=0}^{N-1} \sum_{\lambda=0}^{H-1} \left| \frac{\sin \frac{M}{2} \left(x + \frac{2\varrho\pi}{M} + \frac{2\lambda\pi}{H} \right) \cdot \sin \frac{H}{2} \left(x + \frac{2\varrho\pi}{M} + \frac{2\lambda\pi}{H} \right)}{\sin^2 \frac{1}{2} \left(x + \frac{2\varrho\pi}{M} + \frac{2\lambda\pi}{H} \right)} \right| \right\}. \end{aligned}$$

Let

$$A(x) := \frac{1}{MH} \sum_{\varrho=0}^{N-1} \sum_{\lambda=0}^{H-1} \left| \frac{\sin \frac{M}{2} \left(x + \frac{2\varrho\pi}{M} + \frac{2\lambda\pi}{H} \right) \cdot \sin \frac{H}{2} \left(x + \frac{2\varrho\pi}{M} + \frac{2\lambda\pi}{H} \right)}{\sin^2 \frac{1}{2} \left(x + \frac{2\varrho\pi}{M} + \frac{2\lambda\pi}{H} \right)} \right|.$$

Then, we have

$$\begin{aligned} & \left| \sin \frac{M}{2} \left(x + \frac{2\varrho\pi}{M} + \frac{2\lambda\pi}{H} \right) \right| = \left| \sin \left(\frac{M}{2} x + \varrho\pi + N\lambda\pi \right) \right| \\ &= \left| \sin \frac{M}{2} x \cdot \cos (\varrho + N\lambda)\pi + \cos \frac{M}{2} x \cdot \sin (\varrho + N\lambda)\pi \right| = \left| \sin \frac{M}{2} x \right|, \end{aligned}$$

so

$$A(x) = \frac{\left| \sin \frac{M}{2} x \right|}{MH} \sum_{\varrho=0}^{N-1} \sum_{\lambda=0}^{H-1} \frac{\left| \sin \frac{H}{2} \left(x + \frac{2\varrho\pi}{M} + \frac{2\lambda\pi}{H} \right) \right|}{\sin^2 \frac{1}{2} \left(x + \frac{2\varrho\pi}{M} + \frac{2\lambda\pi}{H} \right)}.$$

Additionally,

$$A(x) = \frac{\left| \sin \frac{M}{2} x \right|}{MH} \sum_{\varrho=0}^{N-1} \sum_{\lambda=0}^{H-1} \frac{1}{\left| \sin \frac{H}{2} \left(x + \frac{2\varrho\pi}{M} + \frac{2\lambda\pi}{H} \right) \right|} \cdot \frac{\sin^2 \frac{H}{2} \left(x + \frac{2\varrho\pi}{M} + \frac{2\lambda\pi}{H} \right)}{\sin^2 \frac{1}{2} \left(x + \frac{2\varrho\pi}{M} + \frac{2\lambda\pi}{H} \right)},$$

where

$$\begin{aligned} & \left| \sin \frac{H}{2} \left(x + \frac{2\varrho\pi}{M} + \frac{2\lambda\pi}{H} \right) \right| = \left| \sin \left(\frac{H}{2} \left(x + \frac{2\varrho\pi}{M} \right) + \lambda\pi \right) \right| \\ &= \left| \sin \frac{H}{2} \left(x + \frac{2\varrho\pi}{M} \right) \cdot \cos \lambda\pi + \cos \frac{H}{2} \left(x + \frac{2\varrho\pi}{M} \right) \cdot \sin \lambda\pi \right| \\ &= \left| \sin \frac{H}{2} \left(x + \frac{2\varrho\pi}{M} \right) \right|, \end{aligned}$$

implying that

$$A(x) = \frac{\left| \sin \frac{M}{2} x \right|}{MH} \sum_{\varrho=0}^{N-1} \frac{1}{\left| \sin \frac{H}{2} \left(x + \frac{2\varrho\pi}{M} \right) \right|} \sum_{\lambda=0}^{H-1} \frac{\sin^2 \frac{H}{2} \left(x + \frac{2\varrho\pi}{M} + \frac{2\lambda\pi}{H} \right)}{\sin^2 \frac{1}{2} \left(x + \frac{2\varrho\pi}{M} + \frac{2\lambda\pi}{H} \right)}.$$

Here, considering Lemma 4.4 in the case $M = H$, the inner sum becomes H^2 , thus

$$A(x) = \frac{H}{M} \left| \sin \frac{M}{2} x \right| \sum_{\varrho=0}^{N-1} \frac{1}{\left| \sin \frac{H}{2} \left(x + \frac{2\varrho\pi}{M} \right) \right|}.$$

Finally, with the substitution $x = \frac{y}{H}$, by $\frac{M}{H} = N$ we have

$$\max_{x \in [0, 2\pi)} A(x) = \max_{y \in [0, 2\pi H)} \left\{ \left| \frac{\sin \frac{N}{2} y}{N} \right| \sum_{\varrho=0}^{N-1} \frac{1}{\left| \sin \left(\frac{y}{2} + \frac{\varrho\pi}{N} \right) \right|} \right\}$$

$$= \max_{x \in [0, 2\pi)} \left\{ \left| \frac{\sin \frac{N}{2}x}{N} \right| \sum_{\varrho=0}^{N-1} \frac{1}{\left| \sin \left(\frac{x}{2} + \frac{\varrho\pi}{N} \right) \right|} \right\}.$$

Now let us have a closer look at the function

$$B_N(x) := \left| \frac{\sin \frac{N}{2}x}{N} \right| \sum_{\varrho=0}^{N-1} \frac{1}{\left| \sin \left(\frac{x}{2} + \frac{\varrho\pi}{N} \right) \right|}.$$

Using simple evaluation one can easily see that B_N is $\frac{2\pi}{N}$ -periodical and

$$B_N\left(\frac{2\pi}{N} - x\right) = B_N(x), \quad x \in \left[0, \frac{2\pi}{N}\right],$$

so B_N is even within each period.

Also observe that $B_N(x) \geq 1$ and $B_N(0) = 1$. Now if the function had a local maximal value on the interval $(0, \frac{\pi}{N})$, then due to its parity there would be at least 4 local extrema on the interval $[0, \frac{2\pi}{N})$, and by its periodicity there would be at least $4N$ extrema on the interval $[0, 2\pi)$, which is not possible. So the function takes its only maximal value on the interval $[0, \frac{2\pi}{N})$ at the point $x = \frac{\pi}{N}$, and due to periodicity

$$\|S_{n,M}^{\varphi_\alpha}\| = \max_{x \in [0, 2\pi)} B_N(x) = \frac{1}{N} \sum_{\varrho=0}^{N-1} \frac{1}{\sin \frac{1+2\varrho}{2N}\pi}.$$

Now suppose that N is not an integer. As above, for the norm of the operator $S_{n,M}^{\varphi_\alpha}$ we have

$$(4.10) \quad \|S_{n,M}^{\varphi_\alpha}\| = \max_{x \in [0, 2\pi)} \left\{ \frac{1}{MH} \sum_{k=0}^{M-1} \left| \frac{\sin \frac{M}{2}(x - x_{k,M}) \sin \frac{H}{2}(x - x_{k,M})}{\sin^2 \frac{x - x_{k,M}}{2}} \right| \right\}.$$

Here we can get the lower bound

$$\|S_{n,M}^{\varphi_\alpha}\| \geq \frac{1}{[N]} \sum_{\varrho=0}^{[N]-1} \frac{1}{\sin \frac{1+2\varrho}{2[N]}\pi}$$

by discarding some members of the sum and taking similar steps as in the previous case.

We can also get the upper bound

$$\|S_{n,M}^{\varphi_\alpha}\| \leq \frac{1}{[N+1]} \sum_{\varrho=0}^{[N]} \frac{1}{\sin \frac{1+2\varrho}{2[N+1]}\pi}$$

in a similar way. $\square$

To complete the proof of Theorem 3.4, let $N \in \mathbb{N}$ and

$$L := \frac{1}{N} \sum_{\varrho=0}^{N-1} \frac{1}{\sin \frac{1+2\varrho}{2N}\pi}.$$

Using that $L_N = \frac{2}{\pi} \log N + O(1)$ (cf. [1, (13)] or [4, p. 108]) we can get the estimation of the theorem. If N is a noninteger value, we obtain the statement in a similar way using part (b) of Lemma 4.5. $\square$

4.5. Proof of Theorem 3.5. Let $\rho_k := \|S_{n_k, M_k}^{\varphi_{\alpha_k}} f - f\|_{\infty}$. Denote by $Q_{M_k - n_k}$ the trigonometric polynomial of degree $M_k - n_k$ for which

$$\|f - Q_{M_k - n_k}\|_{\infty} = E_{M_k - n_k}(f).$$

Consider the inequality

$$(4.11) \quad \begin{aligned} & |(S_{n_k, M_k}^{\varphi_{\alpha_k}} f)(x) - f(x)| \leq \\ & \leq |(S_{n_k, M_k}^{\varphi_{\alpha_k}} f)(x) - Q_{M_k - n_k}(x)| + |Q_{M_k - n_k}(x) - f(x)|. \end{aligned}$$

Using Theorem 3.1 the interpolatory polynomial can be written in the following form:

$$(S_{n_k, M_k}^{\varphi_{\alpha_k}} f)(x) = Q_{M_k - n_k}(x) + (S_{n_k, M_k}^{\varphi_{\alpha_k}}(f - Q_{M_k - n_k}))(x).$$

Using Theorem 3.4 leads to the inequality

$$|(S_{n_k, M_k}^{\varphi_{\alpha_k}} f)(x) - Q_{M_k - n_k}(x)| \leq \left\{ \frac{2}{\pi} \log N + O(1) \right\} \cdot E_{M_k - n_k}(f),$$

which together with (4.11) gives

$$\begin{aligned} \rho_k & \leq \left\{ 1 + \frac{2}{\pi} \log N + O(1) \right\} \cdot E_{M_k - n_k}(f) \\ & = \left\{ \frac{2}{\pi} \log N + O(1) \right\} \cdot E_{M_k - n_k}(f). \quad \square \end{aligned}$$

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